PARTIAL DATA INVERSE PROBLEMS FOR THE NONLINEAR TIME-DEPENDENT SCHRÖDINGER EQUATION

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Contents

Abstract. In this paper we prove the uniqueness and stability in determining a time-dependent nonlinear coefficient $\beta(t,x)$ in the Schrödinger equation $(i\partial_t + \Delta + q(t,x))u + \beta u^2 = 0$, from the boundary Dirichlet-to-Neumann (DN) map. In particular, we are interested in the partial data problem, in which the DN-map is measured on a proper subset of the boundary. We show two results: a local uniqueness of the coefficient at the points where certain type of geometric optics (GO) solutions can reach; and a stability estimate based on the unique continuation property for the linear equation.

1. Introduction

We investigate a partial data inverse problem for the time-dependent Schrödinger equation with a nonlinear term, for example, in modeling the recovery of the nonlinear electromagnetic second order polarization potential from the partial boundary measurements of electromagnetic fields. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded and convex domain with smooth boundary $\partial\Omega$. For $T > 0$, we denote $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \partial\Omega$. Suppose Γ is an open proper subset of the boundary $\partial\Omega$ and denote

$$\Sigma^\sharp := (0,T) \times \Gamma.$$

Key words: Nonlinearity, Inverse problems, Time-dependent Schrödinger equation.

For $q(t,x) \in C^\infty(\overline{Q})$ and $\beta(t,x) \in C^\infty(\overline{Q})$, we consider the nonlinear dynamic Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta + q(t,x))\,u(t,x) + \beta(t,x)u(t,x)^2 & = & 0 & \text{on } Q, \\ u(t,x) & = & f & \text{on } \Sigma, \\ u(t,x) & = & 0 & \text{on } \{0\}\times\Omega, \end{cases} \tag{1.1}$$

where $\Delta u := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the spatial Laplacian.

Based on the well-posedness result in Proposition 2.2, the Dirichlet-to-Neumann (DN) map $\Lambda_{q,\beta}$ is well-defined by

$$\Lambda_{q,\beta} : f \mapsto \partial_\nu u\big|_{\Sigma^\sharp}, \qquad f \in \mathcal{S}_\lambda(\Sigma)$$

for $\lambda > 0$ sufficiently small (see (2.1) for the definition of $\mathcal{S}_\lambda(\Sigma)$, where $\partial_\nu u := \frac{\partial u}{\partial \nu}$ and $\nu(x)$ is the unit outer normal to $\partial\Omega$ at the point $x \in \partial\Omega$. The inverse problem we consider in this paper is the determination of the nonlinear potential $\beta(t,x)$ from the partial DN-map $\Lambda_{q,\beta}$.

1.1. **Main results.** For a set $B \subset \Omega$, we denote $\mathcal{M}_B$ by

$$\mathcal{M}_B := \{g \in C^\infty(\overline{Q}) : \|g\|_{C^r(\overline{Q})} \le m_0, \text{and } g = 0 \text{ on } (0,T)\times B, 1 \le r < \infty\}.$$

for some positive constant m_0. Let $\mathcal{O} \subset \Omega$ be an open neighborhood of the boundary $\partial\Omega$ and $\mathcal{O}' \subset \Omega$ be an open neighborhood of $\Gamma^c := \partial\Omega \setminus \Gamma$.

We define an open subset Ω_Γ of Ω as

$$\Omega_\Gamma := \big\{p \in \Omega \ : \text{there exist } \omega_1, \omega_2 \in \mathbb{S}^{n-1}, \omega_1 \perp \omega_2 \text{ such that } \left((\gamma_{p,\omega_1} \cup \gamma_{p,\omega_2} \cup \gamma_{p,\omega_1+\omega_2}) \cap \partial\Omega\right) \subset \Gamma\big\}, \tag{1.2}$$

where $\gamma_{p,\omega}$ denotes the straight line through a point p in a direction ω in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ is a unit sphere at the origin. The set Ω_Γ consists of those interior points at which three lines in directions ω_1, ω_2 and $\omega_1+\omega_2$ intersect and these three lines must enter and exist Ω through Γ.

Our main results are stated as follows:

Theorem 1.1 (Local uniqueness). *Assume q and β_j are in $C^\infty(\overline{Q})$ for $j = 1, 2$. Suppose $\Lambda_{q,\beta_1}(f) = \Lambda_{q,\beta_2}(f)$ for all $f \in \mathcal{S}_\lambda(\Sigma)$ with support satisfying supp$(f) \subset \Sigma^\sharp$. Then $\beta_1(t,x) = \beta_2(t,x)$ for all $(t,x) \in (0,T)\times\Omega_\Gamma$.*

The result of Theorem 1.1 highly depends on the convexity of the domain Ω in order to recover $\beta(t,\cdot)$ in the region Ω_Γ near the partial boundary Γ.

Theorem 1.2 (Stability estimate). *Assume $\beta_j \in C^\infty(\overline{Q})$ for $j = 1, 2$. Suppose that $(q, \beta_1 - \beta_2) \in \mathcal{M}_\mathcal{O} \times \mathcal{M}_\mathcal{O}$. Let $\Lambda_{q,\beta_j} : \mathcal{S}_\lambda(\Sigma) \to L^2(\Sigma^\sharp)$ be the Dirichlet-to-Neumann maps of the nonlinear Schrödinger equation* (1.1) *associated with β_j for $j = 1, 2$. There exists a sufficiently small $\delta_0 > 0$ so that if the DN maps satisfy*

$$\|(\Lambda_{q,\beta_1} - \Lambda_{q,\beta_2})f\|_{L^2(\Sigma^\sharp)} \le \delta \qquad \text{for all } f \in \mathcal{S}_\lambda(\Sigma),$$

for some $\delta \in (0,\delta_0)$, then for any $0 < T^* < T$, there exist constants $C > 0$ independent of δ and $0 < \sigma < 1$ such that the following stability estimate holds:

$$\|\beta_1 - \beta_2\|_{L^2((0,T^*)\times\Omega)} \le C\left(\delta^{\frac{1}{12}} + |\log(\delta)|^{-\sigma}\right).$$

The logarithmic type stability estimate here is expected since we only take measurements on partial region of the boundary of the domain.

The uniqueness result of Theorem 1.3 follows directly from Theorem 1.1 and Theorem 1.2 by letting $\delta \to 0$. In particular, due to Theorem 1.1, the assumption of $\beta_1 - \beta_2$ can be relaxed to $\mathcal{M}_{\mathcal{O}'}$.

Theorem 1.3 (Global uniqueness)**.** *Suppose that Ω is bounded and strictly convex. Assume $\beta_j \in C^\infty(\overline{Q})$ for $j = 1, 2$. Suppose that $(q, \beta_1 - \beta_2) \in \mathcal{M}_{\mathcal{O}} \times \mathcal{M}_{\mathcal{O}'}$. Let $\Lambda_{q,\beta_j} : \mathcal{S}_\lambda(\Sigma) \to L^2(\Sigma^\sharp)$ be the Dirichlet-to-Neumann maps of the nonlinear Schrödinger equation* (1.1) *with β_j for $j = 1, 2$. If $\Lambda_{q,\beta_1}(f) = \Lambda_{q,\beta_2}(f)$ for all $f \in \mathcal{S}_\lambda(\Sigma)$, then*

$$\beta_1 = \beta_2 \quad \text{in } Q.$$

The nonlinear Schrödinger equation (NLS) in (1.1) can be used to model a basic second harmonic generation process in nonlinear optics. A similar NLS is the Gross-Pitaevskii (GP) equation

$$(i\partial_t + \Delta + q)u + \beta(t, x)|u|^2 u = 0$$

for the single-atom wave function, used in a mean-field description of Bose-Einstein condensates. See [45] for discussions of various NLS models based on integrability and existence of stable soliton solutions, such as the nonlinear term of a saturable one, $|u|^2(1 + |u|^2/u_0^2)^{-1}$ with u_0 a constant, or $(|u|^2 - |u|^4)u$. We remark in Remark 4.3 that our approach can be generalized to power type nonlinearity other than quadratic ones. Similar discussions can be found in [41] for the GP equation.

Similar to those of hyperbolic equations, results related to the determination of coefficients for dynamic Schrödinger equations are usually classified into two categories of time-independent and time-dependent coefficients. For the linear equation, stability estimates for recovering the time-independent electric potential or the magnetic field from the knowledge of the dynamical Dirichlet-to-Neumann map were shown in [2, 5, 6, 7, 9, 13]. A vast literature is devoted for the inverse problems associated to the stationary Schrödinger equation, known under the name of Calderón problem, see [46, 48] for the major results when the DN-map is measured on the whole boundary and see [14, 16, 17, 25] when measured on part of the boundary. The paper [15] by Eskin is known to be the first to show the unique determination of time-dependent electric and magnetic potentials of the Schrödinger equation from the DN-map. Stability for the inverse problem with full boundary measurement was shown in [26, 27, 12]. The stable determination of time-dependent coefficients appearing in the linear Schrödinger equation from partial DN map is then given in [8]. The stability estimate for the problem of determining the time-dependent zeroth order coefficient in a parabolic equation from a partial parabolic Dirichlet-to-Neumann map can be found in [11].

In dealing with the inverse problems for nonlinear PDEs, the first order linearization of the DN-map was introduced in recovering the linear coefficient for the medium, and sometimes the nonlinear coefficients. See [19, 20, 21, 22, 23, 47] for demonstrations for certain semilinear, quasilinear elliptic equations and parabolic equations. Recently the higher order linearization, also called the multifold linearization, of the measurement operators (e.g., the Dirichlet-to-Neumann map or the source-to-solution map) has been applied in determining nonlinear coefficients in more general nonlinear differential equations. For example, based on the scheme, the nonlinear interactions of distorted plane waves were analyzed to recover the metric of a Lorentzian space-time manifold and nonlinear coefficients using the measurements of solutions to nonlinear hyperbolic equations [30, 42, 49]. In contrast the underlying problems for linear hyperbolic equations are still open, see also [10, 42] and the references therein. The method is also applied to study elliptic equations with power-type nonlinearities, including stationary nonlinear Schrödinger equations and magnetic Schrödinger equations, see [28, 29, 31, 32, 33, 38, 39, 43]. A demonstration of the method can be found in [4, 3] on nonlinear Maxwell's equations, in [34, 35] on nonlinear kinetic equations, and in [40] on semilinear wave equations. In [36], we solved an inverse problem for the magnetic Schrödinger equation with nonlinearity in both magnetic and electric potentials using partial DN-map and its nonlocal fractional diffusion version [37]. For the nonlinear dynamic Schrödinger equation considered in this paper, unique determination of time-dependent linear and nonlinear potentials from the knowledge of a source-to-solution map was discussed in [41].

The novelty and contributions of this paper include the following two points. The first one is the construction of gaussian beam approximate solutions to the time-dependent Schrödinger equation in Section 3. Since these solutions are only concentrated near straight lines passing through the observation set Γ, this makes it possible to determine the points in Ω_Γ from the knowledge of partial data. The second one is the GO solutions constructed based on [26, 41] with adaption. Combining with the unique continuation principle in [8] derived through the application of a parabolic Carleman estimate, we are able to prove the stability estimate for the nonlinear coefficient β from the knowledge of the DN map restricted to an arbitrary portion of the boundary. In particular, we also show the uniqueness result for the nonlinear Schrödinger equation with slightly less constraint on the assumption of the unknown coefficient β near $\partial\Omega$ thanks to Theorem 1.1.

Finally, we would like to point out that the nonlinear Schrödinger equation (1.1) was also considered by [41], where both the linear and nonlinear coefficients are uniquely determined from the source-to-solution map. Not only the measurement is different from the DN map we utilize here, but also we establish the stability estimate for the nonlinear coefficient.

The paper is organized as follows. In Section 2, we establish the well-posedness of the direct problem, the initial boundary value problem for our nonlinear time-dependent Schrödinger equation in a bounded domain for well chosen boundary conditions. Then we prove the local uniqueness result Theorem 1.1 in Section 3 by constructing the geometrical optics (GO) solutions for the linear Schödinger equation that concentrate near straight lines intersecting at a point. The higher order (multifold) linearization step is conducted via finite difference expansions in this section to derive the needed integral identity. Then we prove the stability estimate Theorem 1.2 in Section 4 where we implement a more standard type of linear GO solutions and adopt the unique continuation argument to control the boundary term due to the inaccessibility by the partial data measurement. Finally, we present the short proof of Theorem 1.3 for a global uniqueness result by combining assumptions in the previous two theorems.

