Bayesian Calibrated Click-Through Auctions

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Abstract

We study information design in *click-through auctions*, in which the bidders/advertisers bid for winning an opportunity to show their ads but only pay for realized clicks. The payment may or may not happen, and its probability is called the *click-through rate* (CTR). This auction format is widely used in the industry of online advertising. Bidders have private values, whereas the seller has private information about each bidder's CTRs. We are interested in the seller's problem of partially revealing CTR information to maximize revenue. Information design in click-through auctions turns out to be intriguingly different from almost all previous studies in this space since any revealed information about CTRs will never affect bidders' bidding behaviors – they will always bid their true value per click – but only affect the auction's *allocation* and *payment* rule. In some sense, this makes information design effectively a constrained mechanism design problem.

Our first result is an FPTAS to compute an approximately optimal mechanism under a constant number of bidders. The design of this algorithm leverages Bayesian bidder values which help to "smooth" the seller's revenue function and lead to better tractability. The design of this FPTAS is complex and primarily algorithmic. Our second main result pursues the design of "simple" mechanisms that are approximately optimal yet more practical. We primarily focus on the two-bidder situation, which is already notoriously challenging as demonstrated in recent works. When bidders' CTR distribution is symmetric, we develop a simple *prior-free* signaling scheme, whose construction relies on a parameter termed *optimal signal ratio*. The constructed scheme provably obtains a good approximation as long as the maximum and minimum of bidders' value density functions do not differ much.

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1 Introduction

Many of the phenomenal Internet Technology companies are powered by online advertising [25]. When an Internet user browses a webpage, an ad auction may be run to determine which ads to be displayed to this user. Such ad auctions can be either done by the webpage owner or through ad exchange platforms. Large website owners (sellers) sometimes may know the users much better than individual advertisers (bidders) do. It is thus natural for the seller to utilize such information advantage to improve its ad revenue. As an example, the seller's knowledge about the user can be used to better predict an Internet user's *click-through rate* (CTR), which is then used in ad auctions to deliver ad impressions more efficiently.

Interestingly, a recent line of works, starting from Bro Miltersen and Sheffet [14], Emek et al. [26], comes to realize that it may not always be the seller's best interest to use as much information as possible. That is, partially use certain amount – and the right type – of information may be more beneficial for the seller's revenue. This insight has motivated many studies on the algorithmic problem of optimal *information design* – i.e., determining what information to be shared with which bidders – in order to maximize revenue in different auction formats [14, 26, 12, 38, 30, 5].

While there are two main types of bidding strategies in online advertising: autobidding [20, 7, 6] and manual bidding, to better understand the optimal design of information in an auction environment, we follow the rich literature and turn to the cleaner manual bidding where the platform incentivizes buyers to tell the truth. However, different from almost all previous works focusing on stylized auction formats such as the second price auction for independent-value bidders [14, 26, 12, 5] and common-value bidders [38] and Myerson's optimal auction [30], we focus on a different auction format, i.e., the *click-through auction*, which is the main auction format employed by the current online advertising industry [27]. The click-through auction is also widely known as the generalized second-price auction [25] or position auctions [44]. We use the term "click-through auction" as coined by Bergemann et al. [11] to emphasize the sale of *clicks*, since this factor turns out to make information design in click-through auctions significantly different from that in almost all other previously studied auctions. In a click-through auction for selling a single ad position, each bidder i submits a bid b_i expressing his value for each *click* of his ad. Additionally, the seller will estimate a *click through rate* (CTR) r_i for bidder i. The auction runs by ranking bidders according to the *product score* $b_i r_i$, denoted as $b_{(1)}r_{(1)} \geq b_{(2)}r_{(2)} \geq \cdots$, and allocates the item to the bidder with the highest product score $b_{(1)}r_{(1)}$. Crucially, since the auction only sells clicks, the winner does not need to pay *unless a click truly happens* in which case the winner pays the second highest score divided by his own CTR, i.e., $b_{(2)}r_{(2)}/r_{(1)}$. When there is only a single ad slot to allocate, it is straightforward to verify that this auction is truthful, *regardless of the CTR values (even mistakenly estimated)*.[1] As a result, the seller's information advantage of knowing the CTRs *cannot* be exploited to influence the advertisers' bidding behaviors since it is always their best interest to bid the true value per click v_i. In some sense, information design here is effectively a naturally restricted format of mechanism design for multiple items with additive bidder values (which is generally a very difficult question [19]). This crucially differs from the information design task in all previously studied auction formats, in which the seller's information about the item (e.g., the CTR) can be revealed to alter bidders' bidding behaviors. This is intrinsically because bidders pay for realized clicks only thus the probability of receiving a click will not matter in click-through auctions. Note that, even in per-impression ad auctions as studied in [5], bidders pay for impressions but care about clicks or conversions, thus their values do depend on the probability of receiving a click, i.e., the CTR.

[1] Such truthfulness holds only when there is a single slot, which is the case we focus on in this paper. Click-through auctions are well-known to be non-truthful and difficult to analyze when there are multiple slots [25].

Most relevant to ours is the recent work by Bergemann et al. [11], who study the same information design question of optimally revealing the CTR information in click-through auctions. They focused on a simplified setup with *complete information* about the bidders, i.e., the bidders' values are assumed to be fully known to the seller. A natural *calibration* constraint, originating from Foster and Vohra [29], is imposed on their information design, which simply means the disclosed CTR estimation to each bidder has to be consistent with (i.e., equal in expectation) the true CTR of the bidder. They thus call the new model *calibrated click-through auctions*. Notably, the seller is allowed to privately communicate information with each bidder, which is known as private signaling. The main contribution of this paper is to extend the setup of [11] to the more natural and also more widely studied setup of *Bayesian* bidders with independent values. For this reason, We call our model the *Bayesian calibrated click-through auction*.

To our knowledge, Bergemann et al. [11] is the first study of information design in auctions in which the revealed information *cannot* influence bidders' behaviors, but only affect the mechanism itself. This gives rise to an intriguing information design problem – in some sense, it is even in contrast to one's first impression about information design, also known as *persuasion* [33], which seeks to exploit information advantage to influence others' behaviors. In contrast, information design in click-through auctions only affects the final allocation and payment of the mechanism but has no effect on bidders' bidding behaviors. From this perspective, information design here is effectively a form of mechanism design. Indeed, a similar phenomenon was observed by Daskalakis et al. [19], who study the *co-design* of information structure and the auction mechanism. They observe that at the optimal co-design, there is no signaling to bidders whatsoever, and the seller will only use the underlying state to decide the item allocation and payment. Our problem is similar, except that we restrict ourselves to the class of click-through auctions. This restriction is motivated by its practical applications.

Before proceeding to describe our results, we briefly highlight the challenges of information design in our problem. Indeed, the difficulty of information design in strategic games is well documented in previous works [22, 13]. Even just for auctions, Emek et al. [26] shows that computing an optimal information design in a second price auction is NP-hard in general for Bayesian bidder values, though it does become polynomial-time solvable in cases with complete information of bidder values. Unfortunately, such tractability does not transfer to the click-through auctions: in a general environment with complete information about bidder values, even the problem of optimal information design in *two*-bidder click-through auctions is left as an open question in [11]. Bergemann et al. [11] show that when the distributions of CTRs are symmetric, an optimal signaling policy can be characterized. However, back to our Bayesian generalization of their setup, their optimal design for the symmetric information environment is no longer applicable because the design of their signaling scheme relies crucially on knowing the exact identity of the winner for any signal realization, which unfortunately becomes uncertain in our Bayesian setup with random bidder values. This barrier brings challenges, but also brings opportunities to adopt more algorithmic approaches. Next, we elaborate on our findings.

1.1 Results

We study the theoretical aspects of Bayesian calibrated click-through auctions and develop two encouraging positive results.

