Calc. Var. (2024) 63:181
https://doi.org/10.1007/s00526-024-02787-5

Calculus of Variations

Explicit forms for extremals of sharp Sobolev trace inequalities on the unit balls

Cheikh Birahim Ndiaye[1] **· Liming Sun**[2]

Received: 7 February 2023 / Accepted: 2 June 2024

Abstract

We show explicit forms for extremals of some fourth-order sharp Sobolev trace inequalities on the unit balls recently proved by Ache-Chang and Case. We also give a classification result of the bi-harmonic equation on $\mathbb{R}^4_+$ with some conformally covariant boundary conditions. Moreover, we show a classification result for an associated integral equation.

Mathematics Subject Classification Primary 35J40 · 58J32; Secondary 39B05 · 46E35

1 Introduction

Let $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$, $n \geq 1$, be the unit ball with boundary $\partial\mathbb{B}^{n+1} = \mathbb{S}^n$. Recall the following two Sobolev trace inequalities: For any $f \in C^\infty(\mathbb{S}^n)$ and v being a smooth extension of f to $\mathbb{B}^{n+1}$, there hold

- If $n = 1$, then

$$\log\left(\frac{1}{2\pi}\oint_{\mathbb{S}^1} e^f d\sigma\right) \leqslant \frac{1}{4\pi}\int_{\mathbb{B}^2} |\nabla v|^2 dx + \frac{1}{2\pi}\oint_{\mathbb{S}^1} f d\sigma. \tag{1.1}$$

  Moreover, the equality holds if and only if $\Delta v = 0$ and $f = c - \log|1 - \langle z_0, \xi\rangle|$, where c is a constant, $\xi \in \mathbb{S}^1$, and z_0 is some fixed point in the interior of $\mathbb{B}^2$.
- If $n > 1$, then

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}|\mathbb{S}^n|^{1/n}\left(\oint_{\mathbb{S}^n} |f|^{\frac{2n}{n-1}} d\sigma\right)^{\frac{n-1}{n}} \leqslant \int_{\mathbb{B}^{n+1}} |\nabla v|^2 dx + \frac{n-1}{2}\oint_{\mathbb{S}^n} |f|^2 d\sigma. \tag{1.2}$$

Communicated by S. A. Chang.

✉ Cheikh Birahim Ndiaye
cheikh.ndiaye@howard.edu

Liming Sun
lmsun@amss.ac.cn

1 Department of Mathematics Howard University Annex 3, Graduate School of Arts and Sciences, Washington, DC 20059, USA

2 Academy of Mathematics and Systems Science, The Chinese Academy of Sciences, Beijing 100190, China

Published online: 16 July 2024

Moreover, the equality holds if and only if $\Delta v = 0$ and $f = c|1 - \langle z_0, \xi\rangle|^{-(n-1)/2}$, where c is a constant, $\xi \in \mathbb{S}^n$ and $z_0 \in \mathbb{B}^{n+1}$. Here Γ is the standard Gamma function.

The first one (1.1) was proved by Lebedev-Milin [11] and Osgood-Phillips-Sarnak [17]. The second one (1.2) was proved by Lions [14], Escobar [7] and Beckner [2].

A natural question is what the extremal v look like in the unit ball. It is not hard to find that the harmonic extension of $-\log|1 - \langle z_0, \xi\rangle|$ on $\mathbb{B}^2$ is

$$v(\xi) = -\log\left|\frac{\xi}{|\xi|} - |\xi|\omega_0\right|^2 + \log(1 + |\omega_0|^2), \quad \xi \in \mathbb{B}^2 \tag{1.3}$$

and the harmonic extension of $|1 - \langle z_0, \xi\rangle|^{-\frac{n-1}{2}}$ on $\mathbb{B}^{n+1}$ with $n > 1$ is

$$v(\xi) = (1 + |\omega_0|^2)^{\frac{n-1}{2}}\left|\frac{\xi}{|\xi|} - |\xi|\omega_0\right|^{1-n}, \quad \xi \in \mathbb{B}^{n+1} \tag{1.4}$$

where $w_0 = z_0/(1 + \sqrt{1 - |z_0|^2}) \in \mathbb{B}^{n+1}$. To obtain (1.3), one can use the observation

$$\log|\omega_0 - \xi|^2 = \log(1 - 2\omega_0 \cdot \xi + |\omega_0|^2) = \log(1 - \langle z_0, \xi\rangle) + \log(1 + |\omega_0|^2) \tag{1.5}$$

because $z_0 = 2\omega_0/(1 + |\omega_0|^2)$. One notices that $\log|\omega_0 - \xi|^2$ is a harmonic function with a pole in $\mathbb{B}^2$. Using explicit expressions of Green's function on the unit ball (for instance, see [9]) to annihilate the singularity, we can find the explicit forms for the extremal v. This approach also works for (1.4).

Ache and Chang [1] generalized the Lebedev-Milin inequality and its counterpart (1.2) to ones of *order four*. More precisely, let $f \in C^\infty(\mathbb{S}^n)$ and v be a smooth extension of f to the unit ball $\mathbb{B}^{n+1}$. Let η be the outward-pointing unit normal to $\mathbb{S}^n$ and $\bar{\nabla}$ be the gradient in $\mathbb{S}^n$. Then we have the sharp trace inequalities.

- If $n = 3$, then

$$\frac{16\pi^2}{3}\log\left(\frac{1}{2\pi^2}\oint_{\mathbb{S}^3} e^{3f}\,d\sigma\right) \leq \int_{\mathbb{B}^4}(\Delta v)^2\,dx + 2\oint_{\mathbb{S}^3}|\bar{\nabla} f|^2 d\sigma + 8\oint_{\mathbb{S}^3} f\,d\sigma \tag{1.6}$$

  for any v satisfying the homogeneous Neumann boundary condition $\eta v|_{\mathbb{S}^3} = 0$. Equality holds if and only if $f = c - \log|1 - \langle z_0, \xi\rangle|$ where c is a constant, $\xi \in \mathbb{S}^3$, z_0 is some point in $\mathbb{B}^4$ and v satisfies that

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^4, \\ \eta v = 0 & \text{on } \mathbb{S}^3, \\ v = f & \text{on } \mathbb{S}^3. \end{cases} \tag{1.7}$$

- If $n > 3$, then

$$a_n\left(\oint_{\mathbb{S}^n}|f|^{\frac{2n}{n-3}}d\sigma\right)^{\frac{n-3}{n}} \leqslant \int_{\mathbb{B}^{n+1}}|\Delta v|^2 dx + 2\oint_{\mathbb{S}^n}|\bar{\nabla} f|^2 d\sigma + b_n\oint_{\mathbb{S}^n}|f|^2 d\sigma, \tag{1.8}$$

  for any v satisfying the Neumann boundary condition $\eta v|_{\mathbb{S}^n} = -\frac{n-3}{2}f$. Here $a_n = 2\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n-3}{2}\right)}|\mathbb{S}^n|^{3/n}$ and $b_n = (n+1)(n-3)/2$. Equality holds if and only if $f =$

$c\,|1-\langle z_0,\xi\rangle|^{\frac{3-n}{2}}$ and v satisfies that

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^{n+1}, \\ \eta v = -\frac{n-3}{2} v & \text{on } \mathbb{S}^n, \\ v = f & \text{on } \mathbb{S}^n. \end{cases} \tag{1.9}$$

Similar to the second order case, we also want to know what the extremal functions of (1.6) and (1.8) look like in $\mathbb{B}^{n+1}$. We introduce the function $F : \overline{\mathbb{B}^{n+1}} \times \overline{\mathbb{B}^{n+1}} \to \mathbb{R}$ as

$$F(\xi,\omega) = \left| \frac{\xi}{|\xi|} - |\xi|\omega \right|. \tag{1.10}$$

Theorem 1.1 *Given any $z_0 \in \mathbb{B}^{n+1}$, define $\omega_0 = z_0/(1+\sqrt{1-|z_0|^2}) \in \mathbb{B}^{n+1}$.*

(1) *If $n = 3$, then the solution of (1.7) with $f = -\log|1-\langle z_0,\xi\rangle|$ on $\mathbb{S}^3$ is*

$$v(\xi) = -\log F(\xi,\omega_0)^2 + \frac{(1-|\xi|^2)}{2}\left[\frac{1-|\omega_0|^2}{F(\xi,\omega_0)^2} - 1\right] + \log(1+|\omega_0|^2). \tag{1.11}$$

(2) *If $n > 3$, then the solution of (1.9) with $f = |1-\langle z_0,\xi\rangle|^{\frac{3-n}{2}}$ on $\mathbb{S}^n$ is*

$$v(\xi) = \frac{\left(1+|\omega_0|^2\right)^{\frac{n-3}{2}}}{F(\xi,\omega_0)^{n-3}}\left[1 + \frac{(n-3)(1-|\omega_0|^2)(1-|\xi|^2)}{4F(\xi,\omega_0)^2}\right]. \tag{1.12}$$

Proving the above theorem is more complicated than the second-order case because we have one more boundary condition in the fourth-order case. We reformulate the problem on $\mathbb{R}^{n+1}_+$ via the Möbius transformation (see (2.1)), because the extremals of (1.6) and (1.8) (after a change of variable) satisfy simpler equations on $\mathbb{R}^{n+1}_+$. In fact, when $n > 3$, Case [3] and Ngô et al. [16] prove that (1.8) is equivalent to a sharp trace inequality on $\mathbb{R}^{n+1}_+$ whose extremal function satisfies

$$\begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t \Delta u = c u^{\frac{n+3}{n-3}} & \text{on } \partial\mathbb{R}^{n+1}_+, \\ \partial_t u = 0 & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases} \tag{1.13}$$

for some constant $c > 0$. Fortunately, [18] has found the definitive solution to the above equation. One can move everything back to the unit ball through a Möbius transformation.

