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CONTINUOUS MODEL OF OPINION DYNAMICS WITH CONVICTIONS

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Abstract. In this note we study a new kinetic model of opinion dynamics. The model incorporates two forces – alignment of opinions under all-to-all communication driving the system to a consensus, and Rayleigh type friction force that drives each 'player' to its fixed conviction value. The balance between these forces creates a non-trivial limiting outcome.

We establish existence of a global mono-opinion state, whereby any initial distribution of opinions for each conviction value aggregates to the Dirac measure concentrated on a single opinion. We identify several cases where such a state is unique and depends continuously on the initial distribution of convictions. Several regularity properties of the limiting distribution of opinions are presented.

1. **Introduction.** In this note we study regularity and long time behavior of solutions to the following transport equation

$$\partial_t \mu + \partial_y (u(\mu)\mu) = 0, \tag{1}$$

where $\mu = \mu(t, y, \theta)$ is a measure on $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$ for each $t \geqslant 0$, and

$$u(\mu) = \partial_y (W * \mu + \sigma V), \tag{2}$$

$$W(y) = -\frac{1}{2} y^2, \quad V(y, \theta) = \frac{1}{2}\theta y^2 - \frac{1}{p+2} y^{p+2}. \tag{3}$$

Here, σ and p are positive parameters. The variable θ can be thought of as a parameter as well, however, note that the convolution $W * \mu$ couples all the measures together across the family.

The motivation for this particular model is twofold. First, it represents the kinetic counterpart of the corresponding discrete dynamical system:

$$\dot{y}_i = \frac{1}{N} \sum_{k=1}^{N} (y_k - y_i) + \sigma(\theta_i - y_i^p) y_i, \tag{4}$$

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where θ_i's are constant parameters. In fact, the empirical distributions

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{\theta_i} \otimes \delta_{y_i(t)} \tag{5}$$

solve (1) in the weak sense if and only if y_i's solve (4), and formally the mean-field limit $\mu^N \to \mu$ yields a solution to (1). The discrete system (4) was derived in [15] as the effective limiting dynamics of the speeds $y_i = |v_i|$ of agents governed by the corresponding alignment model with all-to-all communication and Rayleigh friction/self-propulsion force

$$\dot{x}_i = v_i, \qquad \dot{v}_i = \frac{1}{N}\sum_{k=1}^N (v_k - v_i) + \sigma(\theta_i - |v_i|^p)v_i. \tag{6}$$

When all velocities v_i belong to a sector of opening less than π, the vectors v_i will dynamically align themselves along one direction $v_i \sim y_i \hat{v}$, where $y_i = |v_i|$, and the evolution of y_i is governed by (4) up to an exponentially decaying force.

The system (6) is a very important example of a collective behavior model of Cucker-Smale type that was introduced in [8, 9] and studied under this particular forcing in the earlier works [7, 13, 15, 17]. Kinetic limits in the context of forced systems including potential interaction and friction/self-propulsion were established in [3, 4, 5, 6]. The first order conservation models of type (1) appeared in the context of aggregation models in the works of Topaz et al [20, 21]. All these works correspond to the non-parametric case, i.e. $\theta = \text{const}$, where friction force appears. The variable θ case, beyond the work [15], was considered more recently in [17] where propagation of chaos with quantified rate was established for sectorial solutions, as described above, to the full Cucker-Smale system.

Our second motivation for this study comes from interpretation of the equation (1) as a continuous model of opinion dynamics. To put it in prospective of a vast existing literature let us compare it to several related models. The classical Hegselmann-Krause model [14] focuses on exchange of opinions only under local environmental averaging protocol – one that is based on interactions of agents with close views. A more elaborate protocol of opinion updates based on randomization of interaction schemes between groups were studied in works of Galam, see [11] and references therein. Equations (1), (4) belong to a class of models that incorporate 'conviction' parameter θ whose role is to pull the opinion of an agent to its value while remaining unchanged. As far as we can trace such models, also called models with 'stubborn' agents, appeared first in the work of Friedkin and Johnsen [10] and later became a staple in many studies on opinion dynamics, see for example [2, 12] and literature therein. In those works, however, the conviction pull is defined by a linear force, which in our notation would correspond to a constant multiple of $\theta_i - y_i$. The model proposed here uses the most basic all-to-all communication rule, but it incorporates the nonlinear conviction force. Phenomenologically it describes the effect of strengthening the pull towards conviction as the latter becomes more extreme. Such a model is necessarily not Galilean invariant and is fully non-linear, which makes the analysis of an 'agreement' or even its existence a challenging problem.

For the discrete variant (4) the problem was addressed in [15] where the model was interpreted as a non-cooperative game in the sense of Nash [16]. The limiting state of opinions is characterized as a Nash equilibrium – an agreement deviation from which is of no benefit to any player, although may not necessarily be the most optimal value to anyone. Clearly, such an agreement is not expected to be a perfect consensus due to adherance to convictions. The existence, uniqueness and stability of the equilibrium was proved in [15] using the Brouwer topological degree theory.

Theorem 1.1. *For any positive set of parameters* $(\theta_1,\dots,\theta_N,\sigma)\in\mathbb{R}_+^N\times\mathbb{R}_+$ *there exists a unique stable Nash equilibrium* $\mathbf{y}^*=(y_1^*,\dots,y_N^*)\in\mathbb{R}_+^N$ *of system* (4) *relative to payoffs*

$$p_i(\mathbf{y})=\sigma\left(\frac{1}{2}\theta_i y_i^2-\frac{1}{p+2}y_i^{p+2}\right)-\frac{1}{2}\left(\bar{y}-y_i\right)^2,\quad \bar{y}=\frac{1}{N}\sum_j y_j. \tag{7}$$

Any solutions with positive initial data will remain positive and converge to $\mathbf{y}^*$ *as* $t\to\infty$. *Moreover, if* $\theta_i=\theta_j$ *then* $y_i=y_j$.

The main difficulty in establishing the result is that the natural gradient structure of (4)

$$\dot{\mathbf{y}}=-\nabla\Phi(\mathbf{y})$$

involves energy $\Phi(\mathbf{y})=\sum_{i=1}^N p_i(\mathbf{y})$ that is not globally convex.

The purpose of this present study is to recreate a similar result for the kinetic model (1). First, we justify it as the mean-field model of (4) by establishing the limit $\mu^N\to\mu$. Such analysis is rather standard for first-order models, which is done by proving a general weak-Lipschitzness of the solution map $\mu_0\to\mu_t$ with respect to the Wasserstein-1 metric, [1],

$$\mathcal{W}_1(\mu_t,\nu_t)\leqslant Ce^{ct}\mathcal{W}_1(\mu_0,\nu_0),\quad t>0,$$

see Section 2. However, the details include a quantitative maximum principle of Lemma 2.2 that will be used later in the paper. So, we present the argument in full.

