A HYPERELLIPTIC CURVE MAPPING TO SPECIFIED ELLIPTIC CURVES

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Abstract. If an ordered 13-tuples $(E_1, \dots, E_{13})$ of elliptic curves over $\mathbb{C}$ is sufficiently general, there is no hyperelliptic curve which admits a non-trivial morphism to each of the E_i.

To Moshe Jarden in honor of his 80th birthday

1. Introduction

In this paper, we ask whether, given elliptic curves $E_1, \dots, E_n$ over $\mathbb{C}$, there exists a hyperelliptic curve admitting a non-constant morphism to each E_i.

Let $A := E_1 \times \cdots \times E_n$ and $G := \{\pm 1\} \subset \mathrm{Aut}(A)$. We denote by $\bar{A}$ the Kummer variety A/G. If X is a rational curve on $\bar{A}$, then the inverse image of X must be irreducible, since there are no rational curves on A. The normalization $\tilde{X}$ of the inverse image therefore admits a degree 2 morphism to $\mathbb{P}^1$; on the other hand, it also admits a morphism to each E_i coming from the projection $A \to E_i$. If X is *diagonal*, then these morphisms are non-trivial. To be precise:

Definition 1.1. *Let $A := E_1 \times \cdots \times E_n$ be a product of elliptic curves, and let X be a curve on $\bar{A} := A/\{\pm 1\}$, where the involution maps P to $-P$ for $P \in A$. The curve X is called* diagonal *if the projection $X \to \bar{E}_i$ is nonconstant for each i.*

Conversely, suppose we are given a hyperelliptic curve $\tilde{X}$ and non-constant morphisms $\phi_i \colon \tilde{X} \to E_i$. If w is the hyperelliptic involution on $\tilde{X}$, then all divisors of the form $[\tilde{x}] + [w(\tilde{x})]$ are linearly equivalent, so $\phi_i(\tilde{x}) + \phi_i(w(\tilde{x}))$ is constant in $\tilde{x}$. Therefore, translating each ϕ_i we may assume $\phi_i(\tilde{x}) = -\phi_i(w(\tilde{x}))$, so $(\phi_1, \dots, \phi_n)$ induces a morphism from the rational curve $X := \tilde{X}/\langle w \rangle$ to $\bar{A}$.

For $n = 3$, $\bar{A}$ always contains a diagonal rational curve. For E_1, E_2, E_3 elliptic curves in Legendre form over a ground field k, [Im13] gave an explicit construction of such a curve over k; by lifting a k-point on this rational curve, one obtains a point on $E_1 \times E_2 \times E_3$ defined over a quadratic extension of k. When $k = \mathbb{Q}$ this can be used to simultaneously twist all three curves by the same $d \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ so that all three twists are of positive rank. Recall that the twist of an elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$ by $d \in \mathbb{Q}^\times$ is the elliptic curve $E_d : dy^2 = x^3 + ax + b$, which is isomorphic to E over $\mathbb{C}$ but not, in general, over $\mathbb{Q}$. Given E, the $\mathbb{Q}$-isomorphism class of E_d depends only on the class of $d \pmod{(\mathbb{Q}^\times)^2}$. In what

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follows, whether a subscript specifies one of several elliptic curves, or a twist of a given curve, should be clear from context.

If we could find diagonal rational curves over $\mathbb{Q}$ on every Kummer variety of the form $\bar{A}$, where $A := E_1 \times \cdots \times E_n$ is a product of elliptic curves E_i over $\mathbb{Q}$, it would follow that there are infinitely many square free integers d such that the quadratic twists $(E_1)_d, \ldots, (E_n)_d$ of $E_1, \ldots, E_n$ each have positive rank. Indeed, in the case that $n = 2^m$ and $E_1, \ldots, E_n$ are the twists of a fixed elliptic curve E_0 by all products of $d_1, \ldots, d_m$, such that each of these curves, except perhaps E_0 itself, has positive rank over $\mathbb{Q}$, we could then obtain d_{m+1} so that all $(E_i)_{d_{m+1}}$ has positive rank, so $(E_0)_d$ has positive rank for all non-empty products d of $d_1, \ldots, d_{m+1}$. We could then use induction to prove that for each elliptic curve $E/\mathbb{Q}$, there exists an infinite sequence $d_1, d_2, \ldots, \in \mathbb{Q}^\times$, independent in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, such that E_d has positive rank for every d which is the product of a non-empty subset of terms in the sequence. This, in turn, would imply that for every $(\sigma_1, \ldots, \sigma_m) \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^m$, the rank of E over the fixed field $\bar{\mathbb{Q}}^{\langle \sigma_1, \ldots, \sigma_m \rangle}$, and indeed over the smaller field $\mathbb{Q}(\sqrt{\mathbb{Q}^\times})^{\langle \sigma_1, \ldots, \sigma_m \rangle}$, is infinite.

A well known theorem of Frey and Jarden [FJ74] asserts that for almost all $(\sigma_1, \ldots, \sigma_m) \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})^m$ in the sense of Haar measure, E has infinite rank over $\bar{\mathbb{Q}}^{\langle \sigma_1, \ldots, \sigma_m \rangle}$. One of us conjectured [La03] that this rank is in reality always infinite. At present, this is known unconditionally for elliptic curves with rational 2-torsion [IL13] and modulo the Birch-Swinnerton-Dyer Conjecture [DD09] for elliptic curves in general. The abovementioned approach to proving the conjecture in [La03] was our original motivation for asking whether diagonal rational curves always exist over $\mathbb{C}$. Our result is as follows:

Theorem 1.2. *If $n \geq 13$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is general, then there does not exist a hyperelliptic curve which admits a non-constant morphism to the elliptic curve $y^2 z = x(x-z)(x-\lambda_i z)$ for all i from 1 to n.*

Equivalently, we have:

Theorem 1.3. *If $n \geq 13$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ is general, then there does not exist a diagonal rational curve on the Kummer variety of the product of elliptic curves $y^2 z = x(x-z)(x-\lambda_i z)$.*

By *general* here and below, we mean that we may exclude a countable union of proper subvarieties from the variety parameterizing our family of abelian varieties. For instance, for some $g \geq 1$ and $n \geq 3$, we might consider the universal family of abelian varieties over the fine moduli space $A_{g,n}$ of g-dimensional principally polarized abelian varieties with full level-n structure. In the case of products of n elliptic curves in Legendre form, we can take this parameter variety to be

$$\mathrm{Spec}\ \mathbb{C}[x_1, \ldots, x_n, \frac{1}{x_1(x_1-1)}, \ldots, \frac{1}{x_n(x_n-1)}],$$

which is affine n-space with n pairs of parallel hyperplanes removed. The fact that for large n, a general $\bar{A}$ has no rational curves over $\mathbb{C}$ does not rule out the possibility that in the cases of interest to us (where the E_i are defined over $\mathbb{Q}$ and are twists of one another) such rational curves may exist, but it does rule out constructing a universal family which can then be specialized to the cases of arithmetic interest, thereby rendering this approach less promising.