In the case $n = 3$, we do not find an equivalent formulation of (1.6) on $\mathbb{R}^4_+$. However, we notice that v is an extremal of (1.6) implies that $(\mathbb{B}^4, e^{2v}g^*)$ has vanishing Q-curvature, constant T-curvature and vanishing mean curvature. Here T-curvature is defined by [5] and g^* is the *adapted metric* on $\mathbb{B}^4$ (see [1, Prop 2.2] and [4]). It is interesting that one has to use the adapted metric here. Via Möbius transformation, these geometric conditions are equivalent to the existence of u which satisfies the following equation[1]

$$\begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^4_+, \\ \partial_t \Delta u = 4e^{3u} & \text{on } \partial\mathbb{R}^4_+, \\ \partial_t u = 0 & \text{on } \partial\mathbb{R}^4_+. \end{cases} \tag{1.14}$$

1 Here the coefficient 4 in front of e^{3u} is normalized for the convenience of (1.17). The other solutions to $\partial_t \Delta u = ce^{3u}$ differs from (1.17) by adding a constant.

This equation is similar to (1.13). It is not studied in [18]. Here we continue to classify the solutions of this equation under the *finite volume conditions* (1.15) and (1.16). These are very natural geometric conditions.

Theorem 1.2 *Suppose that* $u \in C^4(\overline{\mathbb{R}^4_+})$ *satisfies* (1.14) *and the following conditions.*

$$(i) \quad \int_{\mathbb{R}^3} e^{3u(x,0)} dx < \infty, \tag{1.15}$$

$$(ii) \quad \int_{\mathbb{R}^4_+} e^{4u(x,t)} dx dt < \infty. \tag{1.16}$$

Then $\lim_{|x|\to\infty} \bar{\Delta} u(x,0)$ *exists and is non-positive, here* $\bar{\Delta}$ *is the Laplacian w.r.t.* $x \in \mathbb{R}^3$ *only. If* $\bar{\Delta} u(x,0) = o(1)$ *or* $u(x,0) = o(|x|^2)$ *as* $|x| \to \infty$, *then there exist* $a \in \mathbb{R}^3$, $\lambda > 0$ *and* $c \le 0$ *such that* $u = u_{a,\lambda}(x,t) + ct^2$ *where*

$$u_{a,\lambda}(x,t) = \log\left(\frac{2\lambda}{(\lambda+t)^2 + |x-a|^2}\right) + \frac{2t\lambda}{(\lambda+t)^2 + |x-a|^2}. \tag{1.17}$$

Remark 1.3 Conditions (i) and (ii) are sharp in the sense that if we remove both of them, then there are other solutions. For instance, $u = \frac{2}{3}t^3$ satisfies (1.14) but violates (1.15) and (1.16). On the other hand $u = u_{a,\lambda} + ct^2$ for any $c > 0$ satisfies (1.14) and (1.15) but violates (1.16).

As a byproduct of our arguments, we have the following corollary.

Corollary 1.4 *Suppose that* $u \in C^4(\overline{\mathbb{R}^4_+})$ *satisfies* (1.15) *and*

$$u(x,t) = \frac{1}{|\mathbb{S}^3|} \int_{\mathbb{R}^3} e^{3u(y,0)} \log \frac{|y|^2}{|x-y|^2 + t^2} dy + u(0,0) \tag{1.18}$$

then there exists $a \in \mathbb{R}^3$ *and* $\lambda > 0$ *such that* $u = u_{a,\lambda}$.

Remark 1.5 Corollary 1.4 and Theorem 1.1 are expected to be used to describe the asymptotic behavior of sequences of conformal metrics with prescribed T-curvature, $Q = 0$ and $H = 0$ on the whole background manifold rather than just at the boundary as available results in the literature provides.

This is one of the motivations why we consider the extensions of extremals of Ache-Chang's inequality.

After the work of Ache-Chang [1, 3] found more general Sobolev trace inequalities of order four. More precisely, keeping the notations as in (1.6) and (1.8),

- If $n = 3$, then

$$\begin{aligned} &\frac{16\pi^2}{3} \log\left(\frac{1}{2\pi^2} \oint_{\mathbb{S}^3} e^{3f} d\sigma\right) + \left(4\pi \oint_{\mathbb{S}^3} |\psi|^3 d\sigma\right)^{\frac{2}{3}} \\ &\le \int_{\mathbb{B}^4} (\Delta v)^2 \, dx + \oint_{\mathbb{S}^3} \left[2|\bar{\nabla} f|^2 + 4\langle \bar{\nabla} f, \bar{\nabla}\psi\rangle - 2\psi^2 + 8f\right] d\sigma. \end{aligned} \tag{1.19}$$

  Here $\psi = \eta v$. Equality holds if and only if $\Delta^2 v = 0$, $f(\xi) = c_0 - \log(1 - \langle z_0, \xi\rangle)$ and $\psi(\xi) = c_1|1 - \langle z_1, \xi\rangle|^{-1}$ for some constant $c_0, c_1 \in \mathbb{R}$ and points $z_0, z_1 \in \mathbb{B}^4$.

– If $n > 3$, then

$$a_n \left(\oint_{\mathbb{S}^n} |f|^{\frac{2n}{n-3}} d\sigma \right)^{\frac{n-3}{n}} + \tilde{a}_n \left(\oint_{\mathbb{S}^n} |\psi|^{\frac{2n}{n-1}} d\sigma \right)^{\frac{n-1}{n}}$$
$$\le \int_{\mathbb{B}^{n+1}} |\Delta v|^2 dx + \oint_{\mathbb{S}^n} \left[2|\bar{\nabla} f|^2 + 4\langle \bar{\nabla} f, \bar{\nabla} \psi \rangle - 2\psi^2 + 2b_n f\psi + b_n |f|^2 \right] d\sigma, \tag{1.20}$$

where a_n and b_n are the same as in (1.8), $\tilde{a}_n = 2\sqrt{\pi}(n-1)(\Gamma(n/2)/\Gamma(n))^{1/n}$ and

$$\psi = \eta v + \frac{n-3}{2} f.$$

Equality holds if and only if $\Delta^2 v = 0$, $f(\xi) = c_0|1 - \langle z_0, \xi \rangle|^{-(n-3)/2}$ and $\psi(\xi) = c_1|1 - \langle z_1, \xi \rangle|^{-(n-1)/2}$ for some constant $c_0, c_1 \in \mathbb{R}$ and points $z_0, z_1 \in \mathbb{B}^{n+1}$.

We also find the explicit forms of extremals of (1.19) and (1.20). It actually suffices to deal with the case $f = 0$.

Theorem 1.6 *Given any $z_1 \in \mathbb{B}^{n+1}$, we define $\omega_1 = z_1/(1 + \sqrt{1 - |z_1|^2}) \in \mathbb{B}^{n+1}$. If either $n = 3$ and the equality of (1.19) holds for v with $f = 0$ and $\psi = |1 - \langle z_1, \xi \rangle|^{-1}$, or $n > 3$ and the equality of (1.20) holds for v with $f = 0$ and $\psi = |1 - \langle z_1, \xi \rangle|^{-(n-1)/2}$, then*

$$v(\xi) = -\frac{1}{2}(1 - |\xi|^2)\frac{(1 + |\omega_1|^2)^{\frac{n-1}{2}}}{F(\xi, \omega_1)^{n-1}}. \tag{1.21}$$

In the proof of the Theorem 1.6, we also pull the equation to the upper half-space and construct the solution directly. Unfortunately, we do not have a classification result to the corresponding Euler-Lagrange equation on the upper half space, which is important for most blow-up analysis leading to such a limiting equation. See the discussions at the end of this paper.

To deal with the most general case of equalities in (1.19) and (1.20) with arbitrary c_0 and c_1, one needs to combine the results of Theorem 1.1 and Theorem 1.6. More precisely, Theorem 1.1 (up to a translation/scaling) gives a solution to $\Delta^2 v = 0$, $f = c_0 - \log(1 - \langle z_0, \xi \rangle)$ or $c_0|1 - \langle z_0, \xi \rangle|^{-(n-3)/2}$ (depending on whether $n = 3$ or $n \ge 3$) and $\psi = 0$. Theorem 1.6 (up to a scaling) gives a solution to $\Delta^2 v = 0$, $f = 0$ and $\psi = c_1|1 - \langle z_1, \xi \rangle|^{-(n-1)/2}$. By the linear property of bi-harmonic functions, the sum of these gives the solution to the equality with arbitrary c_0 and c_1.

Corollary 1.7 *The equality of (1.19) holds with*

$$v = c_0 + \log \frac{1 + |\omega_0|^2}{F(\xi, \omega_0)^2} + \frac{(1 - |\xi|^2)}{2} \left[\frac{1 - |\omega_0|^2}{F(\xi, \omega_0)^2} - 1 \right] - \frac{c_1}{2}(1 - |\xi|^2)\frac{(1 + |\omega_1|^2)}{F(\xi, \omega_1)^2}. \tag{1.22}$$

where $c_0, c_1 \in \mathbb{R}$ and $\omega_0, \omega_1 \in \mathbb{B}^4$. The equality of (1.20) holds with

$$v = c_0 \frac{(1 + |\omega_0|^2)^{\frac{n-3}{2}}}{F(\xi, \omega_0)^{n-3}} \left[1 + \frac{(n-3)(1 - |\omega_0|^2)(1 - |\xi|^2)}{4F(\xi, \omega_0)^2} \right] - \frac{c_1}{2}(1 - |\xi|^2)\frac{(1 + |\omega_1|^2)^{\frac{n-1}{2}}}{F(\xi, \omega_1)^{n-1}}. \tag{1.23}$$

where $c_0, c_1 \in \mathbb{R}$ and $\omega_0, \omega_1 \in \mathbb{B}^{n+1}$.