Our primary focus will be on the analysis of the Nash equilibrium of the continuous model (1). To state the main result let us fix some notation. Let us observe that the θ-marginal given by

$$\mathrm{d}\pi(\theta,t)=\int_{y\in\mathbb{R}_+}\mathrm{d}\mu(y,\theta,t), \tag{8}$$

is conserved $\frac{\mathrm{d}}{\mathrm{d}t}\pi=0$. This is a reflection of the principle that convictions do not change. By the disintegration theorem, see [1], for π-a.e. $\theta\in\mathbb{R}_+$ there is a unique family of probability 'slicing' measures $\{\mu^\theta\}_{\theta\in\mathbb{R}_+}$ such that $\mu=\mu^\theta\otimes\mathrm{d}\pi(\theta)$, that is,

$$\int_\Omega\varphi(y,\theta)\,\mathrm{d}\mu(y,\theta)=\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}\varphi(y,\theta)\,\mathrm{d}\mu^\theta(y)\,\mathrm{d}\pi(\theta),\quad \forall\varphi\in C_0(\Omega). \tag{9}$$

Each measure μ^θ represents distribution of opinions of agents that share the same conviction θ.

Our main result states that each of these slicing measures approaches a mono-opinion state, i.e. a Dirac measure at a fixed point $g(\theta)$ for some smooth strictly increasing function g. In other words,

$$\mu_t\to\delta_{g(\theta)}\otimes\mathrm{d}\pi(\theta),\quad t\to\infty.$$

To put it formally we assume that our initial measure is located within a box compactly inside Ω:

$$\operatorname{supp}\mu_0\subset R_0:=[y_{\min},y_{\max}]\times[\theta_{\min},\theta_{\max}],\qquad y_{\min},\theta_{\min}>0. \tag{10}$$

Theorem 1.2. *Let μ be the measure-valued solution to (1) with initial data satisfying (10). Then there exists a function $g \in C^\infty([\theta_{\min}, \theta_{\max}])$ strictly increasing such that*

$$\sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) \leqslant C e^{-ct}, \quad t > 0, \tag{11}$$

where $C, c > 0$ depend only on μ_0 and the parameters of the model. Moreover, under the assumption

$$\sigma \theta_{\min} > \frac{p+1}{p} \quad or \quad \frac{\theta_{\max}}{\theta_{\min}} < p+1. \tag{12}$$

the map $\pi \to g$ is Lipschitz,

$$\sup_{\theta \in [\theta_{\min}, \theta_{\max}]} |g(\theta) - \tilde{g}(\theta)| \leqslant C \mathcal{W}_1(\pi, \tilde{\pi}). \tag{13}$$

In particular g is unique for each π.

Structurally, the equation (1) can be considered as a fibered gradient system in the sense of [18] where the fibers are parametrized by convictions θ and the free energy is given by

$$\mathcal{E}(\mu) = \frac{1}{2} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} W(x-y) \, \mathrm{d}\mu(y,\theta) \, \mathrm{d}\mu(x,\eta) - \sigma \int_{\mathbb{R}_+^2} V(y,\theta) \, \mathrm{d}\mu(y,\theta). \tag{14}$$

The equation can be written as a gradient dynamics

$$\partial_t \mu = -\partial \mathcal{E}(\mu),$$

where ∂ is understood as a fibered variant of the Fréchet subdifferential relative to a properly defined fibered Wasserstein distance. Without getting further into details one can obtain directly the following energy dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E} = - \int_{\mathbb{R}_+ \times \mathbb{R}_+} |u(\mu)|^2 \, \mathrm{d}\mu(y,\theta). \tag{15}$$

The law demonstrates perpetual descent of the solution down the energy surface and suggests convergence to a local minimum. The general results of this nature were established in [18] under a properly formulated convexity condition on the energy. However, just as in the discrete case, such convexity is not always true in our settings. Therefore, the statement of Theorem 1.2 does not directly follow from the theory developed in [18]. Our method is based on the Lagrangian approach, which involves detailed analysis of asymptotic behavior of characteristics of (1). Let us note that in the discrete case the uniqueness of the limiting state is unconditional. Removing assumptions (12) for the kinetic model remains an open issue.

2. **Well-posedness and mean-field limit.** In this section, we will prove the existence of measure-valued solutions to the equation (1). First of all, let us introduce some notations and definitions. Let $\Omega = \mathbb{R}_+^2$ and denote $\mathcal{P}_0(\Omega)$ the set of probability measures on Ω which have compact support in the interior of Ω.

Definition 2.1. Given $0 \leqslant T < \infty$, a map $\mu : [0,T) \to \mathcal{P}_0(\Omega)$, $t \mapsto \mu_t$, is called a measure-valued solution to (1) with initial data μ_0 if it satisfies the following conditions:

i) μ is weakly* continuous,
ii) For any $\varphi \in C_0^\infty([0,T) \times \Omega)$ and $0 < t < T$,

$$\int_\Omega \varphi(t,y,\theta) \, \mathrm{d}\mu_t(y,\theta) = \int_\Omega \varphi(0,y,\theta) \, \mathrm{d}\mu_0(y,\theta) + \int_0^t \int_\Omega [\partial_s \varphi + u \, \partial_y \varphi] \, \mathrm{d}\mu_s(y,\theta) \, \mathrm{d}s.$$

Let us note that we do not make any specific assumptions about the class of measures we consider as solutions. In particular, μ is purely atomic, see (5) then it is easy to check that the definition of a solution is equivalent to the ODE (4).

To make further notation simpler let us observe that by making the change of variables

$$y \to \sigma^{\frac{1}{p}} y, \qquad \theta \to \sigma\theta, \qquad \mu \to \sigma^{1+\frac{1}{p}}\mu, \tag{16}$$

we can scale out the parameter σ from the equation altogether. So, from now on we can assume that $\sigma = 1$, and be mindful that all the constants that appear later eventually depend on the original parameter σ.

If $\mu : [0,T) \to \mathcal{P}_0(\Omega)$ is a measure-valued solution to (1) with initial data μ_0, by the classical transport theory, μ is a push-forward of μ_0 along characteristics (Y, Θ):

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t,y,\theta) = \int_\Omega (Y' - Y)\,\mathrm{d}\mu_0(y',\theta') + Y(\Theta - Y^p), \quad Y(0,y,\theta) = y, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t,y,\theta) = 0, \quad \Theta(0,y,\theta) = \theta. \tag{18}$$

Note that Θ is not changing in time, so in the equation (17) we can replace Θ by its initial θ and view θ as a parameter.

