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Optimal control for the conformal CR sub-Laplacian obstacle problem

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Abstract. In this paper, we study an optimal control problem associated to the conformal CR sub-Laplacian obstacle problem on a compact pseudo-hermitian manifold. When the CR Yamabe constant is positive, we show that the optimal controls are equal to their associated optimal states and show the existence of a smooth optimal control which induces a conformal contact form with constant Webster scalar curvature.

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1. Introduction

Suppose (M, g) is a closed (i.e. compact without boundary) n-dimensional Riemannian manifold, where $n \geq 3$. As a generalization of the Uniformization Theorem of surfaces, the Yamabe problem is to find a metric conformal to g such that its scalar curvature is constant. This is equivalent to finding a smooth positive solution to

$$L_g u = c u^{\frac{n+2}{n-2}} \quad \text{in } M \tag{1.1}$$

for some constant c, where

$$L_g := -\frac{4(n-1)}{n-2}\Delta_g + R_g$$

is the conformal Laplacian of g with R_g the scalar curvature of g and Δ_g the Laplacian of g. The Yamabe problem was solved in the works of Yamabe [24], Trudinger [22], Aubin [1], and Schoen [21] by finding a smooth minimizer of the Yamabe functional J^g defined by

$$J^g(u) = \frac{\langle u, u\rangle}{\|u\|^2_{L^{\frac{2n}{n-2}}(M,g)}}, \; u \in H^1_+(M, g) = \left\{u \in H^1(M, g), u > 0\right\},$$

where

$$\langle u, v\rangle_g = \int_M \left(\frac{4(n-1)}{n-2}\nabla_g u \cdot \nabla_g v + R_g uv\right) dV_g,\ \ u, v \in H^1_+(M, g).$$

Here, dV_g is the volume form with respect to g, $\nabla_g u$ is the gradient of u with respect to g, $L^p(M, g)$ is the standard Lebesque space of functions which are p-integrable over M with respect to g, $\|\cdot\|_{L^p(M,g)}$ is the standard L^p-norm on $L^p(M, g)$, and $H^1(M, g)$ is the Sobolev space containing functions which are of class L^2 together with their first derivatives with respect to g. See also [2,3,12] and references therein for results related to the Yamabe flow, which is a geometric flow introduced to study the Yamabe problem.

In [19], the second author studied equation (1.1) in the context of Optimal Control Theory. To state the results, recall that the Yamabe constant of (M, g) is defined as

$$\mathcal{Y}(M, [g]) = \inf_{u \in H^1_+(M,g)} J^g(u).$$

Under the assumption $\mathcal{Y}(M, [g]) > 0$, the following optimal control problem for the conformal Laplacian obstacle problem was studied:

$$\text{Find } u_{\min} \in H^1_+(M, g) \text{ such that } I^g(u_{\min}) = \min_{w \in H^1_+(M,g)} I^g(w),$$

where

$$I^g(u) = \frac{\langle u, u\rangle_g}{\|T_g(u)\|^2_{L^{\frac{2n}{n-2}}(M,g)}},\ \ u \in H^1_+(M, g)$$

with

$$T_g(u) = \arg \min_{v \in H^1_+(M,g), v \geq u} \langle v, v\rangle_g$$

where the symbol $\arg \min_{v \in H^1_+(M,g), v \geq u} \langle v, v\rangle_g$ denotes the unique solution to the minimization problem

$$\min_{v \in H^1_+(M,g), v \geq u} \langle v, v\rangle_g.$$

Note that T_g is a map from $H^1_+(M, g)$ to $H^1_+(M, g)$, i.e. $T_g : H^1_+(M, g) \to H^1_+(M, g)$ (c.f. [19, Lemma 3.1]). Note also that it was shown in [19, Lemma 3.1] that unique solution to this minimization problem exists. Let $C^\infty_+(M)$ be the space of all positive smooth functions on M. In [19], the second author proved the following theorem.

Theorem 1.1. *(Theorem 1.1 in [19]) Suppose that* $\mathcal{Y}(M, [g]) > 0$. *Then*
(i) For any $u \in H^1_+(M, g)$,

$$I^g(u) = \min_{v \in H^1_+(M,g)} I^g(v) \implies T_g(u) = u, u \in C^\infty_+(M) \text{ and } R_{g_u} \equiv c$$

for some constant $c > 0$, *where* $g_u = u^{\frac{4}{n-2}} g$.

(ii) There exists $u_{\min} \in C_+^\infty(M)$ *such that*

$$I^g(u_{\min}) = \min_{v \in H_+^1(M,g)} I^g(v) \text{ and } R_{g_{u_{\min}}} = \mathcal{Y}(M,[g])$$

where $g_{u_{\min}} = u_{\min}^{\frac{4}{n-2}} g$.

On the other hand, the converse of Theorem 1.1(i) is also true on the n-dimensional unit sphere equipped with the standard metric g_{S^n}:

Theorem 1.2. (Theorem 1.2 in [19]) *Suppose that* $(M,g) = (S^n, g_{S^n})$*, the* n*-dimensional unit sphere equipped with the standard metric* g_{S^n}*. For* $u \in H_+^1(S^n, g_{S^n})$*,*

$$I^{g_{S^n}}(u) = \min_{v \in H_+^1(S^n, g_{S^n})} I^{g_{S^n}}(v) \text{ is equivalent to } u \in C_+^\infty(S^{2n+1}) \text{ and } R_{\tilde{g}} = c$$

for some constant c*, where* $\tilde{g} = u^{\frac{4}{n-2}} g_{S^n}$*.*

It was also pointed out by the referee that Theorem 1.2 is actually true, provided that (M,g) is a conformally Einstein manifold with positive Yamabe constant, by using the Obata's theorem for conformally Einstein metrics of constant scalar curvature. This has not been mentioned in [19] and we would like to thank the referee for pointing this out.