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2. Well-posedness of the Dirichlet problem

2.1. **Notations.** Let r and s be two non-negative real numbers, m be a non-negative integer and let X be one of Ω, $\partial\Omega$ and Γ. We introduce the following Hilbert spaces:

- the space $L^2(0,T;H^s(X))$ that consists of all measurable functions $f:[0,T]\to H^s(X)$ with norm
$$\|f\|_{L^2(0,T;H^s(X))} := \left(\int_0^T \|f(t,\cdot)\|^2_{H^s(X)}\,dt\right)^{1/2} < \infty;$$
- the Sobolev space
$$H^m(0,T;L^2(X)) := \{f:\ \partial_t^\alpha f \in L^2(0,T;L^2(X)) \quad \text{for } \alpha = 0,1,\dots,m\};$$
and the interpolation
$$H^r(0,T;L^2(X)) = [H^m(0,T;L^2(X)), L^2(0,T;L^2(X))]_\theta, \quad (1-\theta)m = r.$$

We also define the Hilbert space
$$H^{r,s}((0,T)\times X) := H^r(0,T;L^2(X)) \cap L^2(0,T;H^s(X)),$$

whose norm is given by

$$\|f\|_{H^{r,s}((0,T)\times X)} := \left(\int_0^T \|f(t,\cdot)\|^2_{H^s(X)}dt + \|f\|^2_{H^r(0,T;L^2(X))}\right)^{1/2}.$$

For more details on these definitions, we refer to Chapter 1 and Chapter 4 in [44]. In particular, for integer $m \geq 1$, we define

$$\mathcal{H}^m_0(Q) := \{f \in H^m(Q) :\ \partial_t^\alpha f|_{t=0} = 0, \quad \alpha = 0, \cdots, m-1\}.$$

For $\lambda > 0$ we define the subset $\mathcal{S}_\lambda(\Sigma)$ of $H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)$ by

$$\begin{aligned}\mathcal{S}_\lambda(\Sigma) := \Big\{ f \in H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma) :\ \partial_t^m f(0,\cdot) = 0 \text{ on } \partial\Omega \text{ for integers } m < 2\kappa + \frac{3}{2},\\ \text{and} \quad \|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} \leq \lambda \Big\}.\end{aligned} \tag{2.1}$$

2.2. **Well-posedness.** We first show unique existence of the solution to the linear equation and, based on this, we apply the contraction mapping principle to deduce the well-posedness for the nonlinear equation.

Proposition 2.1. *(Well-posedness for the linear equations) Let $\kappa > \frac{n+1}{2}$ be an integer. Suppose $q \in C^\infty(\overline{Q})$. For any $f \in H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)$ satisfying $\partial_t^m f(0,\cdot) = 0$ for $m < 2\kappa + \frac{3}{2}$, there exists a unique solution $u_f \in H^{2\kappa}(Q)$ to the linear system:*

$$\begin{cases} (i\partial_t + \Delta + q)\, u_f = 0 & \text{in } Q,\\ u_f = f & \text{on } \Sigma,\\ u_f = 0 & \text{on } \{0\}\times\Omega, \end{cases} \tag{2.2}$$

and u_f satisfies the estimate

$$\|u_f\|_{H^{2\kappa}(Q)} \leq C\|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}. \tag{2.3}$$

Proof. In light of [[44], Chapter 4, Theorem 2.3], there exists a function $\tilde{u} \in H^{2\kappa+2,2\kappa+2}(Q)$ such that for $0 \leq \alpha < 2\kappa + \frac{3}{2}$,

$$\partial_t^\alpha \tilde{u}(0,\cdot) = 0 \quad \text{in } \Omega, \qquad \tilde{u}|_\Sigma = f, \tag{2.4}$$

and

$$\|\tilde{u}\|_{H^{2\kappa+2}(Q)} \leq C\|\tilde{u}\|_{H^{2\kappa+2,2\kappa+2}(Q)} \leq C\|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}$$

for some positive constant C, depending only on Ω and T, where the first inequality holds by noticing Proposition 2.3 in Chapter 4 in [44]. Let

$$F := -(i\partial_t + \Delta + q)\tilde{u}.$$

Since $\tilde{u} \in H^{2\kappa+2}(Q)$, we get $F \in H^{2\kappa+1,2\kappa}(Q) \subset H^{2\kappa,2\kappa}(Q)$ implying $F \in H^{2\kappa}(Q)$ by using Proposition 2.3 in Chapter 4 in [44] again. In addition, due to (2.4), F has zero initial condition up to 2κ derivative w.r.t. t, which makes $F \in \mathcal{H}^{2\kappa}_0(Q)$. From Lemma 4 of [41], there exists a unique solution u_* to the Schrödinger equation $(i\partial_t + \Delta + q)u_* = F$ with $F|_{t=0} = 0$ and $u_*|_{t=0} = u_*|_\Sigma = 0$. We denote by $\mathcal{L}^{-1}$ the solution operator of this inhomogeneous Dirichlet problem for the linear Schrödinger equation, that is, $\mathcal{L}^{-1}(F) = u_*$. In particular, we have that $\mathcal{L}^{-1} :\ \mathcal{H}^{2\kappa}_0(Q) \to \mathcal{H}^{2\kappa}_0(Q)$ is a bounded linear operator. Therefore, we obtain

$$\|u_*\|_{H^{2\kappa}(Q)} \leq C\|F\|_{\mathcal{H}^{2\kappa}_0(Q)} \leq C\|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)},$$

and $u_f = \tilde{u} + u_* \in H^{2\kappa}(Q)$ satisfies

$$\|u_f\|_{H^{2\kappa}(Q)} \leq \|\tilde{u}\|_{H^{2\kappa}(Q)} + \|u_*\|_{H^{2\kappa}(Q)} \leq C\|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}.$$

□

Proposition 2.2. *(Well-posedness for the nonlinear equation) Let $\kappa > \frac{n+1}{2}$ be an integer. Suppose q and β are in $C^\infty(\overline{Q})$. For any $f \in \mathcal{S}_\lambda(\Sigma)$ (defined in (2.1)) with $\lambda > 0$ sufficiently small, there exists a unique solution $u \in H^{2\kappa}(Q)$ to the problem (1.1) and it satisfies the estimate*

$$\|u\|_{H^{2\kappa}(Q)} \leq C\|f\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}, \tag{2.5}$$

where the constant $C > 0$ is independent of f.

Proof. If u is a solution to (1.1), we set $w := u - u_f$ which will solve

$$\begin{cases} (i\partial_t + \Delta + q)w = -\beta(t,x)(w+u_f)^2 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w = 0 & \text{on } \{0\}\times\Omega, \end{cases} \tag{2.6}$$

where u_f is the solution to (2.2). Or equivalently, w is the solution to

$$w - \mathcal{L}^{-1} \circ \mathcal{K}w = 0,$$

where $\mathcal{K}w := -\beta(t,x)(w+u_f)^2$. For $\kappa > \frac{n+1}{2}$, using the facts that $H^{2\kappa}(Q)$ is a Banach algebra (see [1]) and that $u_f \in \mathcal{H}_0^{2\kappa}(Q)$, we have that $\mathcal{K} : \mathcal{H}_0^{2\kappa} \to \mathcal{H}_0^{2\kappa}$ is bounded.

We define for $a > 0$ (a small parameter to be determined later) the subset

$$X_a(Q) := \{u \in \mathcal{H}_0^{2\kappa}(Q);\ \|u\|_{H^{2\kappa}(Q)} \leq a\}.$$

From (2.3), we deduce

$$\|(\mathcal{L}^{-1}\circ\mathcal{K})w\|_{H^{2\kappa}(Q)} \leq C\|\mathcal{K}w\|_{H^{2\kappa}(Q)} \leq C\left(\|w\|^2_{H^{2\kappa}(Q)} + \|u_f\|^2_{H^{2\kappa}(Q)}\right) \leq C(a^2+\lambda^2) \leq a$$

for $w \in X_a(Q)$ and

$$\begin{aligned} &\|(\mathcal{L}^{-1}\circ\mathcal{K})w_1 - (\mathcal{L}^{-1}\circ\mathcal{K})w_2\|_{H^{2\kappa}(Q)} \leq C\|\mathcal{K}w_1 - \mathcal{K}w_2\|_{H^{2\kappa}(Q)} \\ &\leq C\left(\|w_1\|_{H^{2\kappa}(Q)} + \|w_2\|_{H^{2\kappa}(Q)} + \|u_f\|_{H^{2\kappa}(Q)}\right)\|w_1 - w_2\|_{H^{2\kappa}(Q)} \\ &\leq C(a+\lambda)\|w_1-w_2\|_{H^{2\kappa}(Q)} \\ &\leq K\|w_1-w_2\|_{H^{2\kappa}(Q)}, \qquad \text{for } w_1,\, w_2 \in X_a(Q) \end{aligned}$$

with $K \in (0,1)$ provided that we choose $0 < \lambda < a < 1$ and a small enough. This proves that $\mathcal{L}^{-1}\circ\mathcal{K}$ is a contraction map on $X_a(Q)$, hence there exists a fixed point $w \in X_a(Q)$ as the solution to (2.6). Moreover,

$$\begin{aligned} \|w\|_{H^{2\kappa}(Q)} = \|(\mathcal{L}^{-1}\circ\mathcal{K})w\|_{H^{2\kappa}(Q)} &\leq C\|\mathcal{K}w\|_{H^{2\kappa}(Q)} \\ &\leq C(\|w\|^2_{H^{2\kappa}(Q)} + \|u_f\|^2_{H^{2\kappa}(Q)}) \\ &\leq Ca\|w\|_{H^{2\kappa}(Q)} + C\lambda\|u_f\|_{H^{2\kappa}(Q)}, \end{aligned}$$

which further implies

$$\|w\|_{H^{2\kappa}(Q)} \leq C\lambda\|u_f\|_{H^{2\kappa}(Q)}$$

by choosing a sufficiently small. Combined with (2.3), we eventually obtain (2.5). □

3. Proof of Theorem 1.1

3.1. **Geometrical optics solutions based on gaussian beam quasimodes.** In this section we construct the geometrical optics solutions to the linear Schrödinger equation

$$(i\partial_t + \Delta + q)u = 0,$$

in Q, having the form

$$u(t,x) = e^{i\rho(\Theta(x)-|\omega|^2\rho t)}a(t,x) + r(t,x)$$

and vanishing on part of the boundary, where the leading part $e^{i\rho(\Theta(x)-|\omega|^2\rho t)}a(t,x)$ follows the construction of gaussian beam approximate solutions concentrated near a straight line in direction ω as $\rho \to \infty$. For completeness, we present a detailed adaptation, to our equation, of the construction in [18], which was for the operator $-\Delta_g - s^2$ on its transversal manifold (M,g) and for large complex frequency s. The analogous construction for the wave equation can be found in [24]. For other similar WKB type constructions, we refer the readers to [18, 26, 41].

Let p be a point in Ω and $\omega \in \mathbb{R}^n$ be a nonzero direction. Denote by $\gamma_{p,\omega}$ the straight line through p in direction ω, parametrized by $\gamma_{p,\omega}(s) = p + s\hat{\omega}$ for $s \in \mathbb{R}$, where $\hat{\omega} := \omega/|\omega|$. We can choose $\omega_2, \ldots, \omega_n \in \mathbb{R}^n$ such that $\mathcal{A} = \{\hat{\omega}, \omega_2, \ldots, \omega_n\}$ forms an orthonormal basis of $\mathbb{R}^n$. Under this basis, we identify $x \in \mathbb{R}^n$ by the new coordinate $z = (s, z')$ where $z' := (z_2, \ldots, z_n)$, that is,

$$x = p + s\hat{\omega} + z_2\omega_2 + \ldots + z_n\omega_n.$$

In particular, $\gamma_{p,\omega}(s) = (s, 0, \ldots, 0)$.

We consider the gaussian beam approximate solutions v with ansatz

$$v(t,z) = e^{i\rho(\varphi(z)-|\omega|^2\rho t)}a(t,z;\rho), \quad \rho > 0, \tag{3.1}$$

in the coordinate $(t,z) \in \mathbb{R}^{n+1}$. The aim is to find smooth complex functions φ and a. Let the Schrödinger operator act on v and get

(3.2)
$$e^{-i\rho(\varphi(z)-|\omega|^2\rho t)}(i\partial_t + \Delta + q)v(t,z) = \rho^2(|\omega|^2 - \langle\nabla\varphi, \nabla\varphi\rangle)a + i\rho(2\nabla\varphi\cdot\nabla a + a\Delta\varphi) + (i\partial_t + \Delta + q)a.$$

We *first choose the phase function* $\varphi(z)$. The equation (3.2) suggests that we will choose the complex phase function φ satisfying the eikonal equation

$$\mathcal{E}(\varphi) := \langle\nabla\varphi, \nabla\varphi\rangle - |\omega|^2 = 0 \quad \text{up to } N\text{-th order of } z' \text{ on } \gamma_{p,\omega},$$

that is, $\mathcal{E}(\varphi) = O(|z'|^{N+1})$. We substitute φ of the form

$$\varphi(s,z') = \sum_{k=0}^{N} \varphi_k(s,z'), \qquad \text{where } \varphi_k(s,z') = \sum_{|\alpha'|=k} \frac{\varphi_{k,\alpha'}(s)}{\alpha'!}(z')^{\alpha'}.$$

Here α is an n-dim multi-index $\alpha = (\alpha_1, \alpha') \in \mathbb{Z}_+^n$ with $\alpha' = (\alpha_2, \ldots, \alpha_n)$, and

$$\varphi_0(z) = |\omega|s, \quad \varphi_1(z) = 0.$$

We obtain

$$\begin{aligned}\langle\nabla\varphi, \nabla\varphi\rangle - |\omega|^2 = {} & \underbrace{\left(2|\omega|\partial_s\varphi_2 + \nabla_{z'}\varphi_2\cdot\nabla_{z'}\varphi_2\right)}_{O(|z'|^2)} + \underbrace{\left(2|\omega|\partial_s\varphi_3 + 2\nabla_{z'}\varphi_2\cdot\nabla_{z'}\varphi_3\right)}_{O(|z'|^3)} \\ & + \underbrace{\left(2|\omega|\partial_s\varphi_4 + 2\nabla_{z'}\varphi_2\cdot\nabla_{z'}\varphi_4 + F_4(s,z')\right)}_{O(|z'|^4)} + \cdots + O(|z'|^{N+1}),\end{aligned}$$

where $F_j(s,z')$ is a j^{th} order homogeneous polynomial in z' depending only on $\varphi_2,\dots,\varphi_{j-1}$. Next we look for φ_2 such that the first $O(|z'|^2)$ term vanish. Writing

$$\varphi_2(s,z') = \frac{1}{2}H(s)z'\cdot z',$$

where $H(s) = (H_{ij}(s))_{2\le i,j\le n}$ is a smooth complex symmetric matrix. Then H satisfy the matrix Riccati equation

$$|\omega|\frac{d}{ds}H(s) + H^2(s) = 0. \tag{3.3}$$

Imposing an initial condition $H(0) = H_0$, where H_0 is a complex symmetric matrix with positive definite imaginary part $\mathrm{Im}H_0$, by [[24] Lemma 2.56], there exists a unique smooth complex symmetric solution $H(s)$ to (3.3) with positive definite $\mathrm{Im}H(s)$ for all $s\in\mathbb{R}$.

