Incompleteness Theorems for Observables in General Relativity

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The quest for complete observables in general relativity has been a long-standing open problem. We employ methods from descriptive set theory to show that no complete observable on rich enough collections of spacetimes is Borel definable. In fact, we show that it is consistent with the Zermelo-Fraenkel and dependent choice axioms that no complete observable for rich collections of spacetimes exists whatsoever. In a nutshell, this implies that the problem of observables is to "analysis" what the Delian problem was to "straightedge and compass." Our results remain true even after restricting the space of solutions to vacuum solutions. In other words, the issue can be traced to the presence of local degrees of freedom. We discuss the next steps in a research program that aims to further uncover this novel connection between theoretical physics and descriptive set theory.

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From Einstein's century old hole-argument paradox [1–3], to the contemporary programs for quantizing gravity [4–7], the problem of deciding which "functions" of the metric components do not depend on the choice of coordinates has raised technical and epistemological difficulties in the theory of general relativity (GR). This issue has become known as *the problem of observables*.

The quest for complete observables—observables which can discern between any pair of diffeomorphically inequivalent spacetimes—began in the 1950s [4,8–12], and is often discussed in the context of more modern approaches [13–18]. While one can tailor observables for special families of spacetimes [19–21], no nontrivial (nonconstant) observable supported on the collection of all spacetimes has been reported. This, despite a seven-decades-long search since Bergmann famously stated the issue [8–11]. The question arises: why this state of affairs? Notwithstanding some interesting partial negative results in the Hamiltonian formulation of the problem [22,23], a conclusive result that identifies the root of the issue has remained elusive.

In this Letter, we employ methods from descriptive set theory to prove a rather conclusive negative result for the definability of complete observables, at least when no significant constraints on the space of solutions are imposed: *there is no constructive way to build complete observables for full general relativity.* We trace the root cause of this incompleteness phenomenon to a certain ergodic-theoretic behavior that general covariance exhibits on any "rich enough" collection of spacetimes.

Our results do not imply that a theory of quantum gravity cannot be based on definable observables of some kind. Rather, they highlight some of the difficulties when considering large and diverse collections $\mathcal{S}$ of spacetimes, all at once—a problem not unique to GR, but potentially to any other physical theory with a large symmetry group. Indeed, a takeaway is that a physical theory can be extremely useful even when the full space of solutions is too large to admit definable complete observables.

Theorems 1 and 2 provide the precise statements. Both theorems hold for any collection of spacetimes $\mathcal{S}$ that is *rich*—a technical term that we define below. In particular, they both hold when $\mathcal{S}$ is the collection of all Lorentzian manifolds of dimension d for any fixed $d \geq 2$.

Theorem 1.—No concrete observable $f: \mathcal{S} \to R$ is both complete and Borel definable.

The terms appearing in the statement of Theorem 1 will be defined below. In plain language: completeness requires that f distinguishes any two diffeomorphically inequivalent spacetimes by assigning to them different values; concreteness requires that f takes concrete objects as values, e.g. real numbers, invariant scalars, etc; Borel definability requires that f is given by some formula expressible in the language of analysis.

Theorem 1 shows that it is as futile to seek an analytic description for a complete observable, as trying to construct $\sqrt[3]{2}$ using straightedge and compass. This is not to say that complete observables do not "exist." In the extremely abstract sense allowed when utilizing the axiom of choice (AC), complete observables do exist. However,

for a mathematical object to be useful in doing physics, it should also be amenable to some kind of description with analytic tools. In a sense, when an object exists only by the power of AC, then for what concerns physics it is as useful as if it did not exist. From this point of view, the following is even more troubling.

Theorem 2.—The statement "no complete concrete observable for $\mathcal{S}$ exists" is consistent with ZF + DC.

Here ZF stands for the usual Zermelo-Fraenkel axioms of set theory and DC stands for the axiom of dependent choice: a "fragment" of AC that is needed even for basic real analysis on the Euclidean space. Theorem 2 is proved in ZF + AC (ZFC) and it highlights the nonconstructive nature of complete observables: any mathematical proof of the statement that complete observables merely "exist," has to make use of the "full" strength of AC.

