# SOME REMARKS ON THE ERDŐS DISTINCT SUBSET SUMS PROBLEM

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Abstract. Let $\{a_1, \dots, a_n\} \subset \mathbb{N}$ be a set of positive integers, $a_n$ denoting the largest element, so that for any two of the $2^n$ subsets the sum of all elements is distinct. Erdős asked whether this implies $a_n \geq c \cdot 2^n$ for some universal $c > 0$. We prove, slightly extending a result of Elkies, that for any $a_1, \dots, a_n \in \mathbb{R}_{>0}$

$$\int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)^2 \prod_{i=1}^{n} \cos\left(a_i x\right)^2 dx \geq \frac{\pi}{2^n}$$

with equality if and only if all subset sums are 1-separated. This leads to a new proof of the currently best lower bound $a_n \geq \sqrt{2/\pi n} \cdot 2^n$. The main new insight is that having distinct subset sums and $a_n$ small requires the random variable $X = \pm a_1 \pm a_2 \pm \dots \pm a_n$ to be close to Gaussian in a precise sense.

## 1. Introduction

A problem of Erdős [11] is as follows: if $\{a_1, \dots, a_n\} \subset \mathbb{N}$ is a set of positive integers, assumed to be ordered as $a_1 < a_2 < \dots < a_n$, such that for each of the $2^n$ subsets the sum of all elements is unique, does this force $a_n \geq c \cdot 2^n$ for some universal $c > 0$? The problem is quite old. Erdős [13] refers to it as "perhaps my first serious conjecture which goes back to 1931 or 32". Since the sums over all subsets leads to $2^n - 1$ distinct positive integers, one has $\sum_{i=1}^{n} a_i \geq 2^n - 1$ (sharp for the powers of 2) and $a_n \gtrsim 2^n/n$. Currently, the best known bound is

$$a_n \geq (c - o(1))\frac{2^n}{\sqrt{n}}$$

where different estimates for $c$ have been given over the years

| | |
|---|---|
| $c \geq 1/4$ | Erdős and Moser [11] |
| $\geq 2/3^{3/2}$ | Alon and Spencer [2] |
| $\geq 1/\sqrt{\pi}$ | Elkies [10] |
| $\geq 1/\sqrt{3}$ | Bae [3] , Guy [15] |
| $\geq \sqrt{3/2\pi}$ | Aliev [1] |
| $\geq \sqrt{2/\pi}$ | Dubroff, Fox and Xu [9]. |

The literature (see [1]) mentions an unpublished manuscript of Elkies and Gleason also showing $c \geq \sqrt{2/\pi}$. Dubroff, Fox and Xu give two different proofs: one appeals to the Berry-Esseen Theorem, the other uses an isoperimetric principle of

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Harper [17]. In the other direction, we note that the powers of 2, with $a_n = 2^{n-1}$, are not extremal: already in 1968, Conway and Guy [8] (answering another question by Erdős [12]) produced a candidate construction showing that $a_n \leq 2^{n-2}$ is possible (see Bohman [5]). The currently best construction is due to Bohman [6] showing $a_n \leq 0.88008 \cdot 2^{n-2}$, see also [4, 7, 18, 19, 20]. It is an interesting question whether relaxing the condition somewhat can give rise to interesting examples. More concretely, are there sets $\{a_1, \ldots, a_n\} \subset \mathbb{N}$ such that the subset sums attain $(1 - o(1)) \cdot 2^n$ distinct values and $a_n = o(2^n)$?

The main purpose of our paper is to give a new proof of $c \geq \sqrt{2/\pi}$. Many arguments, starting with Erdős and Moser [11], have considered the random walk $X = \pm a_1 \pm a_2 \pm \cdots \pm a_n$, where all signs are chosen independently and uniformly at random. If all subset sums are distinct, then all $2^n$ possible outcomes of the random walk are equally likely and they are all at least distance 2 from each other. A well-known argument (see [3, 11, 15, 18]) exploits this by using

$$n \cdot a_n^2 \geq \sum_{i=1}^{n} a_i^2 = \mathbb{E}(X^2) \geq \frac{2}{2^n} \sum_{k=1}^{2^{n-1}} (2k-1)^2 = \frac{4^n - 1}{3}$$

which shows $c \geq 1/\sqrt{3}$. This was further refined by Dubroff, Fox and Xu [9] who argued, using the Berry-Esseen theorem, that if $a_n^2$ is relatively small compared to $\sum_{i=1}^{n} a_i^2$ (the variance of the random walk), then the random walk is well-described by a Gaussian. Our argument will imply a somewhat converse result: unless the distribution of the random walk is close to a Gaussian (in a sense that will be made precise), the set cannot have distinct subset sums and $a_n$ small. This leads to an interesting reformulation of the Erdős distinct subset sums problem as a problem in probability theory: whether it is possible for random walks with a large variance but relatively small largest stepsize to emulate a Gaussian distribution very well.

## 2. Results

2.1. **Main Results.** We start with a basic analytic characterization of what it means for a set of $n$ positive real numbers to have the property that all subset sums are at least distance 1 from each other (if all numbers are integers, then this is the same as asking them to be distinct).

**Theorem 1.** *Let $a_1, \ldots, a_n > 0$ be positive, real numbers. Then*

$$\int_{\mathbb{R}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos\left(2\pi a_i x\right)^2 dx \geq \frac{1}{2^{n+1}}.$$

*Equality occurs if and only if all subset sums are distance $\geq 1$ from each other.*

This result is very similar to the analytic approach of Elkies [10] based on Laurent series. If all $a_i$ are integers, the product is $2\pi$-periodic which simplifies the integral and recovers the characterization used by Elkies.