For $|\alpha|\ge 3$, in order to make the $O(|z'|^3),\dots,O(|z'|^N)$ terms vanish, one derives first order ODE's for the Taylor coefficients $\varphi_{k,\alpha'}$. By imposing well-chosen initial conditions at $s=0$, we may find all the φ_j, $j=3,\dots,N$.

Next we construct the amplitude function $a(t,z;\rho)$. Let $\chi_\eta\in C_c^\infty(\mathbb{R}^{n-1})$ be a smooth function with $\chi_\eta = 1$ for $|z'|\le\frac{\eta}{2}$ and $\chi_\eta = 0$ for $|z'|\ge\eta$. Let $\iota\in C_0^\infty(0,T)$ be a smooth cut-off function of the time variable. We make the ansatz for the amplitude as

$$a(t,s,z';\rho) = \sum_{j=0}^{N}\rho^{-j}a_j(t,s,z')\chi_\eta(z') = (a_0+\rho^{-1}a_1+\dots+\rho^{-N}a_N)\chi_\eta(z').$$

From (3.2), we should determine a_j from

$$\begin{aligned}
&2\nabla\varphi\cdot\nabla a_0 + a_0\Delta\varphi = 0 && \text{up to } N\text{-th order of } z' \text{ on } \gamma_{p,\omega},\\
&2\nabla\varphi\cdot\nabla a_1 + a_1\Delta\varphi = i(i\partial_t+\Delta+q)a_0 && \text{up to } N\text{-th order of } z' \text{ on } \gamma_{p,\omega},\\
&\qquad\vdots\\
&2\nabla\varphi\cdot\nabla a_N + a_N\Delta\varphi = i(i\partial_t+\Delta+q)a_{N-1} && \text{up to } N\text{-th order of } z' \text{ on } \gamma_{p,\omega}.
\end{aligned} \tag{3.4}$$

so that the terms of $O(\rho^{-k})$ $(k=0,\dots,N)$ vanish up to N-th order of z' on $\gamma_{p,\omega}$. Therefore, we write a_0 to have the form

$$a_0(t,s,z') = \sum_{k=0}^{N}a_0^k(s,z')\iota(t), \quad \text{where} \quad a_0^k(s,z') = \sum_{|\alpha'|=k}\frac{a_0^{k,\alpha'}(s)}{\alpha'!}(z')^{\alpha'}.$$

Here a_0^k is a k^{th} order homogeneous polynomial in z'. The first equation in (3.4) becomes

$$\begin{aligned}
2\nabla\varphi\cdot\nabla a_0 + a_0\Delta\varphi &= \iota(t)\left(2|\omega|\partial_s a_0^0 + a_0^0\Delta_{z'}\varphi_2\right)\\
&\quad + \iota(t)\left(2|\omega|\partial_s a_0^1 + 2\nabla_{z'}\varphi_2\cdot\nabla_{z'}a_0^1 + a_0^1\Delta_{z'}\varphi_2 + a_0^0\Delta_{z'}\varphi_3\right) + \dots + O(|z'|^{N+1}).
\end{aligned} \tag{3.5}$$

Note that $\Delta_{z'}\varphi_2 = tr(H(s))$. In order to let the first bracket vanish, we solve $2|\omega|\partial_s a_0^0(s) + tr(H(s))a_0^0(s) = 0$ with a given initial condition $a_0^0(0) = c_0$ for some constant c_0. For later purpose, we choose $c_0 = 1$ to get

$$a_0^0(s) = e^{-\frac{1}{2|\omega|}\int_0^s tr(H(t))dt}.$$

Similarly, the coefficients of $a_0^1,\dots,a_0^N$ can be determined for the other brackets in (3.5) to vanish. Lastly, we can construct $a_1,\dots,a_N$ in a similar way. More specifically, we can write similar ansatzs for $a_1,\dots,a_N$ with corresponding coefficients $a_j^k(s,z')$ being k^{th} order homogeneous in z'. They can be determined by solving similar equations as for a_0^k, but with nonzero right hand side terms that are homogeneous in z'. Finally, we note that $a_0^{k,\alpha'}$ is smooth which further implies that $a(t,z;\rho)$ is smooth.

So far we have constructed a gaussian beam $v(t,z)$ localized near $\{(z_1,0,\dots,0), z_1\in\mathbb{R}\}$ of the form (3.1) with

$$\varphi(s,z') = |\omega|s + \frac{1}{2}H(s)z'\cdot z' + O(|z'|^3), \quad a(t,s,z') = \chi_\eta(z')(a_0+\rho^{-1}a_1+\cdots+\rho^{-N}a_N)$$

with positive definite $\mathrm{Im}H(s)$.

It is easy to verify that by translation and rotation $\Psi(x)=z$, the function defined by $v(t,\Psi(x))$ with $a(t,\Psi(x))$, still denoted by $v(t,x)$ and $a(t,x)$ respectively, is indeed the gaussian beam localized near the line $\gamma_{p,\omega}$ and satisfy

$$\begin{aligned}&(i\partial_t+\Delta_x+q(t,x))v(t,x) = (i\partial_t+\Delta_z+q)v(t,z)\\ &= e^{i\rho(\varphi(z)-|\omega|^2\rho t)}\Big(\chi_\eta(z')\left(O(|z'|^{N+1})\rho^2+O(|z'|^{N+1})\rho+(i\partial_t+\Delta+q)a_N\rho^{-N}\right)+\rho\widehat{\chi}_\eta(z')\vartheta\Big),\end{aligned} \tag{3.6}$$

where $q(t,x)$ here is the above $q(t,z)$ with $z=\Psi(x)$ (We do not distinguish the names of the functions, e.g. $q(t,x)$ and $q(t,z)$, but only indicate the difference due to transformation by notations of variables (t,x) and (t,z)) and $\widehat{\chi}_\eta(z')$ is a smooth function with $\widehat{\chi}_\eta=0$ for $|z'|<\frac{\eta}{2}$ and $|z'|\geq\eta$, and ϑ vanishes near the geodesic $\gamma_{p,\omega}$. This last term accounts for those derivatives landing on χ_η.

More specifically, we have

$$v(t,x) = e^{i\rho(\Theta(x)-|\omega|^2\rho t)}a(t,x), \tag{3.7}$$

where the phase function is explicitly given by

$$\Theta(x) = \varphi(\Psi(x)) = \omega\cdot(x-p) + \frac{1}{2}\mathcal{H}(x)(x-p)\cdot(x-p) + O(\mathrm{dist}(x,\gamma_{p,\omega})^3),$$

where $\mathcal{H}(x)$ is an $n\times n$ matrix, defined by

$$\mathcal{H}(x) = D\Psi(x)\begin{pmatrix}0 & 0\\ 0 & H((x-p)\cdot\widehat{\omega})\end{pmatrix}(D\Psi(x))^T,$$

and the notation $\mathrm{dist}(x,\gamma_{p,\omega})$ represents the distance between the point x and the line $\gamma_{p,\omega}$. Moreover, based on the properties of H, that is, $\mathrm{Im}H(s)$ is positive definite, combined with the fact that $D\Psi$ is a unitary matrix, we have that there exists a constant $c_0>0$ such that

$$\frac{1}{2}\mathrm{Im}\mathcal{H}(x)(x-p)\cdot(x-p) \geq c_0(\mathrm{dist}(x,\gamma_{p,\omega})^2) \qquad \text{for all } x. \tag{3.8}$$

To summarize, we obtain

Proposition 3.1. *Let $q\in C^\infty(\overline{Q})$ and $\gamma_{p,\omega}$ be a straight line through a point $p\in\Omega$ in direction $\omega\in\mathbb{R}^n$. For any $N>0$ and $\eta>0$, there exists a family of approximate solutions $\{v_\rho\in C^\infty(\overline{Q}),\ \rho>1\}$, supported in $(0,T)\times N_\eta(\gamma_{p,\omega})$ where $N_\eta(\gamma_{p,\omega})$ is an η-neighborhood of $\gamma_{p,\omega}$, such that*

$$\|(i\partial_t+\Delta_x+q)v_\rho\|_{H^1(0,T;L^2(\Omega))} \leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+4}, \tag{3.9}$$

and, for integer $m\geq 0$,

$$\|(i\partial_t+\Delta_x+q)v_\rho\|_{H^m(Q)} \leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+2m+2}, \tag{3.10}$$

where C is a positive constant independent of ρ.

Proof. Take v_ρ as in (3.7). It remains to show (3.9) and (3.10). To begin with, since $\mathrm{Im}(H(s))$ is positive definite, there exists $c_1>0$ so that $\mathrm{Im}(H(s))z'\cdot z'\geq c_1|z'|^2$. Therefore, for $\eta<1$ sufficiently small, in the neighborhood $\{|z'|<\eta\}$ one has

$$|e^{i\rho(\varphi(s,z')-|\omega|^2\rho t)}|\leq e^{-\frac{1}{4}c_1\rho|z'|^2}.$$

The equation (3.6) implies

$$|(i\partial_t+\Delta_x+q)v_\rho|\leq Ce^{-\frac{1}{4}c_1\rho|z'|^2}\left(|z'|^{N+1}\rho^2\chi_\eta(z')+\rho^{-N}\chi_\eta(z')+\rho\widehat{\chi}_\eta(z')\vartheta\right),$$

$$|\partial_t(i\partial_t+\Delta_x+q)v_\rho|\leq Ce^{-\frac{1}{4}c_1\rho|z'|^2}\left(|z'|^{N+1}\rho^4\chi_\eta(z')+\rho^{2-N}\chi_\eta(z')+\rho^3\widehat{\chi}_\eta(z')\vartheta\right).$$

Hence it follows that

$$\begin{aligned}&\|(i\partial_t+\Delta_x+q)v_\rho\|^2_{H^1(0,T;L^2(\Omega))}\\&\leq C\rho^8\int_0^T\|e^{-\frac{1}{4}c_1\rho|z'|^2}|z'|^{N+1}\chi_\eta(z')\|^2_{L^2(\Omega)}dt+C\rho^{-2N+4}\int_0^T\|e^{-\frac{1}{4}c_1\rho|z'|^2}\chi_\eta(z')\|^2_{L^2(\Omega)}dt\\&\quad+C\rho^6\int_0^T\|e^{-\frac{1}{4}c_1\rho|z'|^2}\widehat{\chi}_\eta(z')\vartheta\|^2_{L^2(\Omega)}dt=:J_1+J_2+J_3.\end{aligned}\tag{3.11}$$

Now by changing of variable $z'=\rho^{-\frac{1}{2}}y$ and applying integration by parts, we obtain

$$\begin{aligned}J_1&\leq C\rho^8\int_{|z'|\leq\eta}e^{-\frac{1}{2}c_1\rho|z'|^2}|z'|^{2N+2}dz'\\&\leq C\rho^{-N-1-\frac{n-1}{2}+8}\int_{\mathbb{R}^{n-1}}e^{-\frac{1}{2}c_1|y|^2}|y|^{2N+2}dy\\&\leq C\rho^{-N-1-\frac{n-1}{2}+8},\end{aligned}\tag{3.12}$$

where the constant $C>0$ is independent of ρ. Likewise, we can also deduce

$$J_2\leq C\rho^{-2N-\frac{n-1}{2}+4},\tag{3.13}$$

which is controlled by (3.12) provided ρ is sufficiently large. Moreover, since $\widehat{\chi}_\eta$ is supported in $\frac{\eta}{2}\leq|z'|\leq\eta$, by performing the change of variable $z'=\rho^{-\frac{1}{2}}y$ again, we derive

$$\begin{aligned}J_3&\leq C\rho^6\int_{\frac{\eta}{2}\leq|z'|\leq\eta}e^{-\frac{1}{2}c_1\rho|z'|^2}dz'\\&\leq C\rho^{-\frac{n-1}{2}+6}\int_{\frac{\eta}{2}\rho^{\frac{1}{2}}\leq|y|\leq\eta\rho^{\frac{1}{2}}}e^{-\frac{1}{2}c_1|y|^2}dy\\&\leq C\rho^{-\frac{n-1}{2}+6}e^{-\frac{1}{8}c_1\eta^2\rho}(\eta\rho^{\frac{1}{2}})^{n-1}\\&\leq C\eta^{n-1}e^{-\frac{1}{8}c_1\eta^2\rho}\rho^6,\end{aligned}\tag{3.14}$$

which decays exponentially in ρ (for a fixed η) and is also controlled by (3.12) provided ρ is sufficiently large. Therefore, (3.9) holds by combining (3.11), (3.12), (3.13) and (3.14).