The local well-posedness of the system (17) - (18) follows from the standard fixed point argument for integro-differential equations and local Lipschitzness relative to continuous maps (Y, Θ) of the right hand side. Global well-posedness will follow as soon as we establish a priori bounds on the support of Y.

Our standing assumption on the initial support of μ_0 will always be (10). Let us denote

$$Y_{\max}(t) = \max_{R_0} Y(t,\cdot), \quad Y_{\min}(t) = \min_{R_0} Y(t,\cdot).$$

Note that $y_{\max} = Y_{\max}(0)$ and $y_{\min} = Y_{\min}(0)$.

Lemma 2.2. *For any solution Y to (17) on a time interval $[0,T)$, we have for all $t < T$,*

$$Y_{\max}^p \leqslant \frac{\theta_{\max} y_{\max}^p e^{p\theta_{\max} t}}{\theta_{\max} + y_{\max}^p (e^{p\theta_{\max} t} - 1)}, \tag{19}$$

$$Y_{\min}^p \geqslant \frac{\theta_{\min} y_{\min}^p e^{p\theta_{\min} t}}{\theta_{\min} + y_{\min}^p (e^{p\theta_{\min} t} - 1)}. \tag{20}$$

Proof. Evaluating (17) at a point of maximum on R_0, using Rademacher's lemma (see [19]), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} Y_{\max}^p &= pY_{\max}^{p-1} \underbrace{\int_\Omega (Y' - Y_{\max})\,\mathrm{d}\mu_0(y',\theta')}_{\leqslant 0} + Y_{\max}^p(\theta - Y_{\max}^p) \\ &\leqslant pY_{\max}^p(\theta_{\max} - Y_{\max}^p).\end{aligned}$$

The right hand side of (19) solves the above equation exactly. So, by the classical comparison principle, we obtain (19).

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} Y_{\min}^p &= pY_{\min}^{p-1} \underbrace{\int_\Omega (Y' - Y_{\min})\,\mathrm{d}\mu_0(y',\theta')}_{\geqslant 0} + Y_{\min}^p(\theta - Y_{\min}^p) \\ &\geqslant pY_{\min}^p(\theta_{\min} - Y_{\min}^p).\end{aligned}$$

The comparison principle implies (20). □

The lemma shows that on any finite time interval the characteristics will not leave Ω and in fact the image $Y(t, \operatorname{supp}\mu_0)$ will be compactly embedded in Ω and remain uniformly bounded a priori. Consequently, by extension, the system (17) - (18) is globally well-posed. By the push-forward transport, there is a global measure-valued solution to (1).

Theorem 2.3. *Given any measure $\mu_0 \in \mathcal{P}_0(\Omega)$ with (10) there exists a unique measure-valued solution to (1) with initial condition μ_0 and such that $\operatorname{supp}\mu_t \subset \Omega$ remains bounded and bounded away from $\partial\Omega$ uniformly for all times.*

Let us now show continuity of the map $\mu_0 \to \mu_t$ in weak topology, which is the basis for justification of the mean-field limit.

Lemma 2.4. *Let μ and ν be two measure-valued solutions to (1) with μ_0, ν_0 satisfying (10). Then for any $t > 0$ one has*

$$\mathcal{W}_1(\mu_t, \nu_t) \leqslant C e^{ct} \mathcal{W}_1(\mu_0, \nu_0),$$

where $C, c > 0$ depend on the initial condition and the parameters of the model.

Proof. Denote $L^\infty := L^\infty(R_0)$. Let us also denote by Y the characteristics of μ and by Z the characteristics of ν.

In what follows, C and c are constants which are varying line by line. By the definition of the Wasserstein distance, we have

$$\begin{aligned}
\mathcal{W}_1(\mu_t, \nu_t) &= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(y,\theta)\,\mathrm{d}\mu_t(y,\theta) - \int_\Omega \varphi(y,\theta)\,\mathrm{d}\nu_t(y,\theta) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(Y,\theta)\,\mathrm{d}\mu_0(y,\theta) - \int_\Omega \varphi(Z,\theta)\,\mathrm{d}\nu_0(y,\theta) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}} \leqslant 1} \left| \int_\Omega \varphi(Y,\theta)\,\mathrm{d}\mu_0(y,\theta) - \int_\Omega \varphi(Y,\theta)\,\mathrm{d}\nu_0(y,\theta) \right. \qquad (21) \\
&\qquad\qquad \left. + \int_\Omega [\varphi(Y,\theta) - \varphi(Z,\theta)]\,\mathrm{d}\nu_0(y,\theta) \right| \\
&\leqslant (1 + \|\nabla Y\|_\infty) \mathcal{W}_1(\mu_0, \nu_0) + \int_\Omega |Y - Z|\,\mathrm{d}\nu_0(y,\theta) \\
&\leqslant (1 + \|\nabla Y\|_\infty) \mathcal{W}_1(\mu_0, \nu_0) + \|Y - Z\|_\infty. \qquad (22)
\end{aligned}$$

The proof reduces to the estimation of $\|\nabla Y\|_\infty$ and $\|Y - Z\|_\infty$.

Taking the gradient

$$\nabla Y = (\partial_y Y, \partial_\theta Y)$$

of (17) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla Y = -\nabla Y + \theta \nabla Y + (0, Y) - (p+1) Y^p \nabla Y.$$

Evaluating at a point where $\|\nabla Y\|_\infty$ is achieved, by Rademacher's lemma, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla Y\|_\infty \leqslant -(1-\theta)\|\nabla Y\|_\infty - (p+1) Y^p \|\nabla Y\|_\infty + \|Y\|_\infty. \qquad (23)$$

By (19),

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla Y\|_\infty \leqslant C \|\nabla Y\|_\infty + C,$$

and hence,

$$\|\nabla Y\|_{L^\infty} \leqslant Ce^{ct}. \tag{24}$$

Now let us compute the derivative of $\|Y-Z\|_\infty$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(Y-Z) &= \int_\Omega (Y'-Y)\,\mathrm{d}\mu_0(y',\theta') - \int_\Omega (Z'-Z)\,\mathrm{d}\nu_0(y',\theta') \\ &\quad + (\theta - Y^p)Y - (\theta - Z^p)Z \\ &= \int_\Omega Y'\,\mathrm{d}\mu_0(y',\theta') - \int_\Omega Y'\,\mathrm{d}\nu_0(y',\theta') + \int_\Omega Y'\,\mathrm{d}\nu_0(y',\theta') \\ &\quad - \int_\Omega Z'\,\mathrm{d}\nu_0(y',\theta') + (\theta-1)(Y-Z) - (Y^{p+1}-Z^{p+1}).\end{aligned}$$