Our first result, Theorem 3.1, exhibits a Fully Polynomial Time Approximation Scheme (FPTAS) for computing an approximately revenue-optimal signaling scheme in an *arbitrary* information environment, assuming no point mass in bidders' value distributions. This

FPTAS applies to any constant number of bidders. Interestingly, our FPTAS bypasses the notorious challenge of the complete-information general setup of [11] by relaxing it to the more natural Bayesian valuation setup with smooth value distributions. We use signals of the form $k\epsilon$ for some integer k. The key challenge is to enforce the calibration constraint – i.e., the posterior mean of the CTR conditioned on the signal has to equal the signal itself – in the rounding process. We develop a nontrivial technique to address the challenge. Notably, this FPTAS applies to any auction mechanism, as long as its revenue as a function of signals is Lipschitz continuous.

Our second main result, Theorem 4.2, examines the popular recent computational model of unknown value distributions, also known as *prior-free* setups [21]. We give explicit and efficient construction of simple *prior-free* signaling schemes for *symmetric* information environments (as studied also by Bergemann et al. [11]) with strong approximation guarantees. The constructed signaling scheme is based on a parameter coined the *optimal signal ratio*, which we find is at most 1. This allows us to send larger signals (i.e., larger CTR estimations) to the bidder with a high click-through rate than the bidder with a low click-through rate. The proof mainly consists of (i) establishing a connection between the optimal signaling scheme under the unknown value distribution and the optimal signaling scheme under a uniform value distribution and (ii) providing the approximation ratio of the constructed signaling scheme under a uniform value distribution, which is proved to be 0.995. En route to proving (ii), we prove a more general result (Proposition 4.11) which shows that, if the optimal signal ratio is convex in CTR (which is true for various commonly used distributions), then a signaling scheme with better approximation can be devised. This result may be of independent interest.

1.2 Additional Related Works

Due to the space limit, we briefly discuss the related works here, while additional discussions in detail are in Appendix A. The most relevant literature to our work is information design in auctions. Information design has been adopted to second price auctions [14, 26, 18, 5] and Myerson auctions [30] in various setups. Recently, Bergemann et al. [11] provided characterizations for a symmetric calibrated click-through auction but with complete information of bidder values. Another line of related literature is the sale of information, which selectively reveals information for revenue improvement. There are a series of works in this line [2, 17, 9, 15, 39, 46, 37, 10]. Our work is also related to Bayesian persuasion [34, 1, 4, 24, 3, 45, 31]. We also refer interested readers to a comprehensive survey by Dughmi [23]. Finally, our work is partly related to the literature on automated bidding in auctions [36, 35, 28, 40, 8].

2 Preliminaries

We consider the (arguably more natural) Bayesian version of the *calibrated click-through auction* of [11], which is directly motivated by advertising auctions. At a high level, bidders in this auction are ranked by the products of their *private values* (i.e., the willingness-to-pay per click) and the seller's prediction of *click-through-rate* (*abbr.* CTR). Bergemann et al. [11] consider a basic setup in which bidders' private values are perfectly known to the seller. In this paper, we generalize their problem to the Bayesian setup by assuming that bidders' values are independently drawn from distributions. While each bidder knows his own private value, the seller only knows the distributions of bidder values (though our second main result further removes this assumption). This generalization is more aligned with the rich literature of Bayesian auction design, starting from the seminal work of [41]. Similar to [11], we also aim to design a revenue-maximizing auction. Different from classic auction design, the seller in a click-through auction has an additional knob to tune, i.e., informing bidders about their CTRs through partial information disclosure (also known as *information design* [33]). So the optimal auction requires designs of both *incentives* and *information*.

We now describe the general auction setting. There are n bidders. The private value v_i of bidder (she) $i = 1, 2, 3, \ldots, n$ is independently drawn from some known distribution $F_i(v_i)$ (with a density function $f_i(v_i)$), over some bounded interval $[a, b]$ with $a, b \in R_{\geq 0} \cup \{\infty\}$ and $a < b$. Note that the density functions $f_1(v_1), f_2(v_2), \ldots, f_n(v_n)$ are not necessarily identical. As seen, our problem is a natural Bayesian variant of [11], whose settings can be obtained by allowing $f_i(v_i)$ to be some Dirac δ-function. i.e., a point distribution.

When auctions start, the seller privately observes a CTR vector $r = (r_1, r_2, \ldots, r_n) \in [\underline{r}, 1]^n$ for n bidders, which is drawn from a commonly known joint distribution $\lambda(r)$. $\underline{r} > 0$ is a small positive value close to 0. The CTR r_i represents the probability of bidder i's ad being clicked, which may be estimated by the seller with some machine learning method. For bidder i, assume the marginal probability $\lambda(r_i) \geq \xi$, where ξ is a small positive value since it does not make sense to consider a CTR with a marginal probability arbitrarily close to 0 in practice.

To maximize his revenue, the seller may privately disclose some information about r_i to bidder i. Suppose the seller has the power of commitment[2], and he designs a signaling scheme denoted as π, which is also observed by the bidders. Hence, conditioning on observing CTR vector r, the seller sends signal $s = (s_1, s_2, \ldots, s_n)$ with probability $\pi(s|r)$, where s_i is the signal privately sent to bidder i. In the rest of the paper, we assume that $r_i \in \mathcal{R}$ with $\mathcal{R}$ being a discrete and finite set of CTR values.

Calibrated Click-Through Auction. The calibrated click-through auction was previously studied by Bergemann et al. [11]. Initially, each bidder privately observes her value v_i and the seller privately observes the CTR vector r. The seller then sends a signal s_i to each bidder i privately. Upon receiving the signal s_i, bidder i submits her bid b_i to the auction and the auction determines the winner i^* by selecting the one with the highest product, i.e., $b_{i^*} s_{i^*} = \max_j b_j s_j$, and charges the winner for each realized click by

$$p_{i^*} = \max_{j \neq i^*} \frac{b_j s_j}{s_{i^*}}.$$

The winner only pays when a click is received[2]. Hence, the seller's expected revenue is $r_{i^*} p_{i^*}$. One can easily prove that the auction is truthful[2] for any $s_1, \ldots, s_n$, as it follows the "minimum-bid-to-win" payment rule. Therefore, without loss of generality, we will assume $b_i \equiv v_i$ in the rest of the paper.

Following [11], we adopt the concept *calibration* to information design. That is, given any private signal $s_i = s'_i$, bidder i's posterior estimation of CTR r_i should be equal to s'_i, i.e.,

$$E[r_i | s_i = s'_i] = s'_i. \tag{1}$$

The calibration constraint is a consequence of the revelation principle. Given any signal realization $s_i = s'_i$, bidder i will interpret s'_i through Bayes updates by computing the expected CTR value $E[r_i | s_i = s'_i]$. The calibration constraint simply requires that the signal

[2] More discussions about *pay-per-click*, commitment power and truthfulness are in Appendix B.

itself directly reflects this expected value so that bidders do not need to compute the expected CTR value by themselves. Such revelation-principle-style simplification is widely adopted in information design [34, 24, 33], where signals can without loss of generality be *persuasive* action recommendations. Analogously, in our auction setup, such direct information scheme will directly signal the posterior mean of the signal, i.e., obeying the calibration constraint (1). Hence, any signal s_i must be within $[\underline{r}, 1] \subseteq [0, 1]$ by (1). One important observation in [11] is that if the CTRs and signals are discrete, we can rewrite (1) as

$$\sum_{(r,s):s_i=s'_i} \lambda(r)\pi(s|r)(r_i - s'_i) = 0. \tag{2}$$

Bayesian Calibrated Click-Through Auction. We focus on the seller's information design problem, i.e., to design a signaling scheme π in order to maximize the seller's expected revenue. The problem is formulated as below,

$$\begin{aligned}
\max_{\pi,s} \quad & \sum_r \lambda(r) \sum_s \pi(s|r) \int_v f(v)R(r,v,s)dv \\
\text{subject to} \quad & E[r_i|s_i = s'_i] = s'_i \quad \forall s'_i \in [0,1], \forall i \\
& \pi(s|r) \in [0,1], \sum_s \pi(s|r) = 1, \ \forall r, s \in [0,1]^n,
\end{aligned} \tag{3}$$

where $f(v) = f_1(v_1)f_2(v_2)\cdots f_n(v_n)$ is the density function of $v = (v_1, v_2, \ldots, v_n)$, and the revenue $R(r,v,s) = r_{i^*}\frac{\max_{j\neq i^*} v_j s_j}{s_{i^*}}$. Since both the signals s and the probabilities of sending signals $\pi(s|r)$ are variables, the objective of (3) is non-convex.