**Corollary 1** (Elkies [10])**.** *Let $a_1, \ldots, a_n > 0$ be positive integers. Then*

$$\int_0^1 \prod_{i=1}^{n} \cos\left(2\pi a_i x\right)^2 dx \geq \frac{1}{2^n}$$

*with equality if and only if all subset sums are distinct.*

All cosines in the product are aligned around $x = 0$. A natural approach is thus to bound the contribution coming from a small interval around the origin of length $\sim 1/a_n$. If $a_n$ is too small, that contribution is too large (see Lemma 1) and this was Elkies' original approach to prove $c \geq \sqrt{1/\pi}$ (see Lemma 1). The main novelty of our approach is to analyze the contribution coming from outside that interval. This leads to Corollary 2.

**Corollary 2.** *We have*

$$a_n \geq (1 - o(1)) \cdot \sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}.$$

While Corollary 2 itself does not tell us anything new, the proof establishes a connection to probability theory which will be discussed in §2.2 and §2.3.

2.2. **Proof of Corollary 2: Outline.** We use Theorem 1. The first ingredient is a lower bound on how much the integrand contributes to the integral close to the origin where all the cosines are aligned.

**Lemma 1** (see Elkies [10])**.** *Suppose that $\{a_1, \ldots, a_n\}$ is a subset of the positive real numbers. Then*

$$\int_{|x| \leq \frac{1}{4a_n}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^{n} \cos(2\pi a_i x)^2 dx \geq (1 + o(1)) \cdot \frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{\pi n}}.$$

This Lemma in conjunction with Theorem 1 already shows $c \geq 1/\sqrt{\pi}$. The main new idea is to prove that contributions far away from the origin can also be analyzed and that they also contribute a substantial amount.

**Lemma 2.** *Let $c > 0$. Suppose that $\{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0}$ has 1-separated subset sums and, for some $\varepsilon > 0$, we have $a_n^2 \leq c \cdot n^{-2/3-\varepsilon} \sum_{i=1}^{n} a_i^2$. Then, as $n \to \infty$,*

$$\int_{|x| \geq \frac{1}{4a_n}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^{n} \cos(2\pi a_i x)^2 dx \geq (1 + o(1)) \frac{\sqrt{2} - 1}{2\sqrt{\pi}} \left(\sum_{i=1}^{n} a_i^2\right)^{-1/2}.$$

We note that this lower bound can be bounded from below in terms of $a_n$ using the trivial bound $\sum_{i=1}^{n} a_i^2 \leq n \cdot a_n^2$. Combining this with Theorem 1 and Lemma 1, we see that if all subset sums are 1-separated, then

$$\frac{1}{2^{n+1}} \geq (1 + o(1)) \left(\frac{1}{2} \frac{1}{a_n} \frac{1}{\sqrt{\pi n}} + \frac{\sqrt{2} - 1}{2\sqrt{\pi}} \frac{1}{\sqrt{n} \cdot a_n}\right) = \frac{1 + o(1)}{\sqrt{2\pi}\sqrt{n} \cdot a_n}$$

which shows $c \geq \sqrt{2/\pi}$. Lemma 2 appears to be very technical but contains an interesting idea which will tell us something new. Lemma 2 can be written in a completely different way (Theorem 2) and this alternative formulation is also how we are going to prove Lemma 2.

2.3. **Subset Sums and Gaussian Densities.** Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0}$ be a set of positive reals. As already indicated above, we consider the random variable $X = \sum_{i=1}^{n} \varepsilon_i a_i$ where $\varepsilon_i \in \{-1, 1\}$ independently and with equal likelihood (also known as Rademacher random variables). This random variable is distributed according to some probability measure $\mu$ on $\mathbb{R}$. Note that we can write

$$X = -\sum_{i=1}^{n} a_i + 2 \sum_{\substack{i=1 \\ \varepsilon_i = 1}}^{n} a_i.$$

If the minimal distance between the sum of two different subsets of $A$ is 1, then the minimal distance between any two distinct values of $X$ is two. Moreover, by the subset sum condition, $X$ assumes $2^n$ distinct values which implies

$$\mu = \frac{1}{2^n}\sum_{i=1}^{2^n} \delta_{x_i} \qquad \text{where} \qquad \min_{i\neq j} |x_i - x_j| = 2.$$

Our main question of interest will now be whether $\mu$ is close to a Gaussian (and, if so, in what sense). Consider first a simple example: the set $\left\{1, 2, \ldots, 2^{n-1}\right\}$. It is easy to see that all subset sums are distinct (the uniqueness of binary expansion) and, following the construction, we see that $\mu$ is supported on all $2^n$ odd numbers in $[-2^n, 2^n]$ roughly emulating a uniform distribution over that interval. A uniform distribution is not particularly close to a Gaussian overall. This will now be compared to a better construction: we take the first 22 terms induced by the Conway-Guy sequence [8] (where 22 was chosen so as to be 'large' while still computationally feasible). We end up with a set $\{a_1, \ldots, a_{22}\} \subset \mathbb{N}$ with distinct subset sums and $a_{22} = 1051905 \sim 0.51 \cdot 2^{21}$. The probability distribution of the associated random walk $\mu$ is shown in Figure 1. This is quite a bit closer to a Gaussian than uniform distribution would be. This is not a coincidence.

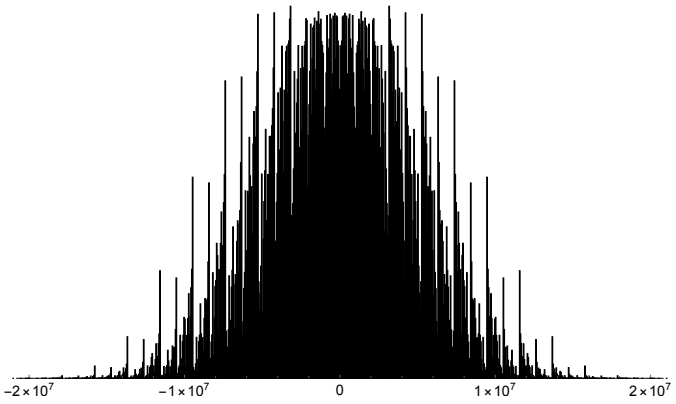


FIGURE 1. A histogram of the discrete measure $\mu$ derived from the first 22 terms from the Conway-Guy sequence.