Similarly, we have the following higher regularity estimate

$$\begin{aligned}
&\|(i\partial_t+\Delta_x+q)v\|^2_{H^m(Q)}\\
&\leq C\rho^{4m+4}\int_0^T\|e^{-\frac14 c_1\rho|z'|^2}|z'|^{N+1}\chi_\eta(z')\|^2_{L^2(\Omega)}dt+C\rho^{-2N+4m}\int_0^T\|e^{-\frac14 c_1\rho|z'|^2}\chi_\eta(z')\|^2_{L^2(\Omega)}dt\\
&\quad+C\rho^{4m+2}\int_0^T\|e^{-\frac14 c_1\rho|z'|^2}\widehat{\chi}_\eta(z')\vartheta\|^2_{L^2(\Omega)}dt\\
&\leq C\rho^{-N-1-\frac{n-1}{2}+4m+4},
\end{aligned}$$

provided ρ is sufficiently large. This completes the proof of (3.10). □

With Proposition 3.1, we can construct the geometrical optics solutions now.

Proposition 3.2. *Let $m>0$ be an even integer and $q\in C^\infty(\overline{Q})$. Given $p\in\Omega$ and $\omega\in\mathbb{R}^n$, suppose that the straight line $\gamma_{p,\omega}$ through p in direction ω satisfies $(\gamma_{p,\omega}\cap\partial\Omega)\subset\Gamma$. Then there exists $\rho_0>1$ such that when $\rho>\rho_0$, the Schrödinger equation $(i\partial_t+\Delta+q)u=0$ admits a solution $u\in H^m(Q)$ of the form*

$$u(t,x)=e^{i\rho(\Theta(x)-|\omega|^2\rho t)}a(t,x)+r(t,x)$$

with boundary value $supp(u|_{(0,T)\times\partial\Omega})\subset\Sigma^\sharp$ and initial data $u|_{t=0}=0$ in Ω (or the final condition $u|_{t=T}=0$ in Ω). Here $\Theta(x)$ and $a(t,x)$ are as in (3.7) and satisfy Proposition 3.1 and the remainder r satisfies the following estimates:

$$\|r\|_{H^m(Q)}\leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+2m+2} \tag{3.15}$$

and

$$\|r\|_{C([0,T],H^2(\Omega))}+\|r\|_{C^1([0,T],L^2(\Omega))}\leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+4}.$$

Proof. We can choose $\eta>0$ small enough such that $(N_\eta(\gamma_{p,\omega})\cap\partial\Omega)\subset\Gamma$. By the previous Proposition 3.1, for $\rho>\rho_0$, we obtain $\Theta(x)$ and $a(t,x)$ correspondingly. By Proposition 3 and Lemma 4 in [41], we obtain the existence of the solution $r\in H^m(Q)$ to

$$\begin{cases}(i\partial_t+\Delta+q)r&=&-(i\partial_t+\Delta+q)v&\text{in } Q,\\ r&=&0&\text{on }\Sigma,\\ r&=&0&\text{on }\{0\}\times\Omega,\end{cases}$$

and the estimate

$$\|r\|_{H^m(Q)}\leq C\|(i\partial_t+\Delta+q)v\|_{H^m(Q)}\leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+2m+2}.$$

Here the last inequality follows from Proposition 3.1. Also, with (3.9), [[26], Lemma 2.3] suggests

$$\|r\|_{C([0,T],H^2(\Omega))}+\|r\|_{C^1([0,T],L^2(\Omega))}\leq C\|(i\partial_t+\Delta+q)v\|_{H^1(0,T;L^2(\Omega))}\leq C\rho^{-\frac{N+1}{2}-\frac{n-1}{4}+4}.$$

□

3.2. **Finite difference.** We introduce the multivariate finite differences, which are approximations to the derivative. We define the second-order mixed finite difference operator D^2 about the zero solution as follows:

$$D^2u_{\varepsilon_1f_2+\varepsilon_2f_2}:=\frac{1}{\varepsilon_1\varepsilon_2}(u_{\varepsilon_1f_1+\varepsilon_2f_2}-u_{\varepsilon_1f_1}-u_{\varepsilon_2f_2}).$$

Note that when $\varepsilon_1=\varepsilon_2=0$, $u_{\varepsilon_1f_2+\varepsilon_2f_2}=0$. We refer the interested readers to [40] for the definitions of higher order finite difference operators. For the purpose of our paper, we only need D^2. To simplify the notation, we denote $u_{\varepsilon_1f_2+\varepsilon_2f_2}$ by $u_{\varepsilon f}$ and define $|\varepsilon|:=|\varepsilon_1|+|\varepsilon_2|$. Then we have the following second order expansion.

Proposition 3.3. *Let $\kappa > \frac{n+1}{2}$ be an integer and $f_j \in H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)$ satisfying $\partial_t^\ell f_j(0,\cdot) = 0$ for $\ell < 2\kappa + \frac{3}{2}$ for $j = 1,2$. For $|\varepsilon| := |\varepsilon_1| + |\varepsilon_2|$ small enough, there exists a unique solution $u_{\varepsilon f} \in H^{2\kappa}(Q)$ to the problem*

$$\begin{cases} (i\partial_t + \Delta + q)u_{\varepsilon f} + \beta u_{\varepsilon f}^2 &=& 0 & \text{in } Q, \\ u_{\varepsilon f} &=& \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on } \Sigma, \\ u_{\varepsilon f} &=& 0 & \text{on } \{0\}\times\Omega. \end{cases}$$

In particular, it admits the following expression:

$$u_{\varepsilon f} = \varepsilon_1 U_1 + \varepsilon_2 U_2 + \frac{1}{2}\left(\varepsilon_1^2 W_{(2,0)} + \varepsilon_2^2 W_{(0,2)} + 2\varepsilon_1\varepsilon_2 W_{(1,1)}\right) + \mathcal{R},$$

where for $j = 1,2$, $U_j \in H^{2\kappa}(Q)$ satisfies the linear equation:

$$\begin{cases} (i\partial_t + \Delta + q)U_j &=& 0 & \text{in } Q, \\ U_j &=& f_j & \text{on } \Sigma, \\ U_j &=& 0 & \text{on } \{0\}\times\Omega, \end{cases} \tag{3.16}$$

and for $k_j \in \{0,1,2\}$ satisfying $k_1 + k_2 = 2$, $W_{(k_1,k_2)} \in H^{2\kappa}(Q)$ is the solution to

$$\begin{cases} (i\partial_t + \Delta + q)W_{(k_1,k_2)} &=& -2\beta U_1^{k_1}U_2^{k_2} & \text{in } Q, \\ W_{(k_1,k_2)} &=& 0 & \text{on } \Sigma, \\ W_{(k_1,k_2)} &=& 0 & \text{on } \{0\}\times\Omega. \end{cases} \tag{3.17}$$

Moreover, the remainder term $\mathcal{R} \in H^{2\kappa}(Q)$ satisfies

$$\|\mathcal{R}\|_{H^{2\kappa}(Q)} \le C\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|^3_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}. \tag{3.18}$$

Proof. The existence of $u_{\varepsilon f} \in H^{2\kappa}(Q)$ is given by Proposition 2.2 when $|\varepsilon| := |\varepsilon_1| + |\varepsilon_2|$ sufficiently small such that $\varepsilon_1 f_1 + \varepsilon_2 f_2 \in \mathcal{S}_\lambda(\Sigma)$. Also, equations (3.16) and (3.17) are both well-posed in $H^{2\kappa}(Q)$, for example by Proposition 4 in [41], for κ as in the assumption ($H^{2\kappa}(Q)$ is a Banach algebra). We denote

$$\tilde{u} := u_{\varepsilon f} - (\varepsilon_1 U_1 + \varepsilon_2 U_2).$$

Then it solves

$$\begin{cases} (i\partial_t + \Delta + q)\tilde{u} &=& -\beta u_{\varepsilon f}^2 & \text{in } Q, \\ \tilde{u} &=& 0 & \text{on } \Sigma, \\ \tilde{u} &=& 0 & \text{on } \{0\}\times\Omega, \end{cases}$$

Applying Lemma 4 in [41] and (2.3) gives that

$$\|\tilde{u}\|_{H^{2\kappa}(Q)} \le C\|\beta u_{\varepsilon f}^2\|_{H^{2\kappa}(Q)} \le C\|u_{\varepsilon f}\|^2_{H^{2\kappa}(Q)} \le C\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|^2_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}. \tag{3.19}$$

From (3.16) we obtain

$$\begin{aligned}(i\partial_t + \Delta + q)u_{\varepsilon f} + \beta u_{\varepsilon f}^2 = &\sum_{j=1,2}\varepsilon_j(i\partial_t + \Delta + q)U_j + \frac{1}{2}\sum_{k_1+k_2=2}\begin{pmatrix} 2 \\ k_1, k_2 \end{pmatrix}\varepsilon_1^{k_1}\varepsilon_2^{k_2}(i\partial_t + \Delta + q)W_{(k_1,k_2)} \\ &+ (i\partial_t + \Delta + q)\mathcal{R} + \beta u_{\varepsilon f}^2.\end{aligned}$$

Then by (3.17), the remainder $\mathcal{R}$ satisfies

$$\begin{cases} (i\partial_t + \Delta + q)\mathcal{R} &=& -\beta u_{\varepsilon f}^2 + \beta(\varepsilon_1 U_1 + \varepsilon_2 U_2)^2 & \text{in } Q, \\ \mathcal{R} &=& 0 & \text{on } \Sigma, \\ \mathcal{R} &=& 0 & \text{on } \{0\}\times\Omega. \end{cases}$$

Then we have that $\mathcal{R} \in H^{2\kappa}(Q)$ exists and satisfies

$$\begin{aligned}\|\mathcal{R}\|_{H^{2\kappa}(Q)} &\le C\| -\beta u_{\varepsilon f}^2 + \beta(\varepsilon_1 U_1 + \varepsilon_2 U_2)^2\|_{H^{2\kappa}(Q)} \\ &\le C\|\tilde{u}\|_{H^{2\kappa}(Q)}\|u_{\varepsilon f} + (\varepsilon_1 U_1 + \varepsilon_2 U_2)\|_{H^{2\kappa}(Q)} \\ &\le C\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|^2_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} \\ &\le C\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|^3_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}\end{aligned}$$

by using the fact that $H^{2\kappa}(Q)$ is a Banach algebra, the equations (3.19), (2.3) and the well-posedness of (3.16). ☐

Remark 3.1. *Based on Proposition 3.3, when one of ε_1 and ε_2 is zero, we have*

$$u_{\varepsilon_1 f_1} = \varepsilon_1 U_1 + \frac{1}{2}\varepsilon_1^2 W_{(2,0)} + \mathcal{R}^{(1)}, \quad u_{\varepsilon_2 f_2} = \varepsilon_2 U_2 + \frac{1}{2}\varepsilon_2^2 W_{(0,2)} + \mathcal{R}^{(2)},$$

where $\mathcal{R}^{(j)}$ is the remainder term of order $O(\varepsilon_j^3)$ for $j = 1, 2$. We can rewrite $u_{\varepsilon f}$ as

$$u_{\varepsilon f} = u_{\varepsilon_1 f_1} + u_{\varepsilon_2 f_2} + \varepsilon_1\varepsilon_2 W_{(1,1)} + \widetilde{\mathcal{R}}, \tag{3.20}$$

where $\widetilde{\mathcal{R}} := \mathcal{R} - \mathcal{R}^{(1)} - \mathcal{R}^{(2)}$. Moreover, we have

$$W_{(1,1)} = D^2 u_{\varepsilon_1 f_1 + \varepsilon_2 f_2} - \frac{1}{\varepsilon_1\varepsilon_2}\widetilde{\mathcal{R}}$$

and also the Neumann data

$$\partial_\nu W_{(1,1)}|_{\Sigma^\sharp} = \frac{1}{\varepsilon_1\varepsilon_2}\left(\Lambda_{q,\beta}(\varepsilon_1 f_1 + \varepsilon_2 f_2) - \Lambda_{q,\beta}(\varepsilon_1 f_1) - \Lambda_{q,\beta}(\varepsilon_2 f_2)\right) - \frac{1}{\varepsilon_1\varepsilon_2}\partial_\nu\widetilde{\mathcal{R}}|_{\Sigma^\sharp}.$$

Through the rest of the paper, we only need to assume $|\varepsilon_1| \sim |\varepsilon_2| \sim |\varepsilon|$, in which case we have $\widetilde{\mathcal{R}} = o(\varepsilon_1\varepsilon_2)$. In fact, from (3.18) we have

$$\|\widetilde{R}\|_{H^{2\kappa}(Q)} \le C(\varepsilon_1+\varepsilon_2)^3\left(\|f_1\|^3_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} + \|f_2\|^3_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}\right)^3.$$

In the case that ε_1 and ε_2 are of different scales such as $|\varepsilon_2| \sim |\varepsilon_1|^k$ for some positive $k > 1$ (or vice versa), more terms can be taken in the expansions of $u_{\varepsilon f}$, $u_{\varepsilon_1 f_1}$ and $u_{\varepsilon_2 f_2}$ to eventually verify that $\widetilde{\mathcal{R}}$ has the norm of order $o(\varepsilon_1\varepsilon_2)$.