Now suppose (M,θ) is a compact pseudohermitian manifold of real dimension $2n+1$ equipped with the contact form θ. The conformal class of θ is denoted by $[\theta]$, i.e.

$$[\theta] = \left\{ u^{\frac{2}{n}}\theta : u \in C_+^\infty(M) \right\}.$$

Hereafter, we set $\theta_u = u^{\frac{2}{n}}\theta$ where $u \in C_+^\infty(M)$. The CR Yamabe problem is to find a contact form $\theta_u \in [\theta]$ such that its Webster scalar curvature R_{θ_u} is constant. Let $2^* = 2 + \dfrac{2}{n}$. Note that if $\theta_u = u^{\frac{2}{n}}\theta$ where $u \in C_+^\infty(M)$, there holds

$$-\left(2 + \frac{2}{n}\right)\Delta_\theta u + R_\theta u = R_{\theta_u} u^{2^*-1}, \tag{1.2}$$

where Δ_θ is the sub-Laplacian of θ. Let

$$L_\theta u = -\left(2 + \frac{2}{n}\right)\Delta_\theta + R_\theta \tag{1.3}$$

be the conformal CR sub-Laplacian of θ. In view of (1.2), the CR Yamabe problem is equivalent to finding $u \in C_+^\infty(M)$ such that

$$L_\theta u = c u^{2^*-1}$$

for some constant c. The CR Yamabe problem was studied in [5,7,10,11,16–18]. Let $S_1^2(M,\theta)$ be the Folland-Stein space (c.f. [9]). As in the Yamabe problem on Riemannian manifolds, one tries to solve the CR Yamabe problem by finding a minimizer of the CR Yamabe functional J^θ defined by

$$J^\theta(u) := \frac{\langle u, u\rangle_\theta}{\|u\|_{L^{2^*}(M,\theta)}}, \ u \in S_1^2(M,\theta)_+ := \big\{u \in S_1^2(M,\theta) : u > 0\big\},$$

where

$$\langle u, v\rangle_\theta = \int_M \left(\left(2+\frac{2}{n}\right)\nabla_\theta u\cdot\nabla_\theta v + R_\theta uv\right)dV_\theta,\ u, v\in S_1^2(M,\theta)_+, \quad (1.4)$$

dV_θ is the volume form with respect to θ, $\nabla_\theta u$ is the sub-gradient of u with respect to θ, $L^p(M,\theta)$ is the standard Lebesque space of functions which are p-integrable over M with respect to θ, and $\|\cdot\|_{L^p(M,\theta)}$ is the standard L^p-norm on $L^p(M,\theta)$, i.e.

$$\|u\|_{L^p(M,\theta)} = \left(\int_M |u|^p dV_\theta\right)^{\frac{1}{p}}.$$

We remark that, unlike the Yamabe problem on Riemannian manifolds, a minimizer may not exist (see [8], in which Cheng-Malchiodi-Yang proved the nonexistence of minimizers on Rossi spheres sufficiently close to the standard CR three-sphere). See also [13–15, 20] for the results related to the CR Yamabe flow, which is a geometric flow introduced to study the CR Yamabe problem.

Recall that the CR Yamabe constant of (M,θ) is defined as

$$\mathcal{Y}(M,[\theta]) = \inf_{u\in S_1^2(M,\theta)_+} J^\theta(u).$$

Inspired by the results in [19], under the assumption that $\mathcal{Y}(M,[\theta])>0$, we study in this paper the following optimal control problem for the conformal CR sub-Laplacian obstacle problem:

$$\text{Find } u_{\min}\in S_1^2(M,\theta)_+ \text{ such that } I^\theta(u_{\min}) = \min_{u\in S_1^2(M,\theta)_+} I^\theta(u),$$

where

$$I^\theta(u) = \frac{\langle u,u\rangle_\theta}{\|T_\theta(u)\|^2_{L^{2^*}(M,\theta)}},\ u\in S_1^2(M,\theta)_+$$

with

$$T_\theta(u) = \arg\min_{v\in S_1^2(M,\theta)_+, v\geq u}\langle v,v\rangle_\theta.$$

Here the symbol

$$\arg\min_{v\in S_1^2(M,\theta)_+, v\geq u}\langle v,v\rangle_\theta$$

denotes the unique solution to the minimization problem (see Lemma 3.1)

$$\min_{v\in S_1^2(M,\theta)_+, v\geq u}\langle v,v\rangle_\theta.$$

Theorem 1.3. *Suppose (M,θ) is a compact pseudohermitian manifold of real dimension $2n+1$ equipped with the contact form θ such that $\mathcal{Y}(M,[\theta])>0$.*

(i) For any $u\in S_1^2(M,\theta)_+$,

$$I^\theta(u) = \min_{v\in S_1^2(M,\theta)_+} I^\theta(v) \implies T_\theta(u) = u, u\in C_+^\infty(M) \text{ and } R_{\theta_u}\equiv c$$

for some constant $c>0$, where $\theta_u = u^{\frac{2}{n}}\theta$.

(ii) If a minimizer for the CR Yamabe problem exists on (M,θ), there exists $u_{\min}\in C^\infty_+(M)$ such that

$$I^\theta(u_{\min})=\min_{v\in S_1^2(M,\theta)_+} I^\theta(v), \tag{1.5}$$

where $\theta_{u_{\min}}=u_{\min}^{\frac{2}{n}}\theta$.

In particular, we have the following:

Corollary 1.4. *Suppose (M,θ) is a compact pseudohermitian manifold of real dimension $2n+1$ equipped with the contact form θ such that $\mathcal{Y}(M,[\theta])>0$. Then (1.5) holds if one of the following is true:*

(i) if $n\geq 2$ and (M,θ) is not spherical,
(ii) if $n=1$ and the CR Paneitz operator of (M,θ) is nonnegative,
(iii) if $n\geq 2$ and (M,θ) is spherical; and when $n=2$, we further assume that the minimum exponent of the integrability of the Green function satisfies $s(M)<1$.

Similar to Theorem 1.2, we have the following:

Theorem 1.5. *Suppose that (M,θ) has positive CR Yamabe constant and θ is a torsion-free pseudo-Einstein contact form. For $u\in S_1^2(M,\theta)_+$,*

$$I^\theta(u)=\min_{v\in S_1^2(M,\theta)_+} I^\theta(v) \iff u\in C^\infty_+(M) \ \ \text{and} \ \ R_{\tilde\theta}=c$$

for some constant c, where $\tilde\theta=u^{\frac{2}{n}}\theta$.

2. Notations and preliminaries

In this section, we fix our notations and collect some well-known facts about CR manifolds, which can be found in [18] for example.

