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Variational theory for the resonant T-curvature equation

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Abstract. In this paper, we study the resonant prescribed T-curvature problem on a compact 4-dimensional Riemannian manifold with boundary. We derive sharp energy and gradient estimates of the associated Euler-Lagrange functional to characterize the critical points at infinity of the associated variational problem under a non-degeneracy on a naturally associated Hamiltonian function. Using this, we derive a Morse type lemma at infinity around the critical points at infinity. Using the Morse lemma at infinity, we prove new existence results of Morse theoretical type. Combining the Morse lemma at infinity and the Liouville version of the Barycenter technique of Bahri–Coron (Commun Pure Appl Math 41–3:253–294, 1988) developed in Ndiaye (Adv Math 277(277):56–99, 2015), we prove new existence results under a topological hypothesis on the boundary of the underlying manifold, the selection map at infinity, and the entry and exit sets at infinity.

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1. Introduction and statement of the results

On a four-dimensional compact Riemannian manifolds with boundary $(\overline{M}, g)$, there exists a fourth-order operator P_g called Paneitz operator discovered by Paneitz [26] and an associated curvature quantity Q_g called Q-curvature introduced by Branson-Oersted [5]. The Paneitz operator P_g and the Q-curvature Q_g are defined in terms of the Ricci tensor Ric_g and the scalar curvature R_g of $(\overline{M}, g)$ by

$$P_g^4 = \Delta_g^2 - div_g\left(\left(\frac{2}{3}R_g g - 2Ric_g\right)\nabla_g\right), \qquad Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|Ric_g|^2),$$

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where div_g is the divergence, ∇_g is the covariant derivative, and Δ_g is the Laplace-Beltrami operator, all with respect to g.

On the other hand, Chang–Qing [7] have discovered an operator P_g^3 which is associated to the boundary ∂M of $\overline{M}$ and a curvature quantity T_g naturally associated to P_g^3. They are defined by the formulas

$$P_g^3 = \frac{1}{2}\frac{\partial\Delta_g}{\partial n_g} + \Delta_{\hat{g}}\frac{\partial}{\partial n_g} - 2H_g\Delta_{\hat{g}} + L_g(\nabla_{\hat{g}}, \nabla_{\hat{g}}) + \nabla_{\hat{g}}H_g.\nabla_{\hat{g}} + \left(F_g - \frac{R_g}{3}\right)\frac{\partial}{\partial n_g}.$$

$$T_g = -\frac{1}{12}\frac{\partial R_g}{\partial n_g} + \frac{1}{2}R_gH_g - < G_g, L_g > +3H_g^3 - \frac{1}{3}tr_g(L_g^3) - \Delta_{\hat{g}}H_g,$$

where $\hat{g}$ is the metric induced by g on ∂M, $\Delta_{\hat{g}}$ is the Laplace-Beltrami operator with respect to $\hat{g}$, $\frac{\partial}{\partial n_g}$ is the inward Neuman operator on ∂M with respect to g, L_g is the second fundamental form of ∂M with respect to g, H_g is the mean curvature of ∂M with respect to $g, R^k_{g,ijl}$ is the Riemann curvature tensor of $(\overline{M}, g)$, $R_{g,ijkl} = g_{mi}R^m_{g,jkl}$ (g_{ij} are the entries of the metric g), $F_g = R^a_{g,nan}$ (with n denoting the index corresponding to the normal direction in local coordinates) and $< G_g, L_g >= \hat{g}^{ac}\hat{g}^{bd}R_{g,anbn}L_{g,cd}$. Moreover, the notation $L_g(\nabla_{\hat{g}}, \nabla_{\hat{g}})$, means $L_g(\nabla_{\hat{g}}, \nabla_{\hat{g}})(u) = \nabla^a_{\hat{g}}(L_{g,ab}\nabla^b_{\hat{g}}u)$. We point out that in all those notations above $i, j, k, l = 1, \ldots 4$ and $a, b, c, d = 1, \ldots 3$, and Einstein summation convention is used for repeated indices.

As the Laplace–Beltrami operator and the Neumann operator on closed surfaces with boundary are conformally covariant, we have that P_g^4 is conformally covariant of bidegree $(0, 4)$ and P_g^3 of bidegree $(0, 3)$. Furthermore, as they govern the transformation laws of the Gauss curvature and the geodesic curvature on compact surfaces with boundary, the couple (P_g^4, P_g^3) does the same for (Q_g, T_g) on a compact four-dimensional Riemannian manifold with boundary $(\overline{M}, g)$. In fact, under a conformal change of metric $g_u = e^{2u}g$, we have

$$\begin{cases} P^4_{g_u} = e^{-4u}P^4_g, \\ P^3_{g_u} = e^{-3u}P^3_g, \end{cases} \quad \text{and} \quad \begin{cases} P^4_g u + 2Q_g = 2Q_{g_u}e^{4u} \text{ in } M, \\ P^3_g u + T_g = T_{g_u}e^{3u} \text{ on } \partial M. \end{cases} \tag{1}$$

Apart from this analogy, we have also an extension of the Gauss–Bonnet identity (2) which is known as the Gauss–Bonnet–Chern formula

$$\int_M \left(Q_g + \frac{|W_g|^2}{8}\right)dV_g + \oint_{\partial M}(T_g + Z_g)dS_g = 4\pi^2\chi(\overline{M}) \tag{2}$$

where W_g denote the Weyl tensor of $(\overline{M}, g)$ and Z_g is given by the following formula

$$\begin{aligned} Z_g = {} & R_gH_g - 3H_gRic_{g,nn} + \hat{g}^{ac}\hat{g}^{bd}R_{g,anbn}L_{g,cd} - \hat{g}^{ac}\hat{g}^{bd}R_{g,acbc}L_{g,cd} \\ & + 6H_g^3 - 3H_g|L_g|^2 + tr_g(L_g^3), \end{aligned}$$

with tr_g denoting the trace with respect to the metric induced on ∂M by g (namely $\hat{g}$) and $\chi(\overline{M})$ the Euler–Poincaré characteristic of $\overline{M}$. Concerning the quantity Z_g, we have that it vanishes when the boundary is totally geodesic and $\oint_{\partial M} Z_g dV_g$ is always conformally invariant, see [7]. Thus, setting

$$\kappa_{(P^4,P^3)} := \kappa_{(P^4,P^3)}[g] := \int_M Q_g dV_g + \oint_{\partial M} T_g dS_g \tag{3}$$

we have that thanks to (2), and to the fact that $|W_g|^2 dV_g$ is pointwise conformally invariant, $\kappa_{(P^4,P^3)}$ is a conformal invariant (which justifies the notation used above). We remark that $4\pi^2$ is the the total integral of the (Q,T)-curvature of the standard four-dimensional Euclidean unit ball $\mathbb{B}^4$.

As was asked in [1], a natural question is whether every compact four-dimensional Riemannian manifold with boundary $(\overline{M}, g)$ carries a conformal metric g_u for which the corresponding Q-curvature Q_{g_u} is zero, the corresponding T-curvature T_{g_u} is a prescribed function and such that $(\overline{M}, g_u)$ has minimal boundary. Thanks to (1), this problem is equivalent to finding a smooth solution to the following BVP:

$$\begin{cases} P_g^4 u + 2Q_g = 0 & \text{in } M, \\ P_g^3 u + T_g = Ke^{3u} & \text{on } \partial M, \\ -\dfrac{\partial u}{\partial n_g} + H_g u = 0 & \text{on } \partial M, \end{cases}$$

where $K : \partial M \longrightarrow \mathbb{R}_+$ is a positive smooth function on ∂M.

Since we are interested to find a metric in the conformal class of g, then we can assume that $H_g = 0$, since this can be always obtained through a conformal transformation of the background metric. Thus, we are led to solve the following BVP with Neumann homogeneous boundary condition:

$$\begin{cases} P_g^4 u + 2Q_g = 0 & \text{in } M, \\ P_g^3 u + T_g = Ke^{3u} & \text{on } \partial M, \\ \dfrac{\partial u}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{4}$$

Defining $\mathcal{H}_{\frac{\partial}{\partial n}}$ as

$$\mathcal{H}_{\frac{\partial}{\partial n}} = \Big\{ u \in W^{2,2}(M) : \ \frac{\partial u}{\partial n_g} = 0 \text{ on } \partial M \Big\},$$

where $W^{2,2}(M)$ denotes the space of functions on M which are square integrable together with their first and second derivatives, and

$$\mathbb{P}_g^{4,3}(u,v) = \left\langle P^4 u, v \right\rangle_{L^2(M)} + 2 \left\langle P_g^3 u, v \right\rangle_{L^2(\partial M)}, \quad u, v \in \mathcal{H}_{\frac{\partial}{\partial n}} \cap W^{4,2}(M),$$

with $W^{4,2}(M)$ denoting the space of functions on M which are square integrable together with their derivatives up to order 4, we have integration by part implies

$$\mathbb{P}_g^{4,3}(u,v) = \int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v \right) dV_g - 2 \int_M Ric_g(\nabla_g u, \nabla_g v) dV_g$$
$$- 2 \oint_{\partial M} L_g(\nabla_{\hat{g}} u, \nabla_{\hat{g}} v) dS_g, \ \ u, v \in \mathcal{H}_{\frac{\partial}{\partial n}} \cap W^{4,2}(M). \tag{5}$$

We observe that the right hand side of (5) is well defined for functions which are just in $\mathcal{H}_{\frac{\partial}{\partial n}}$ and extend the definition $\mathbb{P}_g^{4,3}$ to the full space $\mathcal{H}_{\frac{\partial}{\partial n}}$ by

$$\begin{aligned}\mathbb{P}_g^{4,3}(u,v) = &\int_M \left(\Delta_g u \Delta_g v + \frac{2}{3} R_g \nabla_g u \cdot \nabla_g v\right) dV_g \\ &- 2\int_M Ric_g(\nabla_g u, \nabla_g v) dV_g \\ &- 2\oint_{\partial M} L_g(\nabla_{\hat{g}} u, \nabla_{\hat{g}} v) dS_g, \;\; u, v \in \mathcal{H}_{\frac{\partial}{\partial n}}. \end{aligned} \tag{6}$$

Hence we have $\mathbb{P}_g^{4,3}$ is a well-defined bilinear form on $\mathcal{H}_{\frac{\partial}{\partial n}}$, and we set

$$\ker \mathbb{P}_g^{4,3} := \{u \in \mathcal{H}_{\frac{\partial}{\partial n}} : \;\; \mathbb{P}_g^{4,3}(u,v) = 0, \; \forall v \in \mathcal{H}_{\frac{\partial}{\partial n}}\}. \tag{7}$$

On the other hand, standard regularity theory implies that smooth solutions to (4) can be found by looking at critical points of the geometric functional

$$\mathcal{E}_g(u) = \mathbb{P}_g^{4,3}(u,u) + 4\int_M Q_g u dV_g + 4\oint_{\partial M} T_g u dS_g - \frac{4}{3}\kappa_{(P^4,P^3)} \log \oint_{\partial M} K e^{3u} dS_g, \\ u \in \mathcal{H}_{\frac{\partial}{\partial n}}.$$

As a Liouville type problem, the analytic features of equation (4) and of the associated Euler-Lagrange functional $\mathcal{E}_g$ depend strongly on the conformal invariant $\kappa_{(P^4,P^3)}$. Indeed, depending on whether $\kappa_{(P^4,P^3)}$ is a positive integer multiple of $4\pi^2$ or not, the noncompactness of equation (4) and the way of finding critical points of $\mathcal{E}_g$ changes drastically. As far as existence questions are concerned, we have that problem (4) has been solved in a work of Chang–Qing [8] under the assumption that ker $\mathbb{P}_g^{4,3} \simeq \mathbb{R}$, $\mathbb{P}_g^{4,3}$ is non-negative and $\kappa_{(P^4,\,P^3)} < 4\pi^2$. In [21], we show existence of solutions for (4) under the assumption $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,\,P^3)} \notin 4\pi^2\mathbb{N}^*$.

As a Liouville type problem, the assumption $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} \notin 4\pi^2\mathbb{N}^*$ will be referred to as nonresonant case. This terminology is motivated by the fact that in that situation the set of solutions to some perturbations of Eq. (7) (including it) is compact. Naturally, we call resonant case when $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,\,P^3)} \in 4\pi^2\mathbb{N}^*$. With these terminologies, we have that the works of Chang–Qing [8] and our work in [21] answer affirmatively the question raised above in the nonresonant case. However, for the resonant case, there are no known existence results to the best of our knowledge. For related works dealing with high order conformally invariant equations, see [6–9, 11, 12, 14, 15, 19–25] and the references therein.

In this work, beside existence results for (4), we are interested in a complete variational theory for the boundary value problem (4) in the resonant case, namely when $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} = 4\pi^2 k$ for some $k \in \mathbb{N}^*$. To present the main results of the paper, we need to set first some notation and make some definitions. We define the Hamiltonian function (at infinity)

$\mathcal{F}_K : (\partial M)^k \backslash F_k(\partial M) \longrightarrow \mathbb{R}$ by

$$\mathcal{F}_K((a_1,\ldots,a_k)) := \sum_{i=1}^{k}\left(H(a_i,a_i)+\sum_{j=1,\, j\neq i}^{k} G(a_i,\, a_j)+\frac{2}{3}\log(K(a_i))\right)$$

where $F_k(\partial M)$ denotes the fat Diagonal of $(\partial M)^k$, namely

$$F_k(\partial M) := \{A := (a_1,\ldots,a_k) \in (\partial M)^k :\ \text{there exists } i \neq j \text{ with } a_i = a_j\},$$

G is the Green's function defined by (47), and H is its regular defined as in (49). Furthermore, we define

$$Crit(\mathcal{F}_K) := \{A \in (\partial M)^k \setminus F_k(\partial M),\ A \text{ critical point of } \mathcal{F}_K\}. \tag{8}$$

Moreover, for $A = (a_1,\ldots,a_k) \in (\partial M)^k \backslash F_k(\partial M)$, and $i = 1\ldots k$, we set

$$\mathcal{F}_i^A(x) := e^{3(H(a_i,x)+\sum_{j=1,\, j\neq i}^{k} G(a_j,x))+\frac{1}{3}\log(K(x))}, \tag{9}$$

and define

$$\mathcal{L}_K(A) := -\sum_{i=1}^{k}(\mathcal{F}_i^A)^{\frac{1}{2}} L_{\hat{g}}\left((\mathcal{F}_i^A)^{\frac{1}{6}}\right)(a_i), \tag{10}$$

where

$$L_{\hat{g}} := -\Delta_{\hat{g}} + \frac{1}{8}R_{\hat{g}}$$

is the conformal Laplacian associated to $\hat{g}$. We also set

$$\mathcal{F}_\infty := \{A \in Crit(\mathcal{F}_K) :\ \mathcal{L}_K(A) < 0\}, \tag{11}$$

$$i_\infty(A) := 4k - 1 - Morse(A, \mathcal{F}_K), \tag{12}$$

and define

$$m_i^k := \frac{1}{k!} card\{A \in Crit(\mathcal{F}_K) :\ i_\infty(A) = i\},\ i = 0,\ldots,4k-1, \tag{13}$$

where $Morse(\mathcal{F}_K, A)$ denotes the Morse index of $\mathcal{F}_K$ at A. We point out that for $k \geq 2$, $m_i^k = 0$ for $0 \leq i \leq k-2$.

For $k \geq 2$, we use the notation $B_{k-1}(\partial M)$ to denote the set of formal barycenters of order $k-1$ of ∂M, namely

$$B_{k-1}(\partial M) := \left\{\sum_{i=1}^{k-1}\alpha_i\delta_{a_i},\ a_i \in \partial M,\ \alpha_i \geq 0, i = 1,\ldots,k-1,\ \sum_{i=1}^{k-1}\alpha_i = k\right\}. \tag{14}$$

Furthermore, we define

$$c_p^{k-1} = dim\, H_p(B_{k-1}(\partial M)),\ p = 1,\ldots 4k-5, \tag{15}$$

where $H_p(B_{k-1}(\partial M)$ denotes the p-th homology group of $B_{k-1}(\partial M)$ with $\mathbb{Z}_2$ coefficients. Finally, we say

$$(ND) \text{ holds if } \mathcal{F}_K \text{ is a Morse function and for every } A \in Crit(\mathcal{F}_K),\ \mathcal{L}_K(A) \neq 0. \tag{16}$$

Now, we are ready to state our existence results of Morse theoretical type starting with the *critical* case, namely when $k = 1$.

Theorem 1.1. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4, P^3)} = 4\pi^2$. Assuming that K is a smooth positive function on ∂M such that (ND) holds and the system*

$$\begin{cases} m_0^1 = 1 + x_0, \\ m_i^1 = x_i + x_{i-1}, \quad i = 1, \ldots, 3, \\ 0 = x_3 \\ x_i \geq 0, \qquad\qquad i = 0, \ldots, 3 \end{cases} \tag{17}$$

has no solutions, then K is the T-curvature of a Riemannian metric on $\overline{M}$ conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.

The system (17) not having a solution traduces the violation of a strong Morse type inequalities (*SMTI*) for the critical points at infinity of $\mathcal{E}_g$. Since (*SMTI*) imply Poincare–Hopf type formulas, then we have Theorem 1.1 implies the following Poincare–Hopf index type result.

Corollary 1.2. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4, P^3)} = 4\pi^2$. Assuming that K is a smooth positive function on ∂M such that (ND) holds and*

$$\sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A)} \neq 1, \tag{18}$$

then K is the T-curvature of a Riemannian metric on $\overline{M}$ conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.

The formula (18) says that the Euler characteristic number of the space of variations is different from the total contribution of the true critical points at infinity and is of global character. Localizing the arguments of Corollary 1.2 in the case of the presence of a jump in the Morse index of the critical points of the Hamiltonian function $\mathcal{F}_K$, we have the following extension of Corollary 1.2.

Theorem 1.3. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4, P^3)} = 4\pi^2$ and K be a smooth positive function on ∂M satisfying the non-degeneracy condition (ND). Assuming that there exists a positive integer $1 \leq l \leq 3$ such that*

$$\sum_{A \in \mathcal{F}_\infty,\, i_\infty(A) \leq l-1} (-1)^{i_\infty(A)} \neq 1$$

and

$$\forall A \in \mathcal{F}_\infty, \qquad i_\infty(A) \neq l,$$

then K is the T-curvature of a Riemannian metric on $\overline{M}$ conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.

In the supercritical case, i.e $k \geq 2$, the Euler-Lagrange functional $\mathcal{E}_g$ is not bounded from below, and taking into account the topological contribution of very large negative sublevels of $\mathcal{E}_g$, we have the following analogue of Theorem 1.1.