We start by trying to understand which Gaussian we should compare the distribution $\mu$ to. A Gaussian is uniquely determined by mean and variance. Since $\mu$ is symmetric around the origin, the expectation is $\mathbb{E}X = 0$. Simultaneously, we have an explicit expression for the variance and

$$\mathbb{E}\left(X^2\right) = \mathbb{E}\left(\sum_{i=1}^{n} \varepsilon_i a_i\right)^2 = \mathbb{E}\sum_{i,j=1}^{n} \varepsilon_i \varepsilon_i a_i a_j = \sum_{i=1}^{n} a_i^2.$$

The probability density function of that Gaussian will be abbreviated as

$$\gamma(x) = \frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^{n} a_i^2\right)^{-1/2} \exp\left(-\frac{x^2}{2}\left(\sum_{i=1}^{n} a_i^2\right)^{-1}\right).$$

Note that $\gamma$ is a smooth function while $\mu$ is a singular measure. To facilitate a comparison between the two, we will introduce a smoothed version of $\mu$. Consider the normalized characteristic function $h(x) = (1/2) \cdot \chi_{[-1,1]}$. Since both $\mu$ and $h$ are probability measures, their convolution

$$(h * \mu)(x) = \frac{1}{2^n} \sum_{i=1}^{2^n} \frac{1}{2} \chi_{[x_i - 1, x_i + 1]}(x)$$

is also a probability measure. We observe that $h * \mu$ is a sum of characteristic functions centered at the points $x_i$ at which $\mu$ is supported. Since $\mu$ is distributed over exponentially large scales, smoothing at scale 1 does not change any relevant characteristics. With this language in place, the second main result is as follows.

**Theorem 2.** *Let $c > 0$. Suppose $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$ has 1-separated subset sums and $a_n^2 \leq c \cdot n^{-1/2} \sum_{i=1}^n a_i^2$. Then, as $n \to \infty$, we have*

$$\int_{\mathbb{R}} \left( (h * \mu)(x) - \gamma(x) \right)^2 dx = \int_{|x| \geq \frac{1}{4a_n}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^n \cos\left(2\pi a_i x\right)^2 dx + o(2^{-n}).$$

We emphasize that $h * \mu$ only assumes the values 0 and $2^{-n-1}$ (and the second value is assumed on $2^n$ intervals of length 2). This implies that

$$\int_{\mathbb{R}} (h * \mu)(x)^2 dx = \frac{1}{2^{n+1}} \qquad \text{while} \qquad \int_{\mathbb{R}} \gamma(x)^2 dx = \frac{1}{2\sqrt{\pi}} \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}.$$

We also remark that the probability density of the random walk behaving similarly to a Gaussian was already used by Dubroff, Fox and Xu who invoked the Berry-Esseen theorem. Under a slightly stronger assumption ($a_n^2 \leq c \cdot n^{-2/3-\varepsilon} \sum_{i=1}^n a_i^2$) the Berry-Esseen theorem guarantees that

$$\sup_{x \in \mathbb{R}} \left| \mu\left([-\infty, x]\right) - \int_{-\infty}^x \gamma(y) dy \right| = o(1)$$

which shows convergence of the *cumulative* distribution functions. Theorem 2 establishes that sets with distinct subset sums satisfy (using Theorem 1)

$$\int_{\mathbb{R}} \left( (h * \mu)(x) - \gamma(x) \right)^2 dx \leq \frac{1 + o(1)}{2^n}$$

measuring proximity of the probability density functions in the $L^2-$sense.

2.4. **Concluding Remarks.** Theorem 2 has a fascinating implication insofar as it allows us to reinterpret the Erdős distinct subset sums problem (the general version with real numbers being 1-separated) as a genuine problem in probability theory asking whether particularly excellent random walks exist. More precisely, are there positive real numbers $a_1, \dots, a_n > 0$ such that the random unbiased random walk $X = \pm a_1 \pm a_2 \cdots \pm a_n$ has, simultaneously, (1) a large standard deviation, (2) a small largest element $a_n$ and (3) the ability to approximate the normal distribution very well in a concrete sense?

> **Problem.** Fix $c > 0$. As $n \to \infty$, are there random walks $X = \pm a_1 \pm a_2 \cdots \pm a_n$ such that the largest step size is small compared to the variance
>
> $$\text{largest stepsize} = a_n \leq c \cdot n^{-1/3} \sqrt{\mathbb{V}X}.$$

and, simultaneously, $X$ has a large variance and approximates a Gaussian well in the sense of

$$\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{\mathbb{V}X}} + \int_{\mathbb{R}} \left((h*\mu)(x) - \gamma(x)\right)^2 dx = \frac{1+o(1)}{2^{n+1}} \ ?$$

If there exist $\{a_1, \dots, a_n\} \subset \mathbb{N}$ with distinct subset sums and $a_n \lesssim n^{-1/3-\varepsilon} \cdot 2^n$, then such random walks do indeed exists: this follows from combining Theorem 1, Theorem 2 and the computation carried out after the proof of Lemma 1.