Since $W_{(k_1,k_2)}$ is independent of ε_1 and ε_2, this implies

$$W_{(1,1)} = \lim_{\varepsilon_1,\varepsilon_2\to 0} D^2 u_{\varepsilon_1 f_1+\varepsilon_2 f_2}, \quad \partial_\nu W_{(1,1)}|_{\Sigma^\sharp} = \lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{1}{\varepsilon_1\varepsilon_2}\left(\Lambda_{q,\beta}(\varepsilon f) - \Lambda_{q,\beta}(\varepsilon_1 f_1) - \Lambda_{q,\beta}(\varepsilon_2 f_2)\right). \tag{3.21}$$

in proper norms. For example, in $L^2(\Sigma^\sharp)$, we can derive

$$\begin{aligned}&\left\|\partial_\nu W_{(1,1)}|_{\Sigma^\sharp} - \frac{1}{\varepsilon_1\varepsilon_2}\left(\Lambda_{q,\beta}(\varepsilon f) - \Lambda_{q,\beta}(\varepsilon_1 f_1) - \Lambda_{q,\beta}(\varepsilon_2 f_2)\right)\right\|_{L^2(\Sigma^\sharp)} \\ &= \frac{1}{\varepsilon_1\varepsilon_2}\|\partial_\nu\widetilde{\mathcal{R}}\|_{L^2(\Sigma^\sharp)} \le \frac{1}{\varepsilon_1\varepsilon_2}\|\partial_\nu\widetilde{\mathcal{R}}\|_{H^{2\kappa-\frac{3}{2},2\kappa-\frac{3}{2}}(\Sigma^\sharp)} \le C\frac{1}{\varepsilon_1\varepsilon_2}\|\widetilde{\mathcal{R}}\|_{H^{2\kappa,2\kappa}(Q)} \le C\frac{1}{\varepsilon_1\varepsilon_2}\|\widetilde{\mathcal{R}}\|_{H^{2\kappa}(Q)} \\ &\le C\frac{(\varepsilon_1+\varepsilon_2)^3}{\varepsilon_1\varepsilon_2}\left(\|f_1\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} + \|f_2\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}\right)^3. \end{aligned} \tag{3.22}$$

3.3. **An integral identity.** Let $u_{\ell,\varepsilon f}$ ($\ell = 1,2$) be the small unique solution to the initial boundary value problem for the Schrödinger equation:

$$\begin{cases} (i\partial_t + \Delta + q)u_{\ell,\varepsilon f} + \beta_\ell u_{\ell,\varepsilon f}^2 &= 0 \qquad \text{in } Q,\\ u_{\ell,\varepsilon f} &= \varepsilon_1 f_1 + \varepsilon_2 f_2 \qquad \text{on } \Sigma,\\ u_{\ell,\varepsilon f} &= 0 \qquad \text{on } \{0\}\times\Omega \end{cases}$$

with $\mathrm{supp}(f_j) \subset (0,T)\times\Gamma$ for $j = 1,2$. For $|\varepsilon| := |\varepsilon_1| + |\varepsilon_2|$ small enough, they admit the expansion

$$u_{\ell,\varepsilon f} = \varepsilon_1 U_{\ell,1} + \varepsilon_2 U_{\ell,2} + \frac{1}{2}\left(\varepsilon_1^2 W_{\ell,(2,0)} + \varepsilon_2^2 W_{\ell,(0,2)} + 2\varepsilon_1\varepsilon_2 W_{\ell,(1,1)}\right) + \mathcal{R}_\ell,$$

where $U_{\ell,j}$, $W_{\ell,(k_1,k_2)}$ and $\mathcal{R}_\ell$ are as in Proposition 3.3. Since the linearized equations for both ℓ are the same with the same boundary data f_j, we have

$$U_{1,j} = U_{2,j}, \qquad j = 1,2,$$

denoted by U_j for the rest of the paper.

In addition, let U_0 be the solution of the adjoint problem:

$$\begin{cases} (i\partial_t + \Delta + q)U_0 &= 0 \qquad \text{in } Q,\\ U_0 &= f_0 \qquad \text{on } \Sigma,\\ U_0 &= 0 \qquad \text{on } \{T\}\times\Omega \end{cases} \tag{3.23}$$

with $\mathrm{supp}(f_0) \subset (0,T)\times\Gamma$.

Lemma 3.1. *Let q, $\beta_\ell \in C^\infty(\overline{Q})$ ($\ell = 1,2$) and $\beta := \beta_1 - \beta_2$. Suppose that*

$$\Lambda_{q,\beta_1}(f) = \Lambda_{q,\beta_2}(f)$$

for all $f \in \mathcal{S}_\lambda(\Sigma)$ with supp$(f) \subset \Sigma^\sharp$. Then

$$\int_Q \beta U_1 U_2 \overline{U}_0 \, dxdt = 0. \tag{3.24}$$

Proof. We denote

$$W := W_{2,(1,1)} - W_{1,(1,1)}.$$

By (3.21), we have $\partial_\nu W|_{\Sigma^\sharp} = 0$. After multiplying the equation in (3.17) by $\overline{U}_0$, subtracting and integrating over Q, we have

$$\int_Q 2\beta U_1 U_2 \overline{U}_0 \, dxdt = \int_\Sigma (\overline{U}_0 \partial_\nu W - W\partial_\nu \overline{U}_0)\, d\sigma(x)dt = 0$$

due to that $U_0(T,\cdot) = W(0,\cdot) = 0$, $W|_\Sigma = \partial_\nu W|_{\Sigma^\sharp} = 0$ and $U_0|_\Sigma$ has the support in $\Sigma^\sharp$.

□

3.4. **Proof of Theorem 1.1.** We will show that the coefficient $\beta(t,x)$ can be recovered uniquely for all the points in $(0,T)\times\Omega_\Gamma$.

Proof of Theorem 1.1. For each $p \in \Omega_\Gamma$, choose ω_1, $\omega_2 \in \mathbb{S}^{n-1}$ satisfying the condition in the description of Ω_Γ in (1.2). Set $\omega_0 := \omega_1 + \omega_2$. Based on Proposition 3.2, we can find geometrical optics solutions $U_j = v_j + r_j$, $j = 1,2$ for the problem (3.16) and $U_0 = v_0 + r_0$ for its adjoint problem (3.23) associated to three lines $\gamma_{p,\omega_1}, \gamma_{p,\omega_2}$ and γ_{p,ω_0} respectively. More specifically, we have

$$v_j(t,x) = e^{i\rho(\Theta_j(x) - |\omega_j|^2\rho t)} a^{(j)}(t,x), \quad j = 0,1,2$$

with the phase function

$$\Theta_j(x) = \omega_j \cdot (x-p) + \frac{1}{2}\mathcal{H}_j(x)(x-p)\cdot(x-p) + O(\mathrm{dist}(x,\gamma_{p,\omega})^3).$$

The amplitude functions $a^{(j)}(t,x)|_{\partial\Omega}$ are supported in Γ given $\eta < \eta_0$ for some $\eta_0 > 0$ and the remainder functions $r_j(t,x)$ satisfy (3.15). Let $f_j := U_j|_{\partial\Omega}$ $(j=0,1,2)$. From $\Lambda_{q,\beta_1}(f) = \Lambda_{q,\beta_2}(f)$ on $\mathcal{S}_\lambda(\Sigma)$ with $\mathrm{supp}(f) \subset \Sigma^\sharp$ and Lemma 3.1, we obtain the integral identity (3.24). Plugging in above U_j $(j=0,1,2)$, we obtain

$$0 = \int_Q \beta U_1 U_2 \overline{U}_0 \, dxdt = \int_Q \beta v_1 v_2 \overline{v}_0 \, dxdt + R_1 + R_2 + R_3,$$

where the remainder terms are grouped as

$$R_1 := \int_Q \beta(\overline{r}_0 v_1 v_2 + r_1 v_2 \overline{v}_0 + r_2 v_1 \overline{v}_0) \, dxdt,$$

$$R_2 := \int_Q \beta(\overline{r}_0 r_1 v_2 + \overline{r}_0 r_2 v_1 + r_1 r_2 \overline{v}_0) \, dxdt,$$

$$R_3 := \int_Q \beta r_1 r_2 \overline{r}_0 \, dxdt.$$

When $\kappa > \frac{n+1}{2}$, Proposition 3.2 shows that

$$R_1 + R_2 + R_3 = O(\rho^{-K})$$

for a large $K > \frac{n}{2}$ by choosing N sufficiently large.

Note that $|\omega_1|^2 + |\omega_2|^2 = |\omega_0|^2$. The phase of the product is then given by

$$\Theta_1(x) + \Theta_2(x) - \overline{\Theta}_0(x) = \frac{1}{2}\mathcal{H}(x)(x-p)\cdot(x-p) + \widetilde{h}(x),$$

where $\mathcal{H}(x) := \mathcal{H}_1(x) + \mathcal{H}_2(x) - \overline{\mathcal{H}_0}(x)$ whose imaginary part

$$\mathrm{Im}\mathcal{H}(x) = \mathrm{Im}\mathcal{H}_1(x) + \mathrm{Im}\mathcal{H}_2(x) + \mathrm{Im}\mathcal{H}_0(x).$$

By (3.8), we have

$$\frac{1}{2}\mathrm{Im}\mathcal{H}(x)(x-p)\cdot(x-p) \geq c_0(\mathrm{dist}(x,\gamma_{p,\omega_1})^2 + \mathrm{dist}(x,\gamma_{p,\omega_2})^2) \geq c_0|x-p|^2,$$

which implies $\mathrm{Im}\mathcal{H}$ is positive definite. Also, we have for $|x-p|$ small,

$$|\widetilde{h}(x)| = O(\mathrm{dist}(x,\gamma_{p,\omega_1})^3 + \mathrm{dist}(x,\gamma_{p,\omega_2})^3 + \mathrm{dist}(x,\gamma_{p,\omega_0})^3) = O(|x-p|^3). \tag{3.25}$$

Therefore, for $\eta < \eta_0$ sufficiently small, we shall have

$$\mathrm{Im}(\Theta_1(x) + \Theta_2(x) - \overline{\Theta}_0(x)) \geq \widetilde{c}_0|x-p|^2 \qquad \text{when } |x-p| < \eta. \tag{3.26}$$

Finally, standing on these, we derive

$$\begin{aligned} O(\rho^{-K}) &= \int_Q \beta v_1 v_2 \overline{v}_0 \, dxdt \\ &= \int_0^T \int_{B_{2\eta}(p)} \beta e^{i\rho(\frac{1}{2}\mathcal{H}(x)(x-p)\cdot(x-p) + \widetilde{h}(x))}(\widetilde{a}_0(t,x) + O(\rho^{-1}))\widetilde{\chi}_\eta(x) \, dxdt, \end{aligned}$$

where $\widetilde{a}_0(t,x) = a_0^{(1)} a_0^{(2)} \overline{a}_0^{(0)}(t,x)$ and $\widetilde{\chi}_\eta(x) := \prod_{j=0,1,2} \chi_\eta(z'_{p,\omega_j}(x))$ with $z'_{p,\omega_j}(x)$ being the projection of $x-p$ onto the orthogonal $(n-1)$-dim subspace $\omega_j^\perp = \{\xi \in \mathbb{R}^n : \ \xi \cdot \omega_j = 0\}$. By the change of variable $\tilde{x} = \rho^{\frac{1}{2}}(x-p)$, we have

$$O(\rho^{-K+\frac{n}{2}}) = \int_0^T \int_{B_{2\eta\sqrt{\rho}}(0)} e^{i(\frac{1}{2}\mathcal{H}(\rho^{-\frac{1}{2}}\tilde{x}+p)\tilde{x}\cdot\tilde{x} + \rho\widetilde{h}(\rho^{-\frac{1}{2}}\tilde{x}+p))}(\beta\widetilde{a}_0(t,\rho^{-\frac{1}{2}}\tilde{x}+p) + O(\rho^{-1}))\widetilde{\chi}_\eta(\rho^{-\frac{1}{2}}\tilde{x}+p) \, d\tilde{x}dt.$$

Applying (3.25) and (3.26), and by the dominated convergence theorem, we obtain the limit as $\rho \to \infty$

$$\left(\int_{\mathbb{R}^n} e^{\frac{i}{2}\mathcal{H}(p)\tilde{x}\cdot\tilde{x}}\, d\tilde{x}\right)\left(\int_0^T \beta\widetilde{a}_0(t,p)\, dt\right) = 0,$$

where we use that the pointwise limit of $\rho\widetilde{h}(\rho^{-1/2}\widetilde{x}+p)$ is zero. We can choose the initial condition for H in the matrix Riccati equation such that the first integral is nonzero. Also recall that $a_0^{(j)}(t,p) = \iota(t)$ for $j = 0, 1, 2$ in the constructions of $a_0^{(j)}$, thus

$$\int_0^T \beta(t,p)\iota^3(t)\, dt = 0.$$

Since ι can be chosen to be any smooth cut-off function at the time variable, such as a sequence $\iota_\epsilon(t)$ converging to δ_{t_0} as $\epsilon \to 0$ for a given t_0, this leads to $\beta(t_0, p) = 0$ for arbitrary $t_0 \in (0, T)$. $\square$

To conclude this part, we remark that there are two reasons of the use of Gaussian beams instead of a localized version of geometric optics solutions here: one is that it can be potentially applied to non-Euclidean geometrical settings in future study; the second reason is that although it is possible to use simpler localized geometric optic solutions, we feel the Gaussian beams construction sheds more light on the asymptotic decaying behavior as $\rho \to \infty$ while tracing the effect caused by the shrinkage of δ, the width of the concentration centered at the line.