Evaluating at a point of maximum and noting that $Y^{p+1}-Z^{p+1} = (p+1)\tilde{Y}^p(Y-Z)$ for some $\tilde{Y}$ between Y and Z we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|Y-Z\|_\infty \\ &\leqslant \|\nabla Y\|_\infty \mathcal{W}_1(\mu_0,\nu_0) + (|\theta-1|+1)\|Y-Z\|_\infty - (p+1)\tilde{Y}^p\|Y-Z\|_\infty \\ &\leqslant \|\nabla Y\|_\infty \mathcal{W}_1(\mu_0,\nu_0) + C\|Y-Z\|_\infty.\end{aligned} \tag{25}$$

Combining with (24) and by Grönwall's lemma, it implies that

$$\|Y-Z\|_\infty \leqslant Ce^{ct}\mathcal{W}_1(\mu_0,\nu_0). \tag{26}$$

where c is a constant depending on σ and the supports of μ_0, ν_0 with respect to θ. Therefore, plugging (26) and (24) into (21) we obtain

$$\mathcal{W}_1(\mu_t,\nu_t) \leqslant Ce^{ct}\mathcal{W}_1(\mu_0,\nu_0) \tag{27}$$

which concludes the lemma. □

For any $N \in \mathbb{N}$, if $\{(y_i,\theta_i)\}_{i=1,\dots,N}$ is a solution to the system (4) with the initial conditions $y_i(0) = y_i^0, \theta_i(0) = \theta_i$, then

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{y_i(t)} \otimes \delta_{\theta_i},$$

is a measure-valued solution to (1) with the initial condition

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^N \delta_{y_i^0} \otimes \delta_{\theta_i}.$$

So, if $\mu_0^N \to \mu_0$ weakly, then by Lemma 2.4, $\mu_t^N \to \mu_t$, for any $t > 0$. Which justifies the weak approximation by empirical measures.

This method can be used to give an alternative proof of global existence for (1) without the use of general characteristics Y and simply based on the fact that the discrete system (4) is globally well-posed.

Another proof of Theorem 2.3. Let us pick any weak*-approximation of μ_0 by empirical measures

$$\mu_0^N = \sum_{k=1}^N m_k \delta_{y_k^0} \otimes \delta_{\theta_k} \to \mu_0.$$

Let

$$\mu_t^N := \sum_{k=1}^N m_k \delta_{y_k(t)} \otimes \delta_{\theta_k}.$$

Since μ^N is a measure-valued solution to (1) with the initial data μ_0^N we apply Lemma 2.4 to get

$$\mathcal{W}_1(\mu_t^N, \mu_t^M) \leqslant Ce^T \mathcal{W}_1(\mu_0^N, \mu_0^M), \quad \text{for } N, M > 0,\ t \leqslant T.$$

Hence $\{\mu_t^N\}_N$ is weakly*-Cauchy in the complete metric space $(\mathcal{P}_+(\Omega), \mathcal{W}_1)$, and consequently there is a limit $\mu_t^N \to \mu_t \in \mathcal{P}_+(\Omega)$, and moreover

$$\mathcal{W}_1(\mu_t^N, \mu_t) \leqslant C_T \mathcal{W}_1(\mu_0^N, \mu_0), \quad \text{for } N > 0,\ t \leqslant T. \tag{28}$$

Now we prove the weak*-continuity of the map $t \to \mu_t$. Note that for $\psi \in C_0^\infty(\Omega)$ the sequence $\{\int_\Omega \psi(y,\theta)\, \mathrm{d}\mu_t^N(y,\theta)\}_N$ is uniformly Lipschitz continuous on $[0,T)$. Indeed, for $t \in [0,T)$ and $\Delta t > 0$ with $t + \Delta t \in [0,T)$ we have

$$\begin{aligned}
&\left| \int_\Omega \psi(y,\theta)\, \mathrm{d}\mu_{t+\Delta t}^N(y,\theta) - \int_\Omega \psi(y,\theta)\, \mathrm{d}\mu_t^N(y,\theta) \right| \\
\leqslant & \int_\Omega \left| \psi(Y^N(t+\Delta t),\theta) - \psi(Y^N(t),\theta) \right| \, \mathrm{d}\mu_0^N(y,\theta) \\
\leqslant & \|\nabla\psi\|_\infty \int_\Omega \left| Y^N(t+\Delta t) - Y^N(t) \right| \mathrm{d}\mu_0^N(y,\theta) \\
\leqslant & C\Delta t,
\end{aligned}$$

where Y^N denotes the characteristics of μ^N. For the last inequality we used the uniform Lipschitzness of $\{Y^N\}_N$ on $[0,T)$. Letting $N \to +\infty$, we have

$$\left| \int_\Omega \psi(y,\theta)\, \mathrm{d}\mu_{t+\Delta t}(y,\theta) - \int_\Omega \psi(y,\theta)\, \mathrm{d}\mu_t(y,\theta) \right| \leqslant C\Delta t,$$

which implies the weak*-continuity of the map $t \to \mu_t$.

We will show that this μ is a measure-valued solution to (1) with the given initial μ_0.

Because μ^N is a measure-valued solution, for any test function $\varphi \in C_0^\infty([0,T) \times \Omega)$,

$$\int_\Omega \varphi(t,y,\theta)\, \mathrm{d}\mu_t^N(y,\theta) = \int_\Omega \varphi(0,y,\theta)\, \mathrm{d}\mu_0^N(y,\theta) + \int_0^t \int_\Omega [\partial_s\varphi + u_s^N \partial_y \varphi]\, \mathrm{d}\mu_s^N(y,\theta)\, \mathrm{d}s, \tag{29}$$

where

$$u_s^N = \int_\Omega y'\, \mathrm{d}\mu_s^N(y',\theta') - y + (\theta - y^p)y := P^N(s) + F(y,\theta).$$

All linear terms weakly converge to the natural limits. Since F is a fixed continuous function we also have

$$\int_0^t \int_\Omega F \partial_y \varphi\, \mathrm{d}\mu_s^N(y,\theta)\, \mathrm{d}s \longrightarrow \int_0^t \int_\Omega F \partial_y \varphi\, \mathrm{d}\mu_s(y,\theta)\, \mathrm{d}s \quad \text{as } N \to \infty.$$

Note that the moments $P^N(s)$ is just a sequence of numbers for which we have, by (28),

$$\begin{aligned}
|P^N(s) - P(s)| &= \left| \int_\Omega y'(\, \mathrm{d}\mu_s^N(y',\theta') - \, \mathrm{d}\mu_s(y',\theta')) \right| \leqslant \mathcal{W}_1(\mu_s^N, \mu_s) \\
&\leqslant C_T \mathcal{W}_1(\mu_0^N, \mu_0) \to 0.
\end{aligned}$$

So, $P^N \to P$ uniformly on $[0,T)$. Consequently,

$$\int_0^t \int_\Omega P^N(s)\partial_y \varphi \, \mathrm{d}\mu_s^N(y,\theta)\, \mathrm{d}s \to \int_0^t \int_\Omega P(s)\partial_y \varphi \, \mathrm{d}\mu_s(y,\theta)\, \mathrm{d}s.$$

It follows that μ satisfies (ii). □

3. **Existence and uniqueness of the mono-opinion state.** Let μ be a measure-valued solution to (1) with the initial μ_0. Let π be its time-independent conviction marginal (8).