Note that, considering the constraint on $a_n$ being as small as possible and considering the structure of the first term, it does seem like one would like to have many of the $a_i$ to be roughly comparable to $a_n$. The Conway-Guy [8] sequence has this property: for each $\varepsilon > 0$ at least $n - c_\varepsilon \log n$ terms satisfy $a_i \geq (1-\varepsilon)a_n$. We also observe that for sets of that type, where many of the $a_i$ are comparable in size to $a_n$, one can draw additional information from Theorem 1

$$\int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)^2 \prod_{i=1}^{n} \cos\left(a_i x\right)^2 dx = \frac{\pi}{2^n}.$$

The cosines are all aligned at $x = 0$, the contribution to the integral coming from close to the origin is really just a function of $\sum_{i=1}^n a_i^2$ (see the comment after the proof of Lemma 1) and fairly independent of the arithmetic structure. The next interesting point is $x = \pi/a_n$: if we have $a_i = (1+o(1))a_n$ for many $1 \leq i \leq n-1$, then many of the cosines will still be aligned at $\pi/a_n$. The only way to avoid a large contribution is to have an $a_i \sim (1+o(1))a_n/2$. So it is not inconceivable that Theorem 1 suggests a sort of multi-scale structure as being possibly favorable. The argument can then be continued for $x = k\pi/a_n$ for small $k \in \mathbb{Z}$. As $k$ gets larger, one would expect the cosines to decorrelate.

## 3. Proofs

### 3.1. **Proof of Theorem 1.**

*Proof.* As already mentioned, we will smooth $\mu$ by convolving with the normalized characteristic function $h(x) = (1/2) \cdot \chi_{[-1,1]}$. Since both $\mu$ and $h$ are probability measures, their convolution

$$(h*\mu)(x) = \frac{1}{2^n}\sum_{i=1}^{2^n} \frac{1}{2}\chi_{[x_i-1, x_i+1]}$$

is also a probability measure. We observe that $h*\mu$ is a sum of characteristic functions and its $L^1-$norm is 1. Its $L^2-$norm is minimized if and only if these characteristic functions do not overlap which is equivalent to $\min_{i\neq j}|x_i - x_j| \geq 2$ and therefore, in turn, equivalent to all subset sums being at least distance 1 from

each other. Formally,

$$\begin{aligned}\|h*\mu\|_{L^2}^2 &= \frac{1}{4^n}\int_{\mathbb{R}}\left(\sum_{i=1}^{2^n}\frac{1}{2}\chi_{[x_i-1,x_i+1]}\right)^2 dx\\ &= \frac{1}{4^n}\int_{\mathbb{R}}\sum_{i,j=1}^{2^n}\frac{1}{2}\chi_{[x_i-1,x_i+1]}\frac{1}{2}\chi_{[x_j-1,x_j+1]}dx\\ &\geq \frac{1}{4^n}\int_{\mathbb{R}}\sum_{i=1}^{2^n}\frac{1}{2}\chi_{[x_i-1,x_i+1]}\frac{1}{2}\chi_{[x_i-1,x_i+1]}dx\\ &= \frac{1}{4^n}2^{n-1} = \frac{1}{2^{n+1}}.\end{aligned}$$

This is the only inequality in the entire argument and is attained if and only if all $x_i$ are $2-$separated. Using that the Fourier transform is unitary on $L^2$ and sends convolution to products,

$$\|\mu*h\|_{L^2}^2 = \|\widehat{\mu*h}\|_{L^2}^2 = \|\widehat{\mu}\widehat{h}\|_{L^2}^2 = \int_{\mathbb{R}}\widehat{h}(\xi)^2\widehat{\mu}(\xi)^2 d\xi.$$

It remains to compute the Fourier transforms: the Fourier transform of the characteristic function $h$ is completely explicit

$$\widehat{h}(\xi) = \frac{\sin(2\pi\xi)}{2\pi\xi}.$$

The measure $\mu$ can itself be defined as a convolution

$$\mu = \left(\frac{\delta_{-a_1}}{2}+\frac{\delta_{a_1}}{2}\right)*\left(\frac{\delta_{-a_2}}{2}+\frac{\delta_{a_2}}{2}\right)*\cdots*\left(\frac{\delta_{-a_n}}{2}+\frac{\delta_{a_n}}{2}\right).$$

Using again that the Fourier transform sends convolution to products and

$$\widehat{\left(\frac{\delta_{-a_i}}{2}+\frac{\delta_{a_i}}{2}\right)}(\xi) = \frac{e^{2\pi i(-a_i)\xi}}{2}+\frac{e^{2\pi i a_i\xi}}{2} = \cos(2\pi a_i\xi)$$

leads to

$$\widehat{\mu}(\xi) = \prod_{i=1}^{n}\cos(2\pi a_i\xi).$$

Thus

$$\int_{\mathbb{R}}\left(\frac{\sin 2\pi x}{2\pi x}\right)^2\prod_{i=1}^{n}\cos(2\pi a_i x)^2 dx \geq \frac{1}{2^{n+1}}$$

with equality if and only if all subset sums of $\{a_1,\ldots,a_n\}$ are 1-separated. □

3.2. **Proof of Corollary 1.**

*Proof.* If all $a_i$ are integers, then the product is $1-$periodic and, together with

$$\sum_{k\in\mathbb{Z}}\left(\frac{\sin 2\pi(x-k)}{2\pi(x-k)}\right)^2 = \frac{1+\cos(2\pi x)}{2},$$

this implies

$$2\int_{\mathbb{R}}\left(\frac{\sin 2\pi x}{2\pi x}\right)^2\prod_{i=1}^{n}\cos(2\pi a_i x)^2 dx = \int_0^1(1+\cos(2\pi x))\prod_{i=1}^{n}\cos(2\pi a_i x)^2 dx.$$