Theorem 1.4. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$, and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 2$. Assuming that K is a smooth positive function on ∂M such that (ND) holds and the following system*

$$\begin{cases} 0 = x_0, \\ m_1^k = x_1, \\ m_i^k = c_{i-1}^{k-1} + x_i + x_{i-1}, & i = 2, \ldots, 4k-4, \\ m_i^k = x_i + x_{i-1}, & i = 4k-3, \ldots, 4k-1, \\ 0 = x_{4k-1}, \\ x_i \geq 0, & i = 0, \ldots, 4k-1, \end{cases} \tag{19}$$

has no solutions, then K is the T-curvature of a Riemannian metric on $\overline{M}$ conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.

Remark 1.5. The presence of the number $c_{i-1}^{k-1} = dim H_{i-1}(B_{i-1}(\partial M))$ in (19) account for the contribution of the topology of very negative sublevels of $\mathcal{E}_g$. The relation between the topology of very negative sublevels of the Euler-Lagrange functional of Liouville type problems and the space of formal barycenters was first observed by Djadli–Malchiodi [11].

As in the critical case, we have that Theorem 1.4 implies the following Poincaré–Hopf index type criterion for existence.

Corollary 1.6. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$, and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 2$. Assuming that K is a smooth positive function on ∂M such that (ND) holds and*

$$\frac{1}{k!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A)} \neq \frac{1}{(k-1)!} \Pi_{i=1}^{k-1} (i - \chi(\partial M)), \tag{20}$$

then K is the T-curvature of a Riemannian metric on $\overline{M}$ conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.

As in the critical case, we have that a localization of the arguments of Corollary 1.6 implies the following jumping index type result.

Theorem 1.7. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$, $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 2$, and let K be a smooth positive function on ∂M*

satisfying the non degeneracy condition (ND). Assuming that there exists a positive integer $1 \leq l \leq 4k-1$ *and* $A^l \in \mathcal{F}_\infty$ *with* $i_\infty(A^l) \leq l-1$ *such that*

$$\frac{1}{k!} \sum_{A \in \mathcal{F}_\infty,\, i_\infty(A) \leq l-1} (-1)^{i_\infty(A)} \neq \frac{1}{(k-1)!} \Pi_{j=1}^{k-1}(j - \chi(\partial M))$$

and

$$\forall A \in \mathcal{F}_\infty, \quad i_\infty(A) \neq l,$$

then K *is the* T*-curvature of a Riemannian metric on* $\overline{M}$ *conformally related to* g *with zero* Q*-curvature in* M *and zero mean curvature on* ∂M*.*

Remark 1.8. As already observed in [2], here also and for the same reasons, $\bar{k}$ plays no role in the above results.

The Morse theoretical results stated above depend only the Morse Lemma at infinity around true critical points at infinity (see Lemma 3.24) which justify the condition $\mathcal{L}_K < 0$ in the definition of $\mathcal{F}_\infty$. However, our existence result of algebraic topological type are based on the Morse lemma at infinity around all critical points at infinity. Thus, to state our existence result of algebraic topological type, we need first to introduce the neighborhood of potential critical points at infinity of $\mathcal{E}_g$. In order to do that, we first fix ν to be a positive and small real number, Λ to be a large positive constant, and R to be a large positive constant too. Next, for ϵ small and positive, and $\Theta \geq 0$, we denote by $V(k, \epsilon, \Theta)$ the (k, ϵ, Θ)-*neighborhood of potential critical points at infinity*, namely

$$\begin{aligned} V(k,\epsilon,\Theta) := \{&u \in \mathcal{H}_{\frac{\partial}{\partial n}} : \exists\, a_1, \ldots, a_k \in \partial M, \alpha_1, \ldots, \alpha_k > 0,\ \lambda_1, \ldots, \lambda_k > 0, \\ &\beta_1, \ldots, \beta_{\bar{k}} \in \mathbb{R}, \\ &\|u - \overline{u}_{(Q,T)} - \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} - \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)})\|_{\mathbb{P}^{4,3}} < \epsilon,\ \sum_{i=1}^{k} \alpha_i = k\ \ \alpha_i \geq 1-\nu, \\ &\lambda_i \geq \frac{1}{\epsilon}, i = 1, \ldots, k,\ \frac{2}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \frac{\Lambda}{2}, i,j = 1, \ldots, k,\ |\beta_r| \leq \Theta,\ r = 1, \ldots, \bar{k}, \\ &\text{and } \lambda_i d_{\hat{g}}(a_i, a_j) \geq 4\overline{C}R \text{ for } i \neq j\}, \end{aligned} \tag{21}$$

where $\overline{C}$ is as in (41), the φ_{a_i,λ_i}'s are as in (45), $\bar{k}$ is as in (31), the v_r's are defined as in (32), the $\overline{(v_r)}_{(Q,t)}$'s are as in (29), and $\|\cdot\|_{\mathbb{P}^{4,3}}$ is defined as in (35).

As observed by Chen–Lin [10] for Liouville type problems, the minimization at infinity of Bahri–Coron [3] for Yamabe type problems has the following analogue for our problem. For $\Theta \geq 0$, there exists $\epsilon_0 = \epsilon_0(\Theta)$ small and positive such that $\forall\, 0 < \epsilon \leq \epsilon_0$, we have

$$\begin{aligned} \forall u \in V(k,\epsilon,\Theta), \text{the minimization problem } \min_{B_\epsilon^\Theta} \Bigg\| &u - \overline{u}_{(Q,T)} - \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} \\ &- \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \Bigg\|_{\mathbb{P}^{4,3}} \end{aligned} \tag{22}$$

has a unique solution, up to permutations, where B_ϵ^Θ is defined as follows

$$B_\epsilon^\Theta := \Big\{(\bar\alpha, A, \bar\lambda, \bar\beta) \in \mathbb{R}_+^k \times (\partial M)^k \times \mathbb{R}_+^k \times \mathbb{R}^{\bar k} : \sum_{i=1}^k \alpha_i = k,\, \alpha_i \geq 1-\nu,\, \lambda_i \geq \frac{1}{\epsilon},\\ i = 1, \dots, k,\\ |\beta_r| \leq \Theta, r = 1, \dots, \bar k,\, \lambda_i d_{\hat g}(a_i, a_j) \geq 4\overline{C}R, \quad i \neq j, \quad i, j = 1, \dots, k\Big\}. \tag{23}$$

The selection map s_k is defined by $s_k : V(k,\epsilon,\Theta) \longrightarrow (\partial M)^k/\sigma_k$ as follows

$$s_k(u) := A,\ u \in V(k,\epsilon,\Theta), \quad \text{and} \quad A \text{ is given by (82)}. \tag{24}$$

We denote the critical points at infinity of $\mathcal{E}_g$ by z^∞ and use the notation $M_\infty(z^\infty)$ for their Morse indices at infinity, $W_u(z^\infty)$ for their unstable manifolds and $W_s(z^\infty)$ for their stable manifolds, where z is the corresponding critical point of $\mathcal{F}_K$. Furthermore, we denote by x^∞ the "true" ones, namely $\mathcal{L}_K(x) < 0$ and the y^∞ the "false" ones, namely $\mathcal{L}_K(y) > 0$. Moreover, we define S to be the following invariant set

$$S := \cup_{M_\infty(z_1^\infty),\, M_\infty(z_2^\infty) \geq 4k-4+\bar k} W_u(z_1^\infty) \cap W_s(z_2^\infty). \tag{25}$$

We also define S^∞ to be the part of S at infinity, namely

$$S^\infty := \cup_{M_\infty(z_1^\infty),\, M_\infty(z_2^\infty) \geq 4k-4+\bar k} W_u^\infty(z_1^\infty) \cap W_s(z_2^\infty), \tag{26}$$

where $W_u^\infty(z_1^\infty)$ denotes the restriction of $W_u(z_1^\infty)$ at infinity. Furthermore, we denote by S_-^∞ the exit set from S^∞ starting from a false critical point at infinity y^∞.

Similarly, we denote by S_+^∞ the entry set to S^∞ after having exited S^∞ through a set contained in S_-^∞ and entering into S^∞ through a true critical point at infinity x^∞.

Finally, to state our result of algebraic topological favor in the spirit of Bahri–Coron [3] (as in [23]), we first recall the existence of

$$0 \neq O_{\partial M}^* \in H^3(\partial M). \tag{27}$$

Using (27), we prove

Theorem 1.9. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $H_g = 0$, $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$, and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 2$. Assuming that K is a smooth positive function on ∂M such that (ND) holds and either there is no x^∞ with $M_\infty(x^\infty) = 4k-4+\bar k$ or $s_k^*(O_{\partial M}^*) \neq 0$ in $H^3(S^\infty)$ and $s_k^*(O_M^*) = 0$ in $H^3(S_+^\infty \cup S_-^\infty)$, then K is the T-curvature of a Riemannian metric conformally related to g with zero Q-curvature in M and zero mean curvature on ∂M.*

As in [23], Theorem 1.9 implies the following collorary.

Corollary 1.10. *Let $(\overline{M}, g)$ be a compact 4-dimensional Riemannian manifold with boundary ∂M and interior M such that $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 2$. Assuming that K is a smooth positive function on ∂M*

such that (ND) *holds and that every critical point* x *of* $\mathcal{F}_K$ *of Morse index* 0 *or* 1 *satisfies* $\mathcal{L}_K(x) < 0$*, then* K *is the* T*-curvature of a Riemannian metric conformally related to* g *with zero* Q*-curvature in* M *and zero mean curvature* ∂M*.*

Remark 1.11. As in [23], here also and for the same reasons, the assumption on the Morse indices of Corollary 1.10 imply that the topological assumption of Theorem 1.9 holds.

We describe briefly our strategy to prove Theorem 1.1–Corollary 1.10. The arguments of the proof of Theorem 1.1–1.7 follow the one of [2], while the method of proof of Theorem 1.9 and Corollary 1.10 is the one of [23]. However, the arguments of Ahmedou and Ndiaye [2] and Ndiaye [23] depend on a Harnack type inequality around the standard bubble of the variational problem studied in [2] due to Weinstein–Zhang [27] and its analogue for the problem under study is not known. Such an issue was present in our work [24] and was dealt using the integral blow-up method in our work [19] combined with the argument of Weinstein–Zhang [27]. Here, we use the integral argument in our work [24] to derive the appropriate Harnack type inequality on the boundary. This is possible because of the integral representation (48). One of the main difficulties here is the lack of an explicit formula for the standard bubble of this variational problem. We bypass this issue by using the fact that the nonlinearity is only at the boundary and that an explicit formula for the restriction of the standard bubble of this variational problem on the boundary is known.

The structure of the paper is as follows. In Sect. 2, we collect some notation and preliminary results, like a suitable Green's function G of the $P^4_g(\cdot) + \frac{2}{k}Q_g$ to $P^3_g(\cdot) + \frac{1}{k}T_g$ operator on $\mathcal{H}_{\frac{\partial}{\partial n}}$ and the definition of a family of variational bubbles. In Sect. 3, we carry the blow-up analysis of sequence of vanishing viscosity solutions to (57) and characterize the critical points at infinity of the problem under study. We divide Sect. 3 in 4 subsections. In Sect. 3.1, we recall a local description of blowing-up sequence of solutions of (57), establish a global description of blowing-up sequence of solutions to (57) and use the latter to provide a refined analogue of the deformation lemma of Lucia [16]. In Sect. 3.2, we derive energy and gradient estimates for $\mathcal{E}_g$ at infinity for those u for which their w-part given by (84) is 0. In Sect. 3.3, we perform a finite-dimensional Lyapunow-Schmidt type reduction by using the stablity properties of the standard bubble to show that variationally the w-part in (84) has no contribution. Finally in Sect. 3.4, we use the energy and gradient estimates of Sect. 3.2 and the finite-dimensional reduction in Sect. 3.3 to construct a pseudo-gradient at infinity for $\mathcal{E}_g$ and identify the critical points at infinity of $\mathcal{E}_g$. Combining this with the energy estimates in Sect. 3.2 and the finite-dimensional reduction in Sect. 3.3, we derive a Morse type lemma around the critical points at infinity. In Sect. 4, we present the proof of the existence theorems. We divide Sect. 4 into 3 subsections. In Sect. 4.1, we characterize the topology of very high and very negative sublevels of $\mathcal{E}_g$. In Sect. 4.2, we present the existence results of Morse theoretical type, namely Theorem 1.1-Theorem 1.7. Section 4.3 deals with the proof of the results of algebraic topological type, i.e Theorem 1.9 and Corollary 1.10. Finally, in Sect. 5, we collect some technical lemmas.

2. Notation and preliminaries

In this brief section, we fix our notation, and give some preliminaries. First of all, we recall that $(\overline{M}, g)$ and K are respectively the given underlying compact 4-dimensional Riemannian manifold with boundary ∂M and the prescribed T-curvature function with the following properties:

$$\begin{aligned} &\ker \mathbb{P}^{4,3}_g \simeq \mathbb{R} \quad \text{ and } \quad \kappa_{(P^4,P^3)} = 4k\pi^2 \text{ for some } \; k \in \mathbb{N}^*, \\ &\text{and} \quad K \text{ is a smooth positive function on } \partial M. \end{aligned} \tag{28}$$

The induced metric on ∂M by g will be denoted by $\hat{g} =: g|_{\partial M}$.
In the following, for a Riemmanian metric $\bar{g}$ on ∂M and $p \in \partial M$, we will use the notation $B^{\bar{g}}_p(r)$ to denote the geodesic ball with respect to $\bar{g}$ of radius r and center p. We also denote by $d_{\bar{g}}(x, y)$ the geodesic distance with respect to $\bar{g}$ between two points x and y of ∂M, $exp^{\bar{g}}_x$ the exponential map with respect to $\bar{g}$ at $x \in \partial M$. $inj_{\bar{g}}(\partial M)$ stands for the injectivity radius of $(\partial M, \bar{g})$, $dV_{\bar{g}}$ denotes the Riemannian measure associated to the metric $\bar{g}$. Furthermore, we recall that $\nabla_{\bar{g}}$, $\Delta_{\bar{g}}$, $R_{\bar{g}}$ will denote respectively the covariant derivative, the Laplace-Beltrami operator, and the scalar curvature with respect to $\bar{g}$. For simplicity, we will use the notation $B_p(r)$ to denote $B^{\hat{g}}_p(r)$, namely $B_p(r) = B^{\hat{g}}_p(r)$. $(\partial M)^2$ stands for the cartesian product $\partial M \times \partial M$, while $Diag(\partial M)$ is the diagonal of $(\partial M)^2$.

Similarly, for a Riemmanian metric $\tilde{g}$ on $\overline{M}$, we will use the notation $B^{\tilde{g},+}_p(r)$ to denote the half geodesic ball with respect to $\tilde{g}$ of radius r and center $p \in \partial M$. We also denote by $d_{\tilde{g}}(x, y)$ the geodesic distance with respect to $\tilde{g}$ between two points x and y of $\overline{M}$, $exp^{\tilde{g}}_x$ the exponential map with respect to $\tilde{g}$ at $x \in \partial M$. $inj_{\tilde{g}}(\overline{M})$ stands for the injectivity radius of $(\overline{M}, g)$, $dV_{\tilde{g}}$ denotes the Riemannian measure associated to the metric $\tilde{g}$, and $dS_{\tilde{g}}$ the Riemannian measure associated to $\hat{\tilde{g}} := \tilde{g}|_{\partial M}$, namely $dS_{\tilde{g}} = dV_{\hat{\tilde{g}}}$. Furthermore, we recall that $\nabla_{\tilde{g}}$, $\Delta_{\tilde{g}}$, $R_{\tilde{g}}$ will denote respectively the covariant derivative, the Laplace–Beltrami operator, and the scalar curvature with respect to $\tilde{g}$. For simplicity, we will use the notation $B^+_p(r)$ to denote $B^{g,+}_p(r)$, namely $B^+_p(r) = B^{g,+}_p(r)$, $p \in \partial M$

For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $\theta \in]0, 1[$, $L^p(M)$, $W^{m,p}(M)$, $C^m(\overline{M})$, and $C^{m,\theta}(\overline{M})$ stand respectively for the standard Lebesgue space, Sobolev space, m-continuously differentiable space and m-continuously differential space of Hölder exponent θ, all with respect g. Similarly, $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $\theta \in]0, 1[$, $L^p(\partial M)$, $W^{m,p}(\partial M)$, $C^m(\partial M)$, and $C^{m,\theta}(\partial M)$ stand respectively for the standard Lebesgue space, Sobolev space, m-continuously differentiable space and m-continuously differential space of Hölder exponent θ, all with respect $\hat{g}$.

Given a function $u \in L^1(M) \cap L^1(\partial M)$, we define $\bar{u}_{\partial M}$ and $\overline{u}_{(Q,T)}$ by

$$\bar{u}_{\partial M} = \frac{\oint_{\partial M} u(x) dS_g}{Vol_g(\partial M)},$$

with

$$Vol_g(\partial M) = \oint_{\partial M} dS_g,$$

and

$$\overline{u}_{(Q,T)} = \frac{1}{4k\pi^2} \left(\int_M Q_g u dV_g + \oint_{\partial M} T_g u dS_g \right). \tag{29}$$

For $\epsilon > 0$ and small, $\lambda \in \mathbb{R}_+$, $\lambda \geq \frac{1}{\epsilon}$, and $a \in \partial M$, $O_{\lambda,\epsilon}(1)$ stands for quantities bounded uniformly in λ, and ϵ, and $O_{a,\epsilon}(1)$ stands for quantities bounded uniformly in a and ϵ. For $l \in \mathbb{N}^*$, $O_l(1)$ stands for quantities bounded uniformly in l and $o_l(1)$ stands for quantities which tends to 0 as $l \to +\infty$. For ϵ positive and small, $a \in \partial M$ and $\lambda \in \mathbb{R}_+$ large, $\lambda \geq \frac{1}{\epsilon}, O_{a,\lambda,\epsilon}(1)$ stands for quantities bounded uniformly in a, λ, and ϵ. For ϵ positive and small, $p \in \mathbb{N}^*$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{\epsilon}$ for $i = 1, \ldots, p$, and $A := (a_1, \ldots, a_p) \in (\partial M)^p$ (where $(\mathbb{R}_+)^p$ and $(\partial M)^p$ denotes respectively the cartesian product of p copies of $\mathbb{R}_+$ and ∂M), $O_{A,\bar{\lambda},\epsilon}(1)$ stands for quantities bounded uniformly in A, $\bar{\lambda}$, and ϵ. Similarly for ϵ positive and small, $p \in \mathbb{N}^*$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{\epsilon}$ for $i = 1, \ldots, p$, $\bar{\alpha} := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$, α_i close to 1 for $i = 1, \ldots, p$, and $A := (a_1, \ldots, a_p) \in (\partial M)^p$ (where $\mathbb{R}^p$ denotes the cartesian product of p copies of $\mathbb{R}$, $O_{\bar{\alpha},A,\bar{\lambda},\epsilon}(1)$ will mean quantities bounded from above and below independent of $\bar{\alpha}$, A, $\bar{\lambda}$, and ϵ. For $x \in \mathbb{R}$, we will use the notation $O(x)$ to mean $|x|O(1)$ where $O(1)$ will be specified in all the contexts where it is used. Large positive constants are usually denoted by C and the value of C is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are also denoted by c and their value may varies from formula to formula and also within the same line.