The paper is organized as follows. In Sect. 2, we first give some preliminary on a Möbius transformation which maps the upper half space to the unit ball. Sect.2.1 is devoted to finding the extension of extremals on dimensions five and above and proving Part (2) of Theorem 1.1. The case of dimension four is studied in the subsequent Sect.2.2, and we prove Part (1) of Theorem 1.1. In Sect. 3, we prove the classification theorems about a bi-harmonic equation on $\mathbb{R}^4_+$ with some conformally covariant boundary conditions and an associated integral equation. Theorem 1.2 and Corollary 1.4 is established in this section. Finally, in the last section, we prove Theorem 1.6 and Corollary 1.7 by computing the extension of extremals of Sobolev trace inequalities proved by Case [3].

2 Sobolev trace inequality of order four

Recall that the following Möbius transformation maps the upper half space to the unit ball.

$$\begin{aligned} \mathcal{S}:\mathbb{R}^{n+1}_+=\{X=(x,t)\}&\mapsto \mathbb{B}^{n+1}=\{\xi=(\xi',\xi_{n+1})\}\\ X&\to \frac{2(X+e_{n+1})}{|X+e_{n+1}|^2}-e_{n+1} \end{aligned} \tag{2.1}$$

where $e_{n+1}=(0,\cdots,0,1)\in\mathbb{R}^{n+1}$. Conversely

$$\mathcal{S}^{-1}(\xi)=\frac{2(\xi+e_{n+1})}{|\xi+e_{n+1}|^2}-e_{n+1}. \tag{2.2}$$

It is well-known that

$$\mathcal{S}^*|d\xi|^2=\left(\frac{2}{|X+e_{n+1}|^2}\right)^2|dX|^2. \tag{2.3}$$

$$|x|^2+t^2=\frac{-4\xi_{n+1}}{|\xi'|^2+(\xi_{n+1}+1)^2}+1,\quad |\xi|^2=\frac{-4t}{|x|^2+(t+1)^2}+1. \tag{2.4}$$

Lemma 2.1 *For any $(a,\lambda)\in\mathbb{R}^{n+1}_+$, the following identity holds for $\omega=\mathcal{S}(a,\lambda)\in\mathbb{B}^{n+1}$*

$$\frac{\lambda}{|x-a|^2+|t+\lambda|^2}=\frac{(1-|\omega|^2)}{4}\,\frac{|\xi+e_{n+1}|^2}{F(\xi,\omega)^2}. \tag{2.5}$$

Proof Denote $A=|\xi'|^2+|\xi_{n+1}+1|^2$ for short. We plug in (2.2) to the LHS and achieve

$$\begin{aligned} |\lambda+t|^2+|x-a|^2&=A^{-2}[(2(\xi_{n+1}+1)+(\lambda-1)A)^2+|2\xi'-aA|^2]\\ &=\frac{1}{A^2}[((\lambda-1)^2+|a|^2)A^2+4(\xi_{n+1}+1)(\lambda-1)A-4Aa\cdot\xi'+4A]\\ &=\frac{1}{A}[((\lambda-1)^2+|a|^2)|\xi|^2+2(\lambda^2+|a|^2-1)\xi_{n+1}-4a\cdot\xi'+(\lambda+1)^2+|a|^2]\\ &=\frac{(\lambda+1)^2+|a|^2}{|\xi'|^2+|\xi_{n+1}+1|^2}[|\omega|^2|\xi|^2-2\omega\cdot\xi+1]\\ &=\frac{(\lambda+1)^2+|a|^2}{|\xi'|^2+|\xi_{n+1}+1|^2}\left|\frac{\xi}{|\xi|}-|\xi|\omega\right|^2 \end{aligned}$$

where $\omega = \mathcal{S}(a, \lambda)$ with

$$\omega' = \frac{2a}{(\lambda+1)^2 + |a|^2}, \quad \omega_{n+1} = \frac{2(\lambda+1)}{(\lambda+1)^2 + |a|^2} - 1. \tag{2.6}$$

We also used the following fact

$$|\omega|^2 = \frac{-4\lambda}{(\lambda+1)^2 + |a|^2} + 1. \tag{2.7}$$

Consequently

$$\frac{\lambda}{|\lambda+t|^2 + |x-a|^2} = \frac{1-|\omega|^2}{4} \frac{|\xi'|^2 + |\xi_{n+1}+1|^2}{F(\xi,\omega)^2}. \tag{2.8}$$

□

2.1 Ache-Chang inequality in dimension five and above

In this subsection, we shall consider the case $n > 3$.

Proof of Theorem 1.1 Part (2) Suppose v is the extension. Obviously when $z_0 = 0$, v will be a positive constant. Since v depends continuously on z_0 uniformly, we can suppose for $|z_0|$ small enough that $v > 0$ in $\mathbb{B}^{n+1}$.

We define

$$U(x,t) = v(\mathcal{S}(x,t)) \left(\frac{2}{|x|^2 + (1+t)^2} \right)^{\frac{n-3}{2}} \tag{2.9}$$

Then it is easy to see that

$$\int_{\mathbb{R}^n} |U(x,0)|^{\frac{2n}{n-3}} dx = \oint_{\mathbb{S}^n} |v|^{\frac{2n}{n-3}} d\sigma \tag{2.10}$$

and

$$\partial_t U(x,0) = -\left[\mathcal{S}(x,0) \cdot \nabla v(\mathcal{S}(x,0)) + \tfrac{n-3}{2} v(\mathcal{S}(x,0))\right] \left(\frac{2}{1+|x|^2} \right)^{\frac{n-1}{2}} \tag{2.11}$$

Since $\mathcal{S}(x,0)$ is normal to $\mathbb{S}^n$, thus $\mathcal{S}(x,0) \cdot \nabla v(\mathcal{S}(x,0)) = \eta v(\mathcal{S}(x,0))$. Therefore $\eta v = -\frac{n-3}{2} v$ is equivalent to $\partial_t U(x,0) = 0$ for $\forall\, x \in \mathbb{R}^n$. After some computation (for instance, see Ngô et al. [16, Eq. (4.9)][2]), we obtain

$$\int_{\mathbb{R}^{n+1}_+} |\Delta U|^2 dX = \int_{\mathbb{B}^{n+1}} |\Delta v|^2 dx + 2 \oint_{\mathbb{S}^n} |\bar{\nabla} f|^2 d\sigma + b_n \oint_{\mathbb{S}^n} |f|^2 d\sigma. \tag{2.12}$$

Consequently (1.8) is equivalent to the sharp trace inequality

$$a_n \left(\int_{\mathbb{R}^n} |U(x,0)|^{\frac{2n}{n-3}} dx \right)^{\frac{n-3}{n}} \leqslant \int_{\mathbb{R}^{n+1}_+} |\Delta U(x,t)|^2 dx dt \tag{2.13}$$

[2] The Möbius transformation in [16] is different from ours (see (2.1)) by a negative sign in the ξ_{n+1} coordinate. However, this difference does not affect the energy identity.

for functions U with $\partial_t U(x,0) = 0$. Using (2.9), we know that U is also positive. The positive extremal function of the above inequality satisfies

$$\begin{cases} \Delta^2 U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t \Delta U = c U^{\frac{n+3}{n-3}} & \text{on } \mathbb{R}^n, \\ \partial_t U = 0 & \text{on } \mathbb{R}^n, \end{cases} \tag{2.14}$$

for some $c > 0$. The positive solutions to the above equations have been studied by [18]. It follows from (2.9) that $U(x,t) = o(|x|^2 + t^2)$. Therefore, one can apply [18, Rmk 1.2] to achieve that there exists $\lambda > 0$, $a \in \mathbb{R}^3$ and

$$U(x,t) = c\left(\frac{\lambda}{(\lambda+t)^2 + |x-a|^2}\right)^{\frac{n-3}{2}}\left[1 + \frac{(n-3)t\lambda}{(\lambda+t)^2+|x-a|^2}\right] \tag{2.15}$$

for some constant $c > 0$. Now we plug in the above equation to

$$v(\xi) = U(\mathcal{S}^{-1}(\xi))\left(\frac{2}{|\xi'|^2 + (1+\xi_{n+1})^2}\right)^{\frac{n-3}{2}} \tag{2.16}$$

and using (2.5) and (2.2) to obtain that

$$v(\xi) = c\frac{\left(1-|\omega_0|^2\right)^{\frac{n-3}{2}}}{F(\xi,\omega_0)^{n-3}}\left[1 + \frac{(n-3)(1-|\omega_0|^2)(1-|\xi|^2)}{4F(\xi,\omega_0)^2}\right] \tag{2.17}$$

for some $c > 0$. Here $\omega_0 = \mathcal{S}(a,\lambda)$ and F is defined in (1.10). One can determine c through $c(1-|\omega_0|^2)^{(n-3)/2} = (1+|\omega_0|^2)^{(n-3)/2}$ using $v(\xi) = |1 - \langle z_0, \xi\rangle|^{(3-n)/2}$ on $\mathbb{S}^n$.

Now we want to show that v takes the form in (2.17) for any $z_0 \in \mathbb{B}^{n+1}$, not just for z_0 near the origin. Fixing any $r \in (0,1)$, we define

$$\mathcal{Z}_r := \{z_0 \in \overline{\mathbb{B}^{n+1}}(0,r) : \text{Part (2) of Theorem 1.1 holds true}\}.$$

The previous proof shows that $\mathcal{Z}_r$ contains a neighborhood of 0, thus it is non-empty. Since v depends on z_0 smoothly and is strictly positive, then for any z sufficiently near to z_0, v is also positive. This fact implies that $\mathcal{Z}_r$ is open. Clearly, $\mathcal{Z}_r$ is a close set. Therefore $\mathcal{Z}_r = \overline{\mathbb{B}^{n+1}}(0,r)$. Since this holds for any r, then the proof is complete. □

Remark 2.2 There is a geometric interpretation of the extremals for (1.8) (with the Neumann boundary condition). According to Case [3], the best constant in (1.8) is $Y_{4,1}(\mathbb{B}^{n+1}, \mathbb{S}^n)$. Moreover, the conformal metric $\hat{g} = v^{4/(n-3)} g_{\mathbb{B}^{n+1}}$ will have $\hat{Q}_4 = 0$, $\hat{H} = 0$ and $\hat{T}_3^3 = const > 0$. There is another way to show the extremals v is positive if one uses the fractional GJMS operator P_3 on $\mathbb{S}^n$, which has leading order $(-\Delta_{\mathbb{S}^n})^{3/2}$. The Euler-Lagrange equation implies $P_3 v > 0$. One can apply the result Case and Alice Chang [4, Theorem 1.3] to prove $v > 0$. Here we try to avoid citing this deep result and make the proof self-contained.