To further evaluate the integral, we switch back to exponentials and note that

$$\cos(2\pi a_i x)^2 = \frac{1}{2} + \frac{e^{4\pi i a_i x} + e^{-4\pi i a_i x}}{4}$$

leading to the integral

$$\frac{1}{2^n}\int_0^1 \left(1 + \frac{e^{2\pi i x} + e^{-2\pi i x}}{2}\right)\prod_{i=1}^n \left(1 + \frac{e^{4\pi i a_i x} + e^{-4\pi i a_i x}}{2}\right).$$

Selecting the constant 1 in all terms leads to a contribution of $2^{-n}$. Any other choice of combinations from the big product leads to exponentials of the form $\exp(4\pi i k x)$ where $k \in \mathbb{Z} \setminus \{0\}$ whenever all subset sums are distinct. Thus every other contributions leads to 0 and

$$\int_0^1 \prod_{i=1}^n \cos(2\pi a_i x)^2 dx \geq \frac{1}{2^n}$$

if all subset sums are distinct. Conversely, if not all subset sums are distinct, then there is a corresponding choice of combinations in the product leading to a zero frequency: since all coefficients are nonnegative, we see that the integral will then be larger than $2^{-n}$. □

3.3. **Proof of Lemma 1.**

*Proof, close to Elkies* [10]. Note that, for example, $a_n \geq 2^n/n$, already implies that the interval is very close to the origin where $\sin(2\pi x)/(2\pi x) \sim 1$ and thus

$$\int_{|x|\leq \frac{1}{4a_n}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos(2\pi a_i x)^2 dx = (1 - o(1)))\int_{|x|\leq \frac{1}{4a_n}} \prod_{i=1}^n \cos(2\pi a_i x)^2 dx.$$

On this interval, we have, for all $1 \leq i \leq n-1$ that $\cos(2\pi a_i x) \geq \cos(2\pi a_n x)$ and

$$\int_{|x|\leq \frac{1}{4a_n}} \prod_{i=1}^n \cos(2\pi a_i x)^2 dx \geq \int_{|x|\leq \frac{1}{4a_n}} \cos(2\pi a_n x)^{2n} dx$$

A change of variables and evaluating the integral (see [10]) shows that

$$\begin{aligned}\int_{|x|\leq \frac{1}{4a_n}} \cos(2\pi a_n x)^{2n} dx &= \frac{1}{2\pi a_n}\int_{|x|\leq \frac{\pi}{2}} \cos(x)^{2n} dx \\ &= \frac{1}{2\pi a_n}\frac{\pi}{4^n}\binom{2n}{n} \\ &= (1+o(1))\frac{1}{2}\frac{1}{a_n}\frac{1}{\sqrt{\pi n}},\end{aligned}$$

where evaluating $\int \cos(x)^{2n} dx$ in terms of binomial coefficients is classical [16]. □

This implies a lower bound on $a_n$ since

$$\begin{aligned}\frac{1}{2^{n+1}} &= \int_{\mathbb{R}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos(2\pi a_i x)^2 dx \\ &\geq \int_{|x|\leq \frac{1}{4a_n}} \left(\frac{\sin 2\pi x}{2\pi x}\right)^2 \prod_{i=1}^n \cos(2\pi a_i x)^2 dx \geq (1+o(1))\frac{1}{2}\frac{1}{a_n}\frac{1}{\sqrt{\pi n}}\end{aligned}$$

showing that $a_n \geq 2^n/\sqrt{\pi n}$ which is, in spirit, the original argument of Elkies. We note that, provided $a_n$ is small, i.e. $a_n = o(\sum_{i=1}^n a_i^2)$, one can Taylor expand the cosines and, for $x$ small,

$$\prod_{i=1}^n \cos(2\pi a_i x)^2 dx \sim \exp\left(-4\pi^2 x^2 \sum_{i=1}^n a_i^2\right)$$

which then leads to the slightly refined estimate

$$\int_{|x|\leq \frac{1}{4a_n}} \prod_{i=1}^n \cos(2\pi a_i x)^2 dx dx \geq \frac{(1+o(1))}{2\sqrt{\pi}} \left(\sum_{i=1}^n a_i^2\right)^{-1/2}.$$

At this point, we do not know of any argument that excludes the possibility that $n - o(n)$ of the $a_i$ satisfy $a_i = (1+o(1))a_n$ and this refined estimate does not currently lead to any information different from that provided by the cruder estimate above. Indeed, the Conway-Guy sequence is an example of a set with distinct subset sums and this type of behavior, perhaps extremal configurations do behave like that.

3.4. **Technical Lemma.** The goal of this section is to establish an upper bound on the difference between $\widehat{\mu}$ and the approximating Gaussian measure close to the origin. Lemma 3 will then quickly imply Theorem 2.

**Lemma 3.** *Let $c > 0$. Suppose $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$ has 1-separated subset sums and $a_n^2 \leq c \cdot n^{-1/2} \sum_{i=1}^n a_i^2$. Then, as $n \to \infty$, we have*

$$\int_{|x|\leq \frac{1}{4a_n}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^n \cos(2\pi a_i x) - \exp\left(-2\pi^2 x^2 \sum_{i=1}^n a_i^2\right)\right|^2 dx = o(2^{-n}).$$

*Proof.* The first step is a Taylor expansion around $x = 0$

$$\begin{aligned}
\frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^n \cos(2\pi a_i x) &= \frac{\sin(2\pi x)}{2\pi x} \exp\left(\sum_{i=1}^n \log(\cos(2\pi a_i x))\right) \\
&\leq \frac{\sin(2\pi x)}{2\pi x} \exp\left(\sum_{i=1}^n \log\left(1 - 2\pi^2 a_i^2 x^2 + \mathcal{O}(a_i^4 x^4)\right)\right) \\
&= \frac{\sin(2\pi x)}{2\pi x} \exp\left(\sum_{i=1}^n -2\pi^2 a_i^2 x^2 + \mathcal{O}(a_i^4 x^4)\right) \\
&= e^{\mathcal{O}(x^2 + n a_n^4 x^4)} \exp\left(-2\pi^2 x^2 \sum_{i=1}^n a_i^2\right).
\end{aligned}$$