4. Proof of Theorem 1.2 and Theorem 1.3

4.1. **Geometric optics.** In this section, we will construct the geometric optics (GO) solutions to the Schrödinger equation, similar to the ones used in [26] and [41], and introduce its associated unique continuation principle. Compared to the GO solutions in Proposition 3.2, these are not localized near a straight line.

Following the same ansatz for a GO solution under the global coordinate

$$u(t,x) = e^{i\Phi(t,x)}\left(\sum_{k=0}^{N} \rho^{-k} a_k(t,x)\right) + r(t,x),$$

where we take a simple linear (in x) phase

$$\Phi(t,x) := \rho(x\cdot\omega - \rho|\omega|^2 t),$$

with $\rho > 0$ and $\omega \in \mathbb{R}^n$. Then the terms in the amplitude naturally satisfy

$$\begin{aligned} \omega\cdot\nabla a_0 &= 0, \\ 2i\omega\cdot\nabla a_1 &= -(i\partial_t + \Delta + q)a_0, \\ &\vdots \\ 2i\omega\cdot\nabla a_N &= -(i\partial_t + \Delta + q)a_{N-1}, \end{aligned} \tag{4.1}$$

and the remainder term r satisfies

$$\left\{\begin{array}{rcll} (i\partial_t + \Delta + q)r &=& -\rho^{-N}e^{i\Phi(t,x)}(i\partial_t + \Delta + q)a_N & \text{in } Q, \\ r &=& 0 & \text{on } \Sigma, \\ r &=& 0 & \text{on } \{0\}\times\Omega. \end{array}\right. \tag{4.2}$$

We construct a_0 as follows. Let $0 < T^* < T$ and the function $\theta_h \in C_0^\infty(\mathbb{R})$ satisfy $0 \leq \theta_h \leq 1$ and for $0 < h < \frac{T^*}{4}$,

$$\theta_h(t) = \begin{cases} 0 & \text{in } [0, h] \cup [T^* - h, T^*], \\ 1 & \text{in } [2h,\, T^* - 2h], \end{cases} \tag{4.3}$$

with support in $(h, T^* - h)$ and, moreover, for all $j \in \mathbb{N}$, there exist constants $C_j > 0$ such that

$$\|\theta_h\|_{W^{j,\infty}(\mathbb{R})} \leq C_j h^{-j}. \tag{4.4}$$

We choose

$$a_0(t, x) := \theta_h(t) e^{i(t\tau + x\cdot\xi)}$$

with $\xi \in \omega^\perp$. Then it satisfies

$$a_0(t, x) = 0 \quad \text{ for all } (t, x) \in ((0, h) \cup (T^* - h, T^*)) \times \Omega,$$

and the first equation in (4.1).

Let $y \in \partial\Omega$ and $L := \{x : \omega \cdot (x - y) = 0\}$. Set

$$a_k(t, x + s\omega) = \frac{i}{2} \int_0^s (i\partial_t + \Delta + q) a_{k-1}(t, x + \tilde{s}\omega)\, d\tilde{s}, \qquad x \in L, \quad j = 1, \dots, N. \tag{4.5}$$

Then a_j ($j = 1, \dots, N$) satisfies (4.1) and vanishes on L. The regularity of a_j inherits from a_0, which is smooth both in t and x.

We introduce the notation

$$\langle \tau, \xi \rangle := (1 + \tau^2 + |\xi|^2)^{1/2}, \quad \tau \in \mathbb{R},\, \xi \in \mathbb{R}^n.$$

Proposition 4.1. *Let $\omega \in \mathbb{R}^n$, $N > 0$ and $m > n + 1$ be an integer. Suppose that $q \in C^\infty(\overline{Q})$. Then there exist GO solutions to the Schrödinger equation $(i\partial_t + \Delta + q)u = 0$ in Q of the form*

$$u(t, x) = e^{i\Phi(t,x)} \left(a_0(t, x) + \sum_{k=1}^{N} \rho^{-k} a_k(t, x) \right) + r(t, x), \quad a_0(t, x) = \theta_h(t) e^{i(t\tau + x\cdot\xi)}$$

satisfying the initial condition $u|_{t=0} = 0$ in Ω (or the final condition $u|_{t=T} = 0$ in Ω). Here $a_k \in H^m(Q)$ $(k = 1, \dots, N)$ are given by (4.5) and satisfy

$$\|a_k\|_{H^m(Q)} \leq C \langle \tau, \xi \rangle^{2k+m} h^{-k-m}, \; 0 \leq k \leq N \tag{4.6}$$

for any $\tau \in \mathbb{R}$, $h \in (0, \frac{T^*}{4})$ small enough and $\xi \in \omega^\perp$, where the constant $C > 0$ depending only on Ω and T. The remainder term r satisfies

$$\|r\|_{H^m(Q)} \leq C \rho^{-N+2m} \langle \tau, \xi \rangle^{2N+m+2} h^{-(N+m+1)} \tag{4.7}$$

and

$$\|r\|_{C^1([0,T], L^2(\Omega)) \cap C([0,T], H^2(\Omega))} \leq C \rho^{-N+2} \langle \tau, \xi \rangle^{2N+2} h^{-N-2}$$

for some constant $C > 0$ depending only on Ω and T.

Proof. We show the proof for the case with zero initial condition. The case with zero final condition at T can be justified similarly. For $k = 0$, the estimate (4.6) clearly holds for $m = 0$. For $m = 1$, it is easy to check that $\|\nabla a_0\|_{L^2(Q)} \leq C|\xi|$ and $\|\partial_t a_0\|_{L^2(Q)} \leq C(|\tau| + |\xi| + h^{-1})$ and, therefore, when h is small,

$$\|a_0\|_{H^1(Q)} \leq C \langle \tau, \xi \rangle h^{-1}.$$

Similarly, we can also deduce the bound for $\|a_0\|_{H^m(Q)}$. By induction, assuming that a_{k-1} satisfies

$$\|a_{k-1}\|_{H^m(Q)} \leq C \langle \tau, \xi \rangle^{2k+m-2} h^{-k-m+1}.$$

From (4.5), since we take x-derivative twice and t-derivative on a_{k-1}, the estimate of $\|a_k\|_{H^m(Q)}$ will receive extra $\langle\tau,\xi\rangle^2$ and h^{-1} on top of $\|a_{k-1}\|_{H^m(Q)}$. This leads to (4.6). Note that (4.6) holds for all integer $m \geq 0$.

Now we discuss the existence and estimates of r to the problem (4.2). From Proposition 3 and Lemma 4 in [41], since $e^{i\Phi}(i\partial_t + \Delta + q)a_N \in \mathcal{H}_0^m$ with even integer $m > \frac{n+1}{2}$, there exists a solution r to (4.2) so that

$$\|r\|_{H^m(Q)} \leq C\rho^{-N}\|e^{i\Phi}(i\partial_t + \Delta + q)a_N\|_{H^m(Q)} \leq C\rho^{-N+2m}\langle\tau,\xi\rangle^{2N+m+2}h^{-(N+m+1)}.$$

In addition, from Lemma 2.3 in [26], one can also derive

$$\begin{aligned}\|r\|_{C^1([0,T],L^2(\Omega))\cap C([0,T],H^2(\Omega))} &\leq C\rho^{-N}\|e^{i\Phi}(i\partial_t + \Delta + q)a_N\|_{H^1(0,T,L^2(\Omega))}\\ &\leq C\rho^{-N}\rho^2(\|a_N\|_{H^2(0,T,L^2(\Omega))} + \|a_N\|_{H^1(0,T,H^2(\Omega))})\\ &\leq C\rho^{-N+2}\langle\tau,\xi\rangle^{2N+2}h^{-N-2}.\end{aligned}$$

$\square$

Remark 4.1. *The choice of a_0 is quite flexible as long as $\omega\cdot\nabla a_0 = 0$ is fulfilled. This flexibility is essential in the reconstruction of the unknown coefficient β since it will help eliminate the unwanted terms in the integral identity in order to obtain the Fourier transform of β, see Section 4.4 for more detailed computations and explanations.*

For our purpose, we will also need the GO solution with a simple choice $a_0(t,x) = \theta_h(t)$ where $\theta_h(t)$ is given by (4.3). That is, there exist GO solutions to the Schrödinger equation $(i\partial_t + \Delta + q)u = 0$ in Q of the form

$$u(t,x) = e^{i\Phi(t,x)}\left(\theta_h(t) + \sum_{k=1}^N \rho^{-k}a_k(t,x)\right) + r(t,x),$$

satisfying the initial condition $u|_{t=0} = 0$ in Ω (or the final condition $u|_{t=T} = 0$ in Ω). From (4.1), we obtain $\omega\cdot\nabla a_1 = -\frac{1}{2}\partial_t\theta_h(t)$, implying

$$a_1(t,x) = -\frac{1}{2|\omega|^2}\partial_t\theta_h(t)x\cdot\omega$$

with $a_1(t,x) = 0$ on $\omega^\perp$. The rest of $a_k \in H^m(Q)$ $(k = 2,\ldots,N)$ are given by (4.5) and one can verify

$$\|a_k\|_{H^m(Q)} \leq Ch^{-k-m},\ 0 \leq k \leq N \tag{4.8}$$

and

$$\|r\|_{H^m(Q)} \leq C\rho^{-N+2m}h^{-(N+m+1)}, \qquad \|r\|_{C^1([0,T],L^2(\Omega))\cap C([0,T],H^2(\Omega))} \leq C\rho^{-N+2}h^{-N-2} \tag{4.9}$$

for some constant $C > 0$ depending only on Ω and T. Note that under this construction a_k $(k = 0,\cdots,N)$ all vanish on $((0,h)\cup(T-h,T))\times\Omega$.

4.2. **Unique continuation property (UCP).** Recall that $\mathcal{O}\subset\Omega$ is an open neighborhood of $\partial\Omega$. Let $\mathcal{O}_j$ $(j = 1,2,3)$ denote the open subsets of $\mathcal{O}$ such that $\overline{\mathcal{O}}_{j+1}\subset\mathcal{O}_j$, $\overline{\mathcal{O}}_j\subset\mathcal{O}$. Set $\Omega_j := \Omega\setminus\overline{\mathcal{O}}_j$ and $Q_j := (0,T)\times\Omega_j$. We will need the following lemma of UCP and its corollary for the linear Schrödinger equation. The lemma follows directly from [8] by setting the magnetic potential to be zero.

Lemma 4.1 (Unique continuation property)**.** *Suppose that $q \in \mathcal{M}_{\mathcal{O}}$. Let $\tilde{w} \in H^{1,2}(Q)$ be a solution to the following system*

$$\begin{cases} (i\partial_t + \Delta + q)\tilde{w} &= g_0 & \text{in } Q, \\ \tilde{w} &= 0 & \text{on } \Sigma, \\ \tilde{w} &= 0 & \text{on } \{0\} \times \Omega, \end{cases} \tag{4.10}$$

where $g_0 \in L^2(Q)$ and $\operatorname{supp}(g_0) \subset (0,T) \times (\Omega \setminus \mathcal{O})$. Then for any $T^* \in (0,T)$, there exist $C > 0, \gamma^* > 0, m_1 > 0, \mu_1 < 1$ such that the following estimate holds

$$\|\tilde{w}\|_{L^2((0,T^*)\times(\Omega_3\setminus\Omega_2))} \leq C\left(\gamma^{-\mu_1}\|\tilde{w}\|_{H^{1,1}(Q)} + e^{m_1\gamma}\|\partial_\nu \tilde{w}\|_{L^2(\Sigma^\sharp)}\right),$$

for any $\gamma > \gamma^*$. Here the constants C, m_1 and μ_1 depend on Ω, $\mathcal{O}$, T^* and T.