2.2 Ache-Chang inequality on dimension four

Recall the Paneitz operator defined on a smooth compact Riemannian manifold (X^{n+1}, g) for $n \geq 3$,

$$(L_4)_g = \left(-\Delta_g\right)^2 + \delta_g\left(\left(4P_g - (n-1)J_g g\right)(\nabla\cdot,\cdot)\right) + \frac{n-3}{2}(Q_4)_g \tag{2.18}$$

where δ denotes divergence, ∇ denotes gradient on functions, P_g the Schouten tensor $P_g = \frac{1}{n-1}\left(Ric_g - J_g g\right)$, $J_g = \frac{1}{2n}R_g$, R_g is the scalar curvature of the metric g and Q_4 is the Q-curvature

$$(Q_4)_g = -\Delta_g J_g + \frac{n+1}{2}J_g^2 - 2\left|P_g\right|_g^2.$$

In the following, we shall write $L_4 = (L_4)_g$ for short when the background metric is understood.

When $n = 3$, we have the following conformal invariance property of L_4 and Q_4,

$$\begin{aligned}(L_4)_{\hat{g}}\, U &= e^{-4\tau}\,(L_4)_g\,(U),\\ (Q_4)_{\hat{g}} &= e^{-4\tau}\left((L_4)_g\tau + (Q_4)_g\right),\end{aligned} \tag{2.19}$$

for any smooth function U on X^4 and $\hat{g} = e^{2\tau}g$.

When X^4 has boundary M, Chang and Qing [5] derived a conformally covariant boundary operator P_3^b and associated T-curvature T_3. Suppose h is the induced metric of (X^4, g) on M. Let us use $\bar{\Delta}$ and $\bar{\nabla}$ denote the Laplacian and connection on (M, h). Assume A is the second fundamental form of M and $H = tr_h A$ is the mean curvature. Then

$$P_3^b(u) = -\frac{1}{2}\eta\Delta u - \bar{\Delta}\eta u + \left\langle A - \frac{2}{3}Hh, \bar{\nabla}^2 u\right\rangle + \frac{1}{3}\langle\bar{\nabla}H, \bar{\nabla}u\rangle - (\mathrm{Ric}(\eta,\eta) - 2J)\eta u$$

$$T_3 = \frac{1}{2}\eta J - \frac{1}{3}\bar{\Delta}H + JH - \langle R(\eta,\cdot,\eta,\cdot), A\rangle - \frac{1}{3}\,\mathrm{tr}\,A^3 + \frac{1}{9}H^3.$$

Here $\eta = \eta_g$ is the outward-pointing unit normal to M.

If $\hat{g} = e^{2\tau}g$, we have the transformation laws

$$\begin{aligned}\left(P_3^b\right)_{\hat{g}} U &= e^{-3\tau}\left(P_3^b\right)_g U,\\ (T_3)_{\hat{g}} &= e^{-3\tau}\left(\left(P_3^b\right)_g \tau + (T_3)_g\right).\end{aligned} \tag{2.20}$$

We also have the relation of mean curvature

$$H_{\hat{g}} = e^{-\tau}\left(H_g + n\eta_g\tau\right). \tag{2.21}$$

For the model case $(\mathbb{B}^4, \mathbb{S}^3, g_0)$ where g_0 is the Euclidean metric, one has (for instance, see [1, (6.6)])

$$(P_3^b)_{g_0} = -\frac{1}{2}\eta\Delta - \bar{\Delta}\eta - \bar{\Delta}, \quad (T_3)_{g_0} = 2, \quad H_{g_0} = 3. \tag{2.22}$$

On $\mathbb{B}^4$, there is a special metric $g^* = e^{1-|\xi|^2}g_0$ which has nice properties. It is called *adapted metric* in [4] (also appeared in [8]). Under this metric g_*, $\mathbb{S}^3$ is totally geodesic and

$$(Q_4)_{g^*} = (Q_4)_{g_0} = 0, \tag{2.23}$$

$$(T_3)_{g^*} = (T_3)_{g_0} = 2. \tag{2.24}$$

Now we are ready to prove the main theorem of this subsection.

Proof of Theorem 1.1 part (1) Suppose that v is the extension. Then v will be an extremal function for the (1.6). It is easy to see that the Euler-Lagrange equation of (1.6) is

$$\begin{cases}\Delta^2 v = 0 & \text{in } \mathbb{B}^4,\\ -\eta\Delta v - 2\bar{\Delta}v + 4 = 8\pi^2\left(\int_{\mathbb{S}^3} e^{3v}\right)^{-1} e^{3v} & \text{on } \mathbb{S}^3,\\ \eta v = 0 & \text{on } \mathbb{S}^3.\end{cases} \tag{2.25}$$

Here $\Delta = \Delta_{g_0}$ and $\eta = \eta_{g_0}$. Let $g = e^{2v}g^* = e^{2v+1-|\xi|^2}|d\xi|^2$, here g^* is the so-called *adapted metric*. Denote $\tau = v + (1-|\xi|^2)/2$. The first line of (2.25) implies $(L_4)_{g_0}\tau = \Delta_{g_0}^2\tau = \Delta_{g_0}^2 v = 0$. Then applying (2.19) with $\tau = v + (1-|\xi|^2)/2$, the first line of (2.25) is equivalent to

$$(Q_4)_g = e^{-4\tau}((L_4)_{g_0}\tau + (Q_4)_{g_0}) = 0. \tag{2.26}$$

Applying (2.20) with $\tau = v + (1-|\xi|^2)/2$, the second line of (2.25) is equivalent to

$$(T_3)_g = e^{-3\tau}((P_3^b)_{g_0}\tau + (T_3)_{g_0}) = e^{-3v}\left(-\frac{1}{2}\eta_{g_0}\Delta_{g_0}v - \bar{\Delta}_{g_0}v + 2\right) = const > 0 \tag{2.27}$$

where we have used (2.22), $\tau = v$ on $\mathbb{S}^4$ and $\eta_{g_0}v = 0$. Applying (2.21) with the same τ as before, the third line of (2.25) is equivalent to

$$H_g = e^{-\tau}(H_{g_0} + 3\eta_{g_0}\tau) = e^{-v}(3 + 3(\eta_{g_0}v - 1)) = 0. \tag{2.28}$$

Combining the above analysis, (2.25) is equivalent to

$$(Q_4)_g = 0, \quad (T_3)_g = const > 0, \quad H_g = 0. \tag{2.29}$$

Using Möbius transformation (2.1), we can find w such that $(\mathbb{B}^4 \setminus \{(0,0,0,-1)\}, g)$ is isometric to $(\mathbb{R}^4_+, e^{2w}(|dx|^2 + dt^2))$ through

$$\mathcal{S}^*(e^{2v+1-|\xi|^2}|d\xi|^2) = e^{2w}(|dx|^2 + dt^2) \tag{2.30}$$

where v and w are related by

$$w = v\circ\mathcal{S} + \frac{1}{2} - \frac{1}{2}\frac{|x|^2 + (t-1)^2}{|x|^2 + (t+1)^2} + \log\frac{2}{x^2 + (1+t)^2}, \tag{2.31}$$

$$v = w\circ\mathcal{S}^{-1} - \frac{1-|\xi|^2}{2} + \log\frac{2}{|\xi'|^2 + (1+\xi_4)^2}. \tag{2.32}$$

By the isometry, we can think of (2.29) as referring to $\mathcal{S}^*g$ on $\mathbb{R}^4_+$. Thus using $|dx|^2 + dt^2$ as the background metric and the conformal properties of L_4, P_3^b and H, we rewrite (2.29) as the following

$$\begin{cases} \Delta^2 w = 0 & \text{in } \mathbb{R}^4_+, \\ \partial_t \Delta w = ce^{3w} & \text{on } \mathbb{R}^3, \\ \partial_t w = 0 & \text{on } \mathbb{R}^3, \end{cases} \tag{2.33}$$

for some constant $c > 0$. Moreover, isometry also implies

$$\int_{\mathbb{R}^4_+} e^{4w(x,t)}dxdt = \mathrm{vol}(\mathbb{B}^4, g) < \infty, \quad \int_{\mathbb{R}^3_+} e^{3w(x,0)}dx = \mathrm{vol}(\mathbb{S}^3, g|_{\mathbb{S}^3}) < \infty. \tag{2.34}$$

The solution to (2.33) with (2.34) has been studied by Theorem 1.2. Since v is smooth on $\overline{\mathbb{B}^4}$, then (2.31) leads to $w(x,t) = o(|x|^2 + t^2)$ as $|x| + t \to \infty$. Therefore, it follows from Theorem 1.2 that there exists $\lambda > 0$, $a \in \mathbb{R}^3$ and a constant C such that

$$w(x,t) = \log\frac{2\lambda}{(\lambda+t)^2 + |x-a|^2} + \frac{2t\lambda}{(\lambda+t)^2 + |x-a|^2} + C. \tag{2.35}$$

Plugging in (2.5) with $\omega_0 = \mathcal{S}(a, \lambda)$ and (2.2) to (2.32), one obtains

$$\begin{aligned} v(\xi) &= \log F(\xi, \omega_0)^{-2} + \frac{(1-|\omega_0|^2)(1-|\xi|^2)}{2F(\xi,\omega_0)^2} - \frac{1}{2}(1-|\xi|^2) + C \\ &= -\log F(\xi,\omega_0)^2 + \frac{1-|\xi|^2}{2}\left[\frac{1-|\omega_0|^2}{F(\xi,\omega_0)^2} - 1\right] + C \end{aligned}$$

The precise value of C can be determined through $v(\xi) = -\log|1 - \langle z_0, \xi\rangle|$ for $\xi \in \mathbb{S}^3$. This completes the proof. □

Remark 2.3 The above method also applies to the harmonic case. The proof is simpler in that case because the adapted metric g^* for $\mathbb{B}^{n+1}$ is identical to the Euclidean metric g_0 (see [1, Rmk 2.4]).