The goal is to bound

$$X = \int_{|x|\leq \frac{1}{4a_n}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^n \cos(2\pi a_i x) - \exp\left(-2\pi^2 x^2 \sum_{i=1}^n a_i^2\right)\right|^2 dx$$

which, considering asymptotic expansion, can be bounded as

$$X \leq \int_{|x|\leq \frac{1}{4a_n}} \left| e^{\mathcal{O}(x^2 + n a_n^4 x^4)} - 1\right|^2 \exp\left(-4\pi^2 x^2 \sum_{i=1}^n a_i^2\right) dx.$$

This bound by itself is a little bit too crude but is reasonably close to the origin: note that the integrand is the product of two functions the first of which is small

for small values of $x$ and the second of which is small for large $x$. This suggests splitting the integral into the regions: for some $0 < \delta < 1/(4a_n)$ to be optimized later, we write $I_1 = \{x : |x| \leq \delta\}$ and let $I_2 = \{x : \delta \leq |x| \leq 1/(4a_n)\}$. Provided $\delta^2 + na_n^2\delta^4 = \mathcal{O}(1)$, we can estimate the integral over $I_1$ as

$$\begin{aligned} Y &= \int_{I_1} \left| e^{\mathcal{O}(x^2 + na_n^4 x^4)} - 1 \right|^2 \exp\left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx \\ &\leq \mathcal{O}(\delta^2 + na_n^4\delta^4) \cdot \int_{\mathbb{R}} \exp\left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx \\ &= \mathcal{O}(\delta^2 + na_n^4\delta^4) \cdot \frac{1}{2\sqrt{\pi}} \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}. \end{aligned}$$

We use a different type of expansion for the second region: note that, for $|x| < \pi/2$,

$$\log\left(\cos\left(x\right)\right) \leq -\frac{x^2}{2}$$

and thus, for $|x| \leq 1/(4a_n)$, $\cos\left(2\pi a_i x\right) \leq \exp\left(-2\pi^2 x^2 a_i^2\right)$ from which we deduce

$$\forall |x| \leq \frac{1}{4a_n} \qquad 0 \leq \prod_{i=1}^n \cos\left(2\pi a_i x\right) \leq \exp\left( -2\pi^2 x^2 \sum_{i=1}^n a_i^2 \right).$$

Therefore the contribution of the integrand to $X$ over $I_2$, which is

$$Z = \int_{I_2} \left| \exp\left( -2\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) - \frac{\sin\left(2\pi x\right)}{2\pi x} \prod_{i=1}^n \cos\left(2\pi a_i x\right) \right|^2 dx,$$

can be trivially bounded from above by

$$\begin{aligned} Z &\leq \int_{I_2} \exp\left( -2\pi^2 x^2 \sum_{i=1}^n a_i^2 \right)^2 dx \leq \frac{2}{\delta} \int_\delta^\infty x \exp\left( -4\pi^2 x^2 \sum_{i=1}^n a_i^2 \right) dx \\ &= \frac{2}{\delta} \frac{1}{8\pi^2 \sum_{i=1}^n a_i^2} \exp\left( -4\delta^2\pi^2 \sum_{i=1}^n a_i^2 \right) \leq \frac{1}{\delta} \frac{1}{\sum_{i=1}^n a_i^2} \exp\left( -\delta^2 \sum_{i=1}^n a_i^2 \right). \end{aligned}$$

We want all error estimates to be $o(2^{-n})$ and achieve this by setting

$$\delta = \alpha_n \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}$$

with $\alpha_n$ an arbitrarily slowly growing sequence (think of $\alpha_n = \log\log\log n$). We start by checking whether our first asymptotic expansion is valid in this regime, i.e. whether $\delta^2 + na_n^2\delta^4 = \mathcal{O}(1)$. Moser's estimate implies $\sum_{i=1}^n a_i^2 \gtrsim 4^n$ and thus

$$\delta^2 + na_n^2\delta^4 = \mathcal{O}(\alpha_n^2 4^{-n}) + \mathcal{O}(na_n^2\alpha_n^4 4^{-2n}) = o(2^{-n}).$$

The next step is an estimate on $Y$. Importing our upper bound on $a_n$ shows

$$Y \leq \mathcal{O}(\delta^2 + na_n^4\delta^4) \left( \sum_{i=1}^n a_i^2 \right)^{-1/2} \lesssim na_n^4\alpha_n^4 \left( \sum_{i=1}^n a_i^2 \right)^{-5/2} \lesssim n^{-\varepsilon} \left( \sum_{i=1}^n a_i^2 \right)^{-1/2}$$

which is $\mathcal{O}(n^{-\varepsilon}2^{-n}) = o(2^{-n})$. Finally, for the last error term,

$$Z \leq \frac{1}{\delta}\frac{1}{\sum_{i=1}^n a_i^2}\exp\left(-\delta^2\sum_{i=1}^n a_i^2\right) \leq 10\frac{2^{-n}}{\alpha_n}e^{-\alpha_n^2} = o(2^{-n}).$$