From now on, (M,θ) is the background compact pseudohermitian manifold of real dimension $2n+1$ equipped with the contact form θ. As before, $2^*=2+\dfrac{2}{n}$. Recall the CR Yamabe functional J^θ and its subcritical approximation J_p^θ, $1\leq p<2^*-1$ are given by

$$J^\theta(u)=\frac{\langle u,u\rangle_\theta}{\|u\|^2_{L^{2^*}(M,\theta)}} \tag{2.1}$$

and

$$J_p^\theta(u)=\frac{\langle u,u\rangle_\theta}{\|u\|^2_{L^{p+1}(M,\theta)}}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_\theta$ is defined as in (1.4). Therefore, we can see that

$$J^\theta_{2^*-1}=J^\theta. \tag{2.3}$$

It follows from the definition in (2.2) that

$$J_p^\theta(\lambda u)=J_p^\theta(u) \ \text{ for any } \lambda>0, u\in S_1^2(M,\theta)_+. \tag{2.4}$$

If $\theta_w = w^{\frac{2}{n}}\theta$ where $w \in C^\infty_+(M)$, we have the following transformation rules:

$$wS^2_1(M, \theta_w)_+ = S^2_1(M, \theta)_+, \tag{2.5}$$

$$dV_{\theta_w} = w^{2^*} dV_\theta, \tag{2.6}$$

$$\langle u, u\rangle_{\theta_w} = \langle wu, wu\rangle_\theta \ \text{ for } u \in S^2_1(M, \theta_w), \tag{2.7}$$

$$J^{\theta_w}(u) = J^\theta(wu), \ u \in S^2_1(M, \theta_w). \tag{2.8}$$

When the CR Yamabe constant $\mathcal{Y}(M, [\theta])$ is positive, we define the CR Yamabe optimal obstacle functional I^θ and its subcritical approximation I^θ_p $(1 \le p < 2^* - 1)$:

$$I^\theta(u) = \frac{\langle u, u\rangle_\theta}{\|T_\theta(u)\|^2_{L^{2^*}(M,\theta)}} \tag{2.9}$$

and

$$I^\theta_p(u) = \frac{\langle u, u\rangle_\theta}{\|T_\theta(u)\|^2_{L^{p+1}(M,\theta)}} \tag{2.10}$$

In particular, we have

$$I^\theta_{2^*-1} = I^\theta. \tag{2.11}$$

3. Obstacle problem for the conformal CR sub-Laplacian

In this section, we study the obstacle problem for the conformal CR sub-Laplacian $L_{\tilde\theta}$ (recall its definition in (1.3)) with $\tilde\theta \in [\theta]$ under the assumption $\mathcal{Y}(M, [\theta]) > 0$. More precisely, we consider the minimization problem

$$\min_{v \in S^2_1(M,\tilde\theta)_+, v \ge u} \langle v, v\rangle_{\tilde\theta}. \tag{3.1}$$

We have the following:

Lemma 3.1. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$ and $\tilde\theta \in [\theta]$. For $u \in S^2_1(M, \tilde\theta)_+$, there exists a unique $T_{\tilde\theta}(u) \in S^2_1(M, \tilde\theta)_+$ such that*

$$\|T_{\tilde\theta}(u)\|^2_{\tilde\theta} = \min_{v \in S^2_1(M,\tilde\theta)_+, v \ge u} \|v\|^2_{\tilde\theta}.$$

Hereafter, $\|v\|^2_{\tilde\theta} = \langle v, v\rangle_{\tilde\theta}$.

Proof. Since $\mathcal{Y}(M, [\theta]) > 0$ by assumption, $L_{\tilde\theta} \ge 0$ and $\ker L_{\tilde\theta} = \{0\}$. Thus $\langle \cdot, \cdot\rangle_{\tilde\theta}$ defines a inner product on $S^2_1(M, \tilde\theta)$, which induces a norm $\|\cdot\|_{\tilde\theta}$, equivalent to the standard $S^2_1(M, \tilde\theta)$-norm on $S^2_1(M, \tilde\theta)_+$. Hence, as in the classical obstacle problem for the sub-Laplacian $\Delta_{\tilde\theta}$, the lemma now follows from standard argument in the Calculus of Variations (see [4] for example). □

We now study some properties of the state map $T_{\tilde\theta} : S^2_1(M, \tilde\theta)_+ \to S^2_1(M, \tilde\theta)_+$. First, we have the following:

Proposition 3.2. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$ and $\tilde\theta \in [\theta]$. Then the state map $T_{\tilde\theta} : S^2_1(M, \tilde\theta)_+ \to S^2_1(M, \tilde\theta)_+$ is idempotent, i.e.*

$$T^2_{\tilde\theta} = T_{\tilde\theta}.$$

Proof. Let $v \in S_1^2(M,\tilde{\theta})_+$ be such that $v \geq T_{\tilde{\theta}}(u)$. Since $T_{\tilde{\theta}}(u) \geq u$, we have $v \geq u$. Therefore, by definition of $T_{\tilde{\theta}}$, we have

$$\|v\|_{\tilde{\theta}} \geq \|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}.$$

Hence, since $T_{\tilde{\theta}}(u) \in S_1^2(M,\tilde{\theta})_+$ with $T_{\tilde{\theta}}(u) \geq T_{\tilde{\theta}}(u)$, we have

$$\|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}} \geq \|T_{\tilde{\theta}}(T_{\tilde{\theta}}(u))\|_{\tilde{\theta}}.$$

By uniqueness, we have

$$T_{\tilde{\theta}}(T_{\tilde{\theta}}(u)) = T_{\tilde{\theta}}(u),$$

as required. □

The following lemma shows that $T_{\tilde{\theta}}$ is positively homogeneous.

Lemma 3.3. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $\tilde{\theta} \in [\theta]$. Then for any $\lambda > 0$, there holds*

$$T_{\tilde{\theta}}(\lambda u) = \lambda T_{\tilde{\theta}}(u)$$

for all $u \in S_1^2(M,\tilde{\theta})_+$.