Importantly, *both theorems above hold even if we restrict $\mathcal{S}$ to be the family of vacuum solutions (Solutions with vanishing stress-energy tensor and cosmological constant $\Lambda = 0$. For $\Lambda \neq 0$ we refer to the discussion.) on* $\mathbb{R}^4$. That is, the problem can be traced to the local degrees of freedom present in the vacuum theory. This is a feature that is particular to four dimensions, and perhaps higher. Indeed, it is in sharp contrast to the vacuum theory on $\mathbb{R}^3$, which trivially admits complete observables, as its only geodesically complete solution is the Minkowski spacetime. Another important point is that the above results immediately extend to incompleteness theorems for *countable families* of concrete and definable observables.

Theorems 1 and 2 follow from Lemma 4, a stronger but more technical version of Theorem 1. All three results are proved for an arbitrary space of solutions $\mathcal{S}$ which is "rich," see below. In Theorem 3 we show that the family of gravitational plane waves is rich. This implies that the vacuum sector of solutions is also rich. The proof of Theorem 3 is given in the Supplemental Material [24], where we also show that the family of Robertson-Walker spacetimes in any dimension $d \geq 2$ is rich.

Our results do not imply that all questions regarding the definability of observables have been addressed. In closing, we discuss their reach and speculate on strategies for trying to circumvent incompleteness, by relaxing the notion of "observables" or by restricting the space $\mathcal{S}$ of "acceptable" solutions. We put forth a series of open problems which aim to form the backbone of a research program to determine the intrinsic complexity of general covariance and help identify quantization procedures that could be implemented constructively.

The problem of observables.—Originating in the work of Bergmann [8,10,11], the problem of observables refers to the problem of identifying those "*functions (or functionals) of field variables that are invariant with respect to coordinate transformations*" [12]. Formally, an observable for a collection $\mathcal{S}$ of metric component fields is any function $f\colon \mathcal{S} \to R$ to a set R, so that for all $g_{\mu\nu}, \tilde{g}_{\rho\sigma} \in \mathcal{S}$

$$g_{\mu\nu} \simeq_{\text{diff}} \tilde{g}_{\rho\sigma} \Rightarrow f(g_{\mu\nu}) = f(\tilde{g}_{\rho\sigma}). \tag{1}$$

We write $\tilde{g}_{\rho\sigma} \simeq_{\text{diff}} g_{\mu\nu}$ whenever there exists a smooth change of coordinates $\tilde{x}^\xi = \tilde{x}^\xi(x^\eta)$ so that

$$g_{\mu\nu}(x^\eta) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma}(\tilde{x}^\xi). \tag{2}$$

The goal in Bergmann's program was to piece together a complete family of observables. That is, enough observables to tell apart different geometries represented in $\mathcal{S}$, similarly to how Komar mass [25] classifies Schwarzschild spacetimes. Since the notions of concretness and defibability below are closed under countable products, we can always replace a list $f_1, \ldots f_n, \ldots$ of observables with a single observable $f = \otimes_n f_n$. Hence, it suffices to consider completeness in the context of a single observable.

Completeness.—An observable $f\colon \mathcal{S} \to R$ is complete for $\mathcal{S}$ if, for all $g_{\mu\nu}, \tilde{g}_{\rho\sigma} \in \mathcal{S}$, we can strengthen (1) to

$$g_{\mu\nu} \simeq_{\text{diff}} \tilde{g}_{\rho\sigma} \Leftrightarrow f(g_{\mu\nu}) = f(\tilde{g}_{\rho\sigma}). \tag{3}$$

Without imposing any further restrictions on the "concreteness" of the range R and the "definability" of f, any space of solutions $\mathcal{S}$ admits a complete observable. For example, one can always take R to be the "abstract" collection of all equivalence classes represented in $\mathcal{S}$

$$[g_{\mu\nu}]_{\text{diff}} := \{\tilde{g}_{\rho\sigma} \in \mathcal{S} : \tilde{g}_{\rho\sigma} \simeq_{\text{diff}} g_{\mu\nu}\}, \tag{4}$$

and consider the complete observable that is given by the assignment $g_{\mu\nu} \mapsto [g_{\mu\nu}]_{\text{diff}}$. Or, one can take $R = \mathbb{R}$ to be the more "concrete" space of all real numbers, and use AC to build a complete $\mathbb{R}$-valued observable.