This proves Lemma 3. □

3.5. **Proof of Theorem 2.**

*Proof.* Theorem 2 is a relatively easy consequence of Lemma 3. Recall that

$$\gamma(x) = \frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^n a_i^2\right)^{-1/2}\exp\left(-\frac{x^2}{2}\left(\sum_{i=1}^n a_i^2\right)^{-1}\right)$$

and $\widehat{\gamma}(x) = \exp(-2\pi^2x^2\sum_{i=1}^n a_i^2)$. Using the Fourier transform we get that

$$X = \int_{\mathbb{R}}|(h*\mu)(x) - \gamma(x)|^2dx = \int_{\mathbb{R}}|\widehat{(h*\mu)}(x) - \widehat{\gamma}(x)|^2dx$$

can be written as

$$X = \int_{\mathbb{R}}\left|\frac{\sin(2\pi x)}{2\pi x}\prod_{i=1}^n\cos(2\pi a_i x) - \exp\left(-2\pi^2x^2\sum_{i=1}^n a_i^2\right)\right|^2 dx.$$

Lemma 3 implies that if $\{a_1,\dots,a_n\} \subset \mathbb{R}_{>0}$ has 1-separated subset sums and $a_n^2 \leq c\cdot n^{-1/2}\sum_{i=1}^n a_i^2$, then

$$\int_{|x|\leq\frac{1}{4a_n}}\left|\frac{\sin(2\pi x)}{2\pi x}\prod_{i=1}^n\cos(2\pi a_i x) - \exp\left(-2\pi^2x^2\sum_{i=1}^n a_i^2\right)\right|^2 dx = o(2^{-n})$$

implying that, by splitting the integral into $\{|x| \leq 1/(4a_n)\}$ and $\{|x| \geq 1/(4a_n)\}$,

$$X = \int_{|x|\geq\frac{1}{4a_n}}\left|\frac{\sin(2\pi x)}{2\pi x}\prod_{i=1}^n\cos(2\pi a_i x) - \exp\left(-2\pi^2x^2\sum_{i=1}^n a_i^2\right)\right|^2 dx + o(2^{-n}).$$

Using the upper bound on $a_n$

$$\begin{aligned}\int_{|x|\geq 1/(4a_n)}\exp\left(-4\pi^2x^2\sum_{i=1}^n a_i^2\right)dx &\leq 8a_n\int_{1/(4a_n)}^\infty x\exp\left(-4\pi^2x^2\sum_{i=1}^n a_i^2\right)dx\\ &\leq \frac{a_n}{\sum_{i=1}^n a_i^2}\exp\left(-\frac{\pi^2}{4}\frac{\sum_{i=1}^n a_i^2}{a_n^2}\right)\\ &\lesssim \frac{1}{n^{1/4}}\frac{1}{(\sum_{i=1}^n a_i^2)^{1/2}}e^{-c\sqrt{n}}.\end{aligned}$$

With Moser's estimate $\sum_{i=1}^n a_i^2 \gtrsim 4^n$ one deduces

$$\left\|\exp\left(-2\pi^2x^2\sum_{i=1}^n a_i^2\right)\chi_{|x|\geq\frac{1}{4a_n}}\right\|_{L^2(\mathbb{R})} \lesssim e^{-c\sqrt{n}}\frac{1}{2^n}.$$

Using the triangle inequality in $L^2$, we see that

$$Z = \left\| \left[ \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) - \exp\left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right] \chi_{|x| \geq \frac{1}{4a_n}} \right\|_{L^2(\mathbb{R})}$$
$$= \left\| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) \chi_{|x| \geq \frac{1}{4a_n}} \right\|_{L^2(\mathbb{R})} + \mathcal{O}\left( e^{-c\sqrt{n}} \frac{1}{2^n} \right).$$

Squaring both sides and using Theorem 1, we deduce

$$X = Z^2 + o(2^{-n}) = \int_{|x| \geq \frac{1}{4a_n}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos(2\pi a_i x)^2 dx + o(2^{-n}).$$

□

3.6. **Proof of Lemma 2.** Throughout this proof, we will abbreviate

$$\gamma(x) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \exp\left( -\frac{x^2}{2} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1} \right)$$

for the Gaussian approximating $\mu$. Before proving Lemma 2, we quickly recall the Berry-Esseen theorem which, in our setting, says that

$$\sup_{x \in \mathbb{R}} \left| \mu([-\infty, x]) - \int_{-\infty}^{x} \gamma(x) dx \right| \leq \frac{\sum_{i=1}^{n} a_i^3}{\left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}}.$$

Assuming that $a_n^2 = \mathcal{O}\left( n^{-2/3-\varepsilon} \sum_{i=1}^{n} a_i^2 \right)$, one can bound this by

$$\frac{\sum_{i=1}^{n} a_i^3}{\left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}} \lesssim \frac{n \cdot a_n^3}{\left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}} \lesssim n^{-\frac{3\varepsilon}{2}} = o(1).$$

The way we will use this information is that, for any interval $J \subset \mathbb{R}$

$$\left| \mu(J) - \int_J \gamma(x) dx \right| = o(1).$$

Note that this argument was also used by Dubroff, Fox and Xu [9] for $J$ an interval centered at the origin whose length is proportional to a small multiple of the standard deviation of the Gaussian. We will quickly summarize their short argument at an appropriate place in the proof of Lemma 2.

*Proof of Lemma 2.* We start the argument with a lower bound on

$$X = \int_{\mathbb{R}} |(\mu * h)(x) - \gamma(x)|^2 dx.$$

Taking a Fourier transform,

$$X = \int_{\mathbb{R}} \left| \frac{\sin(2\pi x)}{2\pi x} \prod_{i=1}^{n} \cos(2\pi a_i x) - \exp\left( -2\pi^2 x^2 \sum_{i=1}^{n} a_i^2 \right) \right|^2 dx.$$

We split the integral into two regions: $|x| \leq 1/(4a_n)$ and the remaining region. Lemma 3 implies that the integral over the first region is $o(2^{-n})$, it remains to

analyze the integral over the second region. Arguing exactly as in the proof of Theorem 2, we deduce that

$$X = \int_{|x| \geq \frac{1}{4a_n}} \left( \frac{\sin 2\pi x}{2\pi x} \right)^2 \prod_{i=1}^{n} \cos\left(2\pi a_i x\right)^2 dx + o(2^{-n}).$$