To put the geometric question in a broader context, consider an abelian variety $A/\mathbb{C}$, which need not be a product of elliptic curves, together with a finite subgroup G of $\mathrm{Aut}(A)$. One can ask when the quotient A/G can be expected to have rational curves. At one end of the spectrum, A/G could have uncountably many rational curves. A celebrated result of Looijenga [Lo76] asserts that E^n/W is a weighted projective space when E is an elliptic curve and W is the Weyl group of a root system of rank n acting on E^n through its action on the root lattice. Kollár and Larsen [KL09] showed that in general A/G is uniruled if and only if G fails the *Reid-Tai condition*. This means that there exists an automorphism $g \in G$ whose eigenvalues on $\mathrm{Lie}(A)$ are $e^{2\pi i x_1}, \dots, e^{2\pi i x_n}$ where $0 \le x_1 \le x_2 \le \dots \le x_n < 1$, and $0 < \sum x_i < 1$. Im and Larsen [IL15] showed that if $\sum x_i = 1$, A/G must still have at least one rational curve.

At the other end of the spectrum, there is the classically known fact that A itself contains no rational curves. A major advance was Pirola's theorem [Pi89] that for a general abelian variety of dimension $n \ge 3$, the Kummer variety $\bar{A}$ has no rational curves. His proof exploits the fact that such a variety can be assumed to be simple, and, indeed, we have already noted that his statement is not true for general products of three elliptic curves by [Im13]. A new idea is therefore needed to prove our result.

Our proof makes use of the "easy" direction of the proof of Belyi's theorem, the rigidity of rational curves on Kummer varieties, and a theorem, which may be of independent interest, asserting that a genus 0 field cannot be the compositum of linearly disjoint extensions of $\mathbb{C}(t)$, each ramified over 6 or more valuations. As a corollary, we prove that if F is a finite extension of $\mathbb{C}(t)$ and L and M are intermediate fields such that LM is a field of genus 0, then, possibly after exchanging L and M, M is isomorphic as a $\mathbb{C}(t)$-extension to an extension of a $\mathbb{C}(t)$-subfield L' of the Galois closure L^c of $L/\mathbb{C}(t)$ which is ramified over at most 5 valuations of L'.

The strategy is as follows. Suppose there really is a diagonal rational curve X in the Kummer variety of $\bar{A}$, where the factors E_i of A form a general 13-tuple of elliptic curves. We choose 7-element subsets S and T of $\{1, 2, \dots, 13\}$ such that $S \cap T = \{1\}$. Defining A^S and A^T to be the products of E_i indexed by $i \in S$ and $i \in T$ respectively, we map X into diagonal rational curves X^S and X^T on the Kummer varieties $\bar{A}^S$ and $\bar{A}^T$, and map each of these onto the Kummer variety of E_1. By construction, the function field of X, regarded as a finite extension of the function field of $\bar{E}_1$, contains the function fields L and M of X^S and X^T respectively. Let L^c denote the Galois closure of $L/\mathbb{C}(t)$ and L' denote a field between $\mathbb{C}(t)$ and L^c. The number of degrees of freedom needed to choose A^S is $|S| = 7$, and so number of degrees of freedom needed to specify X is bounded above by 12. However, X maps non-trivially to 13 general elliptic curves, which requires 13 degrees of freedom.

Our method gives, in principle, a new proof of Pirola's theorem for $\dim A \ge 13$. We illustrate the idea by showing something new, namely, that the Kummer varieties corresponding to general points on certain connected Shimura varieties of unitary type have no rational curves. Our strategy is to find a set of abelian varieties, isogenous to products of elliptic curves, which are dense in the desired Shimura varieties.

Throughout this paper, unless otherwise specified, a *variety* will be an integral separated scheme of finite type over $\mathbb{C}$, and a *curve* will be a 1-dimensional variety. We define a *hyperelliptic curve* to be a non-singular curve with a distinguished automorphism of order 2 with rational quotient (in particular, we allow genus 1).

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2. Linearly disjoint subfields of genus 0 fields

In this section we bound the ramification of linearly disjoint extensions of $\mathbb{C}(t)$ whose compositum has genus 0. Our estimate is almost certainly not optimal. Throughout this paper, a *valuation* will always have value group $\mathbb{Z}$ and will always be trivial on the constant field $\mathbb{C}$.

Theorem 2.1. *Let L and M be linearly disjoint extensions of $K := \mathbb{C}(t)$. If $[L : K] \geq [M : K]$ and $L \otimes M$ has genus 0, then M is ramified over at most 5 valuations of K.*

Proof. Let $x_1, \ldots, x_n$ be the valuations of K over which $L \otimes M$ is ramified. Let $y_{i1}, y_{i2}, \ldots$ (resp. $z_{i1}, \ldots$) be the valuations of L (resp. M) lying over x_i, and let the ramification indices for y_{ij} and z_{ik} be l_{ij} and m_{ik} respectively. If $d_L := [L : K]$ and $d_M := [M : K]$, then for $1 \leq i \leq n$,

$$\sum_j l_{ij} = d_L, \ \sum_k m_{ik} = d_M. \tag{1}$$

By Lüroth's theorem, L and M are of genus 0, so

$$\sum_i \sum_j (l_{ij} - 1) = 2d_L - 2, \ \sum_i \sum_k (m_{ik} - 1) = 2d_M - 2 \tag{2}$$

by the Riemann-Hurwitz formula.

The number of valuations of $L \otimes M$ lying over both y_{ij} and z_{ik} is (l_{ij}, m_{ik}), the greatest common divisor of l_{ij} and m_{ik}, and the ramification indices equal $[l_{ij}, m_{ik}]$, their least common multiple. As $L \otimes M$ has genus 0, the Riemann-Hurwitz formula gives

$$2d_L d_M - 2 = \sum_i \sum_{j,k} (l_{ij}, m_{ik})([l_{ij}, m_{ik}] - 1) = \sum_i \sum_{j,k} (l_{ij} m_{ik} - (l_{ij}, m_{ik})).$$

By (1) and (2), this can be rewritten as

$$\begin{aligned} \sum_i \sum_{j,k} (l_{ij} - 1) m_{ik} &+ \sum_i \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})) \\ &= \sum_i d_M \sum_j (l_{ij} - 1) + \sum_i \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})) \\ &= d_M (2d_L - 2) + \sum_i \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})), \end{aligned}$$

so

$$\sum_i \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})) = 2d_M - 2.$$

Let n_M denote the number of valuations x_i over which M is ramified. Renumbering if necessary, we may assume these are $x_1, \ldots, x_{n_M}$. For $1 \le i \le n_M$,

$$\begin{aligned}\sum_{j,k}(m_{ik} - (l_{ij}, m_{ik})) &\ge \sum_{\{j \mid l_{ij}=1\}} \sum_k (m_{ik} - 1) \ge |\{j \mid l_{ij} = 1\}| \\ &\ge \sum_j (l_{ij} - 2(l_{ij} - 1)) = d_L - 2\sum_j (l_{ij} - 1).\end{aligned}$$