3 Classification of the solution to a bi-harmonic equation

In this section, we will prove Theorem 1.2. The strategy is to separate the nonlinear effect, by subtracting a function constructed from nonlinear boundary conditions. Such trick has been used by [10, 18]. The resulting linear fourth-order equation can be classified under the finite volume condition. The proof here is greatly inspired by Lin [13], who initiated the classification of some conformal bi-harmonic equation on $\mathbb{R}^4$.

Given any $f \in L^1(\mathbb{R}^3)$, we define v for $(x, t) \in \mathbb{R}^4_+$

$$v(x,t) = \frac{1}{|\mathbb{S}^3|}\int_{\mathbb{R}^3} f(y)\log\frac{|y|^2}{|x-y|^2+t^2}dy.$$

Lemma 3.1 *For any $f(y) \in C^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, one has*

$$\begin{cases} \Delta^2 v = 0 & in \ \ \mathbb{R}^4_+, \\ \partial_t \Delta v = 4f & on \ \ \partial\mathbb{R}^4_+, \\ \partial_t v = 0 & on \ \ \partial\mathbb{R}^4_+. \end{cases} \tag{3.1}$$

Proof Using the Lebesgue dominating theorem, it is easy to see $\partial_t v(x, 0) = 0$ and for any $t > 0$

$$\Delta v(x,t) = \frac{-4}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|^2+t^2}dy \tag{3.2}$$

$$\partial_t\Delta v(x,t) = \frac{8}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}\frac{tf(y)}{(|x-y|^2+t^2)^2}dy \tag{3.3}$$

$$\Delta^2 v = 0. \tag{3.4}$$

Note that $\frac{2}{|\mathbb{S}^3|}\frac{t}{(|x-y|^2+t^2)^2}$ is the Possion kernel of Δ on $\mathbb{R}^4_+$ (see [18, Lemma 2.2]). Then one has

$$\lim_{t\to 0^+}\partial_t\Delta v(x,t) = 4f(y). \tag{3.5}$$

□

Now suppose that u satisfies the assumptions of Theorem 1.2. In the following, we will denote

$$v(x,t)=\frac{1}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}e^{3u(y,0)}\log\frac{|y|^2}{|x-y|^2+t^2}dy. \tag{3.6}$$

Lemma 3.2 *For $v(x,t)$ defined in (3.6), there exists some constant $C>0$ such that*

$$v(X)\geq-\alpha\log(1+|X|)-C \tag{3.7}$$

where

$$\alpha=\frac{2}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}e^{3u(y,0)}dy. \tag{3.8}$$

Proof The proof is essentially contained in [13, Lemma 2.1 and 2.4] and [15]. For readers' convenience, we present it here.

For $|X|\geq 4$, we decompose $\mathbb{R}^3=A_1\cup A_2$, where $A_1=\{y||(y,0)-X|\leq|X|/2\}$ and $A_2=\{y||(y,0)-X|\geq|X|/2\}$. For $y\in A_1$, one has $|y|\geq|X|-|X-(y,0)|\geq|X|/2\geq|X-(y,0)|$. Consequently, we have $\log|y|/|X-(y,0)|\geq 0$ and

$$\int_{A_1}e^{3u(y,0)}\log\frac{|y|^2}{|x-y|^2+t^2}dy\geq 0. \tag{3.9}$$

For $y\in A_2$, one has $|X-(y,0)|\leq|X||y|$ if $|y|\geq 2$ and $\log|X-(y,0)|\leq\log|X|+C$ if $|X|\geq 4$ and $|y|\leq 2$. Thus

$$v(x)\geq\frac{1}{|\mathbb{S}^3|}\int_{A_2}e^{3u(y,0)}\log\frac{|y|^2}{|x-y|^2+t^2}dy \tag{3.10}$$

$$\geq-\frac{2}{|\mathbb{S}^3|}\log|X|\int_{A_2}e^{3u(y,0)}dy+\frac{1}{|\mathbb{S}^3|}\int_{|y|\leq 2}e^{3u(y,0)}\log\frac{|y|^2}{|X-(y,0)|^2}dy \tag{3.11}$$

$$\geq-\alpha\log|X|-C. \tag{3.12}$$

For $|X|\leq 4$, since v is continuous, we have $v(X)\geq -C$ for some constant $C>0$. Combining the two cases, we have (3.7). □

Lemma 3.3 *Suppose u satisfies (1.14), (1.15) and (1.16). Then there exists a constant $C_1\geq 0$ such that*

$$\Delta u(x,t)=-\frac{4}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}\frac{e^{3u(y,0)}}{|x-y|^2+t^2}dy-C_1. \tag{3.13}$$

Moreover, there exist constants $c_*\leq 0$, $a_i\leq 0$, $i=1,2,3$ such that

$$u-v=c_*t^2+\sum_{i=1}^{3}a_i(x_i-x_i^0)^2+c_0. \tag{3.14}$$

Proof Denote $w=u-v$ where v is defined in (3.6). Then w satisfies

$$\begin{cases}\Delta^2 w=0 & \text{in } \mathbb{R}^4_+,\\ \partial_t\Delta w=0 & \text{on } \partial\mathbb{R}^4_+,\\ \partial_t w=0 & \text{on } \partial\mathbb{R}^4_+.\end{cases} \tag{3.15}$$

We extend w by $w(x,t) = w(x,-t)$ for $t < 0$. We denote $\hat{w}$ this new function on $\mathbb{R}^4$. It follows from the mean value property of harmonic functions that $\Delta\hat{w}$ is smooth is $\mathbb{R}^4$. Consequently, $\hat{w}$ is also smooth in $\mathbb{R}^4$.

It follows from Lemma 3.2 that for $t > 0$,

$$w(x,t) = u(x,t) - v(x,t) \le u(x,t) + \alpha \log(1 + |(x,t)|) + C. \tag{3.16}$$

Thus $\hat{w}(X) \le \hat{u}(X) + \alpha \log(1 + |X|) + C$, where $\hat{u}$ is the even extension of u to $\mathbb{R}^4$. By Pizzetti's formula (see [15]), we have

$$\frac{r^2}{8}\Delta\hat{w}(X_0) = \fint_{\partial B_r(X_0)} \hat{w}\,d\sigma - \hat{w}(X_0). \tag{3.17}$$

By Jensen's inequality

$$\exp\left(\frac{r^2}{2}\Delta\hat{w}(X_0)\right) \le e^{-4\hat{w}(X_0)} \exp\left(4\fint_{\partial B_r(X_0)} \hat{w}\,d\sigma\right) \tag{3.18}$$

$$\le e^{-4\hat{w}(X_0)} \fint_{\partial B_r(X_0)} e^{4\hat{w}}\,d\sigma. \tag{3.19}$$

Since $\hat{w}(X) \le \hat{u}(X) + \alpha \log(1 + |X|) + C$ and (1.16), then $r^{3-4\alpha}\exp(\frac{r^2}{2}\Delta\hat{w}(X_0)) \in L^1[1,\infty)$. Thus $\Delta\hat{w}(X_0) \le 0$ for all $x_0 \in \mathbb{R}^4$. By Liouville's Theorem, $\Delta\hat{w}(X) \equiv -C_1$ in $\mathbb{R}^4$ for some constant $C_1 \ge 0$.

For bi-harmonic functions, one has the following fact that (for instance, see [15, eq. (14)])

$$|D^3\hat{w}|(X_0) \le \frac{C}{r^3}\fint_{B_r(X_0)} |\hat{w}|\,d\sigma \tag{3.20}$$

holds for some universal constant C. Note that

$$\fint_{B_r(X_0)} \hat{w}_+\,d\sigma \le \fint_{B_r(X_0)} \hat{u} + \alpha \log(1 + |X|)\,d\sigma + C$$

$$\le \fint_{B_r(X_0)} [\frac{1}{4}e^{4\hat{u}} + C\log r]\,d\sigma + C$$

Using (1.16), we have $r^{-3}\fint_{B_r(X_0)} \hat{w}_+\,d\sigma \to 0$ as $r \to \infty$. Consequently,

$$|D^3\hat{w}|(X_0) \le \frac{C}{r^3}\fint_{B_r(X_0)} |\hat{w}|\,d\sigma = \frac{C}{r^3}\fint_{B_r(X_0)} [2\hat{w}_+ - 2\hat{w}]\,d\sigma$$

$$= o(1) - \frac{2C}{r^3}\fint_{B_r(X_0)} \hat{w}\,d\sigma.$$

However, (3.17) and $\Delta\hat{w}(X) = -C_1$ implies that

$$\fint_{B_r(X_0)} \hat{w}\,d\sigma = O(r^2). \tag{3.21}$$

Inserting this to the previous inequality and letting $r \to \infty$, one must have $|D^3\hat{w}|(X_0) = 0$. Therefore $\hat{w}$ is a polynomial of degree at most 2. By the boundary condition of w and even symmetry of $\hat{w}$, one has $\hat{w} = c_*t^2 + p(x)$ where $p(x)$ has a degree at most 2.