We say $\mu \in \mathbb{R}$ is an eigenvalue of the P^4_g to P^3_g operator on $\mathcal{H}_{\frac{\partial}{\partial n}}$ if there exists $0 \neq v \in W^{2,2}(M)$ such that

$$\begin{cases} P^4_g v = 0 & \text{in } M, \\ P^3_g v = \mu v & \text{on } \partial M, \\ \dfrac{\partial v}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{30}$$

By abuse of notation, we call v in (30) an eigenfunction associated to μ. We call $\bar{k}$ the number of negative eigenvalues (counted with multiplicity) of the P^4_g to P^3_g operator on $\mathcal{H}_{\frac{\partial}{\partial n}}$. We point out that $\bar{k}$ can be zero, but it is always finite. If $\bar{k} \geq 1$, then we will denote by $E_- \subset \mathcal{H}_{\frac{\partial}{\partial n}}$ the direct sum of the eigenspaces corresponding to the negative eigenvalues of the P^4_g to P^3_g operator on $\mathcal{H}_{\frac{\partial}{\partial n}}$. The dimension of E_- is of course $\bar{k}$, i.e

$$\bar{k} = \dim E_-. \tag{31}$$

On the other hand, we have the existence of a basis of eigenfunctions $v_1, \ldots, v_{\bar{k}}$ of E_- satisfying

$$\begin{cases} P_g^4 v_r = 0 & \text{in } M, \\ P_g^3 v_r = \mu_r v_r & \text{on } \partial M, \\ \dfrac{\partial v_r}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{32}$$

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{\bar{k}} < 0 < \mu_{\bar{k}+1} \leq \cdots, \tag{33}$$

where μ_r's are the eigenvalues of the operator P_g^4 to P_g^3 on $\mathcal{H}_{\frac{\partial}{\partial n}}$ counted with multiplicity. We define $\mathbb{P}^{4,3}_{g,+}$ by

$$\mathbb{P}^{4,3}_{g,+}(u,v) = \mathbb{P}^{4,3}_g(u,v) - 2\sum_{r=1}^{\bar{k}} \mu_r \left(\oint_{\partial M} u v_r dS_g\right)\left(\oint_{\partial M} v v_r dS_g\right). \tag{34}$$

$\mathbb{P}^{4,3}_{g,+}$ is obtained by just reversing the sign of the negative eigenvalue of $\mathbb{P}^{4,3}_g$. We set also

$$\|u\|_{\mathbb{P}^{4,3}} := \sqrt{\mathbb{P}^{4,3}_{g,+}(u,u)}, \quad \text{and} \quad \langle u, v\rangle_{\mathbb{P}^{4,3}} = P^{4,3}_{g,+}(u,v), \tag{35}$$

where $\mathbb{P}^{4,3}_{g,+}$ is defined as in (34). We have $\langle\cdot,\cdot\rangle_{\mathbb{P}^{4,3}}$ is a scalar product on $\{u \in \mathcal{H}_{\frac{\partial}{\partial n}} : \ u_{(Q,T)} = 0\}$. We can choose $v_1, \ldots, v_{\bar{k}}$ so that they constitute a $\langle\cdot,\cdot\rangle_{\mathbb{P}^{4,3}}$-orthonormal basis for E_-. We denote by $\nabla^{\mathbb{P}^{4,3}}$ the gradient with respect to $\langle\cdot,\cdot\rangle_{\mathbb{P}^{4,3}}$.

For $t > 0$, we define the following perturbed functional

$$\begin{aligned}(\mathcal{E}_g)_t(u) :=& \ \mathbb{P}^{4,3}(u,u) + 4t\int_M Q_g u dV_g + 4t\oint_{\partial M} T_g u dS_g \\ & -\frac{4}{3}t\kappa_{(P^4,P^3)} \log \oint_M K e^{3u} dS_g, \\ & u \in \mathcal{H}_{\frac{\partial}{\partial n}}.\end{aligned} \tag{36}$$

$\bar{B}^{\bar{k}}_r$ will stand for the closed ball of center 0 and radius r $\underbrace{\text{in }\mathbb{R}}^{\bar{k}}$. $\mathbb{S}^{\bar{k}-1}$ will denote the boundary of $\bar{B}^{\bar{k}}_1$. Given a set X, we define $X \times \bar{B}^{\bar{k}}_1$ to be the cartesian product $X \times \bar{B}^{\bar{k}}_1$ where the tilde means that $X \times \partial B^{\bar{k}}_1$ is identified with $\partial B^{\bar{k}}_1$.

In the sequel also, $(\mathcal{E}_g)^c$ with $c \in \mathbb{R}$ will stand for $(\mathcal{E}_g)^c := \{u \in \mathcal{H}_{\frac{\partial}{\partial n}} : \mathcal{E}_g(u) \leq c\}$. For X a topological space, $H_*(X)$ will denote the singular homology of X, $H^*(X)$ for the cohomology, and $\chi(X)$ the Euler characteristic of X, all with $\mathbb{Z}_2$ coefficients.

As above, in the general case, namely $\bar{k} \geq 0$, for ϵ small and positive, $\bar{\beta} := (\beta_1, \ldots, \beta_{\bar{k}}) \in \mathbb{R}^{\bar{k}}$ with β_i close to 0, $i = 1, \ldots, \bar{k}$) (where $\mathbb{R}^{\bar{k}}$ is the empty set when $\bar{k} = 0$), $\bar{\lambda} := (\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{\epsilon}$ for $i = 1, \ldots, p$, $\bar{\alpha} := (\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$, α_i close to 1 for $i = 1, \ldots, p$, and $A := (a_1, \ldots, a_p) \in (\partial M)^p$, $p \in \mathbb{N}^*$, $w \in \mathcal{H}_{\frac{\partial}{\partial n}}$ with $\|w\|_{\mathbb{P}^{4,3}}$ small, $O_{\bar{\alpha},A,\bar{\lambda},\bar{\beta},\epsilon}(1)$ will stand quantities

bounded independent of $\bar{\alpha}$, A, $\bar{\lambda}$, $\bar{\beta}$, and ϵ, and $O_{\bar{\alpha},A,\bar{\lambda},\bar{\beta},w,\epsilon}(1)$ will stand quantities bounded independent of $\bar{\alpha}$, A, $\bar{\lambda}$, $\bar{\beta}$, w and ϵ.
For point $b \in \mathbb{R}^3$ and λ a positive real number, we define $\delta_{b,\lambda}$ by

$$\delta_{b,\lambda}(y) := \log\left(\frac{2\lambda}{1+\lambda^2|y-b|^2}\right), \quad y \in \mathbb{R}^3. \tag{37}$$

The functions $\delta_{b,\lambda}$ verify the following equation

$$(-\Delta_{\mathbb{R}^3})^{\frac{3}{2}}\delta_{b,\lambda} = 2e^{3\delta_{b,\lambda}} \text{ in } \mathbb{R}^3. \tag{38}$$

Using the existence of conformal Fermi coordinates, we have that, for $a \in \partial M$ there exists a function $u_a \in C^\infty(\overline{M})$ such that

$$g_a = e^{2u_a}g \text{ verifies } detg_a(x) = 1 + O(d_{g_a}(x,a)^m) \text{ for } x \in B_a^{g_a,+}(\varrho_a). \tag{39}$$

with $0 < \varrho_a < \min\{\frac{inj_{g_a}(\overline{M})}{10}, \frac{inj_{\hat{g}_a}(\partial M)}{10}\}$. Moreover, we can take the families of functions u_a, g_a and ϱ_a such that

$$\text{the maps } a \longrightarrow u_a,\, g_a \text{ are } C^1 \text{ and } \varrho_a \geq \varrho_0 > 0, \tag{40}$$

for some small positive ϱ_0 satisfying $\varrho_0 < \min\{\frac{inj_g(M)}{10}, \frac{inj_{\hat{g}}(\partial M)}{10}\}$, and

$$\begin{aligned}
&\|u_a\|_{C^4(\overline{M})} = O_a(1), \quad \frac{1}{\overline{C}^2}g \leq g_a \leq \overline{C}^2 g,\\
&u_a(x) = O_a(d^2_{\hat{g}_a}(a,x)) = O_a(d^2_{\hat{g}}(a,x)) \quad \text{for} \quad x \in \ B_a^{\hat{g}_a}(\varrho_0) \supset B_a\left(\frac{\varrho_0}{2\overline{C}}\right), \text{ and}\\
&u_a(a) = 0, \quad R_{\hat{g}_a}(a) = 0, \quad \frac{\partial u_a}{\partial n_g}(a) = 0,
\end{aligned} \tag{41}$$

for some large positive constant $\overline{C}$ independent of a. For $a \in \partial M$, and $r > 0$, we set

$$exp_a^a := exp_a^{\hat{g}_a} \text{ and } B_a^a(r) := B_a^{\hat{g}_a}(r). \tag{42}$$

Now, for $0 < \varrho < \frac{\varrho_0}{4}$ where ϱ_0 is as in (40), we define a smooth cut-off function satisfying the following properties:

$$\begin{cases}\chi_\varrho(t) = t & \text{for } t \in [0,\varrho],\\ \chi_\varrho(t) = 2\varrho & \text{for } t \geq 2\varrho,\\ \chi_\varrho(t) \in [\varrho, 2\varrho] & \text{for } t \in [\varrho, 2\varrho].\end{cases} \tag{43}$$

Using the cut-off function χ_ϱ, we define for $a \in \partial M$ and $\lambda \in \mathbb{R}_+$ the function $\hat{\delta}_{a,\lambda}$ as follows

$$\hat{\delta}_{a,\lambda}(x) := \log\left(\frac{2\lambda}{1+\lambda^2\chi^2_\varrho(d_{\hat{g}_a}(x,a))}\right). \tag{44}$$

For every $a \in \partial M$ and $\lambda \in \mathbb{R}_+$, we define $\varphi_{a,\lambda}$ to be the unique solution of

$$\begin{cases} P_g^4 \varphi_{a,\lambda} + \frac{2}{k} Q_g = 0 \text{ in } M, \\ P_g^3 \varphi_{a,\lambda} + \frac{1}{k} T_g = 4\pi^2 \frac{e^{3(\hat{\delta}_{a,\lambda} + u_a)}}{\oint_M e^{3(\hat{\delta}_{a,\lambda} + u_a)} dS_g} \text{ in } \partial M, \\ \frac{\partial \varphi_{a,\lambda}}{\partial n_g} = 0, \\ \overline{\phi_{a,\lambda}}_{(Q,T)} = 0. \end{cases} \tag{45}$$

Next, let $S(a,x)$, $(a,x) \in \partial M \times \overline{M}$ be defined by

$$\begin{cases} P_g^4 S(a,\cdot) + \frac{2}{k} Q_g(\cdot) = 0 \text{ in } M, \\ P_g^3 S(a,\cdot) + \frac{1}{k} T_g(\cdot) = 4\pi^2 \delta_a(\cdot), \text{ on } \partial M, \\ \frac{\partial S(a,\cdot)}{\partial n_g} = 0 \text{ on } \partial M, \\ \int_M S(a,x) Q_g(x) dV_g(x) = 0. \end{cases} \tag{46}$$

Then

$$G(a,\cdot) = S(a,\cdot)|_{\partial M}. \tag{47}$$

is a Green's function of the $P_g^4 + \frac{2}{k} Q_g(\cdot)$ to $P_g^3 + \frac{1}{k} T_g(\cdot)$ operator on $\mathcal{H}_{\frac{\partial}{\partial n}}$. Thus, we have the integral representation: $\forall u \in \mathcal{H}_{\frac{\partial}{\partial n}}$ such that $P_g^4 u + \frac{2}{k} Q_g = 0$,

$$u(x) - \overline{u}_{(Q,T)} = \frac{1}{4\pi^2} \oint_{\partial M} G(x,y) P_g^3 u(y), \quad x \in \partial M. \tag{48}$$

Moreover, G decomposes as follows (see [21])

$$G(a,x) = \log\left(\frac{1}{\chi_\varrho^2(d_{\hat{g}_a}(a,x))}\right) + H(a,x), \tag{49}$$

where H is the regular part of G. Furthermore, we have

$$G \in C^\infty((\partial M)^2 - Diag(\partial M)), \quad \text{and} \quad H \in C^{3,\beta}((\partial M)^2) \ \forall \beta \in (0,1). \tag{50}$$

By symmetry of H, we have

$$\frac{\partial \mathcal{F}(a_1,\ldots,a_k)}{\partial a_i} = \frac{2}{3} \frac{\nabla_{\hat{g}} \mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)}, \quad i = 1,\ldots,k. \tag{51}$$

Next, setting

$$l_K(A) := \sum_{i=1}^k \left(\frac{\Delta_{\hat{g}} \mathcal{F}_i^A(a_i)}{(\mathcal{F}_i^A(a_i))^{\frac{1}{3}}} - \frac{3}{4} R_{\hat{g}}(a_i) (\mathcal{F}_i^A(a_i))^{\frac{2}{3}} \right), \tag{52}$$

we have

$$l_K(A) = 6 \mathcal{L}_K(A), \quad \forall A \in Crit(\mathcal{F}_K). \tag{53}$$

For $k \geq 2$, we denote by $B_k(\partial M)$ the set of formal barycenters of ∂M of order k, namely

$$B_k(\partial M) := \left\{ \sum_{i=1}^k \alpha_i \delta_{a_i}, a_i \in \partial M, \alpha_i \geq 0, i = 1,\ldots,k, \ \sum_{i=1}^k \alpha_i = k \right\}. \tag{54}$$

Finally, we set

$$A_{k,\bar{k}} := B_k(\widetilde{\partial M}) \times \bar{B}_1^{\bar{k}}, \tag{55}$$

and

$$A_{k-1,\bar{k}} := B_{k-1}(\widetilde{\partial M}) \times \bar{B}_1^{\bar{k}}, \tag{56}$$

with $B_{k-1}(\partial M)$ as in (14).

3. Blow-up analysis and critical points at infinity

This section deals with the blowup analysis of sequences of vanishing viscosity solutions of the type

$$\begin{cases} P_g^4 u_l + 2t_l Q_g = 0 & \text{in } M, \\ P_g^3 u_l + t_l T_g = t_l K e^{3u} & \text{on } \partial M, \\ \dfrac{\partial u_l}{\partial n_g} = 0 & \text{on } \partial M. \end{cases} \tag{57}$$

with $t_l \to 1$ under the assumption $\ker \mathbb{P}_g^{4,3} \simeq \mathbb{R}$ and $\kappa_{(P^4,P^3)} = 4k\pi^2$ with $k \geq 1$ and their use to characterize the critical points at infinity of $\mathcal{E}_g$.

3.1. Blow-up analysis

The local behaviour of blowing up sequences of solutions of (57) is quite well understood. In fact, in [21], we prove the following lemma.

Lemma 3.1. *Assuming that (u_l) is a blowing up sequence of solutions to (57), then up to a subsequence, there exists k converging sequence of points $(x_{i,l})_{l\in\mathbb{N}}$, $x_{i,l} \in \partial M$ with limits $x_i \in \partial M$, $i = 1,\dots,k$, k sequences $(\mu_{i,l})_{l\in\mathbb{N}}$ $i = 1,\dots,k$ of positive real numbers converging to 0 such that the following hold:*
(a)

$$\frac{d_{\hat{g}}(x_{i,l}, x_{j,l})}{\mu_{i,l}} \longrightarrow +\infty \quad i \neq j \;\; i,j = 1,\dots,k \quad \text{and}$$
$$t_l K(x_{i,l}) \mu_{i,l}^3 e^{3u_l(x_{i,l})} e^{-3\log 2} = 2.$$

(b)

$$v_{i,l}(x) = u_l(exp^g_{x_{i,l}}(\mu_{i,l} x)) - u_l(x_{i,l}) + \log 2 \longrightarrow V(x) \;\; \text{in } C^4_{loc}(\mathbb{R}^4_+),$$
$$V|_{\mathbb{R}^3}(x) := \log\left(\frac{2}{1+|x|^2}\right).$$

(c) *There exists* $C > 0$ such that $\inf_{i=1,\dots,k} d_{\hat{g}}(x_{i,l}, x)^3 e^{3u_l(x)} \leq C \;\; \forall x \in \partial M$, $\forall l \in \mathbb{N}$.
(d)

$$t_l K e^{3u_l} dS_g \to 4\pi^2 \sum_{i=1}^k \delta_{x_i} \quad \text{in the sense of measure,} \quad \text{and}$$
$$\lim_{l\to+\infty} \oint_{\partial M} t_l K e^{3u_l} dS_g = 4\pi^2 k.$$

(e)

$$u_l - \overline{(u_l)}_{Q,T} \to \sum_{i=1}^{k} G(x_i, \cdot) \ \ in \ \ C^3_{loc}(\partial M - \{x_1, \dots, x_k\}), \ \ \overline{(u_l)}_{Q,T} \to -\infty.$$

As a Liouville type problem, the following Harnack type inequality is sufficient to get the global description of blowing up sequences of solutions needed to describe the critical points at infinity of $\mathcal{E}_g$.