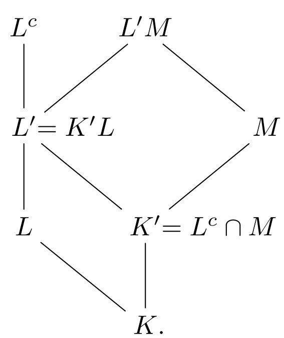
Summing over i, we obtain

$$\begin{aligned}2d_L - 2 \ge 2d_M - 2 &= \sum_{i=1}^{n} \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})) \\ &\ge \sum_{i=1}^{n_M} \sum_{j,k} (m_{ik} - (l_{ij}, m_{ik})) \\ &\ge n_M d_L - 2 \sum_{i=1}^{n_M} \sum_j (l_{ij} - 1) \\ &\ge n_M d_L - 2(2d_L - 2) > (n_M - 4)d_L - 2,\end{aligned}$$

which implies $n_M \le 5$. □

Corollary 2.2. *Let $K = \mathbb{C}(t)$, let F be a finite extension of K, and let L and M be intermediate fields of F/K. If LM has genus 0 and $[L : K] \ge [M : K]$, then there exists a diagram $K \to K' \to M$ where K' is a subfield of the Galois closure L^c of L/K, and M is ramified over ≤ 5 valuations of K'.*

Proof. Let $K' := L^c \cap M$ and $L' := K'L$. We therefore have the following diagram of fields:



The compositum

$$L'M = (K'L)M = L(K'M) = LM$$

has genus 0. As L^c/K' is Galois and $L^c \cap M = K'$, it follows that L^c and M are linearly disjoint over K', so $L' \subset L^c$ and M are likewise linearly disjoint over K'. Finally,

$$[L' : K'] = \frac{[L' : K]}{[K' : K]} \ge \frac{[L : K]}{[K' : K]} \ge \frac{[M : K]}{[K' : K]} = [M : K'].$$

Therefore, Theorem 2.1 applies to L' and M regarded as extensions of K'. It follows that M is ramified over at most 5 valuations of K'. □

3. Families of curves

In this section, we prove some generic properties of families of curves which are needed in the following section to prove the main theorem.

Proposition 3.1. *Let $\phi\colon X \to S$ be a morphism of varieties. Then, after replacing S by a suitable open subvariety, there exists a projective morphism of varieties $Y \to S$ such that for all closed points $s \in S$ for which X_s is a curve, Y_s is a projective non-singular curve birationally equivalent to X_s.*

Proof. We may assume that the set of points $s \in S$ such that X_s is a curve is dense in S, since otherwise we can replace S by an open subvariety over which the statement is trivial. Without loss of generality, we may assume S is affine. By [Gr66, Th. 9.7.7], we may assume that the fibers of ϕ are geometrically irreducible. By [Gr66, Prop. 9.5.5], we may assume that they have dimension 1. If U denotes an open subset of X, by Chevalley's constructibility theorem [Gr64, Cor. 1.8.5], the points $s \in S$ for which $U \cap X_s$ is non-empty (and therefore birationally equivalent to X_s) forms a constructible set which is dense in S, so by shrinking S to a smaller affine open subset, we may replace X by an open subset of U. In particular, by [Gr67, Prop. 17.7.11 (ii)], we may assume that $X \to S$ is smooth.

Let A be the coordinate ring of S, K the fraction field of A, and η the generic point of S. Thus X_η is a non-singular variety, so it is an open subvariety of a projective variety $\bar{X}_\eta \subset \mathbb{P}^n_K \subset \mathbb{P}^n_A$. Let $\bar{X}_A$ denote the Zariski closure of $\bar{X}_\eta$ in $\mathbb{P}^n_A$, endowed with the structure of reduced closed subscheme. As X is reduced, the morphism $X_\eta \to \bar{X}_\eta$ extends uniquely to a morphism $X \to \bar{X}_A$. Replacing S with an open affine subscheme, we may assume that $\bar{X}_A$ is smooth of relative dimension 1, and for all closed points $s \in S$, $X_s \to (\bar{X}_A)_s$ is an open immersion [Gr66, Prop. 9.6.1 (x)], so the two curves are birationally equivalent, and we are done. □

Proposition 3.2. *Let $Y_i \to X_i$ be a countable family of morphisms of varieties whose fibers have dimension ≤ 1. If $(E_1, \ldots, E_n)$ is a general n-tuple of elliptic curves over $\mathbb{C}$ and $\dim X_i \leq n-1$ for all i, then for all $x \in X_i$, there exists j such that every morphism from the normalization $\widetilde{(Y_i)_x}$ of the fiber of Y_i over x to E_j is locally constant.*

Proof. We use induction on n. For $n = 1$, $X_i = \operatorname{Spec} \mathbb{C}$, and each $\tilde{Y_i}$ is a non-singular curve. There is a non-constant morphism $\tilde{Y_i} \to E_1$ if and only if E_1 is a quotient of the Jacobian variety of the non-singular compactification of $\tilde{Y_i}$. The Poincaré reducibility theorem [Mu70, §19, Th. 1] asserts that every abelian variety is isogenous to a product of simple abelian varieties. Fixing one such isogeny, the only simple abelian varieties which are quotients of the original variety are those which are isogenous to one of the simple factors. Since a general elliptic curve does not lie in any fixed isogeny class, and therefore it does not admit a non-constant morphism from any fixed non-singular curve.

We may always replace one of the X_j by an open subvariety, at the cost of introducing an additional morphism to the set $\{X_i \to Y_i\}_i$. Since countable intersections of general subsets are general, we can always reduce to the case of a single morphism $Y \to X$.

We may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, where A and B are integral domains finitely generated over $\mathbb{C}$, and $Y \to X$ is dominant. Let K denote the fraction field of A. Then there exists a finite extension K'/K such that every irreducible component of the spectrum of $B' := B \otimes_K K'$ is geometrically irreducible. Let A' denote the integral closure of A in K'. As $\mathbb{C}$ is a Nagata ring, A' is a finitely generated $\mathbb{C}$-algebra. Replacing A and B by A' and any irreducible component of Spec B' respectively, we may assume that the generic fiber of $Y \to X$ is geometrically irreducible. Replacing X by an affine open subset, we may assume that all fibers of $Y \to X$ are geometrically irreducible. By Proposition 3.1, we may assume that $Y \to X$ is projective.

If the generic fiber Y_η admits a non-constant morphism to E_n (regarded, by extension of scalars, as an elliptic curve over K), then E_n is a quotient of the Jacobian variety of Y_η. We may therefore assume that every morphism $Y_\eta \to E_n$ is constant. By [Gr61, §4], $\mathrm{Hom}_X(Y, X \times E_n)$ is an open subscheme of a countable union of quasi-projective varieties. By [Gr66, Prop. 9.6.1 (ii)], a constructible subset of this consists of dominant (or, equivalently, non-constant) morphisms, and by Chevalley's theorem, the image of each such set under the map to X is a constructible subset of X which does not contain the generic point. It is therefore contained in a finite union of proper closed subvarieties of dimension $\leq n-2$. The union of all such subvarieties is countable, and the proposition follows by induction.