To rule out such extreme "solutions" to the problem of observables, we will next require that observables are concrete and definable. For these notions, as well as for a few more technical points later on, we will need some nomenclature from descriptive set theory [26].

Elements of descriptive set theory.—Let X be a topological space and let $A \subseteq X$. Then, A is nowhere dense if the complement of its closure is dense in X; meager if it is a countable union of nowhere dense sets; comeager if its complement is meager; Borel if it is in the smallest σ-algebra of subsets of X that contains the open sets; Baire-measurable if it is in the smallest σ-algebra of subsets of X that contains both the open and the nowhere dense subsets of X. A map $f\colon X \to Y$ between topological spaces is Borel—respectively, Baire-measurable—if so is $f^{-1}(U)$, for every open $U \subseteq Y$. We are particularly interested in Polish spaces, where these notions are well behaved. A Polish space is a topological space X whose topology is separable and completely metrizable.

The Borel structure on $\mathcal{S}$.—Let $\text{Ein}(M)$ denote the collection of all smooth spacetimes supported on a smooth

manifold M. In what follows, we assume that $\mathcal{S}$ is a subset of $\mathrm{Ein}(M)$, for some fixed M. We denote by τ the C^∞ compact-open topology on $\mathcal{S}$. Specifically, let $C^\infty(M,N)$ be the Polish space of all smooth maps $M \to N$ between two manifolds M, N endowed with the C^∞ compact-open topology. A basic open $U_{f,K,n,\varepsilon} \subseteq C^\infty(M,N)$ consists of all $g \in C^\infty(M,N)$ whose derivatives up to degree n on the compact $K \subseteq M$ are ε-close with those of f [27]. With the usual identifications we view $\mathcal{S}$ as a subset of $C^\infty(M,N)$, where $N := (TM \otimes TM)^*$. Then $\mathcal{S}$ inherits from $C^\infty(M,N)$ the C^∞ compact-open topology. This topology induces on $\mathcal{S}$ the σ-algebra of Borel sets on which we base the notion of definable observables below.

Theorem 1 is a statement about the Borel sets on $\mathrm{Ein}(M)$, so it implicitly relies on τ. Note that τ is a rather weak topology and of no obvious physical relevance. Here, it is used as a convenient "basis" for spanning the σ algebra of Borel sets, which is a more robust structure. For example, any stronger Polish topology $\tau' \supseteq \tau$ on $\mathrm{Ein}(M)$ induces the exact same Borel sets and all physical spacetimes, i.e., sets of the form (4), are themselves Borel (These follow from [26] (Exercise 15.4) and [28] (Proposition 3.1.10).). On the other hand, once $\mathcal{S}$ has been established to be rich, the conclusion of Theorem 2 is agnostic both on the topology and the Borel structure on $\mathcal{S}$.

Concreteness.—An observable $f\colon \mathcal{S} \to R$ is concrete if it takes values in a Polish space. Restricting R to be a Polish space is a generic requirement. For instance, setting R to be either of the Polish spaces $\mathbb{R}$ or $C^\infty(M,\mathbb{R})$ we recover classical definitions of observables [11,12]. However, our definition of concreteness allows observables to take values in much more general spaces, as Polish spaces include spaces of distributions, separable Banach spaces, as well as a vast array of more "exotic" objects like the Cantor set. Restricting R to be a Polish space is natural as well from the viewpoint of descriptive set theory, which considers Polish spaces to be "well behaved" incarnations of uncountable sets. This is because their points are controlled by a countable dense subset, similarly to how the rationals control the reals.

Definability.—A concrete observable $f\colon \mathcal{S} \to R$ is *Borel definable* if it is a Borel map when $\mathcal{S}$ is endowed with the C^∞ compact-open topology. These are exactly those observables which admit a description by an explicit formula in the language of analysis, in the following sense.