Let us derive the equation for μ^θ. By Definition 2.1 and (9), for any $\varphi \in C_0^\infty([0,T)\times\Omega)$ and $0<t<T$ one has

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \varphi(t,y,\theta)\,\mathrm{d}\mu_t^\theta(y)\,\mathrm{d}\pi(\theta) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \varphi(0,y,\theta)\,\mathrm{d}\mu_0^\theta(y)\,\mathrm{d}\pi(\theta) + \int_0^t \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} [\partial_s\varphi + u_s \partial_y \varphi]\,\mathrm{d}\mu_s^\theta(y)\,\mathrm{d}\pi(\theta)\,\mathrm{d}s.$$

It implies that for π-almost every θ, the probability measure μ^θ is a measure-valued solution with the initial μ_0^θ to the equation

$$\partial_t \mu^\theta + \partial_y\left[u\mu^\theta\right] = 0, \tag{30}$$

where

$$u(t,y,\theta) = \int_\Omega (z-y)\,\mathrm{d}\mu_t^\eta(z)\,\mathrm{d}\pi(\eta) + (\theta - y^p)y.$$

Note that the family of equations are all coupled through the velocity u, but otherwise represent transport of each individual slicing measure μ^θ. The characteristics that transport μ^θ, denoted Y_θ are nothing but $Y_\theta(t,y) = Y(t,y,\theta)$ as defined by (17). We will view them, however, as individual trajectories satisfying the coupled system

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_\theta = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (Y'_{\theta'} - Y_\theta)\,\mathrm{d}\mu_0^{\theta'}(y')\,\mathrm{d}\pi(\theta') + (\theta - Y_\theta^p)Y_\theta. \tag{31}$$

In particular we will derive an individual comparison bound from below as an alternative to global (20).

Lemma 3.1. *For any $\theta \in [\theta_{\min}, \theta_{\max}]$ such that $\theta > 1$ one has*

$$Y_\theta^p(t,y) \geqslant \frac{y^p(\theta-1)e^{p(\theta-1)t}}{(\theta-1) + y^p(e^{p(\theta-1)t}-1)}, \quad \forall t \geqslant 0,\ \forall y > 0. \tag{32}$$

Proof. To achieve (32) we decouple the system (31) by ignoring the entire coupling term

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'}\,\mathrm{d}\mu_0^{\theta'}(y')\,\mathrm{d}\pi(\theta') \geqslant 0.$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_\theta^p \geqslant p\,(\theta - 1 - Y_\theta^p)\,Y_\theta^p. \tag{33}$$

The lemma follows from the comparison principle. □

Let us note that in principle the statement of the lemma holds for any $\theta - 1$, but it is most meaningful when the parameter is positive in view of the universal support from below for all characteristics (20).

3.1. **Mono-opinion state.** In the next step we will show that for each $\theta \in \operatorname{supp} \pi$, the slicing measure μ^θ will converge to a Dirac measure in Wasserstein distance with different rates depending on θ.

Lemma 3.2. *Let μ be the measure-valued solution to (1) satisfying (10) and π being the conviction marginal (8). Then there exists a function $g \in \operatorname{Lip}[\theta_{\min}, \theta_{\max}]$ such that*

$$\sup_{\theta \in \operatorname{supp} \pi} \mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) \leqslant Ce^{-ct}, \quad t > 0, \tag{34}$$

where $C, c > 0$ depend only on μ_0 and the parameters of the model.

Proof. Differentiating the characteristic equation (31) we obtain

$$\partial_t \partial_y Y_\theta = (\theta - 1)\partial_y Y_\theta - (p+1)Y_\theta^p \partial_y Y_\theta. \tag{35}$$

In what follows we denote $L^\infty = L^\infty(R_0)$. By Rademacher's lemma, at a point of maximum y such that $(y, \theta) \in R_0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y Y_\theta\|_\infty = (\theta - 1)\|\partial_y Y_\theta\|_\infty - (p+1)Y_\theta^p \|\partial_y Y_\theta\|_\infty. \tag{36}$$

Let us first consider the stable case when $\theta - 1 \leqslant \varepsilon_0$, with $\varepsilon_0 > 0$ to be determined later. Using (20) we find that $Y_\theta^p \geqslant c_0$, which is determined only by the initial condition and the parameters of the model. Plugging in (36), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y Y_\theta\|_\infty \leqslant \varepsilon_0 \|\partial_y Y_\theta\|_\infty - (p+1)c_0\|\partial_y Y_\theta\|_\infty \leqslant -\varepsilon_0 \|\partial_y Y_\theta\|_\infty \tag{37}$$

by setting $\varepsilon_0 = \frac{(p+1)c_0}{2}$.

For the unstable case $\theta - 1 \geqslant \varepsilon_0$, the inequality (32) implies that

$$\begin{aligned} Y_\theta^p &\geqslant \frac{y^p(\theta-1)e^{p(\theta-1)t}}{(\theta-1) + y^p e^{p(\theta-1)t}} = \theta - 1 - \frac{(\theta-1)^2}{(\theta-1) + y^p e^{p(\theta-1)t}} \\ &\geqslant \theta - 1 - (\theta-1)^2 y^{-p} e^{-p(\theta-1)t}. \end{aligned}$$

Therefore, in this case we have

$$Y_\theta^p \geqslant \theta - 1 - c_1 e^{-c_2 t}, \tag{38}$$

where $c_1, c_2 > 0$ depend only on the initial condition and parameters of the model. Hence,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y Y_\theta\|_\infty &\leqslant (\theta-1)\|\partial_y Y_\theta\|_\infty - (p+1)(\theta - 1 - c_1 e^{-c_2 t})\|\partial_y Y_\theta\|_\infty \\ &\leqslant \left(-p\varepsilon_0 + (p+1)c_1 e^{-c_2 t}\right)\|\partial_y Y_\theta\|_\infty. \end{aligned}$$