Corollary 4.1. *Let $q \in \mathcal{M}_{\mathcal{O}}$, and $\tilde{w} \in H^{1,2}(Q)$ a solution of* (4.10) *where $g_0 \in L^2(Q)$ and $\operatorname{supp}(g_0) \subset (0,T) \times (\Omega \setminus \mathcal{O})$ such that $\partial_\nu \tilde{w} = 0$ on $\Sigma^\sharp$. Then $\tilde{w} = 0$ in $(0,T) \times (\Omega_3 \setminus \Omega_2)$.*

4.3. **The integral identity.** In this section, we derive the needed integral identity to prove the stability estimate in Theorem 1.2. We denote

$$Q^* := (0,T^*) \times \Omega \quad \text{ for } 0 < T^* < T.$$

Recall the notation $u_{\ell,\varepsilon f}$ ($\ell = 1,2$) that denotes the small unique solution to the initial boundary value problem

$$\begin{cases} (i\partial_t + \Delta + q)u_{\ell,\varepsilon f} + \beta_\ell u_{\ell,\varepsilon f}^2 &= 0 & \text{in } Q^*, \\ u_{\ell,\varepsilon f} &= \varepsilon_1 f_1 + \varepsilon_2 f_2 & \text{on } \Sigma, \\ u_{\ell,\varepsilon f} &= 0 & \text{on } \{0\} \times \Omega, \end{cases}$$

where $f_1, f_2 \in H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)$ and $|\varepsilon| := |\varepsilon_1| + |\varepsilon_2|$ is sufficiently small such that $\varepsilon_1 f_1 + \varepsilon_2 f_2 \in \mathcal{S}_\lambda(\Sigma)$. Also, let U_j and $W_{\ell,(1,1)}$ be the solutions to the equations (3.16) and (3.17), respectively. In addition, let U_0 be the solution of the adjoint problem,

$$\begin{cases} (i\partial_t + \Delta + q)U_0 &= 0 & \text{in } Q^*, \\ U_0 &= f_0 & \text{on } \Sigma, \\ U_0 &= 0 & \text{on } \{T^*\} \times \Omega \end{cases}$$

for some $f_0 \in H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)$ to be specified later. We also introduce a smooth cut-off function $\chi \in C^\infty(\overline{\Omega})$ satisfying $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 0 & \text{in } \mathcal{O}_3, \\ 1 & \text{in } \overline{\Omega} \setminus \mathcal{O}_2, \end{cases}$$

and denote $W := W_{2,(1,1)} - W_{1,(1,1)}$, which solves

$$\begin{cases} (i\partial_t + \Delta + q)W &= 2\beta U_1 U_2 & \text{in } Q^*, \\ W &= 0 & \text{on } \Sigma, \\ W &= 0 & \text{on } \{0\} \times \Omega, \end{cases} \tag{4.11}$$

where $\beta = \beta_1 - \beta_2$. As we will see below, by applying this cut-off function χ to W, whose Neumann data is not necessary zero, we have a control of the energy near the boundary using UCP. First, we obtain the following key integral identity.

Lemma 4.2. *Suppose that $\beta := \beta_1 - \beta_2 \in \mathcal{M}_{\mathcal{O}}$. Let U_j and W be as above. Then*

$$\int_{Q^*} 2\beta U_1 U_2 \overline{U}_0 + [\Delta, \chi] W \overline{U}_0 \, dx dt = 0, \tag{4.12}$$

where $[\Delta, \chi] := \Delta\chi - \chi\Delta$ is the commutator bracket.

Proof. Let $W^*(t,x) := \chi(x)W(t,x)$. Note that since $\beta_1-\beta_2=0$ in $[0,T]\times\mathcal{O}$ and $\chi=1$ in $\overline{\Omega}\setminus\mathcal{O}$ (a subset of $\overline{\Omega}\setminus\mathcal{O}_2$), we have

$$\chi(\beta_1-\beta_2)=\beta_1-\beta_2 \quad \text{in } Q.$$

This implies that the function W^* satisfies

$$\begin{cases} (i\partial_t+\Delta+q)W^* &=& 2\beta U_1U_2+[\Delta,\chi]W & \text{in } Q^*,\\ W^* &=& 0 & \text{on } \Sigma,\\ W^* &=& 0 & \text{on } \{0\}\times\Omega. \end{cases}$$

In particular, we have

$$W^*|_\Sigma=\partial_\nu W^*|_\Sigma=0.$$

We multiply the first equation above by $\overline{U}_0$ and then integrate over Q^*. Using the condition $\overline{U}_0|_{t=T^*}=W|_{t=0}=0$, we finally obtain

$$\int_{Q^*} 2\beta U_1U_2\overline{U}_0+[\Delta,\chi]W\overline{U}_0\,dxdt=\int_\Sigma(\overline{U}_0\partial_\nu W^*-W^*\partial_\nu\overline{U}_0)\,d\sigma(x)dt=0.$$

□

4.4. **Proof of the stability estimate (Theorem 1.2).** Below we derive a series of estimates to prove the final stability result in Theorem 1.2. We choose to plug in GO solutions U_j, $j=0,1,2$ as in Proposition 4.1 and Remark 4.1. Specifically, we take

$$U_j(t,x):=v_j(t,x)+r_j(t,x)=e^{i\Phi_j(t,x)}\left(a_0^{(j)}+\sum_{k=1}^N\rho^{-k}a_k^{(j)}(t,x)\right)+r_j(t,x), \qquad j=0,1,2,$$

where the phase function Φ_j are of the form

$$\Phi_j(t,x)=\rho\left(x\cdot\omega_j-\rho|\omega_j|^2t\right)$$

with the vectors ω_1, ω_2 and ω_0 satisfying

$$\omega_1+\omega_2=\omega_0,\quad |\omega_1|^2+|\omega_2|^2=|\omega_0|^2. \tag{4.13}$$

The leading amplitudes $a_0^{(j)}$ are given by

$$a_0^{(1)}(t,x)=a_0^{(2)}(t,x)=\theta_h(t),\quad a_0^{(0)}(t,x)=\theta_h(t)e^{i(\tau t+x\cdot\xi)},$$

where $\tau\in\mathbb{R}$ and $\xi\in\omega_0^\perp$.

Substituting $U_j=v_j+r_j$ $(j=0,1,2)$ into the first term on the left-hand side of the identity (4.12), we get

$$\int_{Q^*}2\beta U_1U_2\overline{U}_0\,dxdt=\int_{Q^*}2\beta v_1v_2\overline{v}_0\,dxdt+R_1+R_2+R_3, \tag{4.14}$$

where the remainder terms are grouped into

$$R_1:=2\int_{Q^*}\beta(\overline{r}_0v_1v_2+r_1v_2\overline{v}_0+r_2v_1\overline{v}_0)\,dxdt,$$

$$R_2:=2\int_{Q^*}\beta(\overline{r}_0r_1v_2+\overline{r}_0r_2v_1+r_1r_2\overline{v}_0)\,dxdt,$$

$$R_3:=2\int_{Q^*}\beta r_1r_2\overline{r}_0\,dxdt.$$

We have the following asymptotics.

Lemma 4.3. *There exist $\rho_0 > 1$ and $1 > h_0 > 0$ such that for $\rho > \rho_0$ and $0 < h < h_0$ such that*

$$2\int_{Q^*} \beta v_1 v_2 \overline{v}_0 \, dxdt = 2\int_{Q^*} \beta a_0^{(1)} a_0^{(2)} \overline{a}_0^{(0)} \, dxdt + I, \tag{4.15}$$

where

$$|I| \leq C\left(\rho^{-1}\langle \tau, \xi\rangle^{2N} h^{-N} + \rho^{-2}\langle \tau, \xi\rangle^{2N} h^{-2N} + \rho^{-3}\langle \tau, \xi\rangle^{2N+m} h^{-3N-3m}\right),$$

for any $\tau \in \mathbb{R}$ and $\xi \in \omega_0^\perp$. Here $m > n+1$ is the Sobolev order as in Proposition 4.1 and the positive constant C depends on Q^*, N, and β.

Proof. By the definition of v_j, we have the identity

$$2\int_{Q^*} \beta v_1 v_2 \overline{v}_0 \, dxdt = 2\int_{Q^*} \beta a_0^{(1)} a_0^{(2)} \overline{a}_0^{(0)} \, dxdt + I_1 + I_2 + I_3,$$

where we used the conditions (4.13) to get $\Phi_1 + \Phi_2 - \Phi_0 = 0$. Here the rest $O(\rho^{-1})$ terms are grouped into

$$I_1 := 2\int_{Q^*} \beta \left[a_0^{(1)} a_0^{(2)} \left(\sum_{k=1}^N \rho^{-k} \overline{a}_k^{(0)}\right) + a_0^{(1)} \overline{a}_0^{(0)} \left(\sum_{k=1}^N \rho^{-k} a_k^{(2)}\right) + a_0^{(2)} \overline{a}_0^{(0)} \left(\sum_{k=1}^N \rho^{-k} a_k^{(1)}\right)\right] dxdt,$$

$$\begin{aligned} I_2 := 2\int_{Q^*} \beta \Bigg[& a_0^{(1)} \left(\sum_{k=1}^N \rho^{-k} a_k^{(2)}\right)\left(\sum_{k=1}^N \rho^{-k} \overline{a}_k^{(0)}\right) + a_0^{(2)} \left(\sum_{k=1}^N \rho^{-k} a_k^{(1)}\right)\left(\sum_{k=1}^N \rho^{-k} \overline{a}_k^{(0)}\right) \\ & + \overline{a}_0^{(0)} \left(\sum_{k=1}^N \rho^{-k} a_k^{(1)}\right)\left(\sum_{k=1}^N \rho^{-k} a_k^{(2)}\right)\Bigg] dxdt, \end{aligned}$$

and

$$I_3 := 2\int_{Q^*} \beta \left(\sum_{k=1}^N \rho^{-k} a_k^{(1)}\right)\left(\sum_{k=1}^N \rho^{-k} a_k^{(2)}\right)\left(\sum_{k=1}^N \rho^{-k} \overline{a}_k^{(0)}\right) dxdt.$$

Let us estimate each I_j. To this end, it is sufficient to control the first term in each I_j since the other terms can be handled similarly.

The first term in I_1 is controlled by

$$\begin{aligned} & 2\left|\int_{Q^*} \beta a_0^{(1)} a_0^{(2)} \left(\sum_{k=1}^N \rho^{-k} \overline{a}_k^{(0)}\right) dxdt\right| \\ \leq & C\|\beta\|_{L^\infty(Q^*)} \|a_0^{(1)}\|_{L^\infty(Q^*)} \|a_0^{(2)}\|_{L^\infty(Q^*)} \left(\sum_{k=1}^N \rho^{-k} \|\overline{a}_k^{(0)}\|_{L^2(Q^*)}\right) \\ \leq & C\sum_{k=1}^N \rho^{-k} \|\overline{a}_k^{(0)}\|_{L^2(Q^*)} \leq C\rho^{-1} \|\overline{a}_N^{(0)}\|_{L^2(Q^*)} \leq C\rho^{-1}\langle \tau, \xi\rangle^{2N} h^{-N}, \end{aligned}$$

by (4.6) for sufficiently large $\rho > 1$ and small $h < 1$, where C depending on Q^*, N and β. Similarly, the second term and the third term are less than $C\rho^{-1}h^{-N}$ by applying (4.8) instead. Combining these estimates together gives

$$|I_1| \leq C\rho^{-1}\langle \tau, \xi\rangle^{2N} h^{-N}. \tag{4.16}$$

For I_2, applying Hölder's inequality, the first term is controlled by

$$\begin{aligned}
&2\left|\int_{Q^*}\beta a_0^{(1)}\left(\sum_{k=1}^N \rho^{-k}a_k^{(2)}\right)\left(\sum_{k=1}^N \rho^{-k}\overline{a}_k^{(0)}\right)dxdt\right| \\
&\leq C\|\beta\|_{L^\infty(Q^*)}\|a_0^{(1)}\|_{L^\infty(Q^*)}\left(\sum_{k=1}^N \rho^{-k}\|a_k^{(2)}\|_{L^2(Q^*)}\right)\left(\sum_{k=1}^N \rho^{-k}\|\overline{a}_k^{(0)}\|_{L^2(Q^*)}\right) \\
&\leq C\left(\sum_{k=1}^N \rho^{-k}h^{-k}\right)\left(\sum_{k=1}^N \rho^{-k}\langle\tau,\xi\rangle^{2k}h^{-k}\right) \\
&\leq C\rho^{-2}\langle\tau,\xi\rangle^{2N}h^{-2N},
\end{aligned}$$

by using (4.6) and (4.8) again. Similarly, the second and the third terms share the same bound. Therefore we have

$$|I_2|\leq C\rho^{-2}\langle\tau,\xi\rangle^{2N}h^{-2N}. \tag{4.17}$$

Finally, since $m>n+1$, we can control I_3 by

$$\begin{aligned}
|I_3| &\leq C\|\beta\|_{L^\infty(Q^*)}\left(\sum_{k=1}^N \rho^{-k}\|a_k^{(1)}\|_{H^m(Q^*)}\right)\left(\sum_{k=1}^N \rho^{-k}\|a_k^{(2)}\|_{H^m(Q^*)}\right)\left(\sum_{k=1}^N \rho^{-k}\|\overline{a}_k^{(0)}\|_{H^m(Q^*)}\right) \\
&\leq C\rho^{-3}\|a_N^{(1)}\|_{H^m(Q^*)}\|a_N^{(2)}\|_{H^m(Q^*)}\|a_N^{(0)}\|_{H^m(Q^*)} \\
&\leq C\rho^{-3}\langle\tau,\xi\rangle^{2N+m}h^{-3N-3m} \qquad (4.18)
\end{aligned}$$

Combining (4.16), (4.17), and (4.18) completes the proof. ☐

Lemma 4.4. *Then there exists $\rho_0>1$ and $0<h_0<1$ such that the three remainder terms satisfy the following estimates:*

$$|R_1|\leq C\rho^{-N+2m}\langle\tau,\xi\rangle^{2N+m+2}h^{-3N-3m-1},$$

$$|R_2|\leq C\rho^{-2N+4m}\langle\tau,\xi\rangle^{2N+m+2}h^{-3N-3m-2},$$

and

$$|R_3|\leq C\rho^{-3N+6m}\langle\tau,\xi\rangle^{2N+m+2}h^{-3N-3m-3}$$

for $\rho>\rho_0$, $0<h<h_0$, $\tau\in\mathbb{R}$ and $\xi\in\omega_0^\perp$, where the positive constant C depends on Q^*, N, and β. Here $m>n+1$ is the Sobolev order as in Proposition 4.1.