□

Proposition 3.3. *Let $\phi\colon X \to S \times \mathbb{P}^1$ denote a morphism of varieties such that for every closed point $s \in S$, $\phi_s\colon X_s \to \mathbb{P}^1$ is a non-constant morphism of curves. We denote by $L_s/K_s = \mathbb{C}(t)$ the corresponding extension of function fields. After replacing S with an open subvariety, there exists a finite group G and for each subgroup G' a variety $X_{G'}$ and a morphism $\phi_{G'}\colon X_{G'} \to S \times \mathbb{P}^1$ with the following properties:*

1. *If $G_1 \subset G_2 \subset G$, then there exists a morphism $X_{G_1} \to X_{G_2}$ whose composition with ϕ_{G_2} is ϕ_{G_1}.*
2. *ϕ_G is an open immersion.*
3. *For each G' and each closed point s, the maps of fibers $(X_{\{1\}})_s \to (X_{G'})_s \to \mathbb{P}^1$ correspond to the field extensions $K_s \subset (L_s^c)^{G'} \subset L_s^c$.*

Proof. We may assume S is affine and let A denote the coordinate ring of $S \times \mathbb{A}^1 \subset S \times \mathbb{P}^1$. Replacing X by an affine open subset Spec B and then shrinking S if necessary, we may assume the fibers of $X \to S$ are non-empty and therefore birationally equivalent to the fibers of the original morphism $X \to S$. Thus, ϕ corresponds to an injective homomorphism $A \to B$.

If K is the fraction field of A, since ϕ is dominant and quasi-finite, $L := B \otimes_A K$ is a finite dimensional K-algebra. Since L is contained in the fraction field of B, it is an integral domain, and it follows that it is a field.

Let L^c denote the Galois closure of L/K, and let $G := \mathrm{Gal}(L^c/K)$. Then there exists a surjective homomorphism

$$\Phi\colon L^{\otimes n} = L \otimes_K \cdots \otimes_K L \to L^c,$$

where the n tensor factors are indexed by the elements of G, and Φ is equivariant with respect to the permutation action of the Galois group G on the tensor factors. Let B^c denote $\Phi(B^{\otimes n})$, which is an $A[G]$-algebra. As B is a finitely generated $\mathbb{C}$-algebra, the same is true of B^c.

Each $G' \subset G$ corresponds to some intermediate field $K' := (L^c)^{G'}$. We define $B(G')$ to be the ring of G'-invariants in B^c. By a theorem of Hilbert, $B(G')$ is a finitely generated $\mathbb{C}$-algebra. As $\mathbb{C}[G']$ is semisimple and A is a $\mathbb{C}$-algebra, tensoring over A with any A-algebra commutes with taking G'-invariants. In particular,

$$B(G') \otimes_A K = (B^c)^{G'} \otimes_A K = (B^c \otimes_A K)^{G'} = \Phi(L^{\otimes n})^{G'} = (L^c)^{G'} = K'.$$

We have $A \subset B(G)$, so the morphism Spec $B(G) \to$ Spec A is birational. Therefore, replacing A by $A[1/f]$ for some non-zero $f \in A$, we may assume $A = B(G)$. Likewise, if $G' = \mathrm{Gal}(L^c/L)$, we have $B \subset B(G')$, and shrinking S, and therefore Spec A and Spec B, if necessary, we may assume this is an equality. Let $X_{G'} =$ Spec $B(G')$ for all $G' \subset G$. Thus we have properties (1) and (2).

By [Gr66, Th. 9.7.7], shrinking S if necessary, we may assume $B(G') \otimes_A k$ is a field for every residue field k of A and every subgroup $G' \subset G$. Moreover, $B(G') \otimes_A k = (B^c \otimes_A k)^{G'}$. Thus, $B^c \otimes_A k$ is a Galois extension of k with group G, $B(G') \otimes_A k$ is a k-subfield of $B^c \otimes_A k$, and all k-subfields of $B^c \otimes_A k$ arise in this way.

We apply this in the case that $k = K_s \cong \mathbb{C}(t)$ is the function field of the preimage in A of a closed point of s. Thus, we have that $B^c \otimes_A k$ is the Galois closure of $B \otimes_A k$ over k and has Galois group G, and

$$B^c \otimes_A k = L_s^c,\ B \otimes_A k = (L_s^c)^{\mathrm{Gal}(L^c/L)} = L_s,\ k = (L_s^c)^G = K_s.$$

This gives (3).

□

Proposition 3.4. *Let S and X be finite-dimensional varieties and $X \to S$ a projective morphism whose fibers are non-singular curves. Let m be a positive integer. Then there exist a countable collection of morphisms $T_i \to S$ with $\dim T_i \leq m + \dim S$ and morphisms $Z_i \to T_i \times_S X$ such that for every closed point $s \in S$ and every non-constant morphism $C \to X_s$ from a projective non-singular curve C, which is ramified over $\leq m$ points of X_s, there exist i and $t \in (T_i)_s$ such that $(Z_i)_t$ is isomorphic to C as a scheme over X_s.*

Proof. Let X^m denote the m-fold fiber power of X relative to S and $X^m_\circ$ the open subscheme for which no two coordinates are equal. We can regard $X^{m+1}_\circ \subset X \times_S X^m_\circ$ as the universal family of open subsets of the fibers of $X \to S$, punctured at the m points indexed by $X^m_\circ$.

For any $g \geq 0$, we let $C_g \to M_g$ denote any projective morphism of varieties over $\mathbb{C}$ such that every fiber is a projective non-singular curve of genus g and every isomorphism class of curves appears a positive finite number of times. For instance, for $g \geq 2$, we may take the universal curve over the moduli space of curves of genus g with level 3 Jacobi structure [DM69, (5.14)], which is projective by [DM69, Th. 1.2], while S and X are connected by [DM69, Th. 5.15]. For $g = 1$, we may take the relative projective curve

$$\mathrm{Proj}\ \mathbb{C}[x, y, z]/(y^2 z - x(x - z)(x - \lambda z)) \to \mathrm{Spec}\ \mathbb{C}[\lambda, \frac{1}{\lambda(\lambda - 1)}].$$

Next we apply [Gr61, §4] to construct the scheme

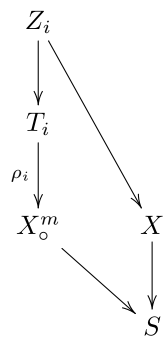
$$\mathrm{Hom}_{X^m_\circ \times M_g}(X^m_\circ \times C_g, (X \times_S X^m_\circ) \times M_g) \tag{3}$$

of morphisms from curves of $C_g \to M_g$ to curves of $X \to S$ as a countable union of quasi-projective varieties. The condition of being unramified relative to

$$X_\circ^{m+1} \times M_g \subset (X \times_S X_\circ^m) \times M_g$$

defines a constructible subset of (3) [Gr67, Prop. 17.7.11 (iii)].