Given CTR vector r and signal s, denote the seller's expected revenue as $R(r,s) = \int_v f(v)R(r,v,s)dv$. Specifically, $R(r,s)$ for 2 bidders is

$$\begin{aligned}
R(r,s) &= \int_v f(v)R(r,v,s)dv \\
&= \int_a^b r_1\frac{v_2 s_2}{s_1}\mathbf{Pr}\left(v_1s_1 \geq v_2s_2|v_2\right) f_2(v_2)dv_2 + \int_a^b r_2\frac{v_1 s_1}{s_2}\mathbf{Pr}\left(v_2s_2 \geq v_1s_1|v_1\right) f_1(v_1)dv_1,
\end{aligned} \tag{4}$$

where $\mathbf{Pr}\left(v_1s_1 \geq v_2s_2|v_2\right)$ is the probability of bidder 1 winning given that bidder 2's value is v_2 (similarly for $\mathbf{Pr}\left(v_2s_2 \geq v_1s_1|v_1\right)$).

By (4), we can see that our problem differs from [11] in a crucial way: With any fixed s, the winner is also fixed in [11] since they have complete information on v, while in our case, either bidder can be the winner with some probability due to the uncertainty in their valuations. This difference somewhat "smooths" our objective function while at the same time brings new challenges.

3 Click-Through Auctions in General Environments

In this section, we present an FPTAS for (3) achieving $1 - O(\epsilon)$ approximation when the number of bidders is a constant. This result hinges on a minor continuity assumption on bidders' value distribution. That is, we assume every distribution F_i has no point mass[3] and has finite second moment, i.e., $\mathbb{E}_{v_i\sim F_i}[v_i^2] < \infty$. Under these two mild technical assumptions, we prove the following theorem.

[3] The no-point-mass assumption alleviates the tie-breaking problem arising in the case of deterministic bidders' values. A discussion on this assumption is in Appendix D.

▶ **Theorem 3.1.** *For any small $\epsilon > 0$, there is an algorithm that computes a (multiplicative) $1 - O(\epsilon)$ approximate signaling scheme in $\mathrm{poly}(|\mathcal{R}|^n, (\frac{n}{\epsilon})^n)$ time where n is the number of bidders and $|\mathcal{R}|$ is the size of set $\mathcal{R}$ of CTR values.*

To prove Theorem 3.1, we resort to discretizing the strategy space. Our starting point is a reformulation of Program (3) as a linear program with infinite dimension, and then convert the infinite-dimensional program into a finite-dimensional one by discretizing the signal space. The main technical challenge is to prove that the solution to the discrete program approximates the optimal solution to the original infinite-dimensional one. This is more involved than typical rounding approaches and hinges on the following two key properties:

a). For any signaling scheme, the value of the discrete objective is sufficiently close to the original objective. This is a consequence of the following Lipschitz continuity of revenue function $R(r, s)$, whose proof is deferred to Appendix C.

▶ **Lemma 3.2.** *$R(r, s)$ is Lipschitz continuous in $s \in [\underline{r}, 1]^n$ with some constant C_n, for any given CTR $r = (r_1, r_2, \ldots, r_n)$.*

b). For any solution to the original infinite dimensional program, there exists a corresponding feasible solution to the discrete program that is sufficiently close to the original solution.

These two properties together can ensure the existence of a solution to the discrete program yielding the revenue that approximates the optimal solution of the original infinite dimensional program. Thus by solving the discrete program, we can get an approximately optimal solution.

The major challenge here is to prove property b), as any naïve rounding of a signaling scheme from the continuous signal space to a discrete space would break the calibration constraints and hence end up with an infeasible rounded signaling scheme to the discrete program. To circumvent the difficulty, we introduce a novel technique to retain all the calibration constraints.

At a high level, we will reserve a small amount of probability mass from the given signaling scheme at the beginning and only apply rounding to the remaining probability mass. After the rounding step, we will redistribute the reserved probability mass to carefully fix all the calibration constraints broken by the rounding step. In particular, one has to be extremely careful to avoid the straightforward discretization structure of the signal space, because otherwise a large fraction of probability mass reservation will be needed for the fixing stage, which leads to a significant revenue loss and fails the $1 - O(\epsilon)$ approximation. The detailed proof of Theorem 3.1 is involved and relegated to Appendix C.2.

It is worthwhile to compare our FPTAS in Theorem 3.1 with a recent FPTAS for multi-channel Bayesian persuasion by Babichenko et al. [4], and highlight their key differences. While both rely on the continuity of the sender's utility functions, their techniques are significantly different. Specifically, the FPTAS of [4] applies to a constant number of states but many receivers (i.e., bidders) whereas our FPTAS applies to many states but a constant number of bidders. Their FPTAS discretizes the space of distributions over the states (thus requires a constant number of states) whereas our FPTAS discretizes the space of posterior CTR mean, which is why we have to adjust the scheme to satisfy the calibration constraints. Our choice of discretizing posterior means is due to its direct usage in click-through auctions. Such a special structure and the resultant challenges of fixing calibration constraint violation – which is the core difficulty in our proof – is not present in the setup of [4].

We remark that the discretization technique applied to Theorem 3.1 is in fact regardless of the expected revenue $R(r, s)$. To retain the calibration constraint and preserve the approximation simultaneously, one basic requirement is that $R(r, s)$ needs to be Lipschitz-continuous in signal s. Therefore, our algorithm can be treated as a framework that can accommodate different auctions (with Lipschitz continuous expected revenue $R(r, s)$) and maintain the calibration of signals. Based on this observation, we consider one simple modified auction: a click-through auction with a reserved price. Reserve prices are widely studied in the literature and commonly used in the industry [43, 42]. Similarly, we only need to show $R(r,s)$ of this modified auction is Lipschitz-continuous for Corollary 3.3 to be true, which is deferred to Appendix C.3.

▶ **Corollary 3.3.** *In a click-through auction where winner pays at least some fixed reserve price p, for any constant number of bidders and any small $\epsilon > 0$, there is an algorithm that computes (multiplicatively) $1 - O(\epsilon)$ approximate signaling scheme with time complexity $poly(|\mathcal{R}|, \frac{1}{\epsilon})$.*

4 Click-Through Auctions in Symmetric Environments

While Theorem 3.1 provides an algorithm that can compute an approximately optimal signaling scheme in fairly general setups, the algorithm relies on solving large-scale linear programs thus may be too costly in reality and also lacks interpretability. In this section, we pursue the design of "simple" signaling schemes through the approximation lens. Given the challenge of the problem in general, we shall restrict our attention to a fundamental special case in this section – i.e., 2 bidders and symmetric environments – which is also a major focus of our preceding work by Bergemann et al. [11]. For this basic case, we first characterize that without calibration constraint, the optimal information design is governed by a single parameter called *optimal signal ratio*, which is at most 1. Such an observation motivates an explicit construction of a simple and *prior-free* signaling scheme, which is close to the optimal when bidder values' probability density function does not fluctuate much.

An environment is *symmetric* [11] if 1) the distribution of the products $v_1r_1, v_2r_2, \dots, v_nr_n$ are exchangeable, and 2) the values v_i's are i.i.d. drawn from a distribution $F(v)$. Hence, the symmetric environment further implies that the distribution of $r_1, r_2, \dots, r_n$ are also exchangeable, i.e., $\lambda(r_1, r_2, \dots, r_n) = \lambda(perm(r_1, r_2, \dots, r_n))$ where $perm(\cdot)$ is any permutation function. For the purpose of easy exposition, we allow r_i to be drawn from a slightly enlarged interval $[0, 1]$, with which the results obtained hold for $r_i \in [\underline{r}, 1]$ as in our original setting.