In either case we obtain, by Grönwall's lemma,

$$\|\partial_y Y_\theta\|_{L^\infty} \leqslant c_3 e^{-c_4 t}. \tag{39}$$

Consequently,

$$|Y_\theta(y,t) - Y_\theta(y',t)| \leqslant c_5 e^{-c_4 t}, \qquad \text{for any } (y,\theta), (y',\theta) \in R_0. \tag{40}$$

We can see that the characteristics are squeezing as t approaches infinity. Since the trajectories are also precompact, for each $\theta \in [\theta_{\min}, \theta_{\max}]$ there exists $g(\theta)$ such that

$$\sup_{y \in [y_{\min}, y_{\max}]} |Y_\theta(y,t) - g(\theta)| \leqslant c_5 e^{-c_4 t}.$$

We compute

$$\begin{aligned}
\mathcal{W}_1(\mu_t^\theta, \delta_{g(\theta)}) &= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \Big| \int_{\mathbb{R}_+} \varphi(y)\,\mathrm{d}\mu_t^\theta(y) - \int_{\mathbb{R}_+} \varphi(y)\delta_{g(\theta)}(y) \Big| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \Big| \int_{\mathbb{R}_+} \varphi(Y_\theta)\,\mathrm{d}\mu_0^\theta(y) - \varphi(g(\theta)) \Big| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \Big| \int_{\mathbb{R}_+} \big(\varphi(Y_\theta) - \varphi(g(\theta))\big)\,\mathrm{d}\mu_0^\theta(y) \Big| \\
&\leqslant \|Y_\theta - g(\theta)\|_\infty.
\end{aligned}$$

The statement (34) follows.

It remains to show that g is a Lipschitz function on $[\theta_{\min}, \theta_{\max}]$. Indeed, computing the evolution of $\partial_\theta Y_\theta$ we obtain

$$\partial_t \partial_\theta Y_\theta = Y_\theta + \big(\theta - 1 - (p+1)Y_\theta^p\big)\partial_\theta Y_\theta.$$

Note that Y_θ remains bounded on R_0 by Lemma 2.2, and the remainder of the equation has the same structure as in (35). So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\theta Y_\theta\|_\infty \leqslant c_1 + (-c_2 + c_3 e^{-c_4 t})\|\partial_\theta Y_\theta\|_\infty.$$

We obtain

$$\|\partial_\theta Y_\theta\|_\infty < C. \tag{41}$$

Consequently,

$$|Y(y,\theta,t) - Y(y,\theta',t)| \leqslant C|\theta - \theta'|.$$

Letting $t \to \infty$ we obtain

$$|g(\theta) - g(\theta')| \leqslant C|\theta - \theta'|.$$

This finishes the proof. □

3.2. **Uniqueness and stability.** The uniqueness of the limiting state follows from the lemma below and holds under either of the two conditions on parameters

$$\theta_{\min} > \frac{p+1}{p} \quad \text{or} \quad \frac{\theta_{\max}}{\theta_{\min}} < p+1. \tag{42}$$

Note that under the change (16) this translates into condition (12).

Lemma 3.3. *Let μ and $\tilde\mu$ be two solutions to (1) starting in a box R_0 and sharing the same conviction measure π. And suppose either of the assumptions (42) hold. Then for any $t \in [0,T)$ one has*

$$\sup_{\theta\in\operatorname{supp}\pi} \mathcal{W}_1(\mu_t^\theta, \tilde\mu_t^\theta) \leqslant c_1 e^{-c_2 t} \sup_{\theta\in\operatorname{supp}\pi} \mathcal{W}_1(\mu_0^\theta, \tilde\mu_0^\theta), \tag{43}$$

where $c_1, c_2 > 0$ depend on the initial data and parameters of the model.

Proof. In what follows $L^\infty := L^\infty([y_{\min}, y_{\max}])$. Denoting $\tilde{Y}_\theta, \tilde{Y}'_\theta$ the characteristics of $\tilde{\mu}^\theta$ starting from y, y' respectively. For fixed $\theta \in \operatorname{supp}\pi$,

$$\begin{aligned}
&\mathcal{W}_1(\mu_t^\theta, \tilde{\mu}_t^\theta) \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(y)\,\mathrm{d}\mu_t^\theta(y) - \int_{\mathbb{R}_+} \varphi(y)\,\mathrm{d}\tilde{\mu}_t^\theta(y) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(Y_\theta)\,\mathrm{d}\mu_0^\theta(y) - \int_{\mathbb{R}_+} \varphi(\tilde{Y}_\theta)\,\mathrm{d}\tilde{\mu}_0^\theta(y) \right| \\
&= \sup_{\|\varphi\|_{\mathrm{Lip}}\leqslant 1} \left| \int_{\mathbb{R}_+} \varphi(Y_\theta)\,\mathrm{d}\mu_0^\theta(y) - \int_{\mathbb{R}_+} \varphi(Y_\theta)\,\mathrm{d}\tilde{\mu}_0^\theta(y) + \int_{\mathbb{R}_+} [\varphi(Y_\theta) - \varphi(\tilde{Y}_\theta)]\,\mathrm{d}\tilde{\mu}_0^\theta(y) \right| \\
&\leqslant \quad \|\partial_y Y_\theta\|_{L^\infty} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}.
\end{aligned}$$

We proved the uniform exponential contraction for $\|\partial_y Y_\theta\|_{L^\infty}$ in (39).

Let us now focus on $\|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Y_\theta - \tilde{Y}_\theta) &= \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}_+} Y'_{\theta'}\,\mathrm{d}\mu_0^{\theta'}(y') - \int_{\mathbb{R}_+} \tilde{Y}'_{\theta'}\,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \right] \mathrm{d}\pi(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (Y_\theta^{p+1} - \tilde{Y}_\theta^{p+1}) \\
&= \int_{\mathbb{R}_+} \Big[\int_{\mathbb{R}_+} Y'_{\theta'}(\,\mathrm{d}\mu_0^{\theta'}(y') - \,\mathrm{d}\tilde{\mu}_0^{\theta'}(y')) + \int_{\mathbb{R}_+} (Y'_{\theta'} - \tilde{Y}'_{\theta'})\,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \Big]\,\mathrm{d}\pi(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (p+1)\hat{Y}_\theta^p (Y_\theta - \tilde{Y}_\theta),
\end{aligned}$$

where $\hat{Y}_\theta$ is between Y_θ and $\tilde{Y}_\theta$. Denote

$$\mathcal{D}(t) = \sup_{\theta \in [\theta_{\min}, \theta_{\max}]} \|Y_\theta - \tilde{Y}_\theta\|_{L^\infty}.$$