We have, therefore, a countable collection of diagrams



such that for each closed point $s \in S$ and each closed point $t \in (T_i)_s$, the morphism $(Z_i)_t \to X_s$ can be ramified over X_s only at those points of X_s which are coordinates of $\rho_i(t)$. Moreover, for any $s \in S$, any morphism of curves to X_s with at most m ramification points arises from at least one T_i and one $t \in (T_i)_s$. Since the set of connected covering spaces of a Riemann surface is countable, every morphism ρ_i has countable fibers. Therefore the ρ_i are quasi-finite, and

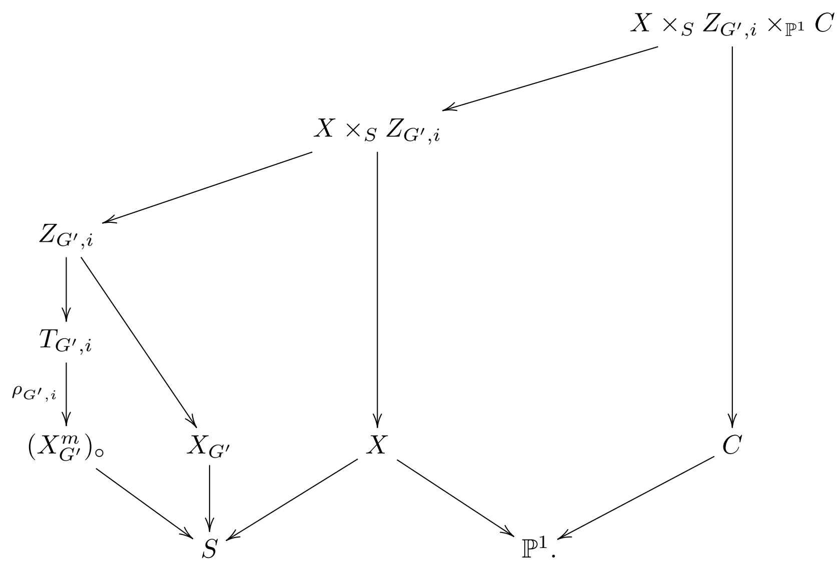
$$\dim T_i \leq \dim X_\circ^m = m + \dim S.$$

□

Proposition 3.5. *Let S and X be varieties and m and n positive integers. Let $X \to S \times \mathbb{P}^1$ be a morphism such that for each $s \in S$, $X_s \to \mathbb{P}^1$ is a morphism of curves. Let $K = \mathbb{C}(t) \to L_s$ denote the extension of function fields corresponding to $X_s \to \mathbb{P}^1$. Let N/K denote a fixed finite extension. Suppose that $n > m + \dim S$, and let $(E_1, \ldots, E_n)$ be a general n-tuple of elliptic curves. Then there does not exist a closed point $s \in S$, an intermediate field K' between K and the Galois closure L_s^c, a finite extension M of K' ramified at $\leq m$ valuations, a field F which is a factor of $L^c \otimes_K M \otimes_K N$, and a curve with function field F admitting a non-constant morphism to each E_i.*

Proof. Using induction on $\dim S$, it suffices to prove this proposition after shrinking S to a suitable open subvariety. Applying Proposition 3.3, there exists a finite group G and morphisms $\phi_{G'}$ for all $G' \subset G$ satisfying conditions (1)–(3). Applying Proposition 3.1 and Proposition 3.4 to the morphisms $\phi_{G'}$ we can obtain a countable sequence of morphisms $Z_{G',i} \to X_{G'} \times_S T_{G',i}$, with $\dim T_{G',i} \leq m + \dim S$, such that the morphisms of curves $(Z_{G',i})_t \to (X_{G'})_s$ are associated to all extensions M/K' ramified at no more than m valuations.

If $C \to \mathbb{P}^1$ is a morphism of curves corresponding to the extension N/K, we have the following diagram:



If $t \in T_{G',i}$ is a closed point mapping to $s \in S$, the fiber of

$$X \times_S Z_{G',i} \times_{\mathbb{P}^1} C \to T_{G',i} \tag{4}$$

over t is $X_s \times_{\mathbb{P}^1} (Z_{G',i})_t \times_{\mathbb{P}^1} C$. The fiber of this 1-dimensional scheme over the generic point of $\mathbb{P}^1$ is Spec $L_s \otimes_K M_t \otimes_K N$, where M_t is an extension of $K' = (L_s^c)^{G'} \supset K$ ramified only over the valuations determined by $\rho_{G',i}(t)$. Applying Proposition 3.2 to the morphism (4), we obtain the proposition.

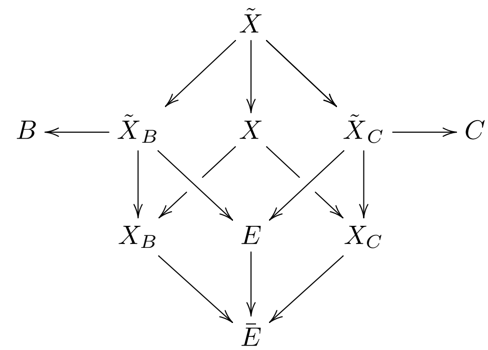
□

4. Products of elliptic curves

Let E be an elliptic curve and B and C abelian varieties, and let $A := E \times B \times C$. Let X be a rational curve in $\bar{A}$ which does not lie in any fiber of the projection maps $\bar{A} \to \bar{E}$, $\bar{A} \to \bar{B}$, or $\bar{A} \to \bar{C}$. Let X_B and X_C denote the projections of X on $\overline{E \times B}$ and $\overline{E \times C}$ respectively. Let $\tilde{X}$, $\tilde{X}_B$, and $\tilde{X}_C$ denote the normalizations of the inverse images of X, X_B, and X_C respectively in $E \times B \times C$, $E \times B$, and $E \times C$; we have already observed that these inverse images are irreducible curves, so $\tilde{X}$, $\tilde{X}_B$, and $\tilde{X}_C$ are hyperelliptic curves.

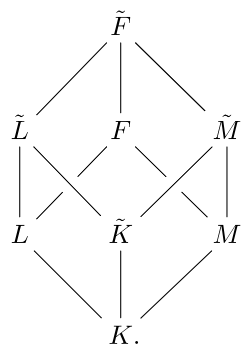
We obtain the following diagram

(5)



and in particular a diagram of function fields

(6)



Note that $[\tilde{K} : K] = [\tilde{L} : L] = [\tilde{M} : M] = 2$. By Lüroth's theorem, $\tilde{K} \not\subset L$ and $\tilde{K} \not\subset M$, so $L\tilde{K} = \tilde{L}$ and $M\tilde{K} = \tilde{M}$. Note also that $LM \subset F$, so LM is of genus 0. Without loss of generality, we assume that $[L : K] \geq [M : K]$. By Corollary 2.2, there exists a K-subfield K' of M K-isomorphic to some K-subfield of L^c such that M is ramified over K' at ≤ 5 valuations. On the other hand, $M\tilde{K} = \tilde{M}$ is the function field of a curve in $E \times C$.