The next argument is completely independent of all the previous arguments: we will derive a lower bound on the same quantity via a completely different argument which will then imply Lemma 2. Recall that

$$X = \int_{\mathbb{R}} \left| (\mu * h)(x) - \gamma(x) \right|^2 dx.$$

$\mu * h$ only assumes the values $\left\{0, 2^{-n-1}\right\}$. Moreover, by the argument above,

$$\sup_{\substack{J \subset \mathbb{R} \\ J \text{ interval}}} \left| \int_J (\mu * h)(x) dx - \int_J \gamma(x) dx \right| = o(1).$$

This leads to an amusing setting: we know that $\mu * h$ approximates the Gaussian in probability over intervals. Simultaneously, $\mu * h$ can only assume two values one of which is 0: thus, the local density of the Gaussian predicts the density of intervals in the region where $\mu * h$ assumes its nonzero value $2^{-n-1}$. An example of what this could look like is shown in Fig. 2. We conclude with a simple proposition.

**Proposition.** *Let $\mu$ be the probability density function of a $\mathcal{N}(0, \sigma^2)$ Gaussian. Let $(\nu_n)_n$ be a sequence of probability density functions such that*

(1) *$\nu_n \to \mu$ in probability: for every interval $J \subset \mathbb{R}$ we have*

$$\lim_{n \to \infty} \int_J \nu_n(x) dx = \int_J \mu(x) dx$$

(2) *and $\nu_n(x)$ only assumes two values $\{0, z_n\}$ for some $z_n > 0$.*

*Then*

$$\liminf_{n \to \infty} \int_{\mathbb{R}} (\mu(x) - \nu_n(x))^2 dx \geq \frac{\sqrt{2} - 1}{2\sqrt{\pi}\sigma}.$$

*Proof of the Proposition.* The density of $\mu$ is simply given by

$$\mu(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left( -\frac{1}{2} \frac{x^2}{\sigma^2} \right).$$

We note that both Properties combined require (by taking $J$ to be a small interval centered around the origin) that

$$\liminf_{n \to \infty} z_n \geq \max_{x \in \mathbb{R}} \mu(x) = \frac{1}{\sqrt{2\pi}\sigma}.$$

Let now $J$ be a small interval centered around $x_0 \in \mathbb{R}$, say $J = (x_0 - \varepsilon, x_0 + \varepsilon)$. The two properties combined tell us what can be expected of $\nu_n$: since

$$\int_J \mu(x) dx = 2\varepsilon \mu(x_0) + \mathcal{O}(\varepsilon^2)$$

we have

$$\lim_{n \to \infty} \int_J \nu_n(x) dx = \int_J \mu(x) dx = 2\varepsilon \mu(x_0) + \mathcal{O}(\varepsilon^2).$$

This allows us to deduce that the fraction $\alpha$ of the interval $J$ where $\nu_n$ assumes the value $z_n$ and the remaining fraction $(1-\alpha)$ where it assumes the value 0 is determined by

$$\alpha z_n = \mu(x_0) + \text{lower order terms.}$$

This tells us that

$$\int_{\mathbb{R}} (\mu(x) - \nu_n(x))^2 dx = (1+o(1)) \int_{\mathbb{R}} \frac{\mu(x)}{z_n} (\mu(x) - z_n)^2 + \left(1 - \frac{\mu(x)}{z_n}\right) \mu(x)^2 dx.$$

The integral algebraically simplifies to

$$\int_{\mathbb{R}} \frac{\mu(x)}{z_n} (\mu(x) - z_n)^2 + \left(1 - \frac{\mu(x)}{z_n}\right) \mu(x)^2 dx = \int_{\mathbb{R}} \mu(x)(z_n - \mu(x)) dx.$$

At this point, we recall that, up to lower order terms, $z_n \geq \mu(0)$. Thus

$$\int_{\mathbb{R}} \mu(x)(z_n - \mu(x)) dx \geq \int_{\mathbb{R}} \mu(x) \left( \frac{1}{\sqrt{2\pi}\sigma} - \mu(x) \right) dx = \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sqrt{\pi}\sigma}$$

which is the desired result. □

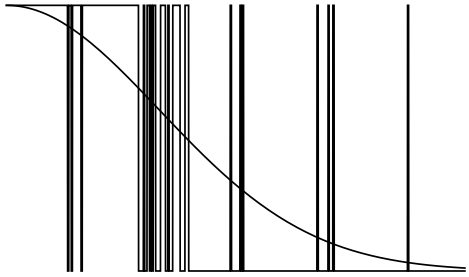

FIGURE 2. A step function assuming only two values approximating a Gaussian density.

At this point can we quickly note, in passing, the original argument of Dubroff, Fox and Xu [9]: the Gaussian attains its maximum density at the origin and therefore

$$\gamma(0) = \frac{1}{\sqrt{2\pi}} \left( \sum_{i=1}^{n} a_i^2 \right)^{-1/2} \leq (1+o(1)) \cdot \|\mu * h\|_{L^\infty} = \frac{1+o(1)}{2^{n+1}}$$

from which one deduces

$$\sqrt{n} \cdot a_n \geq \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2} \geq (1+o(1)) \sqrt{\frac{2}{\pi}} \cdot 2^n.$$

We can now conclude by applying the Proposition. The variance $\sigma$ of the mollified random walk is, up to lower order terms, given by the variance of the random walk which is $\sum_{i=1}^{n} a_i^2$. Thus, applying the Proposition, as $n$ becomes large,

$$X \geq (1+o(1)) \frac{\sqrt{2}-1}{2\sqrt{\pi} \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}}.$$

□

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