Proof. Let $v \in S_1^2(M,\tilde{\theta})_+$ satisfy $v \geq \lambda u$. Since $\lambda > 0$, we have $\lambda^{-1}v \geq u$. Since $\lambda^{-1}v \in S_1^2(M,\tilde{\theta})_+$, it follows from the definition of $T_{\tilde{\theta}}$ that

$$\|\lambda^{-1}v\|_{\tilde{\theta}} \geq \|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}.$$

By the positive homogeneity of $\|\cdot\|_{\tilde{\theta}}$, i.e.

$$\|\lambda v\|_{\tilde{\theta}} = \lambda \|v\|_{\tilde{\theta}} \quad \text{for any } \lambda > 0, \tag{3.2}$$

we obtain

$$\|v\|_{\tilde{\theta}} \geq \|\lambda T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}.$$

Since $\lambda T_{\tilde{\theta}}(u) \in S_1^2(M,\tilde{\theta})_+$ satisfies $\lambda T_{\tilde{\theta}}(u) \geq \lambda u$, by uniqueness we get

$$\lambda T_{\tilde{\theta}}(u) = T_{\tilde{\theta}}(\lambda u),$$

as required. □

Lemma 3.3 implies the following analogue of formula (2.4) for $I_p^{\tilde{\theta}}$.

Corollary 3.4. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$, $\tilde{\theta} \in [\theta]$, and $1 \leq p \leq 2^* - 1$. Then for any $\lambda > 0$, there holds*

$$I_p^{\tilde{\theta}}(\lambda u) = I_p^{\tilde{\theta}}(u)$$

for all $u \in S_1^2(M,\tilde{\theta})_+$.

Proof. By (3.2) and Lemma 3.3, we find

$$I_p^{\tilde{\theta}}(\lambda u) = \frac{\|\lambda u\|_{\tilde{\theta}}^2}{\|T_{\tilde{\theta}}(\lambda u)\|_{\tilde{\theta}}^2} = \frac{\|\lambda u\|_{\tilde{\theta}}^2}{\|\lambda T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}^2} = \frac{\|u\|_{\tilde{\theta}}^2}{\|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}^2} = I_p^{\tilde{\theta}}(u),$$

which proves the assertion. □

4. Transformation rules of $T_{\tilde{\theta}}$ and $I^{\tilde{\theta}}$ for $\tilde{\theta} \in [\theta]$

In this section, we study the transformation rules of $T_{\tilde{\theta}}$ when $\tilde{\theta}$ varies in $[\theta]$. As before, we adopt the notation $\theta_w = w^{\frac{2}{n}}\theta$ whenever $w \in C^\infty_+(M)$.

Lemma 4.1. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $w \in C^\infty_+(M)$. There holds*

$$T_{\theta_w}(u) = w^{-1}T_\theta(wu)$$

for any $u \in S^2_1(M,\theta_w)$.

Proof. Let $v \in S^2_1(M,\theta_w)_+$ with $v \geq u$. It follows from (2.7) that

$$\|v\|_{\theta_w} = \|wv\|_\theta.$$

Since $v \geq u$ and $w > 0$, we have $wv \geq wu$ and $wv, wu \in S^2_1(M,\theta)_+$. Hence, it follows from the definition of T_θ that

$$\|wv\|_\theta \geq \|T_\theta(wu)\|_\theta.$$

Using (2.7) again, we obtain

$$\|T_\theta(wu)\|_\theta = \|w^{-1}T_\theta(wu)\|_{\theta_w}.$$

Combining all these, we obtain

$$\|v\|_{\theta_w} \geq \|w^{-1}T_\theta(wu)\|_{\theta_w}.$$

Now, since $w^{-1}T_\theta(wu) \geq w^{-1}wu = u$ and $w^{-1}T_\theta(wu) \in S^2_1(M,\theta_w)$, by uniqueness, we have

$$T_{\theta_w}(u) = w^{-1}T_\theta(wu)$$

as required. □

The following is an immediate consequence of Lemma 4.1.

Corollary 4.2. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $w \in C^\infty_+(M)$. Then*

$$Fix(T_{\theta_w}) = w^{-1}Fix(T_\theta),$$

where

$$Fix(T_\theta) = \left\{u \in S^2_1(M,\theta) : T_\theta(u) = u\right\}.$$

Proof. It follows from Lemma 4.1 and the definition of fixed point set that

$$u \in Fix(T_{\theta_w}) \iff T_{\theta_w}(u) = u \iff w^{-1}T_\theta(wu) = u \iff wu \in Fix(T_\theta),$$

which proves the assertion. □

Lemma 4.1 also implies the following:

Corollary 4.3. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $w \in C^\infty_+(M)$. Then*

$$\|T_{\theta_w}(u)\|_{L^{2^*}(M,\theta_w)} = \|T_\theta(wu)\|_{L^{2^*}(M,\theta)}$$

for any $u \in S^2_1(M,\theta_w)_+$.

Proof. By (2.6) and Lemma 4.1, we find

$$\begin{aligned}\|T_{\theta_w}(u)\|_{L^{2^*}(M,\theta_w)}^{2^*} &= \int_M |T_{\theta_w}(u)|^{2^*} dV_{\theta_w} = \int_M |w^{-1}T_\theta(wu)|^{2^*} w^{2^*} dV_\theta \\ &= \int_M |T_\theta(wu)|^{2^*} dV_\theta = \|T_\theta(wu)\|_{L^{2^*}(M,\theta)}^{2^*},\end{aligned}$$

which proves the assertion. □

As a consequence of Corollary 4.3, we have the following analogue of formula (2.9) for the CR Yamabe optimal obstacle functional.

Corollary 4.4. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $w \in C_+^\infty(M)$. Then*

$$I^{\theta_w}(u) = I^\theta(wu).$$

Proof. It follows from (2.7) and Corollary 4.3 that

$$I^{\theta_w}(u) = \frac{\|u\|_{\theta_w}^2}{\|T_{\theta_w}(u)\|_{L^{2^*}(M,\theta_w)}^2} = \frac{\|wu\|_\theta^2}{\|T_\theta(wu)\|_{L^{2^*}(M,\theta)}^2} = I^\theta(wu),$$

as required. □

Corollary 4.4 implies the following:

Corollary 4.5. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$ and $w \in C_+^\infty(M)$. Then we have*

$$\inf_{u \in S_1^2(M,\theta_w)_+} I^{\theta_w}(u) = \inf_{u \in S_1^2(M,\theta)_+} I^\theta(u).$$