▶ **Definition 4.1.** *The distribution $F(v)$ (with density $f(v)$) has a monotone hazard rate (MHR) if the "hazard rate" of the distribution $\frac{f(v)}{1-F(v)}$ is increasing in v.*

MHR is a standard assumption widely used in economics [5, 32, 16]. Many distributions are MHR, such as uniform distribution, exponential distribution, gamma distribution, etc. Note that MHR implies regularity proposed by Myerson [41].

▶ **Theorem 4.2.** *There exists a prior-free signaling scheme that can be found in polynomial time and guarantees a (multiplicative) $0.995 \cdot (\underline{f}/\overline{f})^2$-approximation for any continuous MHR distribution $f(v)$, where $\overline{f}$ and $\underline{f}$ are the respective maximum and minimum values of $f(v)$ for $v \in [0, c]$ with any $c > 0$.*

The approximation ratio in Theorem 4.2 achieves its best ratio 0.995 for uniform distribution. More generally, the approximation ratio depends on the term $\underline{f}/\overline{f}$ which intuitively captures how much the distribution deviates from being uniform. Notably, both the algorithm and the signaling scheme require no knowledge about the value distribution F, i.e., F could be unknown to the seller (though Section 4.4 provides a better guarantee when F is known and satisfy certain properties). We remark that the signaling scheme in Theorem 4.2 is *simple* in the sense that its construction is parameterized by a single parameter called *optimal signal* *ratio.* The information design problem thus can be easily optimized by empirically selecting a value that gives the best performance. More importantly, Theorem 4.2 also provides a robust solution to the problem, which can be very useful when it is hard or costly to accurately estimate the distributions of bidder values.

4.1 Key Primitive of the Construction: the Optimal Signal Ratio

Before presenting the proof, we discuss the main ingredient for the construction of the signaling scheme, the *optimal signal ratio.* Given a CTR vector $r = (h, l)$ (without loss of generality, assume $h \geq l$) and a signal pair $s = (s_1, s_2)$, the expected revenue by (4) can be rewritten as (Recall $f(v) = 0, \forall v \notin [0, c]$. Note that although the seller requires no knowledge about $F(v)$ to construct the scheme, the bidders' value should follow some prior $F(v)$.)

$$R(r, s) = \int_0^c (h\frac{v_2 s_2}{s_1}) \int_{\frac{v_2 s_2}{s_1}}^c f(v_1)dv_1 f(v_2)dv_2 + \int_0^c (l\frac{v_1 s_1}{s_2}) \int_{\frac{v_1 s_1}{s_2}}^c f(v_2)dv_2 f(v_1)dv_1. \tag{5}$$

Clearly, from (5), the expected revenue is determined by the signals through their ratio $x = \frac{s_2}{s_1}$, which we call the *signal ratio.* We define the *optimal signal ratio* to be the signal ratio maximizing (5). In the rest of the paper, without loss of generality, we assume $h = 1$. Then the optimal signal ratio only depends on the smaller CTR l, denoted as $x(l)$ with $l \in [0, 1]$.

▶ **Lemma 4.3.** *The optimal signal ratio $l < x(l) \leq 1$ for $l \in [0, 1)$ and $x(1) = 1$.*

Lemma 4.3 is directly implied by Lemma E.1 and E.2 in Appendix E, where more discussion about $x(l)$ can be found. There are two interesting implications from Lemma 4.3. The first interesting implication is that without the calibration constraint, it would be optimal for the seller to only send pairs of signals with the optimal signal ratio where a larger (*resp.* smaller) signal is observed by the bidder with a higher (*resp.* lower) CTR, which motivates our construction of the signaling scheme. The second implication is that by $l < x(l)$ in Lemma 4.3, information design induces more intensive competition between buyers by partial-information revelation (with signal ratio $x(l)$) than full revelation (with signal ratio l). This again explains the observations in [5] that revenue extraction in auctions needs competition and too "fine-grained" targeting information may lead to a thin market. More detailed discussions on the second implication are in Appendix F.

As an application of Lemma 4.3, we observed that $x(l) = 1$ for $l \in [0, 1]$ for any *exponential distribution.* It turns out that in this special case, the revenue-maximizing calibrated signaling scheme is simply to reveal no information, i.e., always sending one calibrated signal pair with $s_1 = s_2 = E[r_i]$. This leads to the following proposition, whose proof is deferred to Appendix G.5.

▶ **Proposition 4.4.** *If the value distribution is an exponential distribution with density $f(v) = \lambda e^{-\lambda v}, v \geq 0, \lambda > 0$, the signaling scheme revealing no information is revenue-optimal.*

4.2 Construction of the *Simple* Signaling Scheme

In this part, we present the construction of a simple signaling scheme. Recall that given the smaller CTR l, the maximum expected revenue is achieved when $\frac{s_2}{s_1} = x(l)$, i.e., the optimal signal ratio. In other words, one can construct an approximately optimal signaling scheme by sending signal pairs of the optimal signal ratio $x(l)$ as frequently as possible, while maintaining the calibration constraints.

Algorithm 1 Construction of *Simple* Signaling Scheme.

1. Given CTR vector $r = (1, l)$, we compute a list of candidate signals $\sigma_0, \sigma_1, \dots, \sigma_{K-1}$, where $K = \lfloor \log_{x(l)} l \rfloor$ such that $x(l)^{K+1} \le l \le x(l)^K$. The signal σ_i is computed as

$$\sigma_K = 1, \sigma_0 = x(l)^K, \sigma_i = \sigma_{i-1} \cdot x(l), \forall i \in [K]. \tag{6}$$

2. Define $p(r, s) = \lambda(r)\pi(s|r)$ as the probability mass of sending signal s conditioning on CTR r. Let $(l, 1)$ and $(1, l)$ send signals (σ_k, σ_{k+1}) and (σ_{k+1}, σ_k), respectively, with the same probability mass

$$p\Big((l,1), (\sigma_k, \sigma_{k+1})\Big) = p\Big((1,l), (\sigma_{k+1}, \sigma_k)\Big) = p_k, \forall k\{1, 2, \dots, K-1\}$$

where p_k is computed as

$$p_k = p_{k-1} \cdot \frac{1-\sigma_k}{\sigma_k - l} = p_0 \prod_{i=1}^{k} \frac{1-\sigma_i}{\sigma_i - l}, \forall k \in \{1, 2, \dots, K-1\}. \tag{7}$$

3. By the above construction, we know $p\Big((l,1), (\sigma_{K-1}, 1)\Big) = p\Big((1,l), (1, \sigma_{K-1})\Big) = p_{K-1}$. To maintain calibration constraint, the seller will additionally send signal (σ_0, σ_0) with $p\Big((l,1), (\sigma_0, \sigma_0)\Big) = p\Big((1,l), (\sigma_0, \sigma_0)\Big) = z$, where

$$z = p_0 \cdot \frac{\sigma_0 - l}{l + 1 - 2\sigma_0}. \tag{8}$$

4. Choose a proper p_0 (other parameters z, p_i are then determined) so that

$$z + p_0 + p_1 + \dots + p_{K-1} = \lambda(r) = \frac{1}{2}$$

Based on the above intuition, we then present the construction of the simple signaling scheme π depicted in Algorithm 1. Since the CTRs of bidders are exchangeable, we can separately consider the signaling scheme for each pair of CTR vectors. Without loss of generality, consider a pair of CTR vectors $(l, 1)$ and $(1, l)$ such that $\lambda(r = (l,1)) = \lambda(r = (1,l)) = \frac{1}{2}$. In particular, we assume $x(l) < 1$ (otherwise, optimality can be easily achieved by revealing no information). With properly chosen parameter p_0, the constructed signaling scheme is calibrated and valid (see Appendix G.1). Note that by (8), the probability mass $z = 0$ if $\sigma_0 = l$, which implies all the signals sent are of the optimal signal ratio. Therefore, the optimal expected revenue is achieved.