From now on, we suppose that B and C are products of elliptic curve factors, and the moduli of the factors of B and C, together with the modulus of E form a general $\dim E \times B \times C$-tuple. To treat families of such varieties, we introduce the following notation.

Let $\Lambda/\mathbb{C}$ denote the complement of $\{0, 1, \infty\}$ in $\mathbb{P}^1_{\mathbb{C}}$. Let n be a positive integer and S a subset of $\{1, \ldots, n\}$ containing 1. For each S we define Λ^S to be the product of copies of Λ indexed by S, so when $S \subset S'$, we have an obvious projection $\Lambda^{S'} \to \Lambda^S$. We define $\pi^S \colon A^S \to \Lambda^S$ to be the universal $|S|$-fold product of elliptic curves with invariant in Λ^S , i.e., the fiber of $A^S \to \Lambda^S$ over $\lambda = (\lambda_{i_1}, \ldots, \lambda_{i_s})$ is $\prod_{j=1}^{s} E(\lambda_{i_j})$. Explicitly,

$$A^S = \text{Proj } \mathbb{C}[x_{i_1}, \ldots, y_{i_s}, y_{i_1}, \ldots, y_{i_s}, z_{i_1}, \ldots, z_{i_s}]/I_{\lambda_{i_1},\ldots,\lambda_{i_s}},$$

where the x, y, and z variables have degree 1, and the ideal $I_{\lambda_{i_1},\ldots,\lambda_{i_s}}$ is generated by the degree-3 elements

$$y_{i_j}^2 z_{i_j} - x_{i_j}(x_{i_j} - z_{i_j})(x_{i_j} - \lambda_{i_j}),\ j = 1, 2, \ldots, s.$$

We denote by $\bar{A}^S \to \Lambda^S$ the family of Kummer varieties associated to this family of abelian varieties. We remark that every abelian scheme $p\colon A \to X$ over a normal base X is projective by a theorem of Grothendieck [Ra70, Th. XI 1.4], so the relative Kummer scheme $\bar{A} \to X$ can be constructed as

$$\operatorname{Proj}_X \bigoplus_{d=0}^{\infty} H^0(X, p_*\mathcal{L}^{\otimes 2d}).$$

By [Ko96, II 2.11,2.12], there exist quasi-projective varieties Y_α^S indexed by α in a countable index set I_S, a $\mathbb{P}^1$-bundle U_α^S over Y_α^S, and a map $f^S\colon \coprod_\alpha U_\alpha^S \to \bar{A}^S$ such that every rational curve on $\bar{A}^S$ lies in the image of a birational map from a fiber of $p_\alpha^S\colon U_\alpha^S \to Y_\alpha^S$. Moreover, distinct fibers of p_α^S have distinct images in $\bar{A}^S$.

We say a $\mathbb{C}$-point y of Y_α^S is *vertical* if the corresponding fiber of p_α^S lies in a single fiber of π^S, i.e., if $\pi^S(f^S((p_\alpha^S)^{-1}(y)))$ is a single point. This is a constructible condition [Gr66, 9.6.1(vii)], so we may partition each Y_α^S into finitely many quasi-projective subvarieties so that each subvariety consists entirely of vertical points or contains no vertical points. Therefore, there exists a countable set J_S indexing quasi-projective varieties Z_β^S, $\mathbb{P}^1$-bundles $q_\beta^S\colon X_\beta^S \to Z_\beta^S$, and morphisms $X_\beta^S \to \bar{A}^S$ such that every rational curve in every fiber of π^S is the image of a fiber of some q_β^S by a birational map. We define $\tilde{X}_\beta^S := X_\beta^S \times_{\bar{A}^S} A^S$, so $\tilde{X}_\beta^S$ is a bundle of curves over Z_β^S with hyperelliptic normalization. We have the following picture:

$$\begin{CD}
\tilde{X}_\beta^S @>>> A^S @>>> A^{\{1\}} \\
@VVV @VVV @VVV \\
X_\beta^S @>{f_\beta^S}>> \bar{A}^S @>>> \bar{A}^{\{1\}} = \Lambda^{\{1\}} \times \mathbb{P}^1 \\
@V{q_\beta^S}VV @VV{\pi^S}V @VV{\pi^{\{1\}}}V \\
Z_\beta^S @>{r_\beta^S}>> \Lambda^S @>>> \Lambda^{\{1\}}
\end{CD}$$

Lemma 4.1. *In the above diagram, for all S and β, r_β^S is quasi-finite.*

Proof. If C is a curve in Z_β^S mapping to a single point $\lambda = (\lambda_{i_1}, \dots, \lambda_{i,s}) \in \Lambda^S$, then

$$(\pi^S)^{-1}(\lambda) = \overline{E(\lambda_{i_1}) \times \cdots \times E(\lambda_{i_s})}$$

has a continuous family of rational curves given by $f_\beta^S((q_\beta^S)^{-1}(c))$, as c varies over C. However, by [Pi89, Theorem 1], every rational curve in a Kummer variety is rigid. $\square$

In particular, $\dim Z_\beta^S \le |S|$ for all S and β. Now, the natural map

$$\tilde{X}_\beta^S \to Z_\beta^S \times_{\Lambda^{\{1\}}} \bar{A}^{\{1\}}$$

is quasi-finite. Note that $\bar{A}^{\{1\}}$ is the universal Kummer variety over the space $\Lambda^{\{1\}}$ of elliptic curves, so $\bar{A}^{\{1\}} \to \Lambda^{\{1\}}$ is the trivial fiber bundle with fiber $\mathbb{P}^1$. Thus, $Z_\beta^S \times_{\Lambda^{\{1\}}} \bar{A}^{\{1\}}$ is the trivial $\mathbb{P}^1$-bundle over Z_β^S. A point $z \in Z_\beta^S$ determines a rational curve $(q_\beta^S)^{-1}(z)$ whose image under f_β^S is a rational curve in $\bar{A}_\lambda^S$, where $\lambda = r_\beta^S(z)$.

We can now prove Theorem 1.2

Proof. It suffices to prove the theorem for $n = 13$. Let $S := \{1, 2, \dots, 7\}$ and $T := \{1, 8, \dots, 13\}$, and assume there is a diagonal rational curve X in $\bar{A}_\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_{13})$. We fix β and assume that X corresponds to a point on $Z_\beta^{S\cup T}$; henceforth, we omit the subscript β from the notation. Let $E := A_{\lambda_1}^{\{1\}}$, $E \times B = A_{\lambda_S}^S$, $E \times C = A_{\lambda_T}^T$, and $E \times B \times C = A_\lambda^{S\cup T}$, and use the notation for curves in (5) and the notation for fields in (6), so K, L and M are the function fields of $\bar{E} = A_{\lambda_1}^{\{1\}}$, $X_S \subset \bar{A}_{\lambda_S}^S$, and $X_T \subset \bar{A}_{\lambda_T}^T$ respectively (which are all isomorphic to $\mathbb{C}(t)$). For a general λ, the degrees $[L : K]$ and $[M : K]$ of $X_B \to \bar{E}$ and $X_C \to \bar{E}$ respectively depend only on β, which is fixed. We assume without loss of generality that $[L : K] \geq [M : K]$.