Proposition 3.2. *Assuming that* u_l *is a blowing up sequence of solutions to* (57)*, then Lemma* 3.1 *holds, and keeping the notations in Lemma* 3.1*, we have that the points* $x_{i,l}$ *are uniformly isolated, namely there exists* $0 < \eta_k < \frac{\varrho_0}{10}$ *[where* ϱ_0 *is as in* (40)*] such that for* l *large enough, there holds*

$$d_{\hat{g}}(x_{i,l}, x_{j,l}) \geq 4\overline{C}\eta_k, \ \ \forall i \neq j = 1, \dots, k. \tag{58}$$

Moreover, the scaling parameters $\lambda_{i,l} := \mu_{i,l}^{-1}$ *are comparable, namely there exists a large positive constant* Λ_0 *such that*

$$\Lambda_0^{-1}\lambda_{j,l} \leq \lambda_{i,l} \leq \Lambda_0\lambda_{j,l}, \quad \forall\, i, j \tag{59}$$

Furthermore, we have that the following estimate around the blow up points holds

$$u_l(y) + \frac{1}{3}\log\frac{t_l K_l(x_{i,l})}{2} = \log\frac{2\lambda_{i,l}}{1 + \lambda_{i,l}^2(d_{\hat{g}_{x_{i,l}}}(y, x_{i,l}))^2} + O(d_{\hat{g}}(y, x_{i,l})), \ \ \forall\, y \in B^{\hat{g}}_{x_{i,l}}(\eta). \tag{60}$$

To prove Proposition 3.2, as it is standard for Liouville type problems, one starts with the uniform isolation of blowing-up points. Indeed, we have

Lemma 3.3. *Assuming that* $(u_l)_{l\in\mathbb{N}}$ *is a bubbling sequence of solutions to BVP* (57)*, then keeping the notations in Lemma* 3.1*, we have that the points* $x_{i,l}$ *are uniformly isolated, namely there exists* $0 < \eta_k < \frac{\varrho_0}{10}$ *[where* ϱ_0 *is as in* (40)*] such that for* l *large enough, there holds*

$$d_{\hat{g}}(x_{i,l}, x_{j,l}) \geq 4\overline{C}\eta_k, \quad \forall i \neq j = 1, \dots, k. \tag{61}$$

Proof. The proof use the integral method of Step 4 in [19] and hence we will be skectchy in many arguments. As in [19], we first fix $\frac{1}{3} < \nu < \frac{2}{3}$, and for $i = 1, \dots, k$, we set

$$\bar{u}_{i,l}(r) = Vol_{\hat{g}}(\partial B_{x_i}(r))^{-1}\int_{\partial B_{x_i}(r)} u_l(x) d\sigma_{\hat{g}}(x), \quad \forall\, 0 \leq r < inj_{\hat{g}}(\partial M),$$

and

$$\psi_{i,l}(r) = r^{4\nu} exp(4\bar{u}_{i,l}(r)), \quad \forall\, 0 \leq r < inj_{\hat{g}}(\partial M).$$

Furthermore, as in [19], we define $r_{i,l}$ as follows

$$r_{i,l} := \sup\left\{R_\nu\mu_{i,l} \leq r \leq \frac{R_{i,l}}{2} \text{ such that } \psi_{i,l}^{'}(r) < 0 \text{ in } [R_\nu\mu_{i,l}, r[\right\}; \tag{62}$$

where $R_{i,l} := \min_{j\neq i} d_{\hat{g}}(x_{i,l}, x_{j,l})$. Thus, by continuity and the definition of $r_{i,l}$, we have that

$$\psi_{i,l}^{'}(r_{i,l}) = 0 \tag{63}$$

Now, as in [19], to prove (61), it suffices to show that $r_{i,l}$ is bounded below by a positive constant in dependent of l. Thus, we assume by contradiction that (up to a subsequence) $r_{i,l} \to 0$ as $l \to +\infty$ and look for a contradiction. In order to do that, we use the integral representation formula (48) and argue as in Step 4 of Ndiaye [19] to derive the following estimate

$$\psi_{i,l}^{'}(r_{i,l}) \leq (r_{i,l})^{3\nu-1} exp(\bar{u}_{i,l}(r_{i,l}))\left(3\nu - 2C + o_l(1) + O_l(r_{i,l})\right).$$

with $C > 1$. So from $\frac{1}{3} < \nu < \frac{2}{3}$, $C > 1$ and $r_{i,l} \longrightarrow 0$ as $l \to +\infty$, we deduce that for l large enough, there holds

$$\psi_{i,l}^{'}(r_{i,l}) < 0. \tag{64}$$

Thus, (63) and (64) lead to a contradiction, thereby concluding the proof of (61). Hence, the proof of the Lemma is complete. □

The next step to derive Proposition 3.2 is to establish its weak $O(1)$ -version.

Lemma 3.4. *Assuming that $(u_l)_{l\in\mathbb{N}}$ is a bubbling sequence of solutions to BVP (57), then keeping the notations in Lemmas 3.1 and 3.3, we have that for l large enough, there holds*

$$u_l(x) + \frac{1}{3}\log\frac{t_l K(x_i)}{2} = \log\frac{2\lambda_{i,l}}{1 + \lambda_{i,l}^2 (d_{\hat{g}_{x_i}}(x, x_i))^2} + O(1), \quad \forall\, x \in B_{x_i}^{x_i}(\eta_k), \tag{65}$$

up to choosing η_k smaller than in Lemma 3.3.

Remark 3.5. We point out that the comparability of the scaling parameters $\lambda_{i,l}$'s follows directly from Lemma 3.4.

Proof. We are going to use the method of Ndiaye [24], hence we will be sketchy in many arguments. Like in [24], thanks to Lemma 3.3, we will focus only on one blow-up point and called it $x \in \partial M$. Thus, we are in the situation where there exists a sequence $x_l \in \partial M$ such that $x_l \to x$ with x_l local maximum point for u_l on ∂M and $u_l(x_l) \to +\infty$. Now, we recall $g_x = e^{2u_x} g$ and choose η_1 such that $20\eta_1 < \min\{\varrho_0, \varrho_k, d\}$ with $4d \leq r_{i,l}$ where $r_{i,l}$ is as in the proof of Lemma 3.3. Next, we let $\hat{w}_x$ be the unique solution of the following boundary value problem

$$\begin{cases} P^4_{g_x}\hat{w}_x = P^4_{\hat{g}}u_x & \text{in } M, \\ P^3_{g_x}\hat{w}_x = P^3_{\hat{g}}u_x & \text{on } \partial M, \\ \dfrac{\partial \hat{w}_x}{\partial n_{g_x}} = 0 & \text{on } \partial M, \\ \overline{\hat{w}}_{(Q,T)} = 0. \end{cases} \tag{66}$$

Using standard elliptic regularity theory and (41), we derive

$$\hat{w}_x(y) = O(d_g(y,x)) \text{ in } B_x^{g_x,+}(2\eta_1). \tag{67}$$

On the other hand, using the conformal covariance properties of the Paneitz operator and of the Chang–Qing one, see (1), we have that $\hat{u}_l := u_l - \hat{w}_x$ satisfies

$$\begin{cases} P^4_{g_x}\hat{u}_l + 2\hat{Q}_l = 0 & \text{in } M, \\ P^3_{g_x}\hat{u}_l + \hat{T}_l = t_l K e^{3\hat{u}_l} & \text{on } \partial M, \\ \dfrac{\partial \hat{u}_l}{\partial n_{g_x}} = 0 & \text{on } \partial M. \end{cases}$$

with

$$\hat{Q}_l = t_l e^{-4\hat{w}} Q_g + \frac{1}{2} P^4_{\hat{g}}\hat{w} \quad \text{and} \quad \hat{T}_l = t_l e^{-3\hat{w}} T_g + P^3_{\hat{g}}\hat{w}.$$

Next, as in [24], we are going to establish the classical sup+inf-estimate for $\hat{u}_l$, since thanks (67) all terms coming from $\hat{w}_x$ can be absorbed on the right hand side of (65). Now, we are going to rescale the functions $\hat{u}_l$ around the points x. In order to do that, we define $\varphi_l : B_0^{\mathbb{R}^3}(2\eta_1\mu_l^{-1}) \longrightarrow B_x^{\hat{g}_x}(2\eta_1)$ by the formula $\varphi_l(z) := \mu_l z$ and μ_l is the corresponding scaling parameter given by Lemma 3.1. Furthermore, as in [24], we define the following rescaling of $\hat{u}_l$

$$v_l := \hat{u}_l \circ \varphi_l + \log \mu_l + \frac{1}{3}\log\frac{t_l K(x)}{2}.$$

Using the Green's representation formula and the method of [24], we get

$$v_l(z) + 2\log|z| = O(1), \quad \text{for } z \in \bar{B}_0^{\mathbb{R}^3}\left(\frac{\eta_1}{\mu_l}\right) - B_0^{\mathbb{R}^3}(-\log\mu_l). \tag{68}$$

Now, we are going to show that the estimate (68) holds also in $\bar{B}_0^{\mathbb{R}^3}(-\log\mu_l)$. To do so, we use Lemma 3.1 and the same arguments as in [24] to deduce

$$v_l(z) + 2\log|z| = O(1), \quad \text{for } z \in \bar{B}_0^{\mathbb{R}^3}(-\log\mu_l). \tag{69}$$

Now, combining (68) and (69), we obtain

$$v_l(z) + 2\log|z| = O(1), \quad \text{for } z \in \bar{B}_0^{\mathbb{R}^3}\left(\frac{\eta_1}{\mu_l}\right). \tag{70}$$

Thus scaling back, namely using $y = \mu_l z$ and the definition of v_l, we obtain the desired $O(1)$-estimate. Hence the proof of the Lemma is complete. □

Proof of formula (60) of Proposition 3.2. We are going to use the method of Ndiaye [24], hence we will be sketchy in many arguments. Now, let V_0 be the unique solution of the following conformally invariant integral equation

$$V_0(z) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \log\frac{|y|}{|z-y|} e^{3V_0(y)}dy + \log 2, \quad z \in \mathbb{R}^3,$$
$$V_0(0) = \log 2, \nabla V_0(0) = 0.$$

Next, we set $w_l(z) = v_l(z) - V_0(z)$ for $z \in B_0^{\mathbb{R}^3}(\eta_1\mu_l^{-1})$, and use Lemma 3.4 to infer that

$$|w_l| \leq C \quad \text{in} \quad B_0^{\mathbb{R}^3}(\eta_1\mu_l^{-1}) \tag{71}$$

On the other hand, it is easy to see that to achieve our goal, it is sufficient to show

$$|w_l| \leq C\mu_l|z| \quad \text{in} \quad B_0^{\mathbb{R}^3}(\eta_1\mu_l^{-1}). \tag{72}$$

To show (72), we first set

$$\Lambda_l := \max_{z\in\Omega_l} \frac{|w_l(z)|}{\mu_l(1+|z|)}$$

with

$$\Omega_l = \overline{B}_0^{\mathbb{R}^3}(\eta_1\mu_l^{-1})$$

We remark that to show (72), it is equivalent to prove that Λ_l is bounded. Now, let us suppose that $\Lambda_l \to +\infty$ as $l \to +\infty$, and look for a contradiction. To do so, we will use the method of [24]. For this, we first choose a sequence of points $z_l \in \Omega_l$ such that $\Lambda_l = \frac{|w_l(z_l)|}{\mu_l(1+|z_l|)}$. Next, up to a subsequence, we have that either $z_l \to z^*$ as $l \to +\infty$ (with $z^* \in \mathbb{R}^3$) or $|z_l| \to +\infty$ as $l \to +\infty$. Now, we make the following definition

$$\bar{w}_l(z) := \frac{w_l(z)}{\Lambda_l\mu_l(1+|z_l|)},$$

and have

$$|\bar{w}_l(z)| \leq \left(\frac{1+|z|}{1+|z_l|}\right), \tag{73}$$

and

$$|\bar{w}_l(z_l)| = 1. \tag{74}$$

Now, we consider the case where the points z_l escape to infinity.

Case 1 : $|z_l| \to +\infty$

In this case, using the integral representation (48) with respect to g_x and the method of [24], we obtain

$$\bar{w}_l(z_l) = \frac{1}{2\pi^2}\int_{\Omega_l} \log\frac{|\xi|}{|z_l-\xi|}\left(\frac{O(1)(1+|\xi|)^{-5}}{(1+|z_l|)} + \frac{O(1)(1+|\xi|)^{-5}}{\Lambda_l(1+|z_l|)}\right)d\xi + o(1).$$

Now, using the fact that $|z_l| \to +\infty$ as $l \to +\infty$, one can easily check that

$$\bar{w}_l(z_l) = \frac{1}{2\pi^2}\int_{\Omega_l} \log\frac{|\xi|}{|z_l-\xi|}\left(\frac{O(1)(1+|\xi|)^{-5}}{(1+|z_l|)} + \frac{O(1)(1+|\xi|)^{-5}}{\Lambda_l(1+|z_l|)}\right)d\xi = o(1).$$

Hence, we reach a contradiction to (74).

Now, we are going to show that, when the points $z_l \to z^*$ as $l \to +\infty$, we reach a contradiction as well.

Case 2: $z_l \to z^*$

In this case, using the assumption $z_l \longrightarrow z^*$, the Green's representation formula, and the method of Ndiaye [24], we obtain that up to a subsequence

$$\bar{w}_l \to w \text{ in } C^1_{loc}(\mathbb{R}^3) \text{ as } l \to +\infty, \tag{75}$$

and

$$\begin{aligned}\bar{w}_l(z) = &\frac{1}{2\pi^2}\int_{\Omega_l} \log\frac{|\xi|}{|z-\xi|}\frac{K\circ\varphi_l(\xi)}{K\circ\varphi_l(0)}e^{3\vartheta_l(\xi)}\bar{w}(\xi)d\xi\\ &+\frac{1}{\Lambda_l\mu_l(1+|z_l|)2\pi^2}\int_{\Omega_l}\log\frac{|\xi|}{|z-\xi|}O(\mu_l(1+|\xi|)^{-5})d\xi\\ &+\frac{O(1)+O(|z|)}{\Lambda_l(1+|z_l|)},\end{aligned} \tag{76}$$

where $e^{3\vartheta_l} := \int_0^1 e^{3(sv_l+(1-s)V_0)}ds$. Thus, appealing to (75) and (76), we infer that w satisfies

$$w(z) = \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\log\frac{|\xi|}{|z-\xi|}e^{3V_0(\xi)}w(\xi)d\xi \tag{77}$$

Now, using (73), we have that w satisfies the following asymptotics

$$|w(z)| \le C(1+|z|). \tag{78}$$

On the other hand, from the definition of v_l, it is easy to see that

$$w(0) = 0, \text{ and } \nabla w(0) = 0. \tag{79}$$

So, using (77)–(79), and observing that Lemma 3.7 in [24] holds for dimension 3, we obtain

$$w = 0.$$

However, from (74), we infer that w satisfies also

$$|w(z^*)| = 1 \tag{80}$$

So we reach a contradiction in the second case also. Hence the proof of the lemma is complete.

Because of the lack of understanding of the blowing PS-sequences for Louiville type problems, the role of the PS-sequences can be replaced by the vanishing viscosity solutions of the type of (57) via the following Bahri-Lucia's deformation type lemma.

Lemma 3.6. *Assuming that $a, b \in \mathbb{R}$ such that $a < b$ and there is no critical values of $\mathcal{E}_g$ in $[a,b]$, then there are two possibilities*

(1) *Either*

$$(\mathcal{E}_g)^a \text{ is a deformation retract of } (\mathcal{E}_g)^b.$$

(2) *Or there exists a sequence $t_l \to 1$ as $l \to +\infty$ and a sequence of critical point u_l of $(\mathcal{E}_g)_{t_l}$ verifying $a \le \mathcal{E}_g(u_l) \le b$ for all $l \in \mathbb{N}^*$, where $(\mathcal{E}_g)_{t_l}$ is as in (36) with t replaced by t_l.*

On the other hand, setting

$$V_R(k,\epsilon,\eta) := \Big\{u \in \mathcal{H}_{\frac{\partial}{\partial n}} : \exists a_1,\dots,a_k \in \partial M,\ \lambda_1,\dots,\lambda_k > 0,\ \|u-\overline{u}_{Q,T} - \sum_{i=1}^{k}\varphi_{a_i,\lambda_i}\|_{\mathbb{P}^{4,3}} < \epsilon \quad \lambda_i \geq \frac{1}{\epsilon},\ \frac{2}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \frac{\Lambda}{2}, \quad \text{and} \quad d_{\hat{g}}(a_i,a_j) \geq 4\overline{C}\eta \text{ for } i \neq j\Big\}, \tag{81}$$

where $\overline{C}$ is as in (41), L as in (21), $O(1) := O_{A,\bar{\lambda},u,\epsilon}(1)$ meaning bounded uniformly in $\bar{\lambda} := (\lambda_1,\dots,\lambda_k)$, $A := (a_1,\dots,a_k)$, u, ϵ, we have as in [24] that Proposition 3.2 implies the following one.

Lemma 3.7. *Let ϵ and η be small positive real numbers with $0 < 2\eta < \varrho$ where ϱ is as in (43). Assuming that u_l is a sequence of blowing up critical point of $(\mathcal{E}_g)_{t_l}$ with $\overline{(u_l)}_{Q,T} = 0, l \in \mathbb{N}$ and $t_l \to 1$ as $l \to +\infty$, then there exists $l_{\epsilon,\eta}$ a large positive integer such that for every $l \geq l_{\epsilon,\eta}$, we have $u_l \in V_R(k,\epsilon,\eta)$, and for the definition of $V_R(k,\epsilon,\eta)$, see (81).*

Finally, as in [24], we have that Lemmas 3.6 and 3.7 implies the following one.

Lemma 3.8. *Assuming that ϵ and η are small positive real numbers with $0 < 2\eta < \varrho$, then for $a, b \in \mathbb{R}$ such that $a < b$, we have that if there is no critical values of $\mathcal{E}_g$ in $[a,b]$, then there are two possibilities*

(1) *Either*

$$(\mathcal{E}_g)^a \text{ is a deformation retract of } (\mathcal{E}_g)^b.$$

(2) *Or there exists a sequence $t_l \to 1$ as $l \to +\infty$ and a sequence of critical point u_l of $(\mathcal{E}_g)_{t_l}$ [for its definition see (36)] verifying $a \leq \mathcal{E}_g(u_l) \leq b$ for all $l \in \mathbb{N}^*$ and $l_{\epsilon,\eta}$ a large positive integer such that $u_l \in V_R(k,\epsilon,\eta)$ for all $l \geq l_{\epsilon,\eta}$, and for the definition of $V_R(k,\epsilon,\eta)$, see (81).*

3.2. Energy and gradient estimates at infinity

In this subsection, we present energy and gradient estimates needed to characterize the critical points at infinity of $\mathcal{E}_g$. We start with a parametrization of infinity. Indeed, as a Liouville type problem, we have that for η a small positive real number with $0 < 2\eta < \varrho$, there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ such that $\forall\, 0 < \epsilon \leq \epsilon_0$, we have

$$\forall u \in V_R(k,\epsilon,\eta), \text{the minimization problem } \min_{B_{\epsilon,\eta}} \left\| u - \overline{u}_{Q,T} - \sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} - \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)})\right\|_{\mathbb{P}^{4,3}} \tag{82}$$

has a unique solution, up to permutations, where $B_{\epsilon,\eta}$ is defined as follows

$$B_{\epsilon,\eta} := \Bigg\{ (\bar\alpha, A, \bar\lambda, \bar\beta) \in \mathbb{R}^k \times (\partial M)^k \times (0,+\infty)^k \times \mathbb{R}^{\bar k} : |\alpha_i - 1|\sqrt{\log \lambda_i} \\ \leq \overline{C}\epsilon, \lambda_i \geq \frac{1}{\epsilon}, \quad i = 1,\dots,k, d_{\hat g}(a_i,a_j) \geq 4\overline{C}\eta, i \neq j, |\beta_r| \leq \overline{C}\epsilon, r = 1,\dots,\bar k \Bigg\}. \tag{83}$$

Moreover, using the solution of (82), we have that every $u \in V_R(k,\epsilon,\eta)$ can be written as

$$u - \overline{u}_{(Q,T)} = \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar k} \beta_r (v_r - \overline{v_r}_{(Q,T)}) + w, \tag{84}$$

where w verifies the following orthogonality conditions

$$\overline{w}_{(Q,T)} = \langle \varphi_{a_i,\lambda_i}, w\rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i}, w \right\rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial a_i}, w \right\rangle_{\mathbb{P}^{4,3}} = \langle v_r, w\rangle_{\mathbb{P}^{4,3}} = 0, \\ i = 1,\dots,k, \\ r = 1,\dots,\bar k \tag{85}$$

and the estimate

$$\|w\|_{\mathbb{P}^{4,3}} = O(\epsilon), \tag{86}$$

where here $O(1) := O_{\bar\alpha,A,\bar\lambda,\bar\beta,w,\epsilon}(1)$. Furthermore, the concentration points a_i, the masses α_i, the concentrating parameters λ_i and the negativity parameter β_r in (84) verify also

$$d_{\hat g}(a_i,a_j) \geq 4\overline{C}\eta,\ i \neq j = 1,\dots,k,\ \frac{1}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda\ i,j = 1,\dots,k,\ \lambda_i \geq \frac{1}{\epsilon}, \quad \text{and} \\ \sum_{r=1}^{\bar k} |\beta_r| + \sum_{i=1}^{k} |\alpha_i - 1|\sqrt{\log \lambda_i} = O(\epsilon) \tag{87}$$

with still $O(1)$ as in (86).