▶ Remark 4.5. As mentioned previously, the construction only depends on one parameter, the optimal signal ratio $x(l)$. Besides, the scheme has a clear structure, i.e., computing a geometric series of signals $\{\sigma_i\}$ and then assigning probabilities, both of which can be done efficiently.

We present the following concrete example that describes the constructed signaling scheme.

▶ **Example 4.6.** Let $f(v)$ be the uniform distribution over $[0, 1]$. The click-through rates are $h = 1$ and $l = 0.6$, with $\lambda((h, l)) = \lambda((l, h)) = 0.5$. By solving (5), we obtain the optimal signal ratio $x = \frac{9}{10}$. We consider $k = 4$ levels of signals within $[0.6, 1)$ as $\sigma_0 = (\frac{9}{10})^4, \sigma_1 =$

Table 1 Example of a signaling scheme. Each entry corresponds to $\lambda(r)\pi(s|r)$, i.e., the probability mass of sending signals $s = (\sigma_i, \sigma_j)$ when observing the CTR vector r.

r \ s	(σ_0, σ_0)	(σ_0, σ_1)	(σ_1, σ_2)	(σ_2, σ_3)	(σ_3, σ_4)	(σ_1, σ_0)	(σ_2, σ_1)	(σ_3, σ_2)	(σ_4, σ_3)
(0.6, 1)	z	p_0	p_1	p_2	p_3	0	0	0	0
(1, 0.6)	z	0	0	0	0	p_0	p_1	p_2	p_3

$(\frac{9}{10})^3, \sigma_2 = (\frac{9}{10})^2, \sigma_3 = \frac{9}{10}, \sigma_4 = (\frac{9}{10})^0 = 1$. The signaling scheme is given by Table 1, where the probabilities z and p_0, p_1, p_2, p_3 are determined according to (7) and (8) to keep the construction a calibrated signaling scheme.

Note that all signal pairs except (σ_0, σ_0) follows the optimal signal ratio $x = \frac{9}{10}$. Hence the revenue suboptimality only happens when sending the signal $s = (\sigma_0, \sigma_0)$, of which the probability mass is $z \approx 0.016$. It turns out that the expected revenue of this calibrated signaling scheme is 0.2698. In contrast, by relaxing the calibration constraint, the maximum revenue one can achieve is 0.27 (i.e., always sending signals of the optimal signal ratio 9/10). Since 0.27 is clearly an upper bound of the optimal revenue for any calibrated signaling scheme, the revenue loss of our construction is less than 0.075%.

We remark that the revenue loss decreases very quickly as the number of signals k increases. In this example, if $l = 0.2$ instead, then the optimal signal ratio $x(l) = \frac{4}{5}$ and we can choose $k = 7$, yielding a much smaller probability mass of sending suboptimal signals $z \approx 1.5 \times 10^{-5}$.

4.3 Proof of Theorem 4.2

We first show a special case of Theorem 4.2, which is also a cornerstone for the main proof. That is, if the value distribution is uniform (i.e., $\underline{f}/\overline{f} = 1$), we can design a 0.995-approximate scheme.

▶ **Proposition 4.7.** *Given a uniform distribution supported on* $[0, c]$*, the constructed signaling scheme can achieve at least (multiplicative)* 0.995*-approximation.*

Simple calculation leads to the optimal signal ratio $x(l) = \frac{3+l}{4}$ under the uniform distribution. The proof of Proposition 4.7 is a combination of the following two lemmas. Recall that the constructed signaling scheme sends signal pairs of the optimal signal ratio $x(l)$ with probability $1 - z$ and with the remaining probability z sends the suboptimal signal pair (σ_0, σ_0). Lemma 4.8 bounds the probability z, while Lemma 4.9 lower bounds the revenue approximation when sending out (σ_0, σ_0). We will discuss the proof of Lemma 4.8 in Section 4.4, where a more general result is proved, and defer the proof of Lemma 4.9 to Appendix G.6.

▶ **Lemma 4.8.** *For optimal signal ratio function* $x(l) = \frac{3+l}{4}$*, there exists a signaling scheme whose worst-case approximation is* $1 - z^*$*, where* $z^* \leq 0.04$.

▶ **Lemma 4.9.** *Sending signal* (σ_0, σ_0)*, i.e., the signal ratio is* 1*, is (multiplicatively)* $\frac{8}{9}$*-approximation.*

Proof of Proposition 4.7. Combining Lemma 4.8 and Lemma 4.9, the approximation of the constructed signaling scheme is at least $(1 - z^*) \cdot 1 + z^* \cdot \frac{8}{9} \geq 224/225 \approx 0.995$. ◀

We prove Theorem 4.2 by showing that the signaling scheme we constructed for the uniform distribution is a $0.995\cdot(\underline{f}/\overline{f})^2$-approximation for any value distribution. In particular, sending signal pairs of signal ratio $\frac{3+l}{4}$ is $(\underline{f}/\overline{f})^2$-approximate and sending (σ_0,σ_0) is $\frac{8}{9}\cdot(\underline{f}/\overline{f})^2$-approximate. An analysis similar to Proposition 4.7 finally leads to $0.995\cdot(\underline{f}/\overline{f})^2$ approximation.

Proof of Theorem 4.2. We first derive the upper and lower bound of the expected revenue $R(r,s)$ by connecting to the uniform-distribution case. Given a CTR $r=(1,l)$ and any signal pair $s=(s_1,s_2)$ with ratio $\frac{s_2}{s_1}\le 1$ (by Lemma 4.3, $\frac{s_2}{s_1}\le 1$ is without loss of generality), the upper bound by (5) is computed as

$$
\begin{aligned}
R(r,s) &= \int_0^c \frac{v_2 s_2}{s_1}\int_{\frac{v_2 s_2}{s_1}}^c f(v_1)dv_1 f(v_2)dv_2 + \int_0^{c\cdot\frac{s_2}{s_1}} (l\cdot\frac{v_1 s_1}{s_2})\int_{\frac{v_1 s_1}{s_2}}^c f(v_2)dv_2 f(v_1)dv_1\\
&\le \int_0^c \frac{v_2 s_2}{s_1}\int_{\frac{v_2 s_2}{s_1}}^c \overline{f}dv_1\overline{f}dv_2 + \int_0^{c\cdot\frac{s_2}{s_1}} (l\cdot\frac{v_1 s_1}{s_2})\int_{\frac{v_1 s_1}{s_2}}^c \overline{f}dv_2\overline{f}dv_1\\
&=(\overline{f}c)^2\cdot\Big(\int_0^c \frac{v_2 s_2}{s_1}\int_{\frac{v_2 s_2}{s_1}}^c \frac{1}{c}dv_1\frac{1}{c}dv_2 + \int_0^{c\frac{s_2}{s_1}} (l\cdot\frac{v_1 s_1}{s_2})\int_{\frac{v_1 s_1}{s_2}}^c \frac{1}{c}dv_2\frac{1}{c}dv_1\Big)\\
&=(\overline{f}c)^2\cdot\overline{R}(r,s)
\end{aligned}
$$

where $\overline{R}(r,s)$ computes the expected revenue under a uniform value distribution given CTR vector r and signal pair s. Similarly, the lower bound is $R(r,s)\ge(\underline{f}c)^2\cdot\overline{R}(r,s)$. Since $R(r,s)$ is only related to the signal ratio $x=\frac{s_2}{s_1}$, we rewrite $R(r,s)$ as $R(r,x)$. Similarly, rewrite $\overline{R}(r,s)$ as $\overline{R}(r,x)$. Let $x^*\le 1$ and $\overline{x}\le 1$ (by Lemma 4.3) be the optimal signal ratio for $R(r,x)$ and $\overline{R}(r,x)$, respectively. Then,

$$
\begin{aligned}
&(\underline{f}c)^2\cdot\overline{R}(r,x^*)\le R(r,x^*)\le(\overline{f}c)^2\cdot\overline{R}(r,x^*)\\
&(\underline{f}c)^2\cdot\overline{R}(r,\overline{x})\le R(r,\overline{x})\le(\overline{f}c)^2\cdot\overline{R}(r,\overline{x})
\end{aligned}
$$