Proof. By (2.5) and Corollary 4.4, we find

$$\begin{aligned}\inf_{u \in S_1^2(M,\theta)_+} I^\theta(u) &= \inf_{u \in wS_1^2(M,\theta_w)_+} I^\theta(u) \\ &= \inf_{\overline{u} \in S_1^2(M,\theta_w)_+} I^\theta(w\overline{u}) = \inf_{\overline{u} \in S_1^2(M,\theta_w)_+} I^{\theta_w}(\overline{u})\end{aligned}$$

where the second equality follows from letting $u = w\overline{u}$. This proves the assertion. □

Similar to the CR Yamabe constant $\mathcal{Y}(M,[\theta])$, we have the following:

Definition 4.6. *Suppose that $\mathcal{Y}(M,[\theta]) > 0$. Define*

$$\mathcal{Y}_{oc}(M,\theta) = \inf_{u \in S_1^2(M,\theta)_+} I^\theta(u).$$

Remark 4.7. If $\tilde{\theta} \in [\theta]$, it follows from Corollary 4.5 that

$$\mathcal{Y}_{oc}(M,\tilde{\theta}) = \mathcal{Y}_{oc}(M,\theta).$$

That is to say, it depends only on the conformal class. As a result, we will write $\mathcal{Y}_{oc}(M,[\theta])$.

In the paper of Jerison and Lee [16] the following family of real numbers was introduced (see (6.1) in [16])

$$\mathcal{Y}^p(M,\tilde{\theta}) := \inf_{u\in S_1^2(M,\tilde{\theta})_+} J_p^{\tilde{\theta}}(u), \quad \text{for } \tilde{\theta}\in[\theta], 1\le p\le 2^*-1.$$

Note that $\mathcal{Y}^p(M,\tilde{\theta})$ is a pseudohermitian invariant, but is not a CR invariant. Clearly, $\mathcal{Y}^{2^*-1}(M,\tilde{\theta}) = \mathcal{Y}(M,[\tilde{\theta}])$. Similarly, whenever $\mathcal{Y}(M,[\theta])>0$, we define

$$\mathcal{Y}^p_{oc}(M,\tilde{\theta}) := \inf_{u\in S_1^2(M,\tilde{\theta})_+} I_p^{\tilde{\theta}}(u), \quad \text{for } \tilde{\theta}\in[\theta], 1\le p\le 2^*-1$$

and

$$\mathcal{Y}^{2^*-1}_{oc}(M,\tilde{\theta}) = \mathcal{Y}_{oc}(M,[\tilde{\theta}]).$$

5. Monotonicity formula for $J_p^{\tilde{\theta}}$, $\tilde{\theta}\in[\theta]$ and $1\le p\le 2^*-1$

In this section, we present a monotonicity formula for $J_p^{\tilde{\theta}}$ when passing from u to $T_{\tilde{\theta}}(u)$. We then give some applications on the relation between the ground state of $J_p^{\tilde{\theta}}$ and the fixed point of $T_{\tilde{\theta}}$. The monotonicity formula reads as follows.

Lemma 5.1. *Suppose that $\mathcal{Y}(M,[\theta])>0$, $\tilde{\theta}\in[\theta]$ and $1\le p\le 2^*-1$. For $u\in S_1^2(M,\tilde{\theta})$, there holds*

$$J_p^{\tilde{\theta}}(u) - J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) \ge \frac{1}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}}\left[\|u\|^2_{\tilde{\theta}} - \|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}\right] \ge 0.$$

Proof. By the definition of $J_p^{\tilde{\theta}}$ in (2.1)-(2.3), we compute

$$\begin{aligned} J_p^{\tilde{\theta}}(u) - J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) &= \frac{\|u\|^2_{\tilde{\theta}}}{\|u\|^2_{L^{p+1}(M,\tilde{\theta})}} - \frac{\|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}} \\ &\ge \frac{\|u\|^2_{\tilde{\theta}}}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}} - \frac{\|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}}, \end{aligned}$$

where we have used $T_{\tilde{\theta}}(u)\ge u>0$. It follows from the definition of $T_{\tilde{\theta}}$ that the last expression is nonnegative, which proves the assertion. □

Lemma 5.1 implies the following:

Corollary 5.2. *Suppose that $\mathcal{Y}(M,[\theta])>0$, $\tilde{\theta}\in[\theta]$ and $1\le p\le 2^*-1$. Then for $u\in S_1^2(M,\tilde{\theta})_+$,*

$$J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) \le J_p^{\tilde{\theta}}(u),$$

and equality holds if and only if $u\in Fix(T_{\tilde{\theta}})$.

Proof. The inequality part follows immediately from Lemma 5.1. Now if $J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) = J_p^{\tilde{\theta}}(u)$, it follows from Lemma 5.1 that

$$\|u\|_{\tilde{\theta}}^2 = \|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}^2.$$

Hence, since $u \geq u \in S_1^2(M, \tilde{\theta})_+$, the uniqueness part in Lemma 3.1 implies that

$$u = T_{\tilde{\theta}}(u),$$

which finishes the proof. □

Corollary 5.2 implies that minimizers of $J_p^{\tilde{\theta}}$ on $S_1^2(M, \tilde{\theta})_+$ belong to $Fix(T_{\tilde{\theta}})$:

Corollary 5.3. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$. For $u \in S_1^2(M, \tilde{\theta})_+$,*

$$J_p^{\tilde{\theta}}(u) = \mathcal{Y}^p(M, \tilde{\theta}) \implies u \in Fix(T_{\tilde{\theta}}).$$

Proof. The assumption $J_p^{\tilde{\theta}}(u) = \mathcal{Y}^p(M, \tilde{\theta})$ implies that

$$J_p^{\tilde{\theta}}(u) = \mathcal{Y}^p(M, \tilde{\theta}) \leq J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)).$$

This together with Corollary 5.2 gives $J_p^{\tilde{\theta}}(u) = J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u))$. By Corollary 5.2 again, we can conclude that $u \in Fix(T_{\tilde{\theta}})$. □