Since $w(X) \le u(X) + \alpha \log(1 + |X|) + C$ for $X \in \mathbb{R}^4_+$ and u satisfies (1.16), then $c_* < 0$. Moreover, after an orthogonal transformation, we can assume $p(x) = \sum_{i=1}^3 a_i x_i^2 + b_i x_i + c_0$.

Since $\int_{\mathbb{R}^3} e^{3u(x,0)}dx < \infty$, then we must have $a_i \le 0$ and $b_i = 0$ whenever $a_i = 0$. Thus

$$p(x) = \sum_i a_i(x - x_i^0)^2 + c_0. \tag{3.22}$$

The proof is complete. □

Lemma 3.4 *Suppose that $v(x,t)$ defined in (3.6). For any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ such that for $|X| > R$,*

$$v(X) \le -(\alpha - \varepsilon)\log|X| \tag{3.23}$$

where α is defined in (3.8).

Proof As in the proof of [13, Lemma 2.4], we can show that for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$-v(X) \ge (\alpha - \frac{\varepsilon}{2})\log|X| + \frac{2}{|\mathbb{S}^3|}\int_{B_1(X)\cap\partial\mathbb{R}^4} \log|X - (y,0)|e^{3u(y,0)}dy \tag{3.24}$$

where $B_1(X)$ denotes the ball in $\mathbb{R}^4$ with center X and radius 1. It suffices to prove that the last term is bounded from below independent of X.

Applying Lemma 3.3 and letting $t \to 0^+$ in (3.14), we have

$$u(x,0) = \frac{2}{|\mathbb{S}^3|}\int_{\mathbb{R}^3} e^{3u(y,0)}\log\frac{|y|}{|x-y|}dy + \sum_{i=1}^3 a_i(x_i - x_i^0)^2 + c_0. \tag{3.25}$$

One can compute that

$$\bar{\Delta}u(x,0) = -\frac{2}{|\mathbb{S}^3|}\int_{\mathbb{R}^3}\frac{e^{3u(y,0)}}{|x-y|^2}dy + 2a_1 + 2a_2 + 2a_3. \tag{3.26}$$

Applying [19, Lemma 3.1], the above equation implies that $0 \le -\bar{\Delta}u(x,0) \le C$ for $x \in \mathbb{R}^3$ and some constant C. In fact, although the statement of Lemma 3.1 of [19] is for the case $a_1 = a_2 = a_3 = 0$, the proof still works for all $a_1 \le 0, a_2 \le 0, a_3 \le 0$ with mild changes, as observed in [10, Lemma 18].

Once we have the bound of $\bar{\Delta}u$, then using [19, Lemma 3.2], one can conclude that $u(x,0) \le C$ for $x \in \mathbb{R}^3$ and some constant C. Consequently, the last term in (3.24) is bounded from below independent of X. Thus (3.23) is established. □

Lemma 3.5 *Suppose that u satisfies the assumptions of Theorem 1.2. Then $\lim_{|x|\to\infty}\bar{\Delta}u(x,0)$ exists and is non-positive, here $\bar{\Delta}$ is the Laplacian w.r.t. $x \in \mathbb{R}^3$ only.If $u(x,0) = o(|x|^2)$ or $\bar{\Delta}u(x,0) = o(1)$ as $|x| \to \infty$, then there exist $\lambda > 0$ and $a \in \mathbb{R}^3$ such that*

$$u(x,0) = u_{a,\lambda}(x,0) = \log\left(\frac{2\lambda}{\lambda^2 + |x-a|^2}\right) \tag{3.27}$$

where $u_{a,\lambda}$ is defined in (1.17). Consequently there exists some $c \le 0$ such that

$$u(x,t) = v_{a,\lambda}(x,t) + u_{a,\lambda}(0,0) + ct^2 \tag{3.28}$$

where

$$v_{a,\lambda}(x,t) = \frac{1}{|\mathbb{S}|^3}\int_{\mathbb{R}^3} e^{3u_{a,\lambda}(y,0)}\log\frac{|y|^2}{|x-y|^2+t^2}dy. \tag{3.29}$$

Proof We claim that the first term on the right-hand side of (3.26) converges to 0 when $|x| \to \infty$. In fact, it can be decomposed to

$$\int_{\mathbb{R}^3} \frac{e^{3u(y,0)}}{|x-y|^2} dy = \int_{B_1(x)} \frac{e^{3u(y,0)}}{|x-y|^2} dy + \int_{\mathbb{R}^3 \setminus B_1(x)} \frac{e^{3u(y,0)}}{|x-y|^2} dy.$$

The first term can be bounded as

$$\int_{B_1(x)} \frac{e^{3u(y,0)}}{|x-y|^2} dy \le \left(\int_{B_1(x)} |x-y|^{-\frac{8}{3}} dy \right)^{\frac{3}{4}} \left(\int_{B_1(x)} e^{12u(y,0)} dy \right)^{\frac{1}{4}} \tag{3.30}$$

$$\le C \left(\int_{B_1(x)} e^{12v(y,0)+12\sum_{i=1}^3 a_i (y-x_i^0)^2 + c_0} dy \right)^{\frac{1}{4}} \to 0 \tag{3.31}$$

as $|x| \to \infty$. Here we have used (3.23) and $a_i \le 0$ from Lemma 3.3. By the dominated convergence theorem and (1.15), the second term is going to 0 as $|x| \to \infty$,

$$\int_{\mathbb{R}^3 \setminus B_1(x)} \frac{e^{3u(y,0)}}{|x-y|^2} dy \to 0 \quad \text{as} \quad |x| \to \infty.$$

Thus the claim is proved and consequently

$$\lim_{|x|\to\infty} \bar{\Delta} u(x,0) = 2(a_1 + a_2 + a_3) \le 0. \tag{3.32}$$

Now, if $u(x,0) = o(|x|^2)$, then clearly $a_1 = a_2 = a_3 = 0$. Since Lemma 3.3 says that $a_i \le 0$ for $i = 1, 2, 3$, then if $\bar{\Delta} u(x,0) = o(1)$, then we also get $a_1 = a_2 = a_3 = 0$.

In both cases, we get

$$u(x,0) = \frac{2}{|\mathbb{S}^3|} \int_{\mathbb{R}^3} e^{3u(y,0)} \log \frac{|y|}{|x-y|} dy + c_0. \tag{3.33}$$

The solutions to such an equation have been studied by Xu [19]. More precisely, under the condition (1.15), there exist some $a \in \mathbb{R}^3$ and $\lambda > 0$ such that $u(x,0) = u_{a,\lambda}(x,0)$ where $u_{a,\lambda}$ is defined in (1.17). Consequently, taking $x = 0$ in (3.33), one obtains that

$$c_0 = u_{a,\lambda}(0,0) = \log \left(\frac{2\lambda}{\lambda^2 + |a|^2} \right). \tag{3.34}$$

Noticing (3.6), we introduce the notation $v_{a,\lambda}$ as defined in (3.29). Since we have derived that $a_1 = a_2 = a_3 = 0$, then (3.14) leads to (3.28). □

It seems hard to integrate (3.29) out explicitly. We get around this difficulty by constructing a solution to (1.14) directly.

Lemma 3.6 *For any $\lambda > 0$ and $a \in \mathbb{R}^3$, (1.17) satisfies (1.14).*

Proof For the convenience of notation, we assume $a = 0$. The general case follows from the translation invariance of the equation. Let $u = u_1 + u_2$

$$u_1 = \log \left(\frac{2\lambda}{(\lambda+t)^2 + |x|^2} \right), \quad u_2 = \frac{2\lambda t}{(\lambda+t)^2 + |x|^2}. \tag{3.35}$$

Then it is easy to see

$$\partial_t u_1(x,0) = -\frac{2\lambda}{\lambda^2 + |x|^2} = -\partial_t u_2(x,0) \tag{3.36}$$

Therefore $\partial_t u(x,0) = 0$ for $x \in \mathbb{R}^3$. We continue taking derivatives

$$\Delta u_1 = -\frac{4}{(\lambda+t)^2+|x|^2}, \quad \Delta u_2 = -\frac{8\lambda(t+\lambda)}{\left((\lambda+t)^2+|x|^2\right)^2} \tag{3.37}$$

$$\partial_t \Delta u_1(x,0) = \frac{8\lambda}{\left(\lambda^2+|x|^2\right)^2}, \quad \partial_t \Delta u_2(x,0) = -\frac{8\lambda\left(|x|^2-3\lambda^2\right)}{\left(\lambda^2+|x|^2\right)^3}. \tag{3.38}$$

Therefore

$$\partial_t \Delta u(x,0) = \frac{32\lambda^3}{\left(\lambda^2+|x|^2\right)^3} = 4e^{3u(x,0)}. \tag{3.39}$$

Last, we have

$$\Delta^2 u_1(x,t) = \Delta^2 u_2(x,t) = 0 \tag{3.40}$$

This implies $\Delta^2 u = 0$ in $\mathbb{R}^4$. Thus u satisfies (1.14). It is easy to see that (1.15) holds. □

Corollary 3.7 *One must have $u_{a,\lambda} = v_{a,\lambda} + u_{a,\lambda}(0,0)$ where $u_{a,\lambda}$ is defined in* (1.17) *and $v_{a,\lambda}$ is defined in* (3.29).