At a point of maximum we obtain using (39),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp}\pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \theta\mathcal{D} - (p+1)\min\{Y_\theta^p, \tilde{Y}_\theta^p\}\mathcal{D}.$$

Using (38),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} &\leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp}\pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + \theta\mathcal{D} - (p+1)[\theta - 1 - c_1 e^{-c_2 t}]\mathcal{D} \\
&= c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp}\pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + [p + 1 - p\theta + c_1 e^{-c_2 t}]\mathcal{D}
\end{aligned}$$

The result follows provided $\theta_{\min} > \frac{p+1}{p}$. Alternatively, using the lower bound (20),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta \in \operatorname{supp}\pi} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + [\theta_{\max} - (p+1)\theta_{\min} + c_1 e^{-c_2 t}]\mathcal{D}$$

and the result follows provided $\frac{\theta_{\max}}{\theta_{\min}} < p + 1$. □

Under the stability assumption (42) the limiting states are also stable with respect to perturbation of convictions. So, a small change even in the weak topology of conviction marginal π results in a small change in the limiting mono-opinion state. This can be proved via a minor modification of the argument above.

First, since we will be comparing slicing measures that are technically defined not on the same set let us adopt a convention that if $\theta \notin \operatorname{supp}\pi$, then $\mu^\theta = 0$.

Lemma 3.4. *Let μ and $\tilde{\mu}$ be two measure-valued solutions to (1) with the conviction marginals π and $\tilde{\pi}$, respectively, and parameters satisfying (42). Then for any $t \in [0, T)$ one has*

$$\sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_t^\theta, \tilde{\mu}_t^\theta) \leqslant c_1 e^{-c_2 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + c_3 e^{-c_4 t} + c_5 \mathcal{W}_1(\pi, \tilde{\pi}), \tag{44}$$

where $c_i > 0$ depend only on the initial condition and parameters of the model.

By sending $t \to \infty$ and using that fact that

$$\sup_{\theta\in[\theta_{\min},\theta_{\max}]} |g(\theta) - \tilde{g}(\theta)| = \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\delta_{g(\theta)}, \delta_{\tilde{g}(\theta)}),$$

we obtain the statement (13) of Theorem 1.2.

Proof. We only need to focus on estimation of $\mathcal{D}(t)$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Y_\theta - \tilde{Y}_\theta) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'} \,\mathrm{d}\mu_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \,\mathrm{d}\tilde{\pi}(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (Y_\theta^{p+1} - \tilde{Y}_\theta^{p+1}) \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} Y'_{\theta'} \,\mathrm{d}\mu_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\mu_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\mu_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') \\
&\quad + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \,\mathrm{d}\pi(\theta') - \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \tilde{Y}'_{\theta'} \,\mathrm{d}\tilde{\mu}_0^{\theta'}(y') \,\mathrm{d}\tilde{\pi}(\theta') \\
&\quad + (\theta - 1)(Y_\theta - \tilde{Y}_\theta) - (p+1)\hat{Y}_\theta^p (Y_\theta - \tilde{Y}_\theta).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} &\leqslant c_3 e^{-c_4 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) \\
&\quad + \int_{\mathbb{R}^+} G(\theta')[\,\mathrm{d}\pi - \,\mathrm{d}\tilde{\pi}] + \theta\mathcal{D} - (p+1)\min\{Y_\theta^p, \tilde{Y}_\theta^p\}\mathcal{D},
\end{aligned}$$

where

$$G(\theta) := \int_{\mathbb{R}_+} \tilde{Y}_\theta(y) \,\mathrm{d}\tilde{\mu}_0^\theta(y) = \int_{\mathbb{R}_+} (\tilde{Y}_\theta(y) - \tilde{g}(\theta)) \,\mathrm{d}\tilde{\mu}_0^\theta(y) + \tilde{g}(\theta).$$

Since the first term is bounded exponentially, and $\tilde{g} \in \mathrm{Lip}$, we have

$$\int_{\mathbb{R}^+} G(\theta')[\,\mathrm{d}\pi - \,\mathrm{d}\tilde{\pi}] \leqslant c_1 e^{-c_2 t} + \|\tilde{g}\|_{\mathrm{Lip}} \mathcal{W}_1(\pi, \tilde{\pi}).$$

Coming back to the $\mathcal{D}$-equation and estimating the rest of the right hand side as previously we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D} \leqslant c_3 e^{-c_4 t} \sup_{\theta\in[\theta_{\min},\theta_{\max}]} \mathcal{W}_1(\mu_0^\theta, \tilde{\mu}_0^\theta) + c_1 e^{-c_2 t} + c_5 \|\tilde{g}\|_{\mathrm{Lip}} \mathcal{W}_1(\pi, \tilde{\pi}) - c_6 \mathcal{D}.$$

The result follows. □

4. **Properties of mono-opinion states.** The results of the previous sections establish that for each conviction measure there is at least one (and in some cases only one) limiting distributions of opinions $g \in \mathrm{Lip}[\theta_{\min}, \theta_{\max}]$. Technically it makes material sense to only consider values of g on the $\operatorname{supp} \pi$, but to study analytic properties of g it will be convenience to make full use of its existence on the closed interval $[\theta_{\min}, \theta_{\max}]$.

We have the following equation for g:

$$\int_{\mathbb{R}_+} g(\eta)\, \mathrm{d}\pi(\eta) + (\theta - 1)g(\theta) - g^{p+1}(\theta) = 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}]. \tag{45}$$

Although it is difficult to find the function g explicitly, solutions to (45) exhibit certain universal features.

Remark 4.1. One instance where g is computable is when $p = 1$. Indeed, let

$$\alpha := \int_{\mathbb{R}_+} g(\eta)\, \mathrm{d}\pi(\eta),$$

then by (45) we have

$$g^2 + (1 - \theta)g - \alpha = 0.$$

This second order equation always has a positive solution

$$g = \frac{1}{2}\left(\theta - 1 + \sqrt{(1-\theta)^2 + 4\alpha}\right),$$

for any parameter $\alpha > 0$. Note that this expression is still implicit as α depends on g. But whatever α is we can see in particular that g is strictly increasing and convex.

Let us discuss these properties more systematically.