Remark 5.4. In view of Proposition 3.2 and Corollary 5.2, we can assume without loss of generality that any minimizing sequence $(u_l)_{l\geq 1}$ of $J_p^{\tilde{\theta}}$ on $S_1^2(M, \tilde{\theta})_+$ satisfies

$$u_l \in Fix(T_{\tilde{\theta}}) \ \text{ for all } l \geq 1.$$

Indeed, suppose $(u_l)_{l\geq 1}$ is a minimizing sequence of $J_p^{\tilde{\theta}}$ on $S_1^2(M, \tilde{\theta})_+$. Then $u_l \in S_1^2(M, \tilde{\theta})_+$ and

$$J_p^{\tilde{\theta}}(u_l) \to \inf_{u \in S_1^2(M, \tilde{\theta})_+} J_p^{\tilde{\theta}}(u) \ \text{ as } l \to \infty.$$

By Corollary 5.2 and the fact that $T_{\tilde{\theta}}(u_l) \in S_1^2(M, \tilde{\theta})_+$, we have

$$\inf_{u \in S_1^2(M, \tilde{\theta})_+} J_p^{\tilde{\theta}}(u) \leq J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u_l)) \leq J_p^{\tilde{\theta}}(u_l).$$

These imply

$$J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u_l)) \to \inf_{u \in S_1^2(M, \tilde{\theta})_+} J_p^{\tilde{\theta}}(u) \ \text{ as } l \to \infty.$$

Hence, if we set $\hat{u}_l = T_{\tilde{\theta}}(u_l)$ and use Proposition 3.2, we obtain

$$J_p^{\tilde{\theta}}(\hat{u}_l) \to \inf_{u \in S_1^2(M, \tilde{\theta})_+} J_p^{\tilde{\theta}}(u) \ \text{ as } l \to \infty \ \text{ and } \ \hat{u}_l = T_{\tilde{\theta}}(\hat{u}_l),$$

as desired.

6. Monotonicity formula for $I_p^{\tilde{\theta}}$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$

In this section, we derive a monotonicity formula for $I_p^{\tilde{\theta}}$, similar to the one for $J_p^{\tilde{\theta}}$ derived in the previous section. Moreover, we present some applications for $I_p^{\tilde{\theta}}$, similar to the ones obtained for $J_p^{\tilde{\theta}}$ in the previous section. First, we have the following:

Lemma 6.1. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$. For $u \in S_1^2(M, \tilde{\theta})$, there holds*

$$I_p^{\tilde{\theta}}(u) - I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) = \frac{1}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}} \left[\|u\|^2_{\tilde{\theta}} - \|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}\right] \geq 0.$$

Proof. By the definition of $I^{\tilde{\theta}}$ in (2.9)-(2.11), we compute

$$\begin{aligned} I_p^{\tilde{\theta}}(u) - I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) &= \frac{\|u\|^2_{\tilde{\theta}}}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}} - \frac{\|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}}{\|T^2_{\tilde{\theta}}(u)\|^2_{L^{2^*}(M,\tilde{\theta})}} \\ &= \frac{1}{\|T_{\tilde{\theta}}(u)\|^2_{L^{p+1}(M,\tilde{\theta})}} \left[\|u\|^2_{\tilde{\theta}} - \|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}\right] \end{aligned}$$

where we have used Proposition 3.2 in the last equality. Note that the last expression is nonnegative by the definition of $T_{\tilde{\theta}}$. This proves the assertion. □

We have the following corollary, similar to Corollary 5.2.

Corollary 6.2. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$. For $u \in S_1^2(M, \tilde{\theta})_+$, we have*

$$I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) \leq I_p^{\tilde{\theta}}(u), \tag{6.1}$$

and equality holds if and only if $u \in Fix(T_{\tilde{\theta}})$.

Proof. The inequality (6.1) follows immediately from Lemma 6.1. By Lemma 6.1 again, equality holds in (6.1) if

$$\|u\|^2_{\tilde{\theta}} = \|T_{\tilde{\theta}}(u)\|^2_{\tilde{\theta}}.$$

Hence, since $u \geq u \in S_1^2(M, \tilde{\theta})_+$, the uniqueness part of Lemma 3.1 implies that

$$u = T_{\tilde{\theta}}(u),$$

which finishes the proof. □

As in the previous section, Corollary 6.2 implies that minimizers of $I_p^{\tilde{\theta}}$ belong to $Fix(T_{\tilde{\theta}})$:

Corollary 6.3. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$. For $u \in S_1^2(M, \tilde{\theta})_+$,*

$$I_p^{\tilde{\theta}}(u) = \mathcal{Y}^p_{oc}(M, \tilde{\theta}) \implies u \in Fix(T_{\tilde{\theta}}).$$

Proof. If $I_p^{\tilde{\theta}}(u) = \mathcal{Y}_{oc}^p(M, \tilde{\theta})$, then

$$I_p^{\tilde{\theta}}(u) = \mathcal{Y}_{oc}^p(M, \tilde{\theta}) \leq I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)).$$

Combining this with Corollary 6.2, we gives the equality in (6.1). Therefore, by Corollary 6.2 again, $u \in Fix(T_{\tilde{\theta}})$. □

Remark 6.4. In view of Proposition 3.2 and Corollary 6.2, we can, by using the same argument as in Remark 5.4, assume that any minimizing sequence $(u_l)_{l\geq 1}$ of $I_p^{\tilde{\theta}}$ on $S_1^2(M, \tilde{\theta})_+$ satisfies

$$u_l \in Fix(T_{\tilde{\theta}}) \ \text{ for all } l \geq 1.$$

7. Comparing $\mathcal{Y}^p(M, \tilde{\theta})$ and $\mathcal{Y}_{oc}^p(M, \tilde{\theta})$ where $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$

In this section, under the assumption that $\mathcal{Y}(M, [\theta]) > 0$, we show that, for $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$, $\mathcal{Y}^p(M, \tilde{\theta}) = \mathcal{Y}_{oc}^p(M, \tilde{\theta})$ and $J_p^{\tilde{\theta}}(u) = \mathcal{Y}^p(M, \tilde{\theta}) \iff I_p^{\tilde{\theta}}(u) = \mathcal{Y}_{oc}^p(M, \tilde{\theta})$. As a result, we prove Theorem 1.3.

We start with the following comparison result by showing that $I_p^{\tilde{\theta}} \leq J_p^{\tilde{\theta}}$ and that $I_p^{\tilde{\theta}} = J_p^{\tilde{\theta}}$ on the range of $T_{\tilde{\theta}}$.