The descriptive power of analysis is rooted in its ability to implement limiting procedures. For example, defining the value $f(g_{\mu\nu})$ of an ADM observable f requires 'taking limits' at least once, as it is given as the limit of integrals over a sequence of compact regions of the manifold [19]. Maps whose definition relies on limiting procedures of length two can already be surprisingly complex. For instance, the characteristic map $\chi_{\mathbb{Q}}\colon \mathbb{R} \to \mathbb{R}$ of the rationals can be expressed as $\chi_{\mathbb{Q}}(x) = \lim_n \lim_m \cos^{2m}(\pi n! x)$.

Borel maps are precisely those maps which are attained by allowing iterations of such limiting procedures for any "number" ξ of times, where ξ ranges over the set ω_1 of all countable ordinals [26] (Theorem 24.3).

Rich families.—Theorems 1 and 2 concern any family of solutions $\mathcal{S}$ which is *rich*. For the definition of this notion we recall a few more elements from *invariant* descriptive set theory [28,29].

A Polish group G is a topological group whose topology is Polish. A Polish G space is a continuous action $G \curvearrowright X$ of the Polish group G on a Polish space X. The associated *orbit equivalence relation* $\simeq_G$ on X is given by setting $x \simeq_G y$ if and only if x, y are in the same orbit, i.e., if $Gx = Gy$. We say that $G \curvearrowright X$ is *generically ergodic* if: (1) there is $x \in X$, whose orbit Gx is dense in X; (2) for every $x \in X$, the orbit Gx is meager in X.

A family $\mathcal{S}$ of spacetimes is called *rich*, if there exists a generically ergodic Polish G-space $G \curvearrowright X$ together with a *Borel reduction* r from $(X, \simeq_G)$ to $(\mathcal{S}, \simeq_{\mathrm{diff}})$. That is, a Borel map $r\colon X \to \mathcal{S}$ so that for all $\alpha, \beta \in X$ we have

$$\alpha \simeq_G \beta \Leftrightarrow r(\alpha) \simeq_{\mathrm{diff}} r(\beta). \tag{5}$$

One way for $\mathcal{S}$ to be rich is if the action $\mathrm{Diff}(M) \curvearrowright \mathcal{S}$ of the diffeomorphism group, implementing (2), is itself generically ergodic. In this case, it is very difficult to tell different orbits apart as any open set in $\mathcal{S}$ will be intersected by almost every orbit, and the mental picture which depicts orbits as "curves" should better be replaced with that of a "knotted ball of yarn." That being said, for $\mathcal{S}$ to be rich it is enough for this tangling between orbits to occur just in some "corner" of $\mathcal{S}$.

The family of vacuum solutions.—Before we turn to the proofs of Theorems 1 and 2 we would like to establish that rich families of solutions exist and hence, these theorems are not vacuous. To the reader familiar with these arguments, it is probably not that surprising that rich families exist. Indeed, without imposing any restrictions on the stress-energy tensors of the members of $\mathcal{S}$, one can simply concoct rich families of energy-momentum distributions that generate ergodic behavior within $\mathcal{S}$. An example that illustrates this can be found in the Supplemental Material [24], where we show that the family of cosmological Robertson-Walker spacetimes [30] is rich.

Perhaps what is more surprising is that the problem is already present in the vacuum sector. That is, even when all members of $\mathcal{S}$ have a vanishing stress-energy tensor (we assume the cosmological constant Λ to be zero).

Theorem 3.—Vacuum solutions on $\mathbb{R}^4$ form a rich family.

Theorem 3 implies that any collection of spacetimes which contains the vacuum solutions on $\mathbb{R}^4$ is rich. In particular, this provides yet another proof that the collection of all spacetimes is rich—one which does not rely on the collection of Robertson-Walker spacetimes.

We now sketch the proof of Theorem 3, detailed in the Supplemental Material [24]. Consider the family GPW of gravitational plane waves on $\mathbb{R}^4$. These are all spacetimes which can be written in Brinkmann form [31] as

$$H(u,x,y)du^2 + dudv + dx^2 + dy^2, \tag{6}$$

where H is a smooth map that is quadratic in x, y and satisfies $H_{xx} - H_{yy} = 0$. Since members of GPW are vacuum solutions [32], it suffices to see that GPW is rich.