Because of the translation invariant property of $\mathcal{E}_g$ and the parametrization (84), to derive energy estimate in $V_R(k,\epsilon,\eta)$ we start with the following lemma.

Lemma 3.9. *Assuming that η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), and $0 < \epsilon \leq \epsilon_0$ where ϵ_0 is as in (82), then for $a_i \in M$ concentration points, α_i masses, λ_i concentration parameters ($i = 1,\dots,k$), and β_r negativity parameters ($r = 1,\dots,\bar k$) satisfying (87), we have*

$$\mathcal{E}_g\left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar k} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right) = C_0^k - 8\pi^2 \mathcal{F}_K(a_1,\dots,a_k) \\ +2\sum_{r=1}^{\bar k} \mu_r \beta_r^2 + \sum_{i=1}^{k} (\alpha_i - 1)^2 \left[16\pi^2 \log \lambda_i + 8\pi^2 H(a_i,a_i) + C_1^k \right] \\ + 8\pi^2 \sum_{i=1}^{k} (\alpha_i - 1) \left[\sum_{r=1}^{\bar k} 2\beta_r (v_r - \overline{v_r}_{(Q,T)})(a_i) + \sum_{j=1, j\neq i}^{k} (\alpha_j - 1) G(a_i,a_j) \right]$$

$$
\begin{aligned}
&- \frac{c^1 8\pi^2}{9}\sum_{i=1}^{k}\frac{1}{\lambda_i^2}\left(\frac{\Delta_{\hat{g}_{a_i}}\mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)}-\frac{3}{4}R_{\hat{g}}(a_i)\right)\\
&+ \frac{c^1 8\pi^2}{9}\sum_{i=1}^{k}\frac{\tilde{\tau}_i}{\lambda_i^2}\left(\frac{\Delta_{\hat{g}_{a_i}}\mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)}-\frac{3}{4}R_{\hat{g}}(a_i)\right)\\
&+ \frac{16\pi^2}{3}\sum_{i=1}^{k}\log(1-\tilde{\tau}_i)+O\left(\sum_{i=1}^{k}|\alpha_i-1|^2+\sum_{r=1}^{\bar{k}}|\beta_r|^3+\sum_{i=1}^{k}\frac{1}{\lambda_i^3}\right),
\end{aligned}
$$

where $O(1)$ *means here* $O_{\bar{\alpha},A,\bar{\lambda},\bar{\beta},\epsilon}(1)$ *with* $\bar{\alpha}=(\alpha_1,\ldots,\alpha_k)$, $A:=(a_1,\ldots,a_k)$, $\bar{\lambda}:=(\lambda_1,\ldots,\lambda_k)$, $\bar{\beta}:=(\beta_1,\ldots,\beta_{\bar{k}})$ *and for* $i=1,\ldots,k$,

$$
\tilde{\tau}_i:=1-\frac{k\tilde{\gamma}_i}{\Gamma},\quad \Gamma:=\sum_{i=1}^{k}\tilde{\gamma}_i,\quad \tilde{\gamma}_i:=\tilde{c}_i\lambda_i^{6\alpha_i-3}\mathcal{F}_i^A(a_i)\mathcal{G}_i(a_i),
$$

with

$$
\begin{aligned}
&\tilde{c}_i:=\int_{\mathbb{R}^3}\frac{1}{(1+|y|^2)^{3\alpha_i}}dy\\
&\mathcal{G}_i(a_i):=e^{3((\alpha_i-1)H(a_i,a_i)+\sum_{j=1,j\neq i}^{k}(\alpha_j-1)G(a_j,a_i))}e^{\frac{3}{2}\sum_{j=1,j\neq i}^{k}\frac{\alpha_j}{\lambda_j^2}\Delta_{g_{a_j}}G(a_j,a_i)}\\
&\qquad e^{\frac{3}{2}\frac{\alpha_i}{\lambda_i^2}\Delta_{g_{a_i}}H(a_i,a_i)}\times e^{3\sum_{r=1}^{\bar{k}}\beta_r v_r(a_i)},
\end{aligned}
$$

C_0^k *is a real number depending only on* k, C_1^k *is a real number depending only on* k *and* c^1 *is a positive real number and for the meaning of* $O_{\bar{\alpha},A,\bar{\lambda},\bar{\beta},\epsilon}(1)$.

Proof. The proof is the same as the one Lemma 4.1 in [2] replacing Lemma 10.1–10.4 in [2] by Lemmas 5.1–5.4. □

Concerning the gradient estimates of $\mathcal{E}_g$ in $V_R(k,\epsilon,\eta)$, we have in the directions of the scaling parameters:

Lemma 3.10. *Assuming that* η *is a small positive real number with* $0<2\eta<\varrho$ *where* ϱ *is as in* (43), *and* $\epsilon\le\epsilon_0$ *where* ϵ_0 *is as in* (82), *then for* $a_i\in\partial M$ *concentration points,* α_i *masses,* λ_i *concentration parameters* ($i=1,\ldots,k$) *and* β_r *negativity parameters* ($r=1,\ldots,\bar{k}$) *satisfying* (87), *we have that for every* $r=1,\ldots,k$, *there holds*

$$
\begin{aligned}
&\left\langle \nabla^{\mathbb{P}^{4,3}}\mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i}+\sum_{r=1}^{\bar{k}}\beta_r(v_r-\overline{v_r}_{(Q,T)})\right),\lambda_j\frac{\partial\varphi_{a_j,\lambda_j}}{\partial\lambda_j}\right\rangle_{\mathbb{P}^{4,3}}\\
&\quad=16\pi^2\alpha_j\tau_j-\frac{c^2 8\pi^2}{3\lambda_j^2}\left(\frac{\Delta_{\hat{g}_{a_j}}\mathcal{F}_j^A(a_j)}{\mathcal{F}_j^A(a_j)}-\frac{3}{4}R_{\hat{g}}(a_j)\right)-\frac{16\pi^2}{\lambda_j^2}\tau_j\Delta_{\hat{g}_{a_j}}H(a_j,a_j)\\
&\qquad-\frac{16\pi^2}{\lambda_j^2}\sum_{i=1,i\neq j}^{k}\tau_i\Delta_{\hat{g}_{a_j}}G(a_j,a_i)+\frac{c^2 8\pi^2}{\lambda_j^2}\tau_j\left(\frac{\Delta_{\hat{g}_{a_j}}\mathcal{F}_j^A(a_j)}{\mathcal{F}_j^A(a_j)}-\frac{3}{4}R_{\hat{g}}(a_j)\right)\\
&\qquad+O\left(\sum_{i=1}^{k}|\alpha_i-1|+\sum_{r=1}^{\bar{k}}|\beta_r|^2+\sum_{i=1}^{k}\frac{1}{\lambda_i^3}\right),
\end{aligned}
$$

where $A := (a_1, \ldots, a_k)$, $O(1)$ *is as in Lemma 3.9,* c^2 *is a positive real number, and for* $i = 1, \ldots, k$,

$$\tau_i := 1 - \frac{k\tilde{\gamma}_i}{D}, \quad D := \oint_{\partial M} K(x) e^{3(\sum_{i=1}^k \alpha_i \varphi_{a_i,\lambda_i}(x) + \sum_{r=1}^{\bar{k}} \beta_r v_r(x))} dS_g(x),$$

with $\tilde{\gamma}_i$ *as in Lemma* 3.9.

Proof. The proof is the same as the one Lemma 5.1 in [2] replacing Lemma 10.1–10.4 in [2] by Lemmas 5.1–5.4. □

As in [2], Lemma 3.10 implies the following corollary.

Corollary 3.11. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43)*, and* $0 < \epsilon \le \epsilon_0$ *where* ϵ_0 *is as in* (82)*, then for* $a_i \in \partial M$ *concentration points,* α_i *masses,* λ_i *concentration parameters* ($i = 1, \ldots, k$)*, and* β_r *negativity parameters* ($r = 1, \ldots, \bar{k}$) *satisfying* (87)*, we have*

$$\begin{aligned}
&\left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^k \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), \sum_{i=1}^k \frac{\lambda_i}{\alpha_i} \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle_{\mathbb{P}^{4,3}} \\
&\quad = \sum_{i=1}^k \frac{c^3 8\pi^2}{\lambda_i^2} \left(\frac{\Delta_{\hat{g}_{a_i}} \mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)} - \frac{3}{4} R_{\hat{g}}(a_i) \right) \\
&\qquad + O\left(\sum_{i=1}^k |\alpha_i - 1| + \sum_{r=1}^{\bar{k}} |\beta_r|^2 + \sum_{i=1}^k \tau_i^2 + \sum_{i=1}^k \frac{1}{\lambda_i^3} \right),
\end{aligned}$$

where $A := (a_1, \ldots, a_k)$, $O(1)$ *is as in Lemma* 3.9*,* c^3 *is a positive real number, and for* $i = 1, \ldots, k$, τ_i *is as in Lemma* 3.10.

Proof. The proof uses the strategy of the proof of Corollary 5.2 in [2] replacing Lemma 5.1 in [2] by its counterpart Lemma 3.10. □

For the gradient estimate in the directions of mass concentrations, we have:

Lemma 3.12. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43)*, and* $0 < \epsilon \le \epsilon_0$ *where* ϵ_0 *is as in* (82)*, then for* $a_i \in \partial M$ *concentration points,* α_i *masses,* λ_i *concentration parameters* ($i = 1, \ldots, k$)*,* *and* β_r *negativity parameters* $(r = 1, \ldots, \bar{k})$ *satisfying* (87), *we have that for every* $j = 1, \ldots, k$, *there holds*

$$\left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), \varphi_{a_j,\lambda_j} \right\rangle_{\mathbb{P}^{4,3}}$$

$$= (2 \log \lambda_j + H(a_j, a_j) - C_2) \frac{1}{\alpha_j}$$

$$\left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), \lambda_j \frac{\partial \varphi_{a_j,\lambda_j}}{\partial \lambda_j} \right\rangle_{\mathbb{P}^{4,3}}$$

$$+ \sum_{i=1, i \neq j}^{k} G(a_j, a_i) \left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), \lambda_i \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle_{\mathbb{P}^{4,3}}$$

$$+ 32\pi^2 (\alpha_j - 1) \log \lambda_j + O\left(\sum_{i=1}^{k} |\alpha_i - 1| + \sum_{r=1}^{\bar{k}} |\beta_r| + \sum_{i=1}^{k} |\tau_i| + \sum_{i=1}^{k} \frac{\log \lambda_i}{\lambda_i^3} \right),$$

where $O(1)$ *as as in Lemma* 3.9 *and* C_2 *is a real number.*

Proof. It follows from the same arguments as in Lemma 5.3 in [2]. □

Concerning the gradient estimate in the directions of points of concentrations, we have:

Lemma 3.13. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43), *and* $0 < \epsilon \leq \epsilon_0$ *where* ϵ_0 *is as in* (82), *then for* $a_i \in \partial M$ *concentration points,* α_i *masses,* λ_i *concentration parameters* $(i = 1, \ldots, k)$, *and* β_r *negativity parameters* $(r = 1, \ldots, \bar{k})$ *satisfying* (87), *we have that for every* $j = 1, \ldots, k$, *there holds*

$$\left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), \frac{1}{\lambda_j} \frac{\partial \varphi_{a_j,\lambda_j}}{\partial a_j} \right\rangle_{\mathbb{P}^{4,3}}$$

$$= -\frac{c^2 32\pi^2}{\lambda_j} \frac{\nabla_{\hat{g}} \mathcal{F}_j^A(a_j)}{\mathcal{F}_j^A(a_j)}$$

$$+ O\left(\sum_{i=1}^{k} |\alpha_i - 1|^2 + \sum_{i=1}^{k} |\tau_i|^2 \right)$$

$$+ O\left(\sum_{i=1}^{k} \frac{1}{\lambda_i^2} + \sum_{r=1}^{\bar{k}} |\beta_r|^2 \right),$$

where $A := (a_1, \ldots, a_k)$, $O(1)$ *is as in Lemma* 3.9, c^2 *is as in Lemma* 3.10 *and for* $i = 1, \ldots, k$, τ_i *is as in Lemma* 3.10.

Proof. The proof is the same as the one of Lemma 5.4 in [2]. □

Concerning the gradient estimate in the directions of the negativity parameters, we have:

Lemma 3.14. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43), *and* $0 < \epsilon \le \epsilon_0$ *where* ϵ_0 *is as in* (82), *then for* $a_i \in \partial M$ *concentration points,* α_i *masses,* λ_i *concentration parameters (*$i = 1, \dots, k$*), ad* β_r *negativity parameters (*$r = 1, \dots, \bar{k}$*) satisfying* (87), *we have that for every* $l = 1, \dots, \bar{k}$*, there holds*

$$\left\langle \nabla^{\mathbb{P}^{4,3}} \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \right), v_l - \overline{(v_l)}_{Q,T} \right\rangle_{\mathbb{P}^{4,3}}$$
$$= 4\mu_l \beta_l + O\left(\sum_{i=1}^{k} |\alpha_i - 1| + \sum_{i=1}^{k} |\tau_i| \right)$$
$$+ O\left(\sum_{i=1}^{k} \frac{1}{\lambda_i^2} + \sum_{r=1}^{\bar{k}} |\beta_r|^2 \right),$$

where $O(1)$ *is as in Lemma 3.9 and for* $i = 1, \dots, k$*,* τ_i *is as in Lemma 3.10*

Proof. It follows from the same arguments as in the proof of Lemma 5.5 in [2]. □

3.3. Finite-dimensional reduction

In this subsection, we complete the energy estimate of $\mathcal{E}_g$ on $V_R(k, \epsilon, \eta)$ via Lyapunov finite dimensional type reduction and second variation arguments. First of all, we have:

Proposition 3.15. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43), *and* $0 < \epsilon \le \epsilon_0$ *where* ϵ_0 *is as in* (82) *and* $u = \overline{u}_{Q,T} + \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) + w \in V_R(k, \epsilon, \eta)$ *with* w*, the concentration points* a_i*, the masses* α_i*, the concentrating parameters* λ_i *(*$i = 1, \dots, k$*), and the negativity parameters* β_r *(*$r = 1, \dots, \bar{k}$*) verifying* (85)–(87)*, then we have*

$$\mathcal{E}_g(u) = \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{(v_r)}_{(Q,T)}) \right)$$
$$- f(w) + Q(w) + o(\|w\|^2_{\mathbb{P}^{4,3}}), \tag{88}$$

where

$$f(w) := 16\pi^2 k \frac{\oint_{\partial M} K e^{3\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}} \beta_r v_r} w dS_g}{\oint_{\partial M} K e^{3\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}} \beta_r v_r} dS_g}, \tag{89}$$

and

$$Q(w) := \|w\|^2_{\mathbb{P}^{4,3}} - 24\pi^2 k \frac{\oint_{\partial M} K e^{3\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}} \beta_r v_r} w^2 dS_g}{\oint_{\partial M} K e^{3\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}} \beta_r v_r} dS_g}. \tag{90}$$

Moreover, setting

$$E_{a_i,\lambda_i} := \left\{ w \in \mathcal{H}_{\frac{\partial}{\partial n}} : \langle \varphi_{a_i,\lambda_i}, w \rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i}, w \right\rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial a_i}, w \right\rangle_{\mathbb{P}^{4,3}} = 0, \right.$$

$$\left.\overline{w}_{(Q,T)} = \langle v_r, w\rangle_{\mathbb{P}^{4,3}} = 0,\ r = 1,\dots,\bar{k}, \quad \textit{and} \quad \|w\|_{\mathbb{P}^{4,3}} = O\left(\epsilon\right)\right\}, \tag{91}$$

and

$$A := (a_1,\dots,a_k),\ \bar{\lambda} = (\lambda_1,\dots,\lambda_k), \quad E_{A,\bar{\lambda}} := \cap_{i=1}^{k} E_{a_i,\lambda_i}, \tag{92}$$

we have that, the quadratic form Q is positive definite in $E_{A,\bar{\lambda}}$. Furthermore, the linear part f verifies that, for every $w \in E_{A,\bar{\lambda}}$, there holds

$$f(w) = O\left[\|w\|_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\frac{|\nabla_{\hat{g}}\mathcal{F}_i^A(a_i)|}{\lambda_i} + \sum_{i=1}^{k}|\alpha_i - 1|\log\lambda_i + \sum_{r=1}^{\bar{k}}|\beta_r| + \sum_{i=1}^{k}\frac{\log\lambda_i}{\lambda_i^2}\right)\right], \tag{93}$$

where here $o(1) = o_{\bar{\alpha},A,\bar{\beta},\bar{\lambda},w,\epsilon}(1)$ and $O\left(1\right) := O_{\bar{\alpha},A,\bar{\beta},\bar{\lambda},w,\epsilon}\left(1\right)$.

As in [2], to prove Proposition 3.15, we will need the following three coming lemmas. We start with the following one:

Lemma 3.16. *Assuming the assumptions of Proposition 3.15 and $\gamma \in (0,1)$ small, then for every $q \geq 1$, there holds the following estimates*

$$\oint_{\partial M} K e^{3\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}}\beta_r v_r}|w|^q = O\left(\|w\|^q_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\lambda_i^{3+\gamma}\right)\right), \tag{94}$$

$$\oint_{\partial M} K e^{3\sum_{i=1}^{k}\alpha_i\hat{\delta}_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}}\beta_r v_r}|w|^q = O\left(\|w\|^q_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\lambda_i^{\gamma}\right)\right), \tag{95}$$

$$\oint_{\partial M} e^{3\hat{\delta}_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}}\beta_r v_r} d_{\hat{g}_{a_i}}(a_i,\cdot)|w|^q = O\left(\|w\|^q_{\mathbb{P}^{4,3}}\frac{1}{\lambda_i^{1-\gamma}}\right),\ i = 1,\dots,k, \tag{96}$$

$$\oint_{\partial M} K e^{3\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}}\beta_r v_r} e^{3\theta_w w}|w|^q = O\left(\|w\|^q_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\lambda_i^{3+\gamma}\right)\right), \tag{97}$$

where $\theta_w \in [0,1]$, and

$$\oint_{\partial M} K e^{3\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + 3\sum_{r=1}^{\bar{k}}\beta_r v_r}\left(e^{3w} - 1 - 3w - \frac{9}{2}w^2\right)dV_g = o\left(\|w\|^2_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\lambda_i^3\right)\right) \tag{98}$$

where here $o(1)$ and $O\left(1\right)$ are as in Proposition 3.15.