Proof Given any $a \in \mathbb{R}^3$ and $\lambda > 0$, it is easy to show that $u_{a,\lambda}$ satisfies (1.15) and (1.16). Combining Lemma 3.6, u satisfies the assumption of Theorem 1.2. Apparently $u_{a,\lambda}(x,0) = o(|x|^2)$. Then Lemma 3.5 asserts the existence of $a' \in \mathbb{R}^3$, $\lambda' > 0$ and $c \leq 0$ such that

$$u_{a,\lambda}(x,t) = v_{a',\lambda'}(x,t) + u_{a',\lambda'}(0,0) + ct^2.$$

Applying $\partial_t \Delta$ on the boundary, Lemma 3.1 and (3.29) imply that $u_{a,\lambda}(x,0) = u_{a',\lambda'}(x,0)$. Thus $a = a'$ and $\lambda = \lambda'$. It is easy to see $c = 0$ in this case, because both $u_{a,\lambda}$ and $v_{a',\lambda'}$ are $o(|x|^2 + t^2)$. This completes the proof. □

Proof of Theorem 1.2 It follows from (3.32) in Lemma 3.5 that $\lim_{|x|\to\infty} \bar{\Delta} u(x,0)$ exists and is non-positive. If $u(x,0) = o(|x|^2)$ or $\bar{\Delta} u(x,0) = o(1)$ as $|x| \to \infty$, then Lemma 3.5 applies to this case. Consequently, (3.28) and Corollary 3.7 give that $u = u_{a,\lambda} + ct^2$ for some $a \in \mathbb{R}^3$ and $\lambda > 0$. □

Proof of Corollary 1.4 Taking $t = 0$ in (1.18) implies that u satisfies (3.33), which leads to (3.27). Then Lemma 3.5 holds with $c = 0$ in (3.28). The result follows from Corollary 3.7. □

4 Equality case for more general Sobolev trace inequalities

In this section, we shall study the general Sobolev trace inequalities proved by Case. The result here is not as rich as Ache-Chang's inequality. As did before, we shall write the equations on the upper half space. However, in this general case, we do not have a classification result for the Euler-Lagrange equations (see the discussion at the end), which plays an important role in most of the the blow up analysis giving rise to such limit equation. Nevertheless, to find the extremals it suffices to find a solution with sufficient decay on the upper half space. When we pull back the solution to the unit ball, it won't cause a singularity.

4.1 Extensions of extremals

We shall prove Theorem 1.6 and Corollary 1.7.

Proof of Theorem 1.6 For $n \geq 3$, we want to find the solution of

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^{n+1}, \\ \eta v + \frac{n-3}{2} v = |1 - \langle z_1, \xi \rangle|^{-(n-1)/2} & \text{on } \mathbb{S}^n, \\ v = 0 & \text{on } \mathbb{S}^n \end{cases} \tag{4.1}$$

Note that we do not need to distinguish the cases $n = 3$ and $n \geq 4$.

We will first find the corresponding equations of (4.1) on $\mathbb{R}^{n+1}_+$ using Möbius transformation. We define

$$U(x,t) = v(\mathcal{S}(x,t)) \left(\frac{2}{|x|^2 + (1+t)^2} \right)^{\frac{n-3}{2}}. \tag{4.2}$$

It follows from (2.11) that

$$\partial_t U(x,0) = -|1 - \langle z_1, \xi \rangle|^{-\frac{n-1}{2}} \left(\frac{2}{1+|x|^2} \right)^{\frac{n-1}{2}}. \tag{4.3}$$

For $\xi \in \mathbb{S}^n$, we have $|1 - \langle z_1, \xi \rangle| = F(\omega_1, \xi)^2 (1 + |\omega_1|^2)^{-1}$. Consequently, we use (2.5) to derive that

$$\begin{aligned} \partial_t U(x,0) &= -(1+|\omega_1|^2)^{\frac{n-1}{2}} F(\omega_1, \xi)^{-(n-1)} \left(\frac{|\xi + e_{n+1}|^2}{2} \right)^{\frac{n-1}{2}} \\ &= c(\omega_1) \left(\frac{\lambda}{|x-a|^2 + (t+\lambda)^2} \right)^{\frac{n-1}{2}} \end{aligned} \tag{4.4}$$

where $c(\omega_1) = -\left(2 \frac{1+|\omega_1|^2}{1-|\omega_1|^2} \right)^{(n-1)/2}$ and $(a, \lambda) = \mathcal{S}^{-1}(\omega_1)$. Thus U satisfies the following equations

$$\begin{cases} \Delta^2 U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial_t U = c(\omega_1) \left(\frac{\lambda}{|x-a|^2 + (t+\lambda)^2} \right)^{\frac{n-1}{2}} & \text{on } \mathbb{R}^n, \\ U = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{4.5}$$

Conversely, if U satisfies the above equation and is smooth, then the following v defined by

$$v(\xi) = U(\mathcal{S}^{-1}(\xi)) \left(\frac{2}{|\xi'|^2 + (1+\xi_{n+1})^2} \right)^{\frac{n-3}{2}} \tag{4.6}$$

will satisfy (4.1) on $\overline{\mathbb{B}^{n+1}} \setminus \{-e_{n+1}\}$.

Claim 1 *Fix any $(a, \lambda) \in \mathbb{R}^{n+1}_+$, equations (4.5) have a solution*

$$U(x,t) = c(\omega_1) t \left(\frac{\lambda}{|x-a|^2 + (t+\lambda)^2} \right)^{\frac{n-1}{2}} \tag{4.7}$$

Proof It is easy to verify the boundary conditions. We only need to prove $\Delta^2 U = 0$. This follows from the direct computation. For instance, one can get

$$\Delta U(x,t) = 2(1-n)(t+\lambda)\frac{\lambda^{\frac{n-1}{2}}}{(|x-a|^2+(t+\lambda)^2)^{\frac{n+1}{2}}}. \tag{4.8}$$

□

Now plugging in (4.7) to (4.6) and using (2.5), we obtain

$$v(\xi) = -\frac{1}{2}(1-|\xi|^2)\frac{(1+|\omega_1|^2)^{\frac{n-1}{2}}}{F(\xi,\omega_1)^{n-1}}. \tag{4.9}$$

Note that this v is smooth on whole $\overline{\mathbb{B}^{n+1}}$. Thus it is the desired solution. □

Proof of Corollary 1.7 For any $n \geq 3$, if v is the extremal of (1.19) or (1.20), then Case [3] implies that it satisfies

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^{n+1}, \\ \eta v + \frac{n-3}{2} v = \psi & \text{on } \mathbb{S}^n, \\ v = f & \text{on } \mathbb{S}^n. \end{cases} \tag{4.10}$$

Denote the unique solution to be $v_{f,\psi}$. By the linearity, we must have $v_{f,\psi} = v_{f,0} + v_{0,\psi}$. However, $v_{f,0}$ is obtained from Theorem 1.1 and Theorem 1.6 finds $v_{0,\psi}$. The sum of them will give $v_{f,\psi}$. □

4.2 Discussion

In the end, we shall provide some discussion to the Euler-Lagrange equation for the general Sobolev trace inequality of order four proved by Case. Consider (1.19) and (1.20) for $f = 0$, that is

$$\tilde{a}_n \left(\oint_{\mathbb{S}^n} |\psi|^{\frac{2n}{n-1}} d\sigma \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{B}^{n+1}} |\Delta v|^2 dx - 2 \oint_{\mathbb{S}^n} \psi^2 d\sigma. \tag{4.11}$$

Recall that $\psi = \eta v$ if $f = 0$. Then the Euler-Lagrange equation for the equality (modulo scaling) is

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^{n+1}, \\ \nabla^2 v(\eta,\eta) + (n-2)\eta v = |\eta v|^{\frac{2}{n-1}} \eta v & \text{on } \mathbb{S}^n, \\ v = 0 & \text{on } \mathbb{S}^n. \end{cases} \tag{4.12}$$

Again using Möbius transformation and (2.9), we compute that $U(x,t)$ satisfies the following equations on the upper half space $\mathbb{R}^{n+1}_+$ for $n \geq 3$.

$$\begin{cases} \Delta^2 U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^2_{tt} U = -|\partial_t U|^{\frac{2}{n-1}} \partial_t U & \text{on } \mathbb{R}^n, \\ U = 0 & \text{on } \mathbb{R}^n. \end{cases} \tag{4.13}$$

It is easy to verify that

$$U(x,t) = ct\left(\frac{\lambda}{|x-a|^2+(t+\lambda)^2}\right)^{\frac{n-1}{2}} \tag{4.14}$$

satisfies (4.13) for some suitable constant c and any $(a,\lambda)\in\mathbb{R}^{n+1}_+$.

It is quite possible that (4.13) has many solutions with $\partial_t U$ changes sign on $\partial\mathbb{R}^{n+1}_+$. An interesting question is the classification of the solutions of (4.13) assuming $\partial_t U$ has a fixed sign on $\partial\mathbb{R}^{n+1}_+$ for $n\geq 3$. Here we provide some useful observations. If we in addition assume that U has *sufficient decay* (say $U\in C^4\cap W^{2,2}(\mathbb{R}^{n+1}_+)$), then one can characterize $\partial_t U$ on $\partial\mathbb{R}^{n+1}_+$. More precisely, we denote $\psi=-U_t(x,0)$ and assume $\psi>0$. Since ΔU is a harmonic function on the upper half space, then we apply the Poisson kernel to get

$$\Delta U(x,t)=\mathcal{P}*\psi(X). \tag{4.15}$$

Since $U=0$ on the boundary, then we use Green's representation on the upper half space

$$U(X)=\int_{\mathbb{R}^{n+1}_+}G(X,Y)(\mathcal{P}*\psi)(Y)dY \tag{4.16}$$

where $X=(X,t)$, $Y=(y,s)$. Recall that $\partial_t|_{t=0}G(X,Y)=-\mathcal{P}(x-y,s)$. Consequently

$$\begin{aligned}\psi(x)=-\partial_t U(X)|_{t=0}&=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{n+1}}\mathcal{P}(x-y,s)\mathcal{P}(y-z,s)dyds\right)\psi(z)^{\frac{n+1}{n-1}}dz\\&=\int_{\mathbb{R}^n}\left(\int_0^\infty\mathcal{P}(x-z,2s)ds\right)\psi(z)^{\frac{n+1}{n-1}}dz\\&=\int_{\mathbb{R}^n}\frac{\psi^{\frac{n+1}{n-1}}}{|x-z|^{n-1}}dz.\end{aligned} \tag{4.17}$$

Namely, ψ satisfies an integral equation. The results in [12] and [6] assert that ψ must be $(\lambda/[\lambda^2+|x-a|^2])^{-(n-1)/2}$ up to some constant for $(a,\lambda)\in\mathbb{R}^4_+$. This coincides with the rigidity of equality in (1.19) and (1.20) through the observation (4.4).