As a model of generic ergodicity we will use the Bernoulli shift $\mathbb{Z} \curvearrowright X$, where $X \coloneqq \{0,1\}^{\mathbb{Z}}$ is the space of all integer-indexed sequences of 0,1 endowed with the product topology. The action $\mathbb{Z} \curvearrowright X$ is implemented by $(k,\alpha) \coloneqq k\alpha$, where $(k\alpha)(n) = \alpha(n-k)$. Hence,

$$\alpha \simeq_{\mathbb{Z}} \beta \Leftrightarrow \exists\, k \in \mathbb{Z} \quad \forall\ n \in \mathbb{Z} \alpha(n-k) = \beta(n). \tag{7}$$

To see that $\mathbb{Z} \curvearrowright X$ is generically ergodic, notice that its orbits are countable and that for the random $\alpha \in X$ in the sense of the coin-flip measure, α admits a dense orbit.

We can now associate a smooth map $W_\alpha \colon \mathbb{R} \to \mathbb{R}$ to each $\alpha \in X$, so that W_α reflects the distribution of 0,1's in the sequence α; see Fig. 1. We define a Borel reduction $r \colon X \to \mathrm{GPW}$ by setting $r(\alpha)$ to be the metric with $H(u,x,y) \coloneqq W_\alpha(u)xy$ in (6). The map r is in fact continuous since compact regions of $r(\alpha)$ are determined by finite regions of α. It is straightforward to check that r satisfies the $(\Rightarrow)$ direction of (5). The $(\Leftarrow)$ direction of (5) also holds and is given in the Supplemental Material [24]. This part is more technical and it relies on the theory of Lie symmetries of planes waves from [32–34].

In summary, we showed that the highly tangled orbit structure of the Bernoulli shift is also present in the orbit structure of general covariance, even after restricting to the vacuum sector. Our incompleteness theorems are a consequence of this complex orbit structure.

Incompleteness of observables in general relativity.—We are now ready to see how Theorems 1 and 2 come about. The proofs follow from standard arguments used in invariant descriptive set theory [28,29]. We sketch these arguments here for completeness.

Let $\mathcal{S}$ be a rich family and fix $r \colon X \to \mathcal{S}$ as in (5). Let now $f \colon \mathcal{S} \to R$ be any complete concrete observable and precompose $\hat{f} \coloneqq f \circ r$ to get a map $\hat{f} \colon X \to R$. By (3) and (5), for every $\alpha, \beta \in X$ we have that

$$\alpha \simeq_G \beta \Leftrightarrow \hat{f}(\alpha) = \hat{f}(\alpha). \tag{8}$$

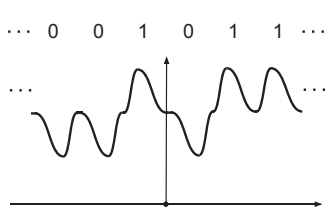

FIG. 1. To each $\alpha \in X$ we associate a smooth W_α.

We will make use of the following classical result. For the proof of its first part, see Ref. [29] (Theorem 3.2) or [28]. The second part of the statement follows directly from the first, as the comeager $C \subseteq X$ cannot be covered by a union of finitely many meager orbits; see, e.g., [26].

Lemma 4.—Let $\hat{f} \colon X \to R$ be a Baire-measurable map which satisfies (8). Then, there exists a comeager set $C \subseteq X$ on which $\hat{f}$ is constant. In particular, there are $\alpha, \beta \in X$ with $\alpha \not\simeq_G \beta$ and $\hat{f}(\alpha) = \hat{f}(\beta)$.

Theorem 1 follows from Lemma 4. Indeed, assume that the complete observable $f \colon \mathcal{S} \to R$ is Borel definable. It follows that the associated $\hat{f}$ above is Baire-measurable and hence, by Lemma 4, we have $\alpha, \beta \in X$ with $\alpha \not\simeq G\beta$ and $\hat{f}(\alpha) = \hat{f}(\beta)$. But this contradicts (8).