Proof. The proof is the same as the one of Lemma 6.2 in [2] replacing Lemma 10.1 by its counterpart Lemma 5.1. □

Still as in [2], the second lemma that we need for the proof of Proposition 3.15 read as follows:

Lemma 3.17. *Assuming the assumptions of Proposition 3.15, then there holds the following estimate*

$$\frac{\oint_{\partial M} Ke^{3\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i}+3\sum_{r=1}^{\bar{k}}\beta_r v_r} w dS_g}{\oint_{\partial M} Ke^{3\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i}+3\sum_{r=1}^{\bar{k}}\beta_r v_r} dS_g} = O\left(\|w\|_{\mathbb{P}^{4,3}}\left(\sum_{i=1}^{k}\frac{|\nabla_{\hat{g}}\mathcal{F}_i^A(a_i)|}{\lambda_i}+\sum_{i=1}^{k}|\alpha_i-1|\log\lambda_i+\sum_{r=1}^{\bar{k}}|\beta_r|+\sum_{i=1}^{k}\frac{\log\lambda_i}{\lambda_i^2}\right)\right). \tag{99}$$

Proof. It follows from the same arguments as in the proof of Lemma 6.3 in [2] replacing Lemma 10.1 by its counterpart Lemma 5.1. □

Finally, as in [2], the third and last lemma that we need for the proof of Proposition 3.15 is the following one.

Lemma 3.18. *Assuming the assumptions of Proposition 3.15, then for every $i = 1, \ldots, k$, there holds*

$$\tau_i = O(\epsilon). \tag{100}$$

Proof. The proof is the same as the one Lemma 6.4 in [2] replacing Lemma 5.1 by Lemma 3.10. □

Proof of Proposition 3.15. It follows from the same arguments as in the proof of Lemma 6.1 in [2] replacing Lemma 6.2–6.4 in [2] by Lemmas 3.16–3.18 and Lemma 10.1 in [2] by Lemma 5.1. Furthermore, Lemma 10.6 and Lemma 10.7 in [2] are replaced by Lemma 5.9 and Lemma 5.10. □

Now, as in [2], we have that Proposition 3.15 implies the following direct corollaries.

Corollary 3.19. *Assuming that η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), $0 < \epsilon \le \epsilon_0$ where ϵ_0 is as in (82) and $u := \sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)})$ with the concentration points a_i, the masses α_i, the concentrating parameters λ_i ($i = 1, \ldots, k$) and the negativity parameters β_r ($r = 1, \ldots, \bar{k}$) satisfying (87), then there exists a unique $\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in E_{A,\bar{\lambda}}$ such that*

$$\mathcal{E}_g\left(u + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\right) = \min_{w\in E_{A,\bar{\lambda}},\, u+w\in V_R(k,\epsilon,\eta)} \mathcal{E}_g(u+w), \tag{101}$$

where $\bar{\alpha} := (\alpha_1, \ldots, \alpha_k)$, $A := (a_1, \ldots, a_k)$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_k)$ and $\bar{\beta} := (\beta_1, \ldots, \beta_k)$.
Furthermore, $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \longrightarrow \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in C^1$ and satisfies the following estimate

$$\frac{1}{C}\|\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\|^2_{\mathbb{P}^{4,3}} \le \left|f\left(\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\right)\right| \le C\|\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\|^2_{\mathbb{P}^{4,3}}, \tag{102}$$

for some large positive constant C independent of $\bar{\alpha}$, A, $\bar{\lambda}$, and $\bar{\beta}$, hence

$$\|\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\|_{\mathbb{P}^{4,3}} = O\left(\sum_{i=1}^{k}\frac{|\nabla_{\hat{g}}\mathcal{F}_i^A(a_i)|}{\lambda_i} + \sum_{i=1}^{k}|\alpha_i - 1|\log\lambda_i + \sum_{r=1}^{\bar{k}}|\beta_r| + \sum_{i=1}^{k}\frac{\log\lambda_i}{\lambda_i^2}\right). \tag{103}$$

Corollary 3.20. *Assuming that η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), $0 < \epsilon \le \epsilon_0$ where ϵ_0 is as in (82), and $u_0 := \sum_{i=1}^{k}\alpha_i^0\varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\bar{k}}\beta_r^0(v_r - \overline{v_r}_{(Q,T)})$ with the concentration points a_i^0, the masses α_i^0, the concentrating parameters λ_i^0 ($i = 1, \ldots, k$) and the negativity parameters β_r^0 ($r = 1, \ldots, \bar{k}$) satisfying (87), then there exists an open neighborhood U of $(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0)$ (with $\bar{\alpha}^0 := (\alpha_1^0, \ldots, \alpha_k^0)$, $A^0 := (a_1^0, \ldots, a_k^0)$, $\bar{\lambda} := (\lambda_1^0, \ldots, \lambda_k^0)$ and $\bar{\beta}^0 := (\beta_1^0, \ldots, \beta_{\bar{k}}^0)$) such that for every $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in U$ with $\bar{\alpha} := (\alpha_1, \ldots, \alpha_k)$, $A := (a_1, \ldots, a_k)$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_k)$, $\bar{\beta} := (\beta_1, \ldots, \beta_{\bar{k}})$, and the a_i, the α_i, the λ_i ($i = 1, \ldots, k$) and the β_r ($r = 1, \ldots, \bar{k}$) satisfying (87), and w satisfying (87) with $\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)}) + w \in V_R(k, \epsilon, \eta)$, we have the existence of a change of variable*

$$w \longrightarrow V \tag{104}$$

from a neighborhood of $\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ to a neighborhood of 0 such that

$$\begin{aligned}\mathcal{E}_g&\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)}) + w\right)\\ &= \mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\right)\\ &\quad + \frac{1}{2}\partial^2\mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i^0\varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\bar{k}}\beta_r^0(v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0)\right)(V, V),\end{aligned} \tag{105}$$

Thus, as in [2], with this new variable, it is easy to see that in $V_R(k, \epsilon, \eta)$ we have a splitting of the variables $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ and V, namely that one can decrease the Euler-Lagrange functional $\mathcal{E}_g$ in the variable V without touching the variable $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ by considering just the flow

$$\frac{dV}{dt} = -V. \tag{106}$$

So, as in [2], and for the same reasons, to develop a Morse theory for $\mathcal{E}_g$ is equivalent to do one for the functional

$$\bar{\mathcal{E}_g}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) := \mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\right), \tag{107}$$

where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $A = (a_1, \ldots, a_k)$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$ and $\bar{\beta} = \beta_1, \ldots, \beta_{\bar{k}}$ with the concentration points a_i, the masses α_i, the concentrating parameters

λ_i $(i = 1, \ldots, k)$ and the negativity parameters β_r $(r = 1, \ldots, \bar{k})$ satisfying (87), and $\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ is as in Corollary 3.19.
Finally, we have the following energy estimate of $\mathcal{E}_g$ on $V_R(k, \epsilon, \eta)$.

Lemma 3.21. *Under the assumptions of Proposition 3.15, $\forall u = \overline{u}_{(Q,T)} + \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) + w \in V_R(k, \epsilon, \eta)$, we have*

$$
\begin{aligned}
\mathcal{E}_g(u) &= \mathcal{E}_g\left(\sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) + w\right) = C_0^k - 8\pi^2 \mathcal{F}_K(a_1, \ldots, a_k) \\
&\quad + 2\sum_{r=1}^{\bar{k}} \mu_r \beta_r^2 + \sum_{i=1}^{k} (\alpha_i - 1)^2 \left[16\pi^2 \log \lambda_i + 8\pi^2 H(a_i, a_i) + C_1^k\right] \\
&\quad - \frac{c^1 8\pi^2}{9} \sum_{i=1}^{k} \frac{1}{\lambda_i^2} \left(\frac{\Delta_{\hat{g}_{a_i}} \mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)} - \frac{3}{4} R_{\hat{g}}(a_i)\right) \\
&\quad + \frac{1}{2} \partial^2 \mathcal{E}_g \left(\sum_{i=1}^{k} \alpha_i^0 \varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\bar{k}} \beta_r^0 (v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0)\right)(V, V) \\
&\quad + 8\pi^2 \sum_{i=1}^{k} (\alpha_i - 1) \left[\sum_{r=1}^{\bar{k}} 2\beta_r \left(v_r - \overline{v_r}_{(Q,T)}\right)(a_i) - \sum_{j=1, j\neq i}^{k} (\alpha_j - 1) G(a_i, a_j)\right] \\
&\quad + \frac{c^1 8\pi^2}{9} \sum_{i=1}^{k} \frac{\tilde{\tau}_i}{\lambda_i^2} \left(\frac{\Delta_{\hat{g}_{a_i}} \mathcal{F}_i^A(a_i)}{\mathcal{F}_i^A(a_i)} - \frac{3}{4} R_{\hat{g}}(a_i)\right) \\
&\quad + \frac{16\pi^2}{3} \sum_{i=1}^{k} \log(1 - \tilde{\tau}_i) + O\left(\sum_{i=1}^{k} |\alpha_i - 1|^2 + \sum_{r=1}^{\bar{k}} |\beta_r|^3 + \sum_{i=1}^{k} \frac{1}{\lambda_i^3}\right. \\
&\quad \left. + \|\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})\|_{\mathbb{P}^{4,3}}^2\right),
\end{aligned}
$$

where $O(1)$ means here $O_{\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}, \epsilon}(1)$ with $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $A := (a_1, \ldots, a_k)$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_k)$, $\bar{\beta} := (\beta_1, \ldots, \beta_{\bar{k}})$ and for $i = 1, \ldots, k$, $\tilde{\tau}_i$ is as in Lemma 3.9. where $\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ is as in Corollary 3.19.

Proof. It follows directly from Lemma 3.9, formula (105) and Proposition 3.15. □

3.4. Morse lemma at infinity

In this subsection, we derive a Morse Lemma at infinity for $\mathcal{E}_g$. As in [2], in order to do that, we first construct a pseudo-gradient for $\bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$, where $\bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ is defined as in (107) exploiting the gradient estimates derived previously. Indeed, we have:

Proposition 3.22. *Assuming that η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), and $0 < \epsilon \leq \epsilon_0$ where ϵ_0 is as in (82), then there exists a pseudogradient $\mathcal{W}_g$ of $\bar{\mathcal{E}}_g(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})$ such that*

(1) *For every $u := \sum_{i=1}^{k} \alpha_i \varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{v_r}_{(Q,T)}) \in V_R(k, \epsilon, \eta)$ with the concentration points a_i, the masses α_i, the concentrating parameters λ_i $(i = 1, \ldots, k)$ and the negativity parameters β_r $(r = 1, \ldots, \bar{k})$*

satisfying (87), there holds

$$\left\langle -\nabla^{\mathbb{P}^{4,3}}\mathcal{E}_g(u),\, \mathcal{W}_g \right\rangle_{\mathbb{P}^{4,3}} \geq c\left(\sum_{i=1}^{k}\frac{1}{\lambda_i^2}+\sum_{i=1}^{k}\frac{|\nabla_{\hat{g}}\mathcal{F}_i^A(a_i)|}{\lambda_i}+\sum_{i=1}^{k}|\alpha_i-1|\right) + c\left(\sum_{i=1}^{k}|\tau_i|+\sum_{r=1}^{\bar{k}}|\beta_r|)\right), \tag{108}$$

and for every $u := \sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i}+\sum_{r=1}^{\bar{k}}\beta_r\lambda(v_r-\overline{v_r}_{(Q,T)})+\bar{w}(\bar{\alpha},A,\bar{\lambda},\bar{\beta}) \in V_R(k,\epsilon,\eta)$ *with the concentration points* a_i, *the masses* α_i, *the concentrating parameters* λ_i $(i=1,\ldots,k)$ *and the negativity parameters* β_r $(r=1,\ldots,\bar{k})$ *satisfying (87), and* $\bar{w}(\bar{\alpha},A,\bar{\lambda},\bar{\beta})$ *is as in (101), there holds*

$$\left\langle -\nabla^{\mathbb{P}^{4,3}}\mathcal{E}_g(u+\bar{w}),\, \mathcal{W}_g+\frac{\partial\bar{w}}{\partial(\bar{\alpha},A,\bar{\lambda},\bar{\beta})} \right\rangle_{\mathbb{P}^{4,3}} \geq c\left(\sum_{i=1}^{k}\frac{1}{\lambda_i^2}+\sum_{i=1}^{k}\frac{|\nabla_{\hat{g}}\mathcal{F}_i^A(a_i)|}{\lambda_i}+\sum_{i=1}^{k}|\alpha_i-1|\right) +c\left(\sum_{i=1}^{k}|\tau_i|+\sum_{r=1}^{\bar{k}}|\beta_r|\right), \tag{109}$$

where c *is a small positive constant independent of* $A := (a_1,\ldots,a_k)$, $\bar{\alpha}=(\alpha_1,\ldots,\alpha_k)$, $\bar{\lambda}=(\lambda_1,\ldots,\lambda_k)$, $\bar{\beta}=(\beta_1,\ldots,\beta_{\bar{k}})$ *and* ϵ.

(2) $\mathcal{W}_g$ *is a* $\|\cdot\|_{\mathbb{P}^{4,3}}$*-bounded vector field and is compactifying outside the region where* A *is very close to a critical point* B *of* $\mathcal{F}_K$ *satisfying* $\mathcal{L}_K(B) < 0$.

Proof. It follows from the same arguments as in the proof of Proposition 8.1 in [2] replacing formulas (52)–(54), Lemma 5.1, Corollary 5.2 and Lemmas 5.3–5.5 in [2] with (51)–(53), Lemma 3.10, Corollary 3.11 and Lemmas 3.12–3.14. Furthermore, Lemmas 4.1, 7.1, and 0.5 in [2] are replaced by Lemmas 3.9, 3.15 and 5.5 □

Now, as in [2], we have that Proposition 3.22 implies the following characterization of the critical points at infinity of $\mathcal{E}_g$.

Corollary 3.23. (1) *The critical points at infinity of* $\mathcal{E}_g$ *correspond to the "configurations"* $\alpha_i = 1$, $\lambda_i = +\infty$, $\tau_i = 0$ $i = 1,\ldots,k$, $\beta_r = 0$, $r = 1,\ldots,\bar{k}$, A *is a critical point of* $\mathcal{F}_K$ *and* $V = 0$, *and we denote them by* z^∞ *with* z *being the corresponding critical point of* $\mathcal{F}_K$.

(2) *The "true" critical points at infinity of* $\mathcal{E}_g$ *are the* z^∞ *satisfying* $\mathcal{L}_K(z) < 0$ *and we denote them by* x^∞ *with* x *being the corresponding critical point of* $\mathcal{F}_K$.

(3) *The "false" critical points at infinity of* $\mathcal{E}_g$ *are the* z^∞ *satisfying* $\mathcal{L}_K(z) > 0$ *and we denote them by* y^∞ *with* y *being the corresponding critical point of* $\mathcal{F}_K$.

(4) *The $\mathcal{E}_g$-energy of a critical point at infinity z^∞ denoted by $\mathcal{J}_g(z^\infty)$ is given by*

$$\mathcal{J}_g(z^\infty) = C_0^k - 8\pi^2 \mathcal{F}_K(z_1, \ldots, z_k) \tag{110}$$

where $z = (z_1, \ldots, z_k)$ and C_0^k is as in Lemma 3.9.

Proof. Point (1)–(3) follow from (53), Lemma 3.8, the discussions right after (105), and Proposition 3.22, while Point (4) follows from Point (1) combined with (103) and Lemma 3.21. □

Finally, we are going to conclude this subsection by establishing an analogue of the classical Morse lemma for both "true" and "false" critical points at infinity. In order to do that, we first remark that, as in [2], the arguments of Proposition 3.22 implies that $V_- := \{u \in V_R(k, \epsilon, \eta) : \, l_K(A) < 0, \forall r \in \{1, \ldots, \bar{k}\} \;\; |\beta_r| \leq 2\tilde{C}_0 \left(\sum_{i=1}^k \frac{|\nabla_{\hat{g}} \mathcal{F}^A(a_i)|}{\lambda_i} + \sum_{i=1}^k |\alpha_i - 1| + \sum_{i=1}^k |\tau_i| + \sum_{i=1}^k \frac{1}{\lambda_i^2} \right), \; \forall i \in \{1, \ldots, k\}$ $|\tau_i| \leq 2\frac{\hat{C}_0}{\lambda_i^2}$, and $\forall i \in \{1, \ldots, k\}$ $\frac{|\nabla_{\hat{g}} \mathcal{F}_i^A(a_i)|}{\lambda_i} \leq 4\frac{C_0}{\lambda_i^2}$ $\}$ and $V_+ := \{u \in V_R(k, \epsilon, \eta) : l_K(A) > 0, \forall r \in \{1, \ldots, \bar{k}\} \;\; |\beta_r| \leq 2\tilde{C}_0 \left(\sum_{i=1}^k \frac{|\nabla_{\hat{g}} \mathcal{F}^A(a_i)|}{\lambda_i} + \sum_{i=1}^k |\alpha_i - 1| + \sum_{i=1}^k |\tau_i| + \sum_{i=1}^k \frac{1}{\lambda_i^2} \right), \; \forall i \in \{1, \ldots, k\}$ $|\tau_i| \leq 2\frac{\hat{C}_0}{\lambda_i^2}$, and $\forall i \in \{1, \ldots, k\}$ $\frac{|\nabla_{\hat{g}} \mathcal{F}_i^A(a_i)|}{\lambda_i} \leq 4\frac{C_0}{\lambda_i^2}$ $\}$ (where $\tilde{C}_0$, $\hat{C}_0$ and C_0 are large positive constants) are respectively a neighborhood of the "true" and "false" critical points at infinity of the variational problem. Hence, as in [2], (103), Corollary 3.20, Lemma 3.21 and classical Morse lemma imply the following Morse type lemma for a "true" critical point at infinity.