However, without any decay condition on U, it is easy to see that ct^3 is also a solution of (4.13) for any constant c.

Furthermore, one can study the Euler-Lagrange equations of (1.19) and (1.20) for general f and ψ. The equations will look more complicated but the results should be expected from some similar analysis in this paper. We shall state the equations and leave the details to the interested reader. More precisely, using Möbius transformation and the results by [3], if $n>3$ then the inequality (1.20) is equivalent to that the following one holds for any $U\in C_c^\infty(\overline{\mathbb{R}^{n+1}})$

$$\begin{aligned}a_n\left(\int_{\mathbb{R}^n}|f|^{\frac{2n}{n-3}}dx\right)^{\frac{n-3}{n}}+\tilde a_n\left(\int_{\mathbb{R}^n}|\psi|^{\frac{2n}{n-1}}dx\right)^{\frac{n-1}{n}}&\leqslant\int_{\mathbb{R}^{n+1}_+}|\Delta U(x,t)|^2dxdt\\&+\int_{\mathbb{R}^n}4\langle\bar\nabla f,\bar\nabla\psi\rangle dx\end{aligned} \tag{4.18}$$

where $f(x)=U(x,0)$, $\psi(x)=-\partial_t U(x,0)$. The Euler-Lagrange equation of the above inequality is

$$\begin{cases}\Delta^2U=0 & \text{in } \mathbb{R}^{n+1}_+,\\ \partial_t\Delta U-2\bar\Delta\psi=2a_n(\int_{\mathbb{R}^n}|f|^{\frac{2n}{n-3}})^{-\frac{3}{n}}|f|^{\frac{6}{n-3}}f & \text{on } \mathbb{R}^n,\\ \partial_{tt}^2U-\bar\Delta f=2\tilde a_n(\int_{\mathbb{R}^n}|\psi|^{\frac{2n}{n-1}})^{\frac{-1}{n}}|\psi|^{\frac{2}{n-1}}\psi & \text{on } \mathbb{R}^n.\end{cases} \tag{4.19}$$

providing $f\not\equiv 0$ and $\psi\not\equiv 0$. If $\psi\equiv 0$, the Euler-Lagrange equation reduces to (2.14) modulo suitable scaling, and to (4.13) if $f\equiv 0$ modulo suitable scaling. We know that (4.19) has explicit solutions which come from the linear combinations of (2.15) and (4.7). We do not know if there is any other solutions.

Now let us consider $n = 3$ case. The Euler-Lagrange equation for (1.19) is

$$\begin{cases} \Delta^2 v = 0 & \text{in } \mathbb{B}^4, \\ -\eta\Delta v - 2\bar{\Delta} v - 2\bar{\Delta}\eta v + 4 = 8\pi^2 \left(\oint_{\mathbb{S}^3} e^{3v}\right)^{-1} e^{3v} & \text{on } \mathbb{S}^4, \\ \Delta v - 2\bar{\Delta} v - 2\eta v = (4\pi)^{\frac{2}{3}} (\oint_{\mathbb{S}^3} |\eta v|^3)^{-\frac{1}{3}} |\eta v| \eta v & \text{on } \mathbb{S}^4 \end{cases} \tag{4.20}$$

provided $\eta v \not\equiv 0$. Here $\eta = \eta_{g_0}$ is the unit outward normal for $(\mathbb{B}^4, g_0)$. If $\eta v \equiv 0$, then the Euler-Lagrange equation reduces to (2.25).

Using the adapted metric g^*, one defines $g = e^{2v} g^* = e^{2v+1-|\xi|^2}|d\xi|^2$. As we did in the Sect. 2.2, the first line of (4.20) is equivalent to $(Q_4)_g = 0$, and the second line is equivalent to $(T_3)_g = const > 0$. In order to interpret the third line of (4.20), we need to introduce the boundary operator B_2^3 and its conformation rule. On a four-dimensional manifold (X^4, g) with boundary M, one can define $B_2^3 u = -\bar{\Delta} u + \nabla^2 u(\eta, \eta) + \frac{1}{3} H \eta u$ for any $u \in C^\infty(X)$ and its associated curvature $T_2^3 = \bar{J} - P(\eta, \eta) + \frac{1}{18} H^2$ (see [3]). Such curvature obeys the conformal transformation rule for $\hat{g} = e^{2\tau} g$ as

$$(T_2^3)_{\hat{g}} = e^{-2\tau}((T_2^3)_g + (B_2^3)_g \tau). \tag{4.21}$$

On $(\mathbb{B}^4, \mathbb{S}^3, g_0)$, we have

$$(B_2^3)_{g_0} u = \Delta u - 2\bar{\Delta} u - 2\eta u, \quad (T_2^3)_{g_0} = 2$$

Using (4.21), the T_2^3 curvature for the conformal metric $g = e^{2v+1-|\xi|^2} g_0$ can be computed as

$$(T_2^3)_g = e^{-2v}\left(2 + B_2^3[v + \tfrac{1}{2}(1 - |\xi|^2)]\right) = e^{-2v} B_2^3 v = \tilde{c}|H_g|H_g$$

for some constant $\tilde{c} > 0$. Here in the last equality, we have used the third line of (4.20) and $H_g = 3e^{-v}\eta_{g_0} v$. Now since $\mathcal{S}^* g = e^{2w}(|dx|^2 + dt^2)$, then the conformal transformation rules (2.19), (2.20) and (4.21) imply that w satisfies

$$\begin{cases} \Delta^2 w = 0 & \text{in } \mathbb{R}^4_+, \\ \partial_t \Delta w + 2\bar{\Delta}\partial_t w = 8\pi^2 (\int_{\mathbb{R}^n} e^{3w})^{-1} e^{3w} & \text{on } \mathbb{R}^3, \\ \partial^2_{tt} w - \bar{\Delta} w = -(4\pi)^{\frac{2}{3}} (\int_{\mathbb{R}^3} |\partial_t w|^3)^{-\frac{1}{3}} |\partial_t w| \partial_t w & \text{on } \mathbb{R}^3. \end{cases} \tag{4.22}$$

We have found some explicit solutions of (4.22), namely, they are the sum of constants, (1.17) and (4.14). We do not know if there are any other solutions.

Acknowledgements L. Sun was partially supported by CAS Project for Young Scientists in Basic Research, Grant No. YSBR–031. C. B. Ndiaye was partially supported by NSF grant DMS–2000164. We would like to thank Jingang Xiong for his helpful suggestions.

References

1. Ache, A.G., Chang, S.-Y.A.: Sobolev trace inequalities of order four. Duke Math. J. **166**(14), 2719–2748 (2017)
2. Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. Math. **138**(1), 213–242 (1993)
3. Case, J.S.: Boundary operators associated with the paneitz operator. Indiana Univ. Math. J. **67**(1), 293–327 (2018)
4. Case, J.S., Chang, S.-Y.A.: On fractional GJMS operators. Commun. Pure Appl. Math. **69**(6), 1017–1061 (2016)

5. Chang, S.-Y.A., Qing, J.: The Zeta Functional Determinants on Manifolds with Boundary. J. Funct. Anal. **147**(2), 327–362 (1997)
6. Chen, W., Li, C., Biao, O.: Classification of solutions for an integral equation. Commun. Pure Appl. Math. **59**(3), 330–343 (2006)
7. Escobar, J.F.: Sharp constant in a sobolev trace inequality. Indiana Univ. Math. J. **37**(3), 687–698 (1988)
8. Fefferman, C., Robin Graham, C.: Q-curvature and Poincaré metrics. Math. Res. Lett. **9**(2–3), 139–151 (2002)
9. Han, Q., Lin, F.: Elliptic partial differential equations, vol. 1. American Mathematical Society, New York (2011)
10. Jin, T., Maalaoui, A., Martinazzi, L., Xiong, J.: Existence and asymptotics for solutions of a non-local q-curvature equation in dimension three. Calc. Var. Partial. Differ. Equ. **52**(3), 469–488 (2015)
11. Lebedev, N.A., Milin, I.M.: On the coefficients of certain classes of analytic functions. Matematicheskii Sbornik **70**(2), 359–400 (1951)
12. Li, Y.Y.: Remark on some conformally invariant integral equations: the method of moving spheres. J. Eur. Math. Soc. **6**(2), 153–180 (2004)
13. Lin, C.-S.: A classification of solutions of a conformally invariant fourth order equation in R^n. Commentarii Mathematici Helvetici **73**(2), 206–231 (1998)
14. Lions, P.-L.: The concentration-compactness principle in the calculus of variations. the limit case, part 1. Revista matemática iberoamericana **1**(1), 145–201 (1985)
15. Martinazzi, L.: Classification of solutions to the higher order liouville's equation on R^{2m}. Math. Z. **263**(2), 307–329 (2009)
16. Ngô, Q.A., Nguyen, V.H., Phan, Q.H.: Higher order sobolev trace inequalities on balls revisited. J. Funct. Anal. **278**(7), 108414 (2020)
17. Osgood, B., Phillips, R., Sarnak, P.: Extremals of determinants of laplacians. J. Funct. Anal. **80**(1), 148–211 (1988)
18. Sun, L., Xiong, J.: Classification theorems for solutions of higher order boundary conformally invariant problems, I. J. Func. Anal. **271**(12), 3727–3764 (2016)
19. Xingwang, X.: Uniqueness and non-existence theorems for conformally invariant equations. J. Funct. Anal. **222**(1), 1–28 (2005)