We now consider the morphism $X^{S\cup T} \to \Lambda^{S\cup T}$ of which X is a fiber. Thus X_B and X_C are fibers of $X_B^{S\cup T} := X^S \times_{\Lambda_S} \Lambda^{S\cup T}$ and $X_C^{S\cup T} := X^T \times_{\Lambda_T} \Lambda^{S\cup T}$ respectively. We have a diagram of morphisms of rational curves over $\Lambda^{S\cup T}$:

$$\begin{array}{ccccc} & & X^{S\cup T} & & \\ & \swarrow & & \searrow & \\ X_B^{S\cup T} & & & & X_C^{S\cup T} \\ & \searrow & & \swarrow & \\ & & \Lambda^{S\cup T} \times \mathbb{P}^1 & & \end{array}$$

Let K denote the function field $\mathbb{C}(t)$ of $\mathbb{P}^1$, and let L and M be the finite extensions of this field associated to a parameter $\lambda \in \Lambda^{S\cup T}$ for which $\bar{A}_\lambda^{S\cup T}$ has a diagonal rational curve. Setting $K' := L^c \cap M$, by Corollary 2.2, M is isomorphic as K extension to an extension of K' ramified over ≤ 5 valuations (or the same thing is true after exchanging the roles of L and M). Let $N := \tilde{K}$ denote the function field of $E(\lambda_1)$, regarded as a degree 2 extension of K. Every direct factor of $L \otimes_K M \otimes_K N$ contains $\tilde{L}$ and $\tilde{M}$, so every projective non-singular curve whose function field is a factor of $L \otimes_K M \otimes_K N$ admits non-trivial morphisms to $E(\lambda_i)$ for $1 \leq i \leq 13$. By Proposition 3.5, this is impossible for general $(E(\lambda_1), \dots, E(\lambda_{13}))$, which proves the theorem by contradiction.

□

5. Families of Abelian Varieties

Let S denote a variety over $\mathbb{C}$ and $\pi \colon A \to S$ an abelian scheme over S of relative dimension g. Let X be a reduced closed subscheme of A which is symmetric in the sense that $X = -X$.

Lemma 5.1. *The condition on $s \in S$ that the fiber X_s generates the abelian variety A_s is constructible.*

Proof. By [Bo91, I Proposition 2.2], X_s generates A_s if and only if the multiplication morphism

$$\mu_s \colon \underbrace{X_s \times \cdots \times X_s}_{2g} \to A_s$$

is surjective. If

$$\mu\colon X^{2g} := \underbrace{X \times_S \cdots \times_S X}_{2g} \to A$$

is the fiberwise multiplication morphism, then the set of $s \in S$ such that the fiber X_s generates the abelian variety A_s is the complement of $\pi(A \setminus \mu(X^{2g}))$, and is therefore constructible by Chevalley's theorem. □

Definition 5.2. *A curve C on the Kummer variety $\bar{A}$ associated to an abelian variety A is* nondegenerate *if the preimage of C in A generates the abelian variety A.*

Lemma 5.3. *If $E_1, \ldots, E_n$ are elliptic curves over $\mathbb{C}$, a non-degenerate rational curve on the Kummer variety of $A = E_1 \times \cdots \times E_n$ is diagonal.*

Proof. Let X denote the inverse image in A of a rational curve C in $\bar{A}$. As C is rational and A does not contain any rational curve, X must consist of a single irreducible (hyperelliptic) component. If C is not diagonal, then the image of C in some $\bar{E}_i$ is a single point, so the image of X in E_i is finite (in fact, a single point). implying that X does not generate A.

□

Lemma 5.4. *If $\pi\colon A \to B$ is a surjective morphism of non-trivial abelian varieties and $\bar{A}$ contains a non-degenerate rational curve C, then $\bar{B}$ contains a non-degenerate rational curve as well.*

Proof. The morphism π determines a morphism $\bar{\pi}\colon \bar{A} \to \bar{B}$. Let $X \subset \bar{A}$ denote the inverse image of C and $Y := \pi(X)$. As π is proper, Y is closed in B; it is also connected and of dimension ≤ 1, so it is either a point or a curve. As X generates A, Y generates B, so Y is a curve. The image of Y in $\bar{B}$ is $\bar{\pi}(C)$. Any 1-dimensional image of a rational curve is again a rational curve. □

Lemma 5.5. *Let S be a variety over $\mathbb{C}$ and $\pi\colon C \to S$ a projective morphism. The set of points $s \in S$ such that the fiber C_s is a curve of genus* 0 *(in particular, geometrically integral) is constructible.*

Proof. By induction on dimension, it suffices to prove this after S is replaced by a suitable open subvariety. By Proposition 3.1, we may therefore assume that the fibers of π are non-singular, and by [Gr66, Th. 11.1.1], we may assume π is flat, so the Hilbert polynomials of the fibers are equal [Gr61, §2]. Therefore, all the curves in the family are of genus 0 or none are. □

Proposition 5.6. *Suppose we have an abelian scheme A of dimension g over a normal base variety $S/\mathbb{C}$. Let $\bar{A} \to S$ be the associated family of Kummer varieties. Then one of the following must hold:*

(1) *There exists a non-empty Zariski open subset $U \subset S$ such that for every $s \in U$, $\bar{A}_s$ has a nondegenerate rational curve.*
(2) *For a general Kummer variety $\bar{A}_s$, there is no nondegenerate rational curve.*

Proof. As in §5, we use [Ko96, II 2.11,2.12] to construct a countable collection of families of vertical rational curves on $\bar{A} \to S$:

$$\begin{array}{ccc} \tilde{X}_\beta & \longrightarrow & A \\ \downarrow & & \downarrow \\ X_\beta & \xrightarrow{f_\beta} & \bar{A} \\ \downarrow & & \downarrow \\ Z_\beta & \xrightarrow{r_\beta} & S. \end{array}$$

By Lemma 5.1, the set N_β of $z \in Z_\beta$ such that $f_\beta((X_\beta)_z)$ is nondegenerate in $\bar{A}_{r_\beta(z)}$ is constructible, so $r_\beta(N_\beta)$ is a constructible subset of S. If it contains the generic point of S, then it contains a dense open subset of S, so condition (1) holds. Otherwise, for each s in the complement of the countable union $\bigcup_\beta \overline{r_\beta(N_\beta)}$, the Kummer variety $\bar{A}_s$ has no nondegenerate rational curve, which implies condition (2).