Lemma 3.24. (Morse lemma at infinity near a "true" one) *Assuming that η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), $0 < \epsilon \leq \epsilon_0$ where ϵ_0 is as in (82) and $u_0 := \sum_{i=1}^k \alpha_i^0 \varphi_{a_i^0, \lambda_i^0} + \sum_{r=1}^{\bar{k}} \beta_r^0 (v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0) \in V_-(k, \epsilon, \eta)$ (where $\bar{\alpha}^0 := (\alpha_1^0, \ldots, \alpha_k^0)$, $A^0 := (a_1^0, \ldots, a_k^0)$, $\bar{\lambda} := (\lambda_1^0, \ldots, \lambda_k^0)$ and $\bar{\beta}^0 := (a_1^0, \ldots, \beta_{\bar{k}}^0)$) with the concentration points a_i^0, the masses α_i^0, the concentrating parameters λ_i^0 ($i = 1, \ldots, k$) and the negativity parameters β_r^0 ($r = 1, \ldots, \bar{k}$) satisfying (87) and furthermore $A^0 \in Crit(\mathcal{F}_K)$, then there exists an open neighborhood U of $(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0)$ such that for every $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in U$ with $\bar{\alpha} := (\alpha_1, \ldots, \alpha_k)$, $A := (a_1, \ldots, a_k)$, $\bar{\lambda} := (\lambda_1, \ldots, \lambda_k)$, $\bar{\beta} := (\beta_1, \ldots, \beta_{\bar{k}})$, and the a_i, the α_i, the λ_i ($i = 1, \ldots, k$) and the β_r ($r = 1, \ldots, \bar{k}$) satisfying (87), and w satisfying (87) with $u = \overline{u}_{(Q,T)} + \sum_{i=1}^k \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{k}} \beta_r (v_r - \overline{(v_r)}_{(Q,T)}) + w \in V_-(k, \epsilon, \eta)$, we have the existence of a change of variable*

$$\begin{aligned}
&\alpha_i \longrightarrow s_i, \quad i = 1, \ldots, k, \\
&A \longrightarrow \tilde{A} = (\tilde{A}_-, \tilde{A}_+) \\
&\lambda_1 \longrightarrow \theta_1, \\
&\tau_i \longrightarrow \theta_i, \quad i = 2, \ldots, k, \\
&\beta_r \longrightarrow \tilde{\beta}_r \\
&V \longrightarrow \tilde{V},
\end{aligned} \tag{111}$$

such that

$$\begin{aligned}\mathcal{E}_g(u) &= \mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r\left(v_r - \overline{(v_r)}_{(Q,T)}\right) + w\right)\\ &= -|\tilde{A}_-|^2 + |\tilde{A}_+|^2 + \sum_{i=1}^{k}s_i^2\\ &\quad - \sum_{r=1}^{\bar{k}}\tilde{\beta}_r^2 + \theta_1^2 - \sum_{i=2}^{k}\theta_i^2 + \|\tilde{V}\|^2 \end{aligned} \tag{112}$$

where $\tilde{A} = (\tilde{A}_-, \tilde{A}_+)$ *is the Morse variable of the map* $\mathcal{J}_g : (\partial M)^k \setminus F((\partial M)^k) \longrightarrow \mathbb{R}$ *which is defined by the right hand side of* (110)*. Hence a "true" critical point at infinity* x^∞ *of* $\mathcal{E}_g$ *has Morse index at infinity*

$$M_\infty(x^\infty) = i_\infty(x) + \bar{k},$$

with i_∞ *as in* (12)*.*

Similarly, and for the same reasons as above, we have the following analogue of the classical Morse lemma for a "false" critical point at infinity.

Lemma 3.25. (Morse lemma at infinity near a "false" one) *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43)*,* $0 < \epsilon \le \epsilon_0$ *where* ϵ_0 *is as in* (82) *and* $u_0 := \sum_{i=1}^{k}\alpha_i^0\varphi_{a_i^0,\lambda_i^0} + \sum_{r=1}^{\bar{k}}\beta_r^0(v_r - \overline{v_r}_{(Q,T)}) + \bar{w}(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0) \in V_+(k,\epsilon,\eta)$ *(where* $\bar{\alpha}^0 := (\alpha_1^0,\ldots,\alpha_k^0)$*,* $A^0 := (a_1^0,\ldots,a_k^0)$*,* $\bar{\lambda} := (\lambda_1^0,\ldots,\lambda_k^0)$ *and* $\bar{\beta}^0 := (a_1^0,\ldots,\beta_{\bar{k}}^0)$*) with the concentration points* a_i^0*, the masses* α_i^0*, the concentrating parameters* λ_i^0 *(*$i = 1,\ldots,k$*) and the negativity parameters* β_r^0 *(*$r = 1,\ldots,\bar{k}$*) satisfying* (87) *and furthermore* $A^0 \in Crit(\mathcal{F}_K)$*, then there exists an open neighborhood* U *of* $(\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0)$ *such that for every* $(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in U$ *with* $\bar{\alpha} := (\alpha_1,\ldots,\alpha_k)$*,* $A := (a_1,\ldots,a_k)$*,* $\bar{\lambda} := (\lambda_1,\ldots,\lambda_k)$*,* $\bar{\beta} := (\beta_1,\ldots,\beta_{\bar{k}})$*, and the* a_i*, the* α_i*, the* λ_i *(*$i = 1,\ldots,k$*) and the* β_r *(*$r = 1,\ldots,\bar{k}$*) satisfying* (87)*, and* w *satisfying* (87) *with* $u = \overline{u}_{(Q,T)} + \sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r(v_r - \overline{(v_r)}_{Q^n}) + w \in V_+(k,\epsilon,\eta)$*, we have the existence of a change of variable*

$$\begin{aligned} &\alpha_i \longrightarrow s_i, \quad i = 1,\ldots,k,\\ &A \longrightarrow \tilde{A} = (\tilde{A}_-, \tilde{A}_+)\\ &\lambda_1 \longrightarrow \theta_1,\\ &\tau_i \longrightarrow \theta_i, \quad i = 2,\ldots,k,\\ &\beta_r \longrightarrow \tilde{\beta}_r\\ &V \longrightarrow \tilde{V}, \end{aligned} \tag{113}$$

such that

$$\mathcal{E}_g(u) = \mathcal{E}_g\left(\sum_{i=1}^{k}\alpha_i\varphi_{a_i,\lambda_i} + \sum_{r=1}^{\bar{k}}\beta_r\left(v_r - \overline{(v_r)}_{(Q,T)}\right) + w\right) = -|\tilde{A}_-|^2 + |\tilde{A}_+|^2$$

$$+\sum_{i=1}^{k} s_i^2 - \sum_{r=1}^{\bar{k}} \tilde{\beta}_r^2 - \sum_{i=1}^{k} \theta_i^2 + \|\tilde{V}\|^2, \tag{114}$$

where $\tilde{A} = (\tilde{A}_-, \tilde{A}_+)$ *is the Morse variable of the map* $\mathcal{J}_g : (\partial M)^k \setminus F((\partial M)^k) \longrightarrow \mathbb{R}$ *which is defined by the right hand side of* (110). *Hence a "false" critical point at infinity* y^∞ *of* $\mathcal{E}_g$ *has Morse index at infinity*

$$M_\infty(y^\infty) = i_\infty(y) + 1 + \bar{k}.$$

4. Proof of existence theorems

In this section, we show how the Morse lemma at infinity implies the main existence results via strong Morse type inequalities or Barycenter technique of Bahri–Coron.

4.1. Topology of vey high and negative sublevels of $\mathcal{E}_g$

We study the topology of very high sublevels of $\mathcal{E}_g$ and its every negative ones. We start with the very high sublevels of $\mathcal{E}_g$ and first derive the following lemma.

Lemma 4.1. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43), *then there exists* $\hat{C}_0^k := \hat{C}_0^k(\eta)$ *such that for every* $0 < \epsilon \leq \epsilon_0$ *where* ϵ_0 *is as in (82), there holds*

$$V(k, \epsilon, \eta) \subset (\mathcal{E}_g)^{\hat{C}_0^k} \setminus (\mathcal{E}_g)^{-\hat{C}_0^k}.$$

Proof. It follows directly from (84)–(87), Proposition 3.15, Lemmas 3.18 and 3.21. □

Next, combining Proposition 3.7 and the latter lemma, we have the following corollary.

Corollary 4.2. *There exists a large positive constant* $\hat{C}_1^k$ *such that*

$$Crit(\mathcal{E}_g) \subset (\mathcal{E}_g)^{\hat{C}_1^k} \setminus (\mathcal{E}_g)^{-\hat{C}_1^k}.$$

Proof. It follows, via a contradiction argument, from the fact that $\mathcal{E}_g$ is invariant by translation by constants, Proposition 3.7, and Lemma 4.1. □

Now, we are ready to characterize the topology of very high sublevels of $\mathcal{E}_g$. Indeed, as in [2] and for the same reasons, we have that Lemma 3.8, Lemma 4.1 and Corollary 4.2 imply the following one which describes the topology of very high sublevels of the Euler-Lagrange functional $\mathcal{E}_g$.

Lemma 4.3. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in (43), then there exists a large positive constant* $L^k := L^k(\eta)$ *with* $L^k > 2\max\{\hat{C}_0^k, \hat{C}_1^k\}$ *such that for every* $L \geq L^k$, *we have that* $(\mathcal{E}_g)^L$ *is a deformation retract of* $\mathcal{H}_{\frac{\partial}{\partial n}}$, *and hence it has the homology of a point, where* $\hat{C}_0^k$ *is as in Lemma 4.1 and* $\hat{C}_1^k$ *is as in Lemma 4.2.*

Next, we turn to the study of the topology of very negative sublevels of $\mathcal{E}_g$ when $k \geq 2$ or $\bar{k} \geq 1$. Indeed, as in [2] and for the same reasons, we have that the well-know topology of very negative sublevels in the *nonresont* case (see [21]), Proposition 3.7, Lemma 4.1 and Corollary 4.2 imply the following lemma which gives the homotopy type of the very negative sublevels of the Euler-Lagrange functional $\mathcal{E}_g$.

Lemma 4.4. *Assuming that $k \geq 2$ or $\bar{k} \geq 1$, and η is a small positive real number with $0 < 2\eta < \varrho$ where ϱ is as in (43), then there exists a large positive constant $L_{k,\bar{k}} := L_{k,\bar{k}}(\eta)$ with $L_{k,\bar{k}} > 2\max\{\hat{C}_0^k, \hat{C}_1^k\}$ such that for every $L \geq L_{k,\bar{k}}$, we have that $(\mathcal{E}_g)^{-L}$ has the same homotopy type as $B_{k-1}(\partial M)$ if $k \geq 2$ and $\bar{k} = 0$, as $A_{k-1,\bar{k}}$ if $k \geq 2$ and $\bar{k} \geq 1$ and as $S^{\bar{k}-1}$ if $k = 1$ and $\bar{k} \geq 1$, where $\hat{C}_0^k$ is as in Lemma 4.1 and $\hat{C}_1^k$ as in Lemma 4.2.*

However, as in [23], to prove Theorem 1.9, we need a further information about the topology of very negative sublevels of $\mathcal{E}_g$. In order to derive that, we first make some definitions. For $p \in \mathbb{N}^*$ and $\lambda > 0$, we define

$$f_p(\lambda) : B_p(\partial M) \longrightarrow \mathcal{H}_{\frac{\partial}{\partial n}}$$

as follows

$$f_p(\lambda)\left(\sum_{i=1}^p \alpha_i \delta_{a_i}\right) := \sum_{i=1}^p \alpha_i \varphi_{a_i,\lambda}, \quad \sigma = \sum_{i=1}^p \alpha_i \delta_{a_i} \in B_p(\partial M), \tag{115}$$

with the $\varphi_{a_i,\lambda}$'s defined by (45). Furthermore, when $\bar{k} \geq 1$, for $\Theta > 0$, we define

$$\Psi_{p,\bar{k}}(\lambda, \Theta) : A_{p,\bar{k}} \longrightarrow \mathcal{H}_{\frac{\partial}{\partial n}} \tag{116}$$

as follows

$$\Psi_{p,\bar{k}}(\lambda, \Theta)(\sigma, s) := \begin{cases} \varphi_s + f_p(\lambda)(\sigma) & \text{for } |s| \leq \frac{1}{4}, \ \sigma \in B_p(\partial M), \\ \varphi_s + f_p(2\lambda - 1 + 4(1-\lambda)|s|)(\sigma) & \text{for } \frac{1}{4} \leq |s| \leq \frac{1}{2}, \ \sigma \in B_p(\partial M), \\ \varphi_s + 2(1 - f_p(1)(\sigma))|s| + 2f_p(1) - 1 & \text{for } |s| \geq \frac{1}{2}, \ \sigma \in B_p(\partial M), \end{cases} \tag{117}$$

where φ_s is defined by the following formula

$$\varphi_s = \Theta \sum_{r=1}^{\bar{k}} s_r (v_r - \overline{(v_r)}_{(Q,T)}), \tag{118}$$

with $s = (s_1, \ldots, s_{\bar{k}})$. As in [23], concerning the $f_p(\lambda)$'s, we have the following estimates.

Lemma 4.5. *Assuming that $p \in \mathbb{N}^*$, then we have*

(1) *If $p < k$, then for every $L > 0$, there exists $\lambda_p^L > 0$ such that for all $\lambda \geq \lambda_p^L$, we have*

$$f_p(\lambda)(B_p(\partial M)) \subset (\mathcal{E}_g)^{-L}.$$

(2) *If* $p = k$*, then there exist* $\hat{C}_k > 0$ *and* $\lambda_k > 0$ *such that for all* $\lambda \geq \lambda_k$*, we have*

$$f_k(\lambda)(B_k(\partial M)) \subset (\mathcal{E}_g)^{\hat{C}_k}.$$

(3) *There exists* $\hat{C}^k > 0$ *such that up to taking* ϵ_0 *smaller, where* ϵ_0 *is given by (82), we have that for every* $0 < \epsilon \leq \epsilon_0$*, there holds*

$$V(k, \epsilon) \subset (\mathcal{E}_g)^{\hat{C}^k}.$$

Proof. It follows from the same arguments as in the proof of Lemma 3.1 in [23] by using Lemmas 5.1, 5.3 and 5.6–5.8. □

Still, as in [23], we have the following estimates for the $\Psi_p(\lambda, \Theta)$'s when $\bar{k} \geq 1$.

Lemma 4.6. *Assuming that* $p \in \mathbb{N}^*$*, then we have*

(1) *If* $1 \leq p < k$*, then for every* $L > 0$*, there exists* $\lambda^L_{p,\bar{k}} > 0$ *and* $\Theta^L_{p,\bar{k}} > 0$ *such that for all* $\lambda \geq \lambda^L_{p,\bar{k}}$*, we have*

$$\Psi_{p,\bar{k}}(\lambda, \Theta^L_{p,\bar{k}})(A_{p,\bar{k}}) \subset (\mathcal{E}_g)^{-L}.$$

(2) *If* $p = k$ *and* $\Theta > 0$*, then there exists* $C^\Theta_{k,\bar{k}} > 0$*,* $\lambda^\Theta_{k,\bar{k}} > 0$*, such that for every* $\lambda \geq \lambda^\Theta_{k,\bar{k}}$*, we have*

$$\Psi_{k,\bar{k}}(\lambda, \Theta)(A_{k,\bar{k}}) \subset (\mathcal{E}_g)^{C^\Theta_{k,\bar{k}}}.$$

(3) *If* $\Theta > 0$*, then there exists* $C^{k,\bar{k}}_\Theta > 0$ *such that up to taking* ϵ_0 *smaller, where* ϵ_0 *is given by (82), we have that for every* $0 < \epsilon \leq \epsilon_0$*, there holds*

$$V(k, \epsilon, \Theta) \subset (\mathcal{E}_g)^{C^{k,\bar{k}}_\Theta}.$$

Proof. It follows from the same arguments as in the proof of Lemma 4.1 in [23] by using Lemma 4.5. □

On the other hand, as in [23], Lemma 4.4 and Lemma 4.5 imply the following one:

Lemma 4.7. *Assuming that* $k \geq 2$*,* $\bar{k} = 0$*, and* $L \geq L_{k.0}$*, then there exists* λ^L_{k-1} *such that for all* $\lambda \geq \lambda^L_{k-1}$*, we have*

$$f_{k-1}(\lambda) : B_{k-1}(\partial M) \longrightarrow (\mathcal{E}_g)^{-L}$$

is well defined and induces an isomorphism in homology.

Furthermore, still as in [23], we have also that Lemmas 4.4 and 4.5 imply the following one:

Lemma 4.8. *Assuming that* $k \geq 2$*,* $\bar{k} \geq 1$*,* $L \geq L_{k,,\bar{k}}$*, then there exists* $\lambda^L_{k-1,\bar{k}} > 0$ *and* $\Theta^L_{k-1,\bar{k}} > 0$ *such that for all* $\lambda \geq \lambda^L_{k-1,\bar{k}}$*, we have*

$$\Psi_{k-1,\bar{k}}(\lambda, \Theta^L_{k-1,\bar{k}}) : A_{k-1,\bar{k}} \longrightarrow (\mathcal{E}_g)^{-L}$$

is well defined and induces an isomorphism in homology.

4.2. Morse theoretical type results

Proof of Theorems 1.1–1.7. The proof is the same as the one of Theorem 1.1-Theorem 1.6 in [2] by using Lemma 3.8, Proposition 3.22, Corollary 3.23, Lemmas 3.24, 4.1, Corollary 4.2, Lemmas 4.3, 4.4 combined with the works of Bahri–Rabinowitz [4], Karell–Karoui [13] and Malchiodi [18]. □

4.3. Algebraic topological type results

In order to carry the algebraic topological argument for existence, as in [23], we need the following lemma.