Simple calculation leads to $(\underline{f}/\overline{f})^2\cdot R(r,x^*)\le(\underline{f}c)^2\cdot\overline{R}(r,x^*)\le(\underline{f}c)^2\cdot\overline{R}(r,\overline{x})\le R(r,\overline{x})$. The second inequality is due to the optimality of $\overline{x}$ to $\overline{R}(r,x)$. The whole inequality implies that if the seller sends signal pair whose ratio is optimal under a uniform value distribution, the approximation ratio is at least $(\underline{f}/\overline{f})^2$. Notice that Lemma 4.9 shows that under a uniform value distribution, sending a signal pair with signal ratio $\frac{s_2}{s_1}=1$ achieves $\frac{8}{9}$ approximation, i.e., $\frac{8}{9}\overline{R}(r,\overline{x})\le\overline{R}(r,1)$. Hence, we have $\frac{8}{9}(\underline{f}/\overline{f})^2\cdot R(r,x^*)\le\frac{8}{9}(\underline{f}c)^2\cdot\overline{R}(r,\overline{x})\le(\underline{f}c)^2\cdot\overline{R}(r,1)\le R(r,1)$, implying that if the seller sends signal pair with ratio equal to 1, then the approximation is $\frac{8}{9}\cdot(\underline{f}/\overline{f})^2$.

Now, we present the $0.995\cdot(\underline{f}/\overline{f})^2$ approximation. Lemma 4.8 shows that under a uniform distribution, the probability mass z of sending signal (σ_0,σ_0) is upper bounded as $z<0.04$, while $1-z>0.96$ probability mass is for sending signal pairs of the optimal signal ratio $\overline{x}$. Hence, if we construct a signaling scheme by assuming the unknown value distribution to be a uniform distribution, it can achieve at least $0.995\cdot(\underline{f}/\overline{f})^2$-approximation, where $0.995\approx 0.04\times\frac{8}{9}+0.96$. ◀

The above proof implies that the *prior-free* signaling scheme is constructed with signal ratio $x(l)=\frac{3+l}{4}$ obtained by assuming the unknown value distribution to be a uniform distribution. One direct result from the proof is that an easier but loose scheme is revealing no information which gives $\frac{8}{9}\cdot(\underline{f}/\overline{f})^2$ approximation. The improved ratio in Theorem 4.2 demonstrates the benefits of strategic information revelation.

4.4 Completing the Last Piece – Approximation Guarantee with Known Distributions

The only missing piece for completing the proof of Theorem 4.2 is the proof of Lemma 4.8, which is for a special linear optimal signal ratio function $x(l) = \frac{3+l}{4}$. In this part, we prove a more general result for any convex $x(l)$, which directly implies Lemma 4.8 with linear $x(l)$. To formally state the general proposition for convex $x(l)$, we need the following definitions.

▶ **Definition 4.10.** *Given that the optimal signal ratio $x(l)$ is a convex function in $l \in [0, 1]$, define the following notations.*

1. *Initial number K_0: it satisfies that i) $x = x(l)^{K_0} \geq l$, and ii) $x = x(l)^{K_0+1}$ crosses the line $x = l$ and intersects at some point (l, l) with $l < 1$.*
2. *Define $S(k, l)$ with $\sigma_0 = x(l)^k$, $\sigma_i = x(l)^{k-i}$ as*

$$S(k, l) = 1 + \frac{l + 1 - 2\sigma_0}{\sigma_0 - l} + \frac{l + 1 - 2\sigma_0}{\sigma_0 - l} \cdot \frac{1 - \sigma_1}{\sigma_1 - l} + \cdots + \frac{l + 1 - 2\sigma_0}{\sigma_0 - l} \cdot \prod_{i=1}^{k-1} \frac{1 - \sigma_i}{\sigma_i - l} \tag{9}$$

3. *Intersection point $l[k]$: given some $k > K_0$, $l[k] \neq 1$ is a solution to the equation $x(l)^k = l$. In another word, $x = x(l)^k$ crosses the line $x = l$ and intersects at the point $(l[k], l[k])$.*

Recall in the construction of the signaling scheme that given a CTR vector $r = (l, 1)$, at most $K = \lfloor \log_{x(l)} l \rfloor$ signals (i.e., σ_i) are within the interval $[l, 1]$. In fact, the initial number K_0 specifies the minimum number of signals constructed within $[l, 1]$ for any $l < 1$. Also, the probability mass z of sending (σ_0, σ_0) can be computed with the defined $S(k, l)$,

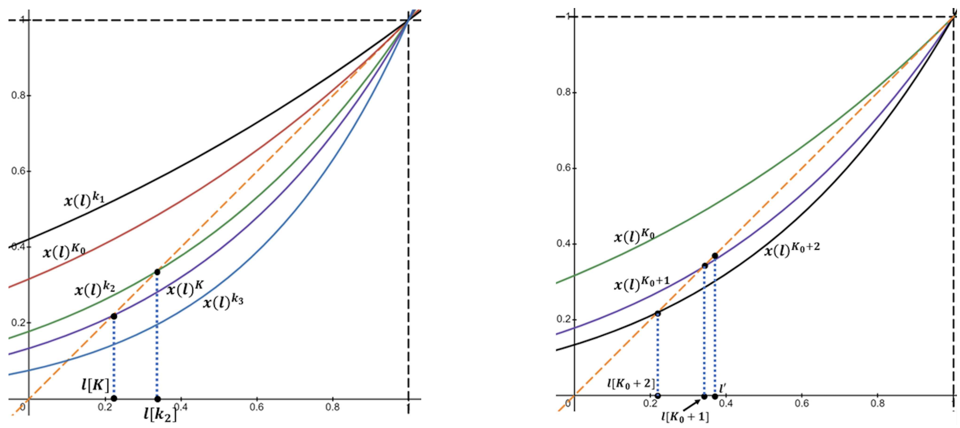
$$z + p_0 + p_1 + \cdots + p_{K-1} = \frac{1}{2} \iff z \cdot S(K, l) = \frac{1}{2} \tag{10}$$

With the above definition, we present the general result.

▶ **Proposition 4.11.** *Assume $x(l)$ is a convex function. There exists a signaling scheme whose worst-case approximation is $1 - z^*$, where $z^* = 1/S(K_0, l[K_0 + 1])$ and $S(K_0, l[K_0 + 1])$ is computed as (9).*

Proposition 4.11 provides a *worst-case* approximation bound for the constructed signaling scheme and shows how it depends on the intrinsic properties of the optimal signal ratio function. Intuitively, by Equation (9), two situations lead to a small z^*: 1) The initial number K_0 is large. In this case, it implies that the seller can send many signal pairs with the optimal signal ratio and thus $S(K_0, l[K_0 + 1])$ grows exponentially; 2) $l[K_0 + 1]$ is close to σ_0 in (9). Later, we will show it is equivalent to that $l[K_0 + 1]$ is close to $x(l[K_0 + 1])^{K_0}$. This case will lead to a large $S(K_0, l[K_0 + 1])$ and thus small z^*. In fact, given the number of signals $k \geq K_0$ and the CTR l, the approximation of the constructed signaling scheme can be calculated similarly as $1 - z$ with $z = 1/S(k, l)$. As we will prove later, the worst-case approximation is obtained only when l is approaching $l[K_0 + 1]$ from above (see Figure 1b for an example). Hence, when l varies, the actual approximation of the constructed scheme may be (much) better than $1 - z^*$.