Theorem 2 follows from Lemma 4 and the fact that there exists, provably from ZFC, a model of ZF + DC in which every map $\hat{f} \colon X \to R$ is Baire-measurable [35,36]. Since Lemma 4 is provable in ZF + DC, all maps in this model satisfy the last statement of Lemma 4, and hence, they have to fail (8). Notice the resemblance of this proof with the usual consistency proof of the first four axioms of Euclid with the negation of the parallel postulate, which uses Euclidean geometry to construct a model of non-Euclidean geometry such as the Poincare disc.

Discusssion.—Similar to Gödel's first incompleteness theorem which shows that no "rich enough" fragment of arithmetic admits a consistent extension that is both complete and computable, Theorems 1 and 2 show that no "rich enough" collection of spacetimes admits an observable that is both complete and definable.

Given the central role that various types of observables play in quantization procedures [4–7], a natural followup question is, how much of Bergmann's program for "*the identification and systematic exploitation of the observables*" [37] can be salvaged, and in what precise form?

Some first attempts to preserve definability while trying to maintain completeness of observables on large collections of spacetimes can be ruled out merely on the basis of how flexible is the notion of concrete observable in Theorems 1 and 2. This includes attempts involving gauge–fixing procedures $s \colon \mathcal{S} \to \mathcal{S} \subseteq \mathrm{Ein}(M)$ which select a single representative $s(g_{\mu\nu}) \in [g_{\mu\nu}]_{\mathrm{diff}}$ from each class (4); or the use of families of observables $\mathcal{F} = \{f_i : i \in I\}$ in place of a single observable f. Indeed, since $\mathrm{Ein}(M)$ is a Polish space, Theorems 1 and 2 imply that no gauge-fixing map s can be definable. Similarly, since the notions of concreteness and definability are closed under countable products, a countable $\mathcal{F}$ can be replaced by the single observable $\otimes_i f_i$. In fact, a technical elaboration shows that our incompleteness results extend to uncountable families of observables, so long as the parametrization $i \mapsto f_i$ is "definable enough".

One could also try to circumvent these issues by endowing $\mathcal{S}$ with a different topology. This would have to be a topology so "fine" that it admits a Borel definable complete observable $f\colon \mathcal{S} \to \mathbb{R}$. While this is certainly doable—for example, one may consider the discrete topology, Theorem 2 raises the question of whether Borel maps in this new topology would be amenable to computations.

On the other hand it is not at all clear—and in fact almost certainly not a good idea—that a successful theory of gravity should predicate on the definability of complete observables over large collections of spacetimes. In fact, similar incompleteness phenomena may occur in electromagnetism, for example, if one takes observables to be all the Poincare-invariant quantities defined on the space of all Maxwell solutions without imposing any boundary conditions. Yet this does not seem it would cause any issues to the theory.

We are left with many questions. How much do we need to restrict the collection of spacetimes $\mathcal{S}$ before it admits definable complete observables? Are there generalizations of the notion of an observable that allow to classify rich collections of spacetimes definably? Is there a formal sense in which general covariance is strictly more complex than the gauge induced, say, by actions of the Poincare group? With such questions in mind, we next discuss two natural directions in which this work could be extended.

Several interesting and physically relevant collections of spacetimes might admit definable complete observables. Consider for example the collections AF, DCD, VS_+, VS_- of all spacetimes on $\mathbb{R}^4$ which are asymptotically flat, maximal globally hyperbolic developments of a Cauchy dataset, and Λ-vacuum solutions with positive or negative cosmological constant Λ, respectively. In the context of Theorems 1,2 it is natural to ask the following:

Problem 5.—Which of the families above are rich?

Preliminary results not provided here suggest that DCD is indeed rich, but this requires some different techniques than the ones presented in this Letter. Moreover, in view of Penrose's method for approximating regions of any spacetime near a null geodesic via plane waves [38] the proof for Theorem 3 is likely generalizable to other families of spacetimes.