□

6. Some Families of Unitary Type

A general reference for the construction in this section is [Sh98, Chapter XXIV].

Let K be an imaginary quadratic field with ring of integers $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$, where $\operatorname{Im}\tau > 0$. Let $p \geq q$ be integers ≥ 13, and let $\Lambda_1 = \mathbb{Z} + \mathbb{Z}\tau_1, \ldots, \Lambda_q = \mathbb{Z} + \mathbb{Z}\tau_q$, $\operatorname{Im}\tau_i > 0$, denote lattices in $\mathbb{C}$. Let E (resp. E_i) denote the elliptic curve over $\mathbb{C}$ whose associated analytic space is isomorphic to $\mathbb{C}/\mathcal{O}$ (resp. $\mathbb{C}/\Lambda_i$). By Theorem 1.2, we may choose the Λ_i such that the Kummer variety of $E_1 \times \cdots \times E_q$ has no diagonal rational curves. Since the property that two elliptic curves are not isogenous is general on the moduli of the curves, we may further assume that $E_1, \ldots, E_q$ are mutually non-isogenous. Define

$$\Lambda := \mathcal{O}^{p-q} \oplus \bigoplus_{i=1}^{q} \operatorname{Hom}(\mathcal{O}, \Lambda_i),$$

and regard $\Lambda \otimes \mathbb{R}$ as a $\mathbb{C}$-vector space V via the given inclusions $\Lambda_i \hookrightarrow \mathbb{C}$ and a choice of embeddings $K \hookrightarrow \mathbb{C}$. Regarding Λ as a free $\mathcal{O}$-module of rank $p+q$ via the obvious action of $\mathcal{O}$ on $\operatorname{Hom}(\mathcal{O}, \Lambda_i)$, and extending by $\mathbb{R}$-linearity to define an action of $\mathcal{O}$ on V, we obtain a natural homomorphism ι from $\mathcal{O}$ to the endomorphism ring of the complex torus V/Λ. It also defines a second complex structure on V, which we call the ι-structure.

We define a Hermitian form H on V as follows. If $v = (a_1, \ldots, a_{p-q}, \alpha_1, \ldots, \alpha_q)$ and $w = (b_1, \ldots, b_{p-q}, \beta_1, \ldots, \beta_q)$, where $\alpha_i, \beta_i \in \operatorname{Hom}(\mathcal{O}, \mathbb{C}) = \operatorname{Hom}(\mathcal{O}, \Lambda_i) \otimes \mathbb{R}$, then

$$H(v, w) = \sum_{i=1}^{p-q} \frac{\bar{a}_i b_i}{\operatorname{Im}\tau} + \sum_{j=1}^{q} \frac{\bar{\alpha}_i(1)\beta_i(1) + \bar{\alpha}_i(\tau)\beta_i(\tau)}{\operatorname{Im}\tau_i}.$$

As $\operatorname{Im}\tau, \operatorname{Im}\tau_i > 0$, this is positive definite. For all $\sigma \in \mathbb{C} \setminus \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$, we have

$$\operatorname{Im} \frac{\overline{(a + b\sigma)}(c + d\sigma)}{\operatorname{Im}\sigma} = ad - bc,$$

so $\operatorname{Im} H$ restricts to a perfect $\mathbb{Z}$-valued anti-symmetric pairing on Λ. It follows that V/Λ is isomorphic to the analytic space of a principally polarized abelian variety $A_{\Lambda_1,\ldots,\Lambda_q}$.

We define a closed subgroup scheme $G/\mathbb{Z}$ of the symplectic group $\mathrm{Sp}_{2p+2q}/\mathbb{Z}$ as follows: For any commutative ring R, let $G(R)$ denote the group of R-linear and $\mathcal{O}$-linear automorphisms of $\Lambda \otimes R$ which respect to the anti-symmetric form

$$\operatorname{Im} H \colon \Lambda \otimes R \times \Lambda \otimes R \to R.$$

In particular, $G(\mathbb{R})$ is the group of automorphisms of the real vector space V which are symplectic with respect to $\operatorname{Im} H$ and $\mathcal{O}$-linear. If H' is the Hermitian form on V determined by $\operatorname{Im} H$ and the ι-structure, then H' has signature (p, q), and $G(\mathbb{R})$ can be regarded as the unitary group $U(p, q)$ defined by H' and the ι-structure. For all $g \in G$, the complex torus $V/g(\Lambda)$ is polarized by the Hermitian form $g(H)$ defined by

$$g(H)(v, w) := H(g^{-1}(v), g^{-1}(w)),$$

so $V/g(\Lambda)$ is isomorphic to the analytic space of a principally polarized abelian variety A_g endowed with the endomorphism structure $\iota_g \colon \mathcal{O} \to \operatorname{End} A_g$.

Let K denote the subgroup of $G(\mathbb{R})$ consisting of elements which are $\mathbb{C}$-linear with respect to the usual complex structure on V. Then $K \cong U(p) \times U(q)$, and $G(\mathbb{R})/K$ is a Hermitian symmetric domain. Let Γ denote a neat congruence subgroup of $G(\mathbb{Z})$. By the Baily-Borel theorem, the quotient $\Gamma\backslash G(\mathbb{R})/K$ is isomorphic to the analytic space associated to a complex non-singular quasi-projective variety S. The variety S parametrizes a family of abelian varieties $\{A_s \mid s \in S(\mathbb{C})\}$ of dimension $p + q$ together with a principal polarization on each A_s, a map $\iota \colon \mathcal{O} \to \operatorname{End} A_s$, and a Γ-level structure. This family comes from an abelian scheme $A \to S$.

Theorem 6.1. *For a general member A_s of the above family of abelian varieties, there are no rational curves on the Kummer variety of A_s.*

Proof. By the weak approximation theorem for semisimple groups, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ in the real topology. It follows that the image of $G(\mathbb{Q})$ is dense in the complex topology in $S(\mathbb{C})$ and therefore Zariski-dense in $S(\mathbb{C})$. By construction, all elements of $G(\mathbb{Q})$ correspond to abelian varieties isogenous to $E_1^2 \times \cdots \times E_q^2 \times E^{p-q}$. Any such abelian variety admits a surjective homomorphism of abelian varieties to $B := E_1 \times \cdots \times E_q$. We have assumed that $\bar{B}$ has no diagonal rational curves. By Lemma 5.3, this implies that $\bar{B}$ has no nondegenerate rational curves, so by Lemma 5.4, the Kummer variety of any abelian variety corresponding to an element of $G(\mathbb{Q})$ has no nondegenerate rational curve.

By Proposition 5.6, it follows that for a general point $s \in S(\mathbb{C})$, the Kummer variety of A_s has no nondegenerate rational curve. Since a general point in $S(\mathbb{C})$ is a simple abelian variety by [LB92, Theorem 9.9.1], we deduce that for a general member A_s of the family, there are no rational curves on the Kummer variety of A_s.

□

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