Lemma 7.1. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$ and $\tilde{\theta} \in [\theta]$. For $u \in S_1^2(M, \tilde{\theta})_+$, we have*

$$I_p^{\tilde{\theta}}(u) \leq J_p^{\tilde{\theta}}(u) \tag{7.1}$$

and

$$I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) = J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)). \tag{7.2}$$

Proof. By definition of $J_p^{\tilde{\theta}}$ and $I_p^{\tilde{\theta}}$ (see (2.1)-(2.3) and (2.9)-(2.11)), we have

$$J_p^{\tilde{\theta}}(u) - I_p^{\tilde{\theta}}(u) = \frac{\|u\|_{\tilde{\theta}}^2}{\|u\|_{L^{p+1}(M,\tilde{\theta})}^2} - \frac{\|u\|_{\tilde{\theta}}^2}{\|T_{\tilde{\theta}}(u)\|_{L^{p+1}(M,\tilde{\theta})}^2}. \tag{7.3}$$

Now (7.1) follows from this and the fact that $T_{\tilde{\theta}}(u) \geq u > 0$. Moreover, we have

$$J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) - I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)) = \frac{\|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}^2}{\|T_{\tilde{\theta}}(u)\|_{L^{p+1}(M,\tilde{\theta})}^2} - \frac{\|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}^2}{\|T_{\tilde{\theta}}^2(u)\|_{L^{p+1}(M,\tilde{\theta})}^2}.$$

This together with $T_{\tilde{\theta}}^2(u) = T_{\tilde{\theta}}(u)$ (see Proposition 3.2) implies (7.2). □

Remark 7.2. Clearly, $T_{\tilde{\theta}}(u) \geq u > 0$ and (7.3) imply that

$$I_p^{\tilde{\theta}}(u) = J_p^{\tilde{\theta}}(u) \iff u = T_{\tilde{\theta}}(u).$$

Proposition 7.3. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$ and $1 \leq p \leq 2^* - 1$. There holds*

$$\mathcal{Y}^p(M, \tilde{\theta}) = \mathcal{Y}_{oc}^p(M, \tilde{\theta}).$$

Proof. By Corollary 5.2, we have

$$\mathcal{Y}^p(M,\tilde{\theta}) = \inf_{u\in S_1^2(M,\tilde{\theta})_+} J_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)).$$

Similarly, by Corollary 6.2, we have

$$\mathcal{Y}_{oc}^p(M,\tilde{\theta}) = \inf_{u\in S_1^2(M,\tilde{\theta})_+} I_p^{\tilde{\theta}}(T_{\tilde{\theta}}(u)).$$

Now the assertion follows from these and (7.2). □

From Proposition 7.3, we have the following:

Proposition 7.4. *Suppose that* $\mathcal{Y}(M,[\theta])>0$, $\tilde{\theta}\in[\theta]$ *and* $1\leq p\leq 2^*-1$. *For* $u\in S_1^2(M,\tilde{\theta})_+$,

$$J_p^{\tilde{\theta}}(u)=\mathcal{Y}^p(M,\tilde{\theta}) \iff I_p^{\tilde{\theta}}(u)=\mathcal{Y}_{oc}^p(M,\tilde{\theta}).$$

Proof. If

$$J_p^{\tilde{\theta}}(u)=\mathcal{Y}^p(M,\tilde{\theta}), \tag{7.4}$$

then $u\in Fix(T_{\tilde{\theta}})$ by Corollary 5.3, i.e. $u=T_{\tilde{\theta}}(u)$. Therefore, (7.2) gives $I_p^{\tilde{\theta}}(u)=J_p^{\tilde{\theta}}(u)$. This together with (7.4) and Proposition 7.3 gives $I_p^{\tilde{\theta}}(u)=\mathcal{Y}_{oc}^p(M,\tilde{\theta})$.

On the other hand, if

$$I_p^{\tilde{\theta}}(u)=\mathcal{Y}_{oc}^p(M,\tilde{\theta}), \tag{7.5}$$

then $u\in Fix(T_{\tilde{\theta}})$ by Corollary 6.3, i.e. $u=T_{\tilde{\theta}}(u)$. Thus, (7.2) implies $J_p^{\tilde{\theta}}(u)=I_p^{\tilde{\theta}}(u)$. Combining this with (7.5) and Proposition 7.3 yields $J_p^{\tilde{\theta}}(u)=\mathcal{Y}^p(M,\tilde{\theta})$. □

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. To prove (i), we assume $u\in S_1^2(M,\theta)_+$ satisfies $I^\theta(u)=\mathcal{Y}_{oc}(M,[\theta])$. By Proposition 7.4, $J^\theta(u)=\mathcal{Y}(M,[\theta])$. Thus, as a critical point of J^θ, u satisfies the CR Yamabe equation (1.2) with $R_{\theta_u}\equiv c$ for some constant $c>0$. The smooth regularity of positive solutions of the CR Yamabe equation (1.2) implies that $u\in C_+^\infty(M)$ (see Theorem 5.15 in [18]). Finally, Corollary 5.3 implies that $u\in Fix_\theta(T_\theta)$. This proves (i).

To prove (ii), let $u_{\min}\in C_+^\infty(M)$ be a minimizer for the CR Yamabe problem on (M,θ), i.e.

$$J^\theta(u_{\min})=\mathcal{Y}(M,[\theta]). \tag{7.6}$$

Therefore, (7.6) and Proposition 7.4 implies that $I^\theta(u_{\min})=\mathcal{Y}_{oc}(M,[\theta])$. This proves (ii). □

Proof of Corollary 1.4. In view of Theorem 1.3(ii), it suffices to proves that a minimizer for the CR Yamabe problem exists on (M,θ) under the assumptions in Corollary 1.4.