The fully invariant observables considered here can be relaxed to "equivariant" types of observables. Let G be a Polish group. A *G-observable for* $\mathcal{S}$ is a Polish G space $G \curvearrowright R$ together with a map $f\colon \mathcal{S} \to R$ so that

$$g_{\mu\nu} \simeq_{\mathrm{diff}} \tilde{g}_{\rho\sigma} \Leftrightarrow f(g_{\mu\nu}) \simeq_G f(\tilde{g}_{\rho\sigma}) \tag{9}$$

holds for all $g_{\mu\nu}, \tilde{g}_{\rho\sigma} \in \mathcal{S}$. Instances of G observables have been considered in the literature before. For example, some modern criticisms to Bergmann's program [39] maintain the use of scalars $R = C^\infty(\mathbb{R}^4, \mathbb{R})$ as values for observables, but replace the equality in the right-hand side of (1) with covariance $\simeq_{\mathrm{diff}}$ of scalars.

Families of spacetimes which are incomplete in the sense of Theorems 1 and 2, may still admit definable G observables, for various groups G. If G can be chosen to have nice representation-theoretic properties, then G observables can still be promoted to operators on a Hilbert space, and hence be used for quantization. An elaboration on Theorem 1 shows that rich families $\mathcal{S}$ do not admit G observables for compact G [28] (Exercise 5.4.5). But, could G be locally compact? Or could G be the unitary group of a separable C^* algebra?

Problem 6.—For which Polish groups G and $\mathcal{S} \subseteq \mathrm{Ein}(\mathbb{R}^4)$ there exists a definable and complete G observable for $\mathcal{S}$?

Investigating Problem 6 may also allow us to formally address whether the problem of observables is "more severe" in general relativity than in, say, electromagnetism, e.g., by taking G to be the gauge group underlying the latter.

Recent breakthroughs [40–43] in the complexity theory of Polish group actions provide sharp tools for investigating the problems listed above. We hope that this work may form the basis of a research program for cross-pollinating theoretical physics and descriptive set theory in order to analyze the complexity of general covariance.

In summary, the roots of the problem of observables run deep and suggest another beautiful connection of mathematics with physics inspired by general relativity. This work opens many unexplored future directions. Perhaps further investigation may also help identify types of quantization recipes that can be implemented definably.

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[1] A. Einstein, Prinzipielles zur verallgemeinerten relativitaetstheorie, Phys. Z. **15**, 176 (1914).

[2] A. Einstein, Die formale grundlage der allgemeinen relativitaetstheorie, Preuss. Akad. der Wiss. **2**, 1030 (1914).

[3] J. Norton, Einstein, the hole argument and the reality of space, in *Measurement, Realism and Objectivity: Essays on Measurement in the Social and Physical Sciences*, edited by J. Forge (Springer, Netherlands, 1987), pp. 153–188.

[4] K. V. Kuchař, Canonical quantum gravity, arXiv:gr-qc/9304012.

[5] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2007).

[6] S. Carlip, Quantum gravity: A progress report, Rep. Prog. Phys. **64**, 885 (2001).

[7] A. Ashtekar, M. Reuter, and C. Rovelli, From general relativity to quantum gravity, in *General Relativity and Gravitation: A Centennial Perspective*, edited by A. Ashtekar, B. K. Berger, J. Isenberg, and M. MacCallum (Cambridge University Press, Cambridge, England, 2015), pp. 553–611.

[8] P. G. Bergmann, Introduction of "true observables" into the quantum field equations, Il Nuovo Cimento (1955–1965) **3**, 1177 (1956).

[9] A. Komar, Construction of a complete set of independent observables in the general theory of relativity, Phys. Rev. **111**, 1182 (1958).

[10] P. G. Bergmann, Observables in general relativity, Rev. Mod. Phys. **33**, 510 (1961).

[11] P. G. Bergmann and A. Komar, Observables and commutation relations, in *Les Theories relativistes de la Gravitation, Royaumont* (Centre National de la Recherche Scientifique, Paris, 1962), pp. 309–325.

[12] P. G. Bergmann and A. I. Janis, Subsidiary conditions in covariant theories, Phys. Rev. **111**, 1191 (1958).

[13] C. Rovelli, Partial observables, Phys. Rev. D **65**, 124013 (2002).

[14] B. Dittrich, Partial and complete observables for canonical general relativity, Classical Quantum Gravity **23**, 6155 (2006).

[15] T. Thiemann, Reduced phase space quantization and Dirac observables, Classical Quantum Gravity **23**, 1163 (2006).