Proposition 4.11 may be of independent interest. If an estimate for the bidders' value distribution is available, the seller then can construct a signaling scheme as in Section 4.2 that has an approximation guarantee indicated by Proposition 4.11. Note that Proposition 4.11 only requires $x(l)$ to be convex. This condition applies to various classic distributions, e.g., (1) the uniform distribution admits a 0.995-approximation, (2) the exponential distribution can have the exact optimal mechanism, (3) the standard Weibull distribution (truncated on



(a) **(b)**

Figure 1 K_0 is initial number. $l[k]$ is the intersection point. $k_1 < K_0 < k_2 < K < k_3$. $x(l)$ is convex.

$[0, 1]$) with parameter $\gamma \geq 2$ (specifically we let $\gamma = 10$) admits approximation ratio ≈ 0.75. In some cases, it may be difficult to write down the explicit formula of $x(l)$ and verify its convexity. We find it relatively easy to compute $l(x)$, the inverse of $x(l)$. Hence, to verify the convexity of $x(l)$, we only need to verify the concavity of $l(x)$. An example of this idea is in Appendix H.

Before presenting the proof of Proposition 4.11, we show some properties for convex $x(l)$.

▶ **Observation 4.12.** *The function $x = x(l)^k$, for any integer $k > 1$ and $l \in [0, 1]$, intersects with line $x = l$ at most twice: one point intersecting is $(1, 1)$, and the other one (if exists) is $(l[k], l[k])$ (called intersection point as in Definition 4.10). Furthermore, when $k > K_0$ increases, $l[k]$ decreases and $\lim_{k\to\infty} l[k] = 0$.*

The idea of Observation 4.12 is depicted in Figure 1a. Given an initial number K_0 and $K = \lfloor \log_{x(l')} l' \rfloor$ for some l', we can observe from the figure that: i) If $k \leq K_0$, the curve $x(l)^k$ will be above line $x = l$; ii) If $K_0 \leq k \leq K$, then $l' \leq l[K]$ and $x(l')^k$ will be above point (l', l'). If $l' = l[K]$, then there are exactly K signals constructed within $[l', 1]$; and iii) If $k > K$, $x(l')^k$ will be below (l', l').

The following key lemma characterizes the monotonicity of convex $x(l)$.

▶ **Lemma 4.13.** *$x(l)$ is a monotone increasing function.*

Now we are ready to show the proof of Proposition 4.11. The high-level idea of the proof is by upper bounding the probability mass z of sending (at most one) *non-optimal* signal pair (σ_0, σ_0), because when sending other signal pairs (σ_i, σ_{i+1}) as constructed in the signaling scheme, the signaling scheme achieves the maximum of expected revenue expressed in (5).

Proof of Proposition 4.11. Without loss of generality, we consider the case where there is only one pair of CTR vectors, $(l, 1)$ and $(1, l)$ such that $\lambda((l, 1)) = \lambda(1, l)) = \frac{1}{2}$. By the definition of symmetric environments, the analysis can be easily generalized to the case of more than two CTR vectors.

The following lemma segments $[0, 1]$ by the intersection points $l[k]$'s, and shows how the probability mass z changes within each segment. Note that starting from $k = K_0 + 1$, $x(l)^k$ crosses the line $x = l$ (see Figure 1b). Then, we define $l[K_0] = 1$.

▶ **Lemma 4.14.** *Given $k \geq K_0 \geq 1$, the probability mass $z > 0$ is decreasing in $l \in (l[k+1], l[k]]$.*

The proof of Lemma 4.14 is in Appendix G.4.The reason we only care about $(l[k+1], l[k]]$ instead of the closed interval $[l[k+1], l[k]]$ is that i) for $l[k+1] < l \leq l[k]$, there will be k signals constructed in the interval $[l, 1]$, and ii) if $l = l[k+1]$, the seller can construct a signaling scheme with a list of $k+1$ signals where all the signal pairs sent have the optimal signal ratio $x(l[k+1])$ and the probability mass $z = 0$.

One implication of Lemma 4.14 is that when there are at most k signals constructed, the probability mass z achieves its maximum when l approaches $l[k+1]$ from above, i.e., $l \downarrow l[k+1]$. Therefore, at the limit $l[k+1]$, the probability mass z is computed with $S(k, l[k+1])$ where $\sigma_0 = x(l[k+1])^k, \sigma_1 = x(l[k+1])^{k-1}, \cdots, \sigma_i = x(l[k+1])^{k-i}$. In another word, when $l = l[k+1]$, instead of constructing a scheme with a list of $k+1$ signals where all the signal pairs sent have signal ratio $x(l[k+1])$, the seller constructs a signaling scheme with a list of k signals starting from $\sigma_0 = x(l[k+1])^k$, which sends signal pair (σ_{i-1}, σ_i) of signal ratio $x(l[k+1])$ and one signal pair (σ_0, σ_0).

By the above analysis, to upper bound z, we only need to compare its values at these limit points $l[k]$'s. Alternatively, we compare $S(k, l[k+1])$ for different k. The following lemma shows that $S(k, l[k+1])$ is increasing in k, whose proof is in Appendix G.7.

▶ **Lemma 4.15.** *Given $k-1 \geq K_0 \geq 1$, we have $S(k, l[k+1]) \geq S(k-1, l[k])$.*

Lemma 4.15 implies that $S(K_0, l[K_0+1])$ is the minimum compared with other $S(k, l[k+1])$ for $k > K_0$. The quantity $S(K_0, l[K_0+1])$ is computed as (9) with

$$\sigma_0 = x(l[K_0+1])^{K_0}, \sigma_1 = x(l[K_0+1])^{K_0-1}, \ldots, \sigma_i = x(l[K_0+1])^{K_0-i}$$

Since both CTR vectors $(1, l)$ and $(l, 1)$ will send signal (σ_0, σ_0), the upper bound of the probability mass of sending signal pair (σ_0, σ_0) in the constructed signaling scheme is $z^* = 1/S(K_0, l[K_0+1])$.

The above analysis generalizes to the case of more than two CTR vectors. Note that given any CTR vector (h, l), in the symmetric environment, we can find one CTR vector (l, h) with equal probability, i.e., $\lambda((h, l)) = \lambda((l, h))$. Hence, we can separately consider these pairs of CTR vectors when designing a signaling scheme and each of them achieves at least $(1 - z^*)$ (multiplicatively) approximation. Therefore, the constructed signaling scheme can achieve at least $(1 - z^*)$ (multiplicatively) approximation. ◀

5 Conclusions and Open Problems

This paper studies the natural Bayesian variant of the calibrated click-through auction of [11]. We focus on the seller's information design problem to maximize the expected revenue. In general environments, we develop an FPTAS to compute an approximately optimal signaling scheme. In a symmetric environment, we give a simple and prior-free signaling scheme with a constant approximation guarantee for not-too-fluctuating value distributions.

Our results raise many interesting questions for future research. Below we discuss some of them. The first is to develop a simple signaling scheme for more than two bidders in symmetric environments. More generally, it is interesting to study more efficient algorithms for optimal signaling. Though our FPTAS for the general model enriches our understanding of the tractability of the problem, such an algorithm may not be ideal from a practitioner's perspective. The second is to consider the worst-case approximation as in Proposition 4.11 but without the convexity assumption. In Appendix I,we provide more discussions about these two open problems.

References

1 Itai Arieli and Yakov Babichenko. Private bayesian persuasion. *Journal of Economic Theory*, 182:185–217, 2019.

2 Moshe Babaioff, Robert Kleinberg, and Renato Paes Leme. Optimal mechanisms for selling information. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC 2012)*, pages 92–109, 2012.

3 Yakov Babichenko and Siddharth Barman. Algorithmic aspects of private bayesian persuasion. In *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017.

4 Yakov Babichenko, Inbal Talgam-Cohen, Haifeng Xu, and Konstantin Zabarnyi. Multi-channel bayesian persuasion. In *Proceedings of 13th Innovations in Theoretical Computer Science Conference (ITCS 2022)*, volume 215, page 11. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

5 Ashwinkumar Badanidiyuru, Kshipra Bhawalkar, and Haifeng Xu. Targeting and signaling in ad auctions. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2018)*, pages 2545–2563, 2018.