Note that if (M,θ) is CR equivalent to $(S^{2n+1},\theta_{S^{2n+1}})$, then a minimizer for the CR Yamabe problem exists (c.f. [18]). Therefore, we assume that (M,θ)

is not CR equivalent to $(S^{2n+1}, \theta_{S^{2n+1}})$. It was shown in [18, Theorem 3.4] that if

$$\mathcal{Y}(M, \theta) < \mathcal{Y}(S^{2n+1}, \theta_{S^{2n+1}}), \tag{7.7}$$

then a minimizer for the CR Yamabe problem exists on (M, θ). Therefore, it suffices to show that (7.7) holds. Now, if assumption (i) in Corollary 1.4 holds, i.e. if $n \geq 2$ and (M, θ) is not spherical, it follows from [16,17] that (7.7) holds. Also, if assumption (ii) in Corollary 1.4 holds, i.e. if $n = 1$ and the CR Panetiz operator of (M, θ) is nonnegative, then it follows from [7, Theorem 1.2] that (7.7) holds. Finally, if assumption (iii) in Corollary 1.4 holds, it follows from [5] that the CR mass of (M, θ) is positive, since (M, θ) is not CR equivalent to $(S^{2n+1}, \theta_{S^{2n+1}})$ by assumption. From this, one can construct a test function to show that (7.7) holds, as shown in the proof of [7, Theorem 1.2]. This finishes the proof. □

8. Obstacle problem and Folland-Stein type inequality

In this section, we discuss some Folland-Stein type inequalities related to the obstacle problem for the conformal CR sub-Laplacian. We then specialize to the case of the CR sphere.

Lemma 8.1. *Suppose that $\mathcal{Y}(M, [\theta]) > 0$, $\tilde{\theta} \in [\theta]$. Then for $u \in S_1^2(M, \tilde{\theta})_+$*

$$\|T_{\tilde{\theta}}(u)\|_{L^{2^*}(M, \tilde{\theta})} \leq \frac{1}{\sqrt{\mathcal{Y}(M, [\theta])}} \|u\|_{\tilde{\theta}}.$$

Proof. Since $\mathcal{Y}(M, [\theta]) > 0$ by assumption, it follows from the definition of $\mathcal{Y}(M, [\theta])$ that

$$\|v\|_{L^{2^*}(M, \tilde{\theta})} \leq \frac{1}{\sqrt{\mathcal{Y}(M, [\theta])}} \|v\|_{\tilde{\theta}} \tag{8.1}$$

for any $v \in S_1^2(M, \tilde{\theta})_+$. For $u \in S_1^2(M, \tilde{\theta})_+$, $T_{\tilde{\theta}}(u) \in S_1^2(M, \tilde{\theta})_+$. Taking $v = T_{\tilde{\theta}}(u)$ in (8.1) yields

$$\|T_{\tilde{\theta}}(u)\|_{L^{2^*}(M, \tilde{\theta})} \leq \frac{1}{\sqrt{\mathcal{Y}(M, [\theta])}} \|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}}. \tag{8.2}$$

By the definition of $T_{\tilde{\theta}}$, we have $\|T_{\tilde{\theta}}(u)\|_{\tilde{\theta}} \leq \|u\|_{\tilde{\theta}}$. The assertion follows from (8.2). □

Proposition 8.2. *Suppose that (M, θ) has positive CR Yamabe constant and θ is a torsion-free pseudo-Einstein contact form. Then for $u \in S_1^2(S^{2n+1}, \theta)_+$*

$$\|T_\theta(u)\|_{L^{2^*}(S^{2n+1}, \theta)} \leq \frac{1}{\sqrt{\mathcal{Y}(M, [\theta])}} \|u\|_\theta, \tag{8.3}$$

and equality holds in (8.3) if and only if

$$u \in C_+^\infty(M) \quad \textit{and} \quad R_{\theta_u} = c$$

for some constant c, where $\theta_u = u^{\frac{2}{n}} \theta$.

Proof. The inequality (8.3) in Proposition 8.2 follows from Lemma 8.1.

To prove the remaining part, we first suppose that u satisfies

$$\|T_\theta(u)\|_{L^{2^*}(M,\theta)} = \frac{1}{\sqrt{\mathcal{Y}(M,[\theta])}}\|u\|_\theta. \tag{8.4}$$

By Proposition 7.3, (8.4) is equivalent to

$$\|T_\theta(u)\|_{L^{2^*}(M,\theta)} = \frac{1}{\sqrt{\mathcal{Y}_{oc}(M,[\theta])}}\|u\|_\theta. \tag{8.5}$$

Hence, it follows from the definition of I^θ in (2.9) that (8.5) is equivalent to

$$I^\theta(u) = \mathcal{Y}_{oc}(M,[\theta]) \tag{8.6}$$

By Proposition 7.4, we see that (8.6) is equivalent to

$$J^\theta(u) = \mathcal{Y}(M,[\theta]). \tag{8.7}$$

It follows from [18, Theorem 5.15] that we must have $u \in C^\infty_+(M)$, and the Webster scalar curvature of $u^{\frac{2}{n}}\theta = \theta_u$ is constant, i.e. $R_{\theta_u} = c$ for some constant c.

To prove the converse, we first note that a minimizer for the CR Yamabe problem exists on (M,θ) under the assumptions in Proposition 8.2. This follows from [18], when (M,θ) is the CR equivalent to $(S^{2n+1},\theta_{S^{2n+1}})$. Suppose now that (M,θ) is not CR equivalent to $(S^{2n+1},\theta_{S^{2n+1}})$. When $n=1$, any torsion-free pseudo-Einstein contact form has vanishing Cartan tensor (this can be seen, for example, by [6, Lemma 2.2]), and hence, combined with the assumption that $\mathcal{Y}(M,[\theta])>0$, (M,θ) is a nontrivial quotient of the sphere. Thus $\mathcal{Y}(M,[\theta]) < \mathcal{Y}(S^3,\theta_{S^3})$. When $n>1$, either (M,θ) a nontrivial quotient of the sphere or the Chern-Moser tensor does not vanish. In either case, [17] implies existence of a minimizer.

Now let $\theta_u \in [\theta]$ have constant Webster scalar curvature. The CR Obata Theorem [23] implies that there is a CR diffeomorphism $\Phi : M \to M$ such that $\theta_u = \Phi^*\theta$. Since a minimizer $\widehat{\theta} \in [\theta]$ of the CR Yamabe problem has constant Webster scalar curvature, there is a CR diffeomorphism $\Psi : M \to M$ such that $\widehat{\theta} = \Psi^*\theta$. Therefore $\theta_u = (\Psi^{-1}\circ\Phi)^*\widehat{\theta}$, and so θ_u minimizes $\mathcal{Y}(M,[\theta])$, i.e. (8.7) holds. As we have argued above, (8.7) is equivalent to (8.4). This completes the proof of Proposition 8.2. □

Proof of Theorem 1.5. This is a direct consequence of Proposition 7.3 and Proposition 8.2. □

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