First, let us consider the extreme values

$$g_{\max} = \max_{[\theta_{\min}, \theta_{\max}]} g(\theta), \qquad g_{\min} = \min_{[\theta_{\min}, \theta_{\max}]} g(\theta).$$

We claim that

$$\theta_{\min} \leqslant g^p_{\min}, \quad g^p_{\max} \leqslant \theta_{\max}. \tag{46}$$

Indeed, the equation (45) can be rewritten as

$$\int_{\mathbb{R}_+} [g(\eta) - g(\theta)]\, \mathrm{d}\pi(\eta) + \theta g(\theta) - g^{p+1}(\theta) = 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}]. \tag{47}$$

Let $\bar{\theta}$ be the point such that $g_{\min} = g(\bar{\theta})$. Since

$$\int_{\mathbb{R}_+} [g(\eta) - g_{\min}]\, \mathrm{d}\pi(\eta) \geqslant 0,$$

by the equation (47), we have

$$\bar{\theta} g_{\min} - g^{p+1}_{\min} \leqslant 0.$$

Therefore,

$$\theta_{\min} \leqslant \bar{\theta} \leqslant g^p_{\min}.$$

Similarly, we have

$$g^p_{\max} \leqslant \theta_{\max}.$$

By (45), we also have that

$$(\theta - 1)g(\theta) - g^{p+1}(\theta) \leqslant 0, \qquad \forall \theta \in [\theta_{\min}, \theta_{\max}].$$

Thus, for each $\theta \in [\theta_{\min}, \theta_{\max}]$ the following estimate holds true

$$g^p(\theta) \geqslant \theta - 1. \tag{48}$$

A more refined estimate will be obtained next.

Lemma 4.2. *Let g be a solution to the equation (45). Then $g \in C^\infty([\theta_{\min}, \theta_{\max}])$, g is strictly increasing on $[\theta_{\min}, \theta_{\max}]$, and for each $\theta \in [\theta_{\min}, \theta_{\max}]$,*

$$g^p(\theta) \geqslant \theta + \pi([\theta, \infty)) - 1. \tag{49}$$

Proof. Since g is Lipschitz we can conclude monotonicity from the sign of the derivative,

$$g' = \frac{g}{1 - \theta + (p+1)g^p}. \tag{50}$$

If $1 \geqslant \theta$, then using (46), it is clear that the denominator is positive, and so $g' > 0$. If $1 < \theta$ we have by the rough bound (48)

$$1 - \theta + (p+1)g^p \geqslant p(\theta - 1) > 0.$$

This establishes monotonicity. Also, since the denominator of (50) is always positive, by bootstrapping this implies $g \in C^\infty([\theta_{\min}, \theta_{\max}])$.

Combining monotonicity with the equation (45) we obtain

$$\int_{\{\eta \geqslant \theta\}} g(\theta)\, \mathrm{d}\pi(\eta) - g(\theta) + [\theta - g^p(\theta)]g(\theta) \leqslant 0.$$

Since $g(\theta) \geqslant 0$ for all $\theta \in [\theta_{\min}, \theta_{\max}]$ we must have

$$\int_{\{\eta \geqslant \theta\}} \mathrm{d}\pi(\eta) - 1 + \theta - g^p(\theta) \leqslant 0.$$

The estimate (49) follows. □

Let us discuss convexity. The second derivative of $g(\theta)$ is given by

$$g'' = \frac{g'[1 - \theta + (p+1)g^p] - g[-1 + p(p+1)g^{p-1}g']}{[1 - \theta + (p+1)g^p]^2}$$

and using (50) to replace g' we obtain

$$g'' = \frac{2(1-\theta)g + (2 + p - p^2)g^{p+1}}{[1 - \theta + (p+1)g^p]^3}. \tag{51}$$

The denominator is always positive, and we note that in view of (48) the numerator is also positive regardless of the range of θ provided $p \leqslant 1$. So, g is globally convex in this case.

In other cases, the convexity may change. In fact for $p = 2$ we have

$$g'' = \frac{2(1-\theta)g}{[1 - \theta + 3g^2]^3}.$$

So, $\theta = 1$ is an inflection point.

For $p > 2$, the solution has no more than one inflection point. This can be seen by solving for $g'' = 0$ in (51). We have

$$2(1 - \theta) = (p^2 - p - 2)g^p.$$

The left hand side is a decreasing function and the right hand side is increasing for $p > 2$. So, the two can meet at most at one point.

The exact value of α depends on g and since the solution is in general not possible to compute explicitly we present in the figure below solutions to (45) with several 'passive' choices of α for illustration.

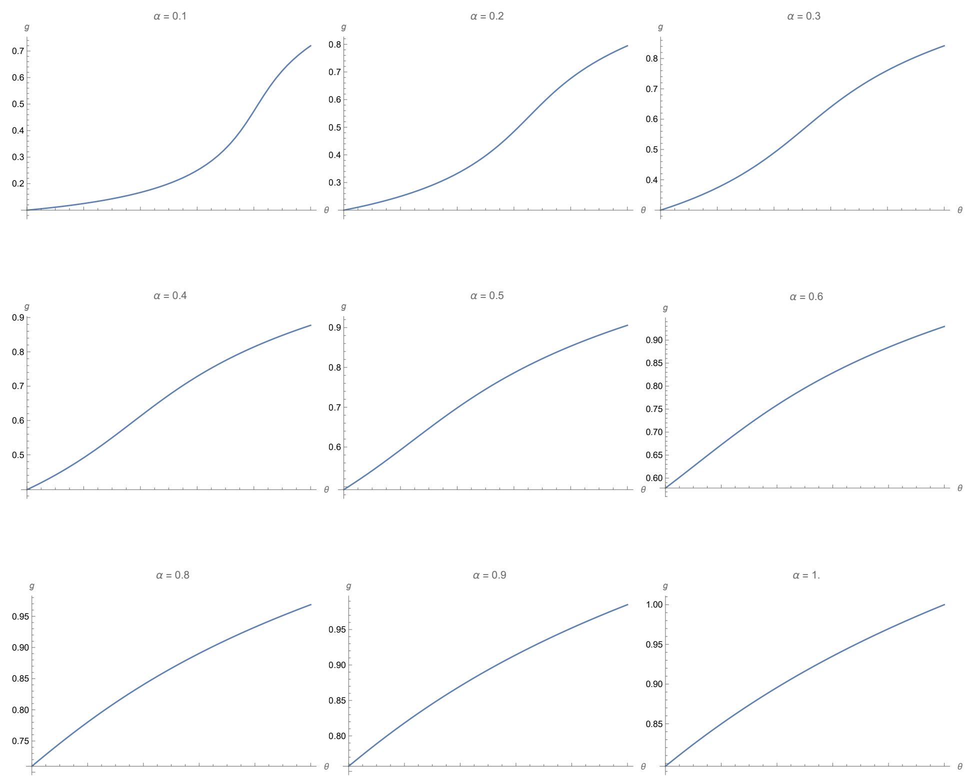

FIGURE 1. The behavior of $g(\theta)$ for the case $p = 6$. Here $\theta \in (0, 1]$ and α change in $(0, 1]$ at discrete steps of 0.1.

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