6 Santiago Balseiro, Yuan Deng, Jieming Mao, Vahab Mirrokni, and Song Zuo. Robust auction design in the auto-bidding world. *Advances in Neural Information Processing Systems*, 34:17777–17788, 2021.

7 Santiago R Balseiro, Yuan Deng, Jieming Mao, Vahab S Mirrokni, and Song Zuo. The landscape of auto-bidding auctions: Value versus utility maximization. In *Proceedings of the 22nd ACM Conference on Economics and Computation*, pages 132–133, 2021.

8 Santiago R Balseiro and Yonatan Gur. Learning in repeated auctions with budgets: Regret minimization and equilibrium. *Management Science*, 65(9):3952–3968, 2019.

9 Dirk Bergemann, Alessandro Bonatti, and Alex Smolin. The design and price of information. *American economic review*, 108(1):1–48, 2018.

10 Dirk Bergemann, Yang Cai, Grigoris Velegkas, and Mingfei Zhao. Is selling complete information (approximately) optimal? In *Proceedings of the 23rd ACM Conference on Economics and Computation, (EC 2022)*, 2022.

11 Dirk Bergemann, Paul Dütting, Renato Paes Leme, and Song Zuo. Calibrated click-through auctions. In *Proceedings of the ACM Web Conference 2022*, pages 47–57, 2022.

12 Dirk Bergemann, Tibor Heumann, Stephen Morris, Constantine Sorokin, and Eyal Winter. Optimal information disclosure in auctions. *American Economic Review: Insights (forthcoming)*, 2022.

13 Umang Bhaskar, Yu Cheng, Young Kun Ko, and Chaitanya Swamy. Hardness results for signaling in bayesian zero-sum and network routing games. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 479–496, 2016.

14 Peter Bro Miltersen and Or Sheffet. Send mixed signals: earn more, work less. In *Proceedings of the 13th ACM Conference on Electronic Commerce (EC 2012)*, pages 234–247, 2012.

15 Yang Cai and Grigoris Velegkas. How to sell information optimally: An algorithmic study. In *Proceedings of the 12th Innovations in Theoretical Computer Science Conference (ITCS 2021)*, volume 185, 2021.

16 Shuchi Chawla, Jason D Hartline, and Robert Kleinberg. Algorithmic pricing via virtual valuations. In *Proceedings of the 8th ACM Conference on Electronic Commerce*, pages 243–251, 2007.

17 Yiling Chen, Haifeng Xu, and Shuran Zheng. Selling information through consulting. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2020)*, pages 2412–2431. SIAM, 2020.

18 Yu Cheng, Ho Yee Cheung, Shaddin Dughmi, Ehsan Emamjomeh-Zadeh, Li Han, and Shang-Hua Teng. Mixture selection, mechanism design, and signaling. In *2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS 2015)*, pages 1426–1445. IEEE, 2015.

19 Constantinos Daskalakis, Christos Papadimitriou, and Christos Tzamos. Does information revelation improve revenue? In *Proceedings of the 17th ACM Conference on Economics and Computation (EC 2016)*, pages 233–250, 2016.

20 Yuan Deng, Negin Golrezaei, Patrick Jaillet, Jason Cheuk Nam Liang, and Vahab Mirrokni. Fairness in the autobidding world with machine-learned advice. *arXiv preprint*, 2022. `arXiv:2209.04748`.

21 Nikhil R Devanur, Bach Q Ha, and Jason D Hartline. Prior-free auctions for budgeted agents. In *Proceedings of the fourteenth ACM conference on Electronic commerce*, pages 287–304, 2013.

22 Shaddin Dughmi. On the hardness of signaling. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 354–363. IEEE, 2014.

23 Shaddin Dughmi. Algorithmic information structure design: a survey. *ACM SIGecom Exchanges*, 15(2):2–24, 2017.

24 Shaddin Dughmi and Haifeng Xu. Algorithmic persuasion with no externalities. In *Proceedings of the 18th ACM Conference on Economics and Computation (EC 2017)*, pages 351–368, 2017.

25 Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. *American economic review*, 97(1):242–259, 2007.

26 Yuval Emek, Michal Feldman, Iftah Gamzu, Renato PaesLeme, and Moshe Tennenholtz. Signaling schemes for revenue maximization. *ACM Transactions on Economics and Computation (TEAC)*, 2(2):1–19, 2014.

27 Seyf Emen. A beginners guide to generalized second-price auctions, 2022. URL: `https://www.topsort.com/post/generalized-second-price-auctions-how#:~:text=The%20second%2Dprice%20auction%20is,bid%20to%20win%20the%20auction`.

28 Yiding Feng, Brendan Lucier, and Aleksandrs Slivkins. Strategic budget selection in a competitive autobidding world. *arXiv preprint*, 2023. `arXiv:2307.07374`.

29 Dean P Foster and Rakesh V Vohra. Calibrated learning and correlated equilibrium. *Games and Economic Behavior*, 21(1-2):40, 1997.

30 Hu Fu, Patrick Jordan, Mohammad Mahdian, Uri Nadav, Inbal Talgam-Cohen, and Sergei Vassilvitskii. Ad auctions with data. In *International Symposium on Algorithmic Game Theory (SAGT 2012)*, pages 168–179. Springer, 2012.

31 Ronen Gradwohl, Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. Algorithms for persuasion with limited communication. In *Proceedings of the 32nd ACM-SIAM Symposium on Discrete Algorithms (SODA 2021)*, pages 637–652, 2021.

32 Jason D Hartline and Tim Roughgarden. Simple versus optimal mechanisms. In *Proceedings of the 10th ACM conference on Electronic commerce (EC 2009)*, pages 225–234, 2009.

33 Emir Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11:249–272, 2019.

34 Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

35 Yoav Kolumbus and Noam Nisan. Auctions between regret-minimizing agents. In *Proceedings of the ACM Web Conference 2022*, pages 100–111, 2022.

36 Yoav Kolumbus and Noam Nisan. How and why to manipulate your own agent: On the incentives of users of learning agents. *Advances in Neural Information Processing Systems*, 35:28080–28094, 2022.

37 Yingkai Li. Selling data to an agent with endogenous information. In *Proceedings of the 23rd ACM Conference on Economics and Computation (EC 2022)*, 2022.

38 Zhuoshu Li and Sanmay Das. Revenue enhancement via asymmetric signaling in interdependent-value auctions. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 2093–2100, 2019.

39 Shuze Liu, Weiran Shen, and Haifeng Xu. Optimal pricing of information. In *Proceedings of the 22nd ACM Conference on Economics and Computation (EC 2021)*, pages 693–693, 2021.

40 Aranyak Mehta and Andres Perlroth. Auctions without commitment in the auto-bidding world. *arXiv preprint*, 2023. `arXiv:2301.07312`.

41 Roger B Myerson. Optimal auction design. *Mathematics of operations research*, 6(1):58–73, 1981.

42 Michael Ostrovsky and Michael Schwarz. Reserve prices in internet advertising auctions: A field experiment. In *Proceedings of the 12th ACM conference on Electronic commerce*, pages 59–60, 2011.

43 Renato Paes Leme, Martin Pal, and Sergei Vassilvitskii. A field guide to personalized reserve prices. In *Proceedings of the 25th international conference on world wide web*, pages 1093–1102, 2016.

44 Hal R Varian. Position auctions. *international Journal of industrial Organization*, 25(6):1163–1178, 2007.

45 Haifeng Xu. On the tractability of public persuasion with no externalities. In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2020)*, pages 2708–2727. SIAM, 2020.

46 Shuran Zheng and Yiling Chen. Optimal advertising for information products. In *Proceedings of the 22nd ACM Conference on Economics and Computation (EC 2021)*, pages 888–906, 2021.