Lemma 4.9. *Assuming that* (ND) *holds,* $s_k^*(O_{\partial M}^*) \neq 0$ *in* $H^3(S^\infty)$ *and* $s_k^*(O_{\partial M}^*) = 0$ *in* $H^3(S_+^\infty \cup S_-^\infty)$*, then there exists* $0 \neq \tilde{O}_{\partial M}^* \in H^3(S)$ *such that*

$$i^*(\tilde{O}_{\partial M}^*) = s_k^*(O_{\partial M}^*),$$

where $i : S^\infty \longrightarrow S$ *is the canonical injection.*

Proof. It follows from the same arguments as in the proof of Lemma 3.6 in [23] by using the analysis of Sect. 3. □

Proof of Theorem 1.9. The proof is the same as the one of Theorem 1.1 in [23] by using the algebraic topological tools (54)–(56), characterization of the critical points at infinity of $\mathcal{E}_g$ established in Sect. 3, and Lemma 4.9. □

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5. Appendix

Lemma 5.1. *Assuming that* ϵ *is positive and small,* $a \in \partial M$ *and* $\lambda \geq \frac{1}{\epsilon}$*, then*

(1)

$$\varphi_{a,\lambda}(\cdot) = \hat{\delta}_{a,\lambda}(\cdot) + \log\frac{\lambda}{2} + H(a,\cdot) + \frac{1}{2\lambda^2}\Delta_{\hat{g}_a} H(a,\cdot) + O\left(\frac{1}{\lambda^3}\right) \quad \text{on } \partial M$$

(2)

$$\lambda\frac{\partial\varphi_{a,\lambda}(\cdot)}{\partial\lambda} = \frac{2}{1+\lambda^2\chi_\varrho^2(d_{\hat{g}_a}(a,\cdot))} - \frac{1}{\lambda^2}\Delta_{\hat{g}_a} H(a,\cdot) + O\left(\frac{1}{\lambda^3}\right) \quad \text{on } \partial M,$$

(3)

$$\frac{1}{\lambda}\frac{\partial\varphi_{a,\lambda}(\cdot)}{\partial a} = \frac{\chi_\varrho(d_{\hat{g}_a}(a,\cdot))\chi_\varrho^{'}((d_{\hat{g}_a}(a,\cdot))}{d_{\hat{g}_a}(a,\cdot)}\frac{2\lambda exp_a^{-1}(\cdot)}{1+\lambda^2\chi_\varrho^2(d_{\hat{g}_a}(a,\cdot))} + \frac{1}{\lambda}\frac{\partial H(a,\cdot)}{\partial a}$$

$$+O\left(\frac{1}{\lambda^3}\right); \text{on } \partial M,$$

where $O(1)$ *means* $O_{a,\lambda,\epsilon}(1)$ *and for it meaning see Sect. 2.*

Lemma 5.2. *Assuming that* ϵ *is small and* d *positive,* $a \in M$, $\lambda \geq \frac{1}{\epsilon}$, *and* $0 < 2\eta < \varrho$ *with* ϱ *as in* (43), *then there holds*

$$\varphi_{a,\lambda}(\cdot) = G(a,\cdot) + \frac{1}{2\lambda^2}\Delta_{\hat{g}_a}G(a,\cdot) + O\left(\frac{1}{\lambda^3}\right) \text{ on } \partial M \setminus B^a_a(\eta),$$

$$\lambda\frac{\partial\varphi_{a,\lambda}(\cdot)}{\partial\lambda} = -\frac{1}{\lambda^2}\Delta_{\hat{g}_a}G(a,\cdot) + O\left(\frac{1}{\lambda^3}\right) \text{ on } \partial M \setminus B^a_a(\eta),$$

and

$$\frac{1}{\lambda}\frac{\partial\varphi_{a,\lambda}(\cdot)}{\partial a} = \frac{1}{\lambda}\frac{\partial G(a,\cdot)}{\partial a} + O\left(\frac{1}{\lambda^3}\right) \text{ on } \partial M \setminus B^a_a(\eta),$$

where $O(1)$ *means* $O_{a,\lambda,\epsilon}(1)$ *and for it meaning see Sect. 2.*

Lemma 5.3. *Assuming that* ϵ *is small and positive,* $a \in \partial M$ *and* $\lambda \geq \frac{1}{\epsilon}$, *then there holds*

$$\mathbb{P}^{4,3}_g\left(\varphi_{a,\lambda}, \varphi_{a,\lambda}\right) = 16\pi^2\log\lambda - 8\pi^2 C_0 + 8\pi^2 H(a,a) + \frac{8\pi^2}{\lambda^2}\Delta_{\hat{g}_a}H(a,a)$$
$$+O\left(\frac{1}{\lambda^3}\right),$$

$$\mathbb{P}^{4,3}_g\left(\varphi_{a,\lambda}, \lambda\frac{\varphi_{a,\lambda}}{\partial\lambda}\right) = 8\pi^2 - \frac{8\pi^2}{\lambda^2}\Delta_{\hat{g}_a}H(a,a) + O\left(\frac{1}{\lambda^3}\right),$$

$$\mathbb{P}^{4,3}_g\left(\varphi_{a,\lambda}, \frac{1}{\lambda}\frac{\varphi_{a,\lambda}}{\partial a}\right) = \frac{8\pi^2}{\lambda}\frac{\partial H(a,a)}{\partial a} + O\left(\frac{1}{\lambda^3}\right),$$

where C_0 *is a positive constant,* $O(1)$ *means* $O_{a,\lambda,\epsilon}(1)$ *and for its meaning see Sect. 2.*

Lemma 5.4. *Assuming that* ϵ *is small and positive* $a_i, a_j \in \partial M$, $d_{\hat{g}}(a_i,a_j) \geq 4\overline{C}\eta$, $0 < 2\eta < \varrho$, $\frac{1}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda$, *and* $\lambda_i, \lambda_j \geq \frac{1}{\epsilon}$, $\overline{C}$ *as in* (41), *and* ϱ *as in* (43), *then there hold*

$$\mathbb{P}^{4,3}_g\left(\varphi_{a_i,\lambda_i}, \varphi_{a_j,\lambda_j}\right) = 8\pi^2 G(a_j,a_i) + \frac{4\pi^2}{\lambda_i^2}\Delta_{\hat{g}_{a_i}}G(a_i,a_j) + \frac{4\pi^2}{\lambda_j^2}\Delta_{\hat{g}_{a_j}}G(a_j,a_i)$$
$$+ O\left(\frac{1}{\lambda_i^3} + \frac{1}{\lambda_j^3}\right),$$

$$\mathbb{P}^{4,3}_g\left(\varphi_{a_i,\lambda_i}, \lambda_j\frac{\partial\varphi_{a_j,\lambda_j}}{\partial\lambda_j}\right) = -\frac{8\pi^2}{\lambda_j^2}\Delta_{\hat{g}_{a_j}}G(a_j,a_i) + O\left(\frac{1}{\lambda_j^3}\right),$$

and

$$\mathbb{P}^{4,3}_g\left(\varphi_{a_i,\lambda_i}, \frac{1}{\lambda_j}\frac{\partial\varphi_{a_j,\lambda_j}}{\partial a_j}\right) = \frac{8\pi^2}{\lambda_j}\frac{\partial G(a_j,a_i)}{\partial a_j} + O\left(\frac{1}{\lambda_j^3}\right),$$

where $O(1)$ *means here* $O_{A,\bar{\lambda},\epsilon}(1)$ *with* $A=(a_i,a_j)$ *and* $\bar{\lambda}=(\lambda_i,\lambda_j)$ *and for the meaning of* $O_{A,\bar{\lambda},\epsilon}(1)$*, see Sect. 2.*

Lemma 5.5. *Assuming that* $\epsilon>0$ *is very small, we have that for* $a\in\partial M$, $\lambda\geq\frac{1}{\epsilon}$*, there holds*

$$\left\|\lambda\frac{\partial\varphi_{a,\lambda}}{\partial\lambda}\right\|_{\mathbb{P}^{4,3}}=\tilde{O}(1), \tag{119}$$

$$\left\|\frac{1}{\lambda}\frac{\partial\varphi_{a,\lambda}}{\partial a}\right\|_{\mathbb{P}^{4,3}}=\tilde{O}(1), \tag{120}$$

and

$$\left\|\frac{1}{\sqrt{\log\lambda}}\varphi_{a,\lambda}\right\|_{\mathbb{P}^{4,3}}=\tilde{O}(1), \tag{121}$$

where here $\tilde{O}(1)$ *means bounded by positive constants form below and above independent of* ϵ*,* a*, and* λ*.*

Lemma 5.6. (1) *If* ϵ *is small and positive,* $a\in\partial M$*,* $p\in\mathbb{N}^*$*, and* $\lambda\geq\frac{1}{\epsilon}$*, then there holds*

$$C^{-1}\lambda^{6p-3}\leq\oint_{\partial M}e^{3p\varphi_{a,\lambda}}dS_g\leq C\lambda^{6p-3}, \tag{122}$$

where C *is independent of* a*,* λ*, and* ϵ*.*

(2) *If* ϵ *is positive and small,* $a_i,a_j\in\partial M$*,* $\lambda\geq\frac{1}{\epsilon}$ *and* $\lambda d_{\hat{g}}(a_i,a_j)\geq 4\overline{C}R$*, then we have*

$$\mathbb{P}^{4,3}_g(\varphi_{a_i,\lambda},\,\varphi_{a_j,\lambda})\,\leq\,8\pi^2G(a_i,a_j)\,+\,O(1), \tag{123}$$

where $O(1)$ *means here* $O_{A,\lambda,\epsilon}(1)$ *with* $A=(a_i,a_j)$*, and for the meaning of* $O_{A,\lambda,\epsilon}(1)$*, see Sect. 2.*

(3) *If* ϵ *is positive and small,* $a_i,a_j\in\partial M$*,* $\lambda_i,\lambda_j\geq\frac{1}{\epsilon}$*,* $\frac{1}{\Lambda}\leq\frac{\Lambda_i}{\lambda_j}\leq\Lambda$ *and* $\lambda_i d_{\hat{g}}(a_i,a_j)\geq 4\overline{C}R$*, then we have*

$$\mathbb{P}^{4,3}_g(\varphi_{a_i,\lambda},\,\varphi_{a_j,\lambda})\,\leq\,8\pi^2G(a_i,a_j)\,+\,O(1), \tag{124}$$

where $O(1)$ *means here* $O_{A,\bar{\lambda},\epsilon}(1)$ *with* $A=(a_i,a_j)$ *and* $\bar{\lambda}=(\lambda_i,\lambda_j)$ *and for the meaning of* $O_{A,\bar{\lambda},\epsilon}(1)$*, see Sect. 2.*

Lemma 5.7. *Let* $p\in\mathbb{N}^*$*,* $\hat{R}$ *be a large positive constant,* ϵ *be a small positive number,* $\alpha_i\geq 0$*,* $i=1,\ldots,p$*,* $\sum_{i=1}^p\alpha_i=k$*,* $\lambda\geq\frac{1}{\epsilon}$ *and* $u=\sum_{i=1}^p\alpha_i\varphi_{a_i,\lambda}$*. Assuming that there exist two positive integer* $i,j\in\{1,\ldots,p\}$ *with* $i\neq j$ *such that* $\lambda d_{\hat{g}}(a_i,a_j)\leq\frac{\hat{R}}{4\overline{C}}$*, where* $\overline{C}$ *is as in (41), then we have*

$$\mathcal{E}_g(u)\,\leq\mathcal{E}_g(v)\,+\,O(\log\hat{R}), \tag{125}$$

with

$$v:=\sum_{k\leq p,k\neq i,j}\alpha_k\varphi_{a_k,\lambda}+(\alpha_i+\alpha_j)\varphi_{a_i,\lambda}.$$

where here $O(1)$ *stand for* $O_{\bar{\alpha},A,\lambda,\epsilon}(1)$*, with* $\bar{\alpha}=(\alpha_1,\ldots,\alpha_p)$ *and* $A=(a_1,\ldots,a_p)$*, and for the meaning of* $O_{\bar{\alpha},A,\lambda,\epsilon}(1)$*, we refer the reader to Sect. 2.*

Lemma 5.8. (1) *If* ϵ *is positive and small,* $a_i, a_j \in \partial M$*,* $\lambda \geq \frac{1}{\epsilon}$ *and* $\lambda d_{\hat{g}}(a_i, a_j) \geq 4\overline{C}R$*, then*

$$\varphi_{a_j,\lambda}(\cdot) = G(a_j, \cdot) + O(1) \ \ in \ B^{a_i}_{a_i}\left(\frac{R}{\lambda}\right),$$

where here $O(1)$ *means here* $O_{A,\lambda,\epsilon}(1)$*, with* $A = (a_i, a_j)$*, and for the meaning of* $O_{A,\lambda,\epsilon}(1)$*, see Sect.* 2*.*

(2) *If* ϵ *is positive and small,* $a_i, a_j \in \partial M$*,* $\lambda_i, \lambda_j \geq \frac{1}{\epsilon}$*,* $\frac{1}{\Lambda} \leq \frac{\Lambda_i}{\lambda_j} \leq \Lambda$*, and* $\lambda_i d_{\hat{g}}(a_i, a_j) \geq 4\overline{C}R$*, then*

$$\varphi_{a_j,\lambda_j}(\cdot) = G(a_j, \cdot) + O(1) \ \ in \ B^{a_i}_{a_i}\left(\frac{R}{\lambda_i}\right),$$

where here $O(1)$ *means here* $O_{A,\bar{\lambda},\epsilon}(1)$*, with* $A = (a_i, a_j)$*,* $\bar{\lambda} = (\lambda_i, \lambda_j)$ *and for the meaning of* $O_{A,\lambda,\epsilon}(1)$*, see Sect.* 2*.*

Lemma 5.9. *There exists* Γ_0 *and* $\tilde{\Lambda}_0$ *two large positive constant such that for every* $a \in \partial M$*,* $\lambda \geq \tilde{\Lambda}_0$*, and* $w \in F_{a,\lambda} := \{w \in \mathcal{H}_{\frac{\partial}{\partial n}}, \overline{w}_{(Q,T)} = \langle \varphi_{a,\lambda}, w\rangle_{\mathbb{P}^{4,3}} = \langle v_r, w\rangle_{\mathbb{P}^{4,3}} = 0,\ r = 1, \ldots, \bar{k}\}$*, we have*

$$\oint_{\partial M} e^{3\hat{\delta}_{a,\lambda}} w^2 dV_{g_a} \leq \Gamma_0 \|w\|^2_{\mathbb{P}^{4,3}}. \tag{126}$$

Lemma 5.10. *Assuming that* η *is a small positive real number with* $0 < 2\eta < \varrho$ *where* ϱ *is as in* (43)*, then there exists a small positive constant* $c_0 := c_0(\eta)$ *and* $\Lambda_0 := \Lambda_0(\eta)$ *such that for every* $a_i \in \partial M$ *concentrations points with* $d_{\hat{g}}(a_i, a_j) \geq 4\bar{C}\eta$ *where* $\bar{C}$ *is as in* (41)*, for every* $\lambda_i > 0$ *concentrations parameters satisfying* $\lambda_i \geq \Lambda_0$*, with* $i = 1, \ldots, k$*, and for every* $w \in E^*_{A,\bar{\lambda}} = \cap_{i=1}^k E^*_{a_i,\lambda_i}$ *with* $A := (a_1, \ldots, a_k)$*,* $\bar{\lambda} := (\lambda_1, \ldots, \lambda_k)$ *and* $E^*_{a_i,\lambda_i} = \{w \in \mathcal{H}_{\frac{\partial}{\partial n}} : \langle \varphi_{a_i,\lambda_i}, w\rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial \lambda_i}, w\right\rangle_{\mathbb{P}^{4,3}} = \left\langle \frac{\partial \varphi_{a_i,\lambda_i}}{\partial a_i}, w\right\rangle_{\mathbb{P}^{4,3}} = \overline{w}_{(Q,T)} = \langle v_r, w\rangle_{\mathbb{P}^{4,3}} = 0,\ r = 1, \ldots, \bar{k}\}$*, there holds*

$$\|w\|^2_{\mathbb{P}^{4,3}} - 6\sum_{i=1}^k \oint_{\partial M} e^{3\hat{\delta}_{a_i,\lambda_i}} w^2 dS_{g_{a_i}} \geq c_0 \|w\|^2_{\mathbb{P}^{4,3}}. \tag{127}$$

References

[1] Ahmedou, M., Kallel S., Ndiaye, C.B.: The resonant boundary Q-curvature problem and boundary-weighted barycenters. arXiv:1604.03745

[2] Ahmedou, M., Ndiaye, C.B.: Morse theory and the resonant Q-curvature problem. arxiv:1409.7919

[3] Bahri, A., Coron, J.M.: On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. Commun. Pure Appl. Math. **41–3**, 253–294 (1988)

[4] Bahri, A., Rabinowitz, P.: Periodic solutions of Hamiltonian systems of 3-body. Ann. Inst. Poincaré, Anal. Non Linéaire **8**, 561–649 (1991)

[5] Branson, T.P., Oersted, B.: Explicit functional determinants in four dimensions. Proc. Amer. Math. Soc. **113**(3), 669–682 (1991)

[6] Chang, S.Y.A., Qing, J., Yang, P.C.: On the Chern–Gauss–Bonnet integral for conformal metrics on R^4. Duke Math. J. **03**(3), 523–544 (2000)

[7] Chang, S.Y.A., Qing, J.: The Zeta Functional Determinants on manifolds with boundary 1. The Formula. J. Funct. Anal. **147**, 327–362 (1997)

[8] Chang, S.Y.A., Qing, J.: The Zeta Functional Determinants on manifolds with boundary II. Extremal Metrics and Compactness of Isospectral Set. J. Funct. Anal. **147**, 363–399 (1997)

[9] Chang, S.Y.A., Qing, J., Yang, P.C.: Compactification of a class of conformally flat 4-manifold. Invent. Math. **142–1**, 65–93 (2000)

[10] Chen, C.C., Lin, C.S.: Topological degree for a mean field equation on Riemann surfaces. Commun. Pure. Appl. Math **56–12**, 1667–1727 (2003)

[11] Djadli, Z., Malchiodi, A.: Existence of conformal metrics with constant Q-curvature. Ann. Math. (2) **168**(3), 813–858 (2008)

[12] Jin, T., Li, Y., Xiong, J.: The Nirenberg problem and its generalizations: a unified approach. Math. Ann. **369**(1–2), 109–151 (2017)

[13] Kallel, S., Karoui, R.: Symmetric joins and weighted barycenters. Adv. Nonlinear Stud. **11**, 117–143 (2011)

[14] Li, Y., Xiong, J.: Compactness of conformal metrics with constant Q-curvature. I. Adv. Math. **345**, 116–160 (2019)

[15] Lin, C.S., Wei, J.: Sharp estimates for bubbling solutions of a fourth order mean field equation. Ann. Sci. Norm. Super. Pisa Cl. Sci. (5) **6**(4), 599–630 (2007)

[16] Lucia, M.: A deformation lemma with an application to a mean field equation. Topol. Methods Nonlinear Anal. **30**(1), 113–138 (2007)

[17] Malchiodi, A.: Compactness of solutions to some geometric fourth-order equations. J. Reine Angew. Math. **594**, 137–174 (2006)

[18] Malchiodi, A.: Morse theory and a scalar field equation on compact surfaces. Adv. Differ. Equ. **13**, 1109–1129 (2008)

[19] Ndiaye, C.B.: Constant Q-curvature metrics in arbitrary dimension. J. Funct. Anal. **251**(1), 1–58 (2007)

[20] Ndiaye, C.B.: Conformal metrics with constant Q-curvature for manifolds with boundary. Commun. Anal. Geom. **16**(5), 1049–1124 (2008)

[21] Ndiaye, C.B.: Constant T-curvature conformal metric on 4-manifolds with boundary. Pac. J. Math. **240**(1), 151–184 (2009)

[22] Ndiaye, C.B.: Q-curvature flow on manifolds with boundary. Math. Z. **269**, 83–114 (2011)

[23] Ndiaye, C.B.: Algebraic topological methods for the supercritical Q-curvature problem. Adv. Math. **277**(277), 56–99 (2015)

[24] Ndiaye, C.B.: Topological methods for the resonant Q-curvature problem in arbitrary even dimension. J. Geom. Phys. **140**, 178–213 (2019)

[25] Ndiaye, C.B., Xiao, J.: An upper bound of the total Q-curvature and its isoperimetric deficit for higher-dimensional conformal Euclidean metrics. Calc. Var. Part. Differ. Equ. **38**(1–2), 1–27 (2010)

[26] Paneitz, S.: A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Preprint (1983)

[27] Weinstein, G., Zhang, L.: The profile of bubbling solutions of a class of fourth order geometric equations on 4-manifolds. J. Funct. Anal. **257**(12), 3895–3929 (2009)

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