[16] S. B. Giddings, D. Marolf, and J. B. Hartle, Observables in effective gravity, Phys. Rev. D **74**, 064018 (2006).

[17] B. Dittrich, Partial and complete observables for Hamiltonian constrained systems, Gen. Relativ. Gravit. **39**, 1891 (2007).

[18] I. Khavkine, Local and gauge invariant observables in gravity, Classical Quantum Gravity **32**, 185019 (2015).

[19] R. L. Arnowitt, S. Deser, and C. W. Misner, The dynamics of general relativity, *Gravitation: An Introduction to Current Research* (Wiley, New York, 1962), p. 227.

[20] J. D. Brown and K. V. Kuchar, Dust as a standard of space and time in canonical quantum gravity, Phys. Rev. D **51**, 5600 (1994).

[21] V. Husain and T. Pawłowski, Time and a physical Hamiltonian for quantum gravity, Phys. Rev. Lett. **108**, 141301 (2012).

[22] C. G. Torre, Gravitational observables and local symmetries, Phys. Rev. D **48**, R2373 (1993).

[23] B. Dittrich, P. A. Höhn, T. A. Koslowski, and M. I. Nelson, Can chaos be observed in quantum gravity?, Phys. Lett. B **769**, 554 (2017).

[24] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.131.171402 for proofs that the families of gravitational plane waves and of Robertson-Walker spacetimes are both rich.

[25] A. Komar, Positive-definite energy density and global consequences for general relativity, Phys. Rev. **129**, 1873 (1963).

[26] A. S. Kechris, *Classical Descriptive Set Theory* (Springer-Verlag, Berlin, 1995).

[27] M. Hirsch, *Differential Topology*, Graduate Texts in Mathematics Vol. 33 (Wiley, New York, 1994).

[28] S. Gao, *Invariant Descriptive Set Theory* (Taylor & Francis, London, 2008).

[29] G. Hjorth, *Classification and Orbit Equivalence Relations* (American Mathematical Society, Providence, 2000), Vol. 75.

[30] S. M. Carroll, *Spacetime and Geometry* (Cambridge University Press, Cambridge, England, 2019).

[31] H. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math. Ann. **94**, 119 (1925).

[32] P. Jordan, J. Ehlers, and W. Kundt, Republication of: Exact solutions of the field equations of the general theory of relativity, Gen. Relativ. Gravit. **41**, 2191 (2009).

[33] R. Sippel and H. Goenner, Symmetry classes OFPP-waves, Gen. Relativ. Gravit. **18**, 1229 (1986).

[34] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2009).

[35] R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, Ann. Math. **92**, 1 (1970).

[36] S. Shelah, Can you take Solovay's inaccessible away?, Isr. J. Math. **48**, 1 (1984).

[37] P. G. Bergmann, Physics and geometry, in *Logic, Methodology and Philosophy of Science, Jerusalem*, edited by Y. Bar-Hillel(North-Holland, Amsterdam, 1965), pp. 343–346.

[38] R. Penrose, Any space-time has a plane wave as a limit, in *Differential Geometry and Relativity: A Volume in Honour of André Lichnerowicz on His 60th Birthday*, edited by M. Cahen and M. Flato (Springer, Netherlands, 1976), pp. 271–275.

[39] J. B. Pitts, What are observables in Hamiltonian Einstein-Maxwell theory?, Found. Phys. **49**, 786 (2019).

[40] G. Hjorth, A dichotomy theorem for being essentially countable, in *Contemporary Mathematics*, edited by A. Blass and Y. Zhang (American Mathematical Society, Providence, RI, 1999, 2005), Vol. 380, pp. 109–128.

[41] M. Lupini and A. Panagiotopoulos, Games orbits play and obstructions to Borel reducibility, Groups, Geom. Dyn. **12**, 1461 (2018).

[42] A. Allison and A. Panagiotopoulos, Dynamical obstructions to classification by (co)homology and other TSI-group invariants, Trans. Am. Math. Soc. **374**, 8793 (2021).

[43] A. S. Kechris, M. Malicki, A. Panagiotopoulos, and J. Zielinski, On Polish groups admitting non-essentially countable actions, Ergodic Theory Dyn. Syst. **42**, 180 (2022).