Proof. Again it is sufficient to evaluate the first term in each R_j. Substituting v_1, v_2, and r_0 into the first term of R_1, we get

$$\int_{Q^*}\beta\overline{r}_0v_1v_2dxdt=\int_{Q^*}\beta\overline{r}_0e^{i\Phi_0}\left(\sum_{k=0}^N \rho^{-k}a_k^{(1)}\right)\left(\sum_{k=0}^N \rho^{-k}a_k^{(2)}\right)dxdt.$$

Since $H^m(Q)$ is an algebra, by (4.7) and (4.8), we have

$$\begin{aligned}
\left|\int_{Q^*}\beta\overline{r}_0v_1v_2dxdt\right| &\leq C\|\beta\|_{L^\infty(Q^*)}\|r_0\|_{H^m(Q^*)}\left(\sum_{k=0}^N \rho^{-k}\|a_k^{(1)}\|_{H^m(Q^*)}\right)\left(\sum_{k=0}^N \rho^{-k}\|a_k^{(2)}\|_{H^m(Q)}\right) \\
&\leq C\rho^{-N+2m}\langle\tau,\xi\rangle^{2N+m+2}h^{-N-m-1}\|a_N^{(1)}\|_{H^m(Q^*)}\|a_N^{(2)}\|_{H^m(Q^*)} \\
&\leq C\rho^{-N+2m}\langle\tau,\xi\rangle^{2N+m+2}h^{-3N-3m-1}.
\end{aligned}$$

The rest terms in R_1 satisfy the same estimate similarly. The same argument also gives the corresponding bounds for R_2 and R_3, using (4.6), (4.7), (4.8) and (4.9). This completes the proof of this lemma. $\square$

Now we are ready to prove an estimate for the Fourier transform of $\beta\theta_h^3(t)$ below.

Lemma 4.5. *Let $m = 2\kappa > n+1$ in Proposition 4.1, $N > 4\kappa + 1$ and $\beta = \beta_1 - \beta_2 \in \mathcal{M}_{\mathcal{O}}$. For $\rho > \rho_0 > 1$ and $1 > h_0 > h > 0$, we have*

$$2\left|\int_{Q^*} \beta\theta_h^3(t)e^{-i(\tau t + x\cdot\xi)}\,dxdt\right| \le \left|\int_{Q^*}[\Delta,\chi]W\overline{U}_0\,dxdt\right| + C\rho^{-1}\langle\tau,\xi\rangle^{2N+2\kappa+2}h^{-3N-6\kappa-3} \tag{4.19}$$

for $\tau \in \mathbb{R}$ and $\xi \in \omega_0^\perp$. Here the constant $C > 0$ is independent of ρ, τ, ξ and h.

Proof. We derive from (4.14), (4.15) with $m = 2\kappa$ and the identity (4.12) that

$$2\int_{Q^*} \beta a_0^{(1)}a_0^{(2)}\overline{a}_0^{(0)}\,dxdt = -\int_{Q^*}[\Delta,\chi]W\overline{U}_0\,dxdt - (I + R_1 + R_2 + R_3). \tag{4.20}$$

With the estimates in Lemma 4.3 and Lemma 4.4, we can further simplify the estimate of $I + R_1 + R_2 + R_3$ into

$$|I + R_1 + R_2 + R_3| \le C\rho^{-1}\langle\tau,\xi\rangle^{2N+2\kappa+2}h^{-3N-6\kappa-3}$$

by noting that $N > 4\kappa+1$, $\rho > 1$, and $\langle\tau,\xi\rangle \ge 1$. The lemma is then proved by recalling that $a_0^{(0)} = \theta_h(t)e^{i(\tau t + x\cdot\xi)}$ and $a_0^{(1)} = a_0^{(2)} = \theta_h(t)$. $\square$

Next we try to estimate the first term on the right hand side of (4.19) in terms of the boundary measurements difference.

Lemma 4.6. *Let $m = 2\kappa > n+1$ in Proposition 4.1 and $\beta = \beta_1 - \beta_2 \in \mathcal{M}_{\mathcal{O}}$. Suppose*

$$\|(\Lambda_{q,\beta_1} - \Lambda_{q,\beta_2})f\|_{L^2(\Sigma^\sharp)} \le \delta \qquad \textit{for all } f \in \mathcal{S}_\lambda(\Sigma).$$

Then for $\rho > \rho_0 > 1$, $1 > h_0 > h > 0$ and $|\varepsilon_1| + |\varepsilon_2|$ sufficiently small, we have

$$\|W\|_{H^{1,1}(Q)} \le C\rho^{8\kappa}h^{-2N-4\kappa-2}$$

and

$$\|\partial_\nu W\|_{L^2(\Sigma^\sharp)} \le \frac{C}{\varepsilon_1\varepsilon_2}\left(\delta + (\varepsilon_1+\varepsilon_2)^3\rho^{12\kappa+12}h^{-3N-6\kappa-9}\right).$$

Proof. Recall that from Remark 4.1, we can derive

$$\|U_j\|_{H^{2s}(Q)} \le C\rho^{4s}h^{-N-2s-1}, \qquad j = 1, 2 \tag{4.21}$$

when $s > \frac{n+1}{2}$. We first take $s = \kappa$. Since the non-homogeneous term of (4.11) is $2\beta U_1U_2 \in H^{2\kappa}$, applying Lemma 4 in [41] yields that

$$\|W\|_{H^{1,1}(Q)} \le \|W\|_{H^{2\kappa}(Q)} \le C\|U_1\|_{H^{2\kappa}(Q)}\|U_2\|_{H^{2\kappa}(Q)} \le C\rho^{8\kappa}h^{-2N-4\kappa-2},$$

where C depends on β, Ω and T.

Below we will estimate $\partial_\nu W = \partial_\nu W_{2,(1,1)} - \partial_\nu W_{1,(1,1)}$. From $f_j = U_j|_\Sigma$, according to (4.21) with $s = \kappa+1$ and Theorem 2.1 (the trace theorem) in [44], we obtain

$$\|f_j\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} \le C\|U_j\|_{H^{2\kappa+2}(Q)} \le C\rho^{4\kappa+4}h^{-N-2\kappa-3}, \tag{4.22}$$

for $j = 1$, 2, where the constant C is independent of f_j.

Denote $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_2 - \widetilde{\mathcal{R}}_1$ where $\widetilde{\mathcal{R}}_\ell$ is the remainder as in (3.20) for $u_{\ell,\varepsilon f}$ ($\ell = 1, 2$). From (3.22) and (4.22), we obtain

$$\begin{aligned}
&\|\partial_\nu W\|_{L^2(\Sigma^\sharp)} \\
&\leq \frac{1}{\varepsilon_1\varepsilon_2}\|\widetilde{\Lambda}(\varepsilon_1 f_1 - \varepsilon_2 f_2) - \widetilde{\Lambda}(\varepsilon_1 f_1) - \widetilde{\Lambda}(\varepsilon_2 f_2)\|_{L^2(\Sigma^\sharp)} + \frac{1}{\varepsilon_1\varepsilon_2}\|\partial_\nu \widetilde{\mathcal{R}}\|_{L^2(\Sigma^\sharp)} \\
&\leq \frac{3}{\varepsilon_1\varepsilon_2}\delta + C\frac{1}{\varepsilon_1\varepsilon_2}(\varepsilon_1+\varepsilon_2)^3\left(\|f_1\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} + \|f_2\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)}\right)^3 \\
&\leq \frac{C}{\varepsilon_1\varepsilon_2}\left(\delta + (\varepsilon_1+\varepsilon_2)^3\rho^{12\kappa+12}h^{-3N-6\kappa-9}\right).
\end{aligned}$$

where $\widetilde{\Lambda} := \Lambda_{q,\beta_1} - \Lambda_{q,\beta_2}$.

□

Lemma 4.7. *Suppose that $q \in \mathcal{M}_{\mathcal{O}}$ and $\beta_1 - \beta_2 \in \mathcal{M}_{\mathcal{O}}$. Then for $N > 0$ large enough there exist $\gamma^* > 0$, $m_1 > 0$, $\rho_0 > 1$ and $0 < h_0 < 1$ such that*

$$\left|\int_{Q^*}[\Delta,\chi]W\overline{U}_0\,dxdt\right| \leq C\langle\tau,\xi\rangle^{2N+4}h^{-4N-6\kappa-12}\left(\gamma^{-\mu_1}\rho^{8\kappa+4} + e^{m_1\gamma}\rho^{12\kappa+16}\left(\varepsilon^{-2}\delta+\varepsilon\right)\right).$$

for $\gamma > \gamma^*$, $\tau \in \mathbb{R}$, $\xi \in \omega_0^\perp$, $\rho > \rho_0$ and $0 < h < h_0$. Moreover, for each $(\tau,\xi) \in \mathbb{R}^{n+1}$, the Fourier transform of $\beta\theta_h^3$ (extended by zero outside Q^*) satisfies

$$\begin{aligned}
|\widehat{\beta\theta_h^3}(\tau,\xi)| \leq C\Big(&\rho^{-1}\langle\tau,\xi\rangle^{2N+2\kappa+2}h^{-3N-6\kappa-3} + \langle\tau,\xi\rangle^{2N+4}\gamma^{-\mu_1}\rho^{8\kappa+4}h^{-4N-6\kappa-12} \\
&+ \langle\tau,\xi\rangle^{2N+4}e^{m_1\gamma}\rho^{12\kappa+16}h^{-4N-6\kappa-12}\left(\varepsilon^{-2}\delta+\varepsilon\right)\Big).
\end{aligned} \tag{4.23}$$

Proof. We choose $\varepsilon_1 = \varepsilon_2 =: \varepsilon$. From Lemma 4.6, we obtain

$$\|\partial_\nu W\|_{L^2(\Sigma^\sharp)} \leq C\left(\varepsilon^{-2}\delta + \varepsilon\rho^{12\kappa+12}h^{-3N-6\kappa-9}\right).$$

By the UCP in Lemma 4.1, there exist $\gamma^* > 0$, $m_1 > 0$ and $\mu_1 < 1$ such that

$$\begin{aligned}
&\left|\int_{Q^*}[\Delta,\chi]W\overline{U}_0\,dxdt\right| \\
&\leq C\|[\Delta,\chi]W\|_{L^2(0,T^*;H^{-1}(\Omega_3\setminus\Omega_2))}\|\overline{U}_0\|_{L^2(0,T^*;H^1(\Omega))} \\
&\leq C\|W\|_{L^2((0,T^*)\times(\Omega_3\setminus\Omega_2))}\|\overline{U}_0\|_{L^2(0,T^*;H^2(\Omega))} \\
&\leq C\left(\gamma^{-\mu_1}\|W\|_{H^{1,1}(Q)} + e^{m_1\gamma}\|\partial_\nu W\|_{L^2(\Sigma^\sharp)}\right)\rho^4\langle\tau,\xi\rangle^{2N+4}h^{-N-3} \\
&\leq C\langle\tau,\xi\rangle^{2N+4}\left(\gamma^{-\mu_1}\rho^{8\kappa+4}h^{-3N-4\kappa-5} + e^{m_1\gamma}\left(\varepsilon^{-2}\rho^4h^{-N-3}\delta + \varepsilon\rho^{12\kappa+16}h^{-4N-6\kappa-12}\right)\right)
\end{aligned}$$

for any $\gamma > \gamma^*$.

Together with Lemma 4.5 this leads to (4.23) for $\xi \in \omega_0^\perp$. Choosing enough ω_0, this ends the proof. □

Proof of Theorem 1.2. Let $\rho = \gamma^{\frac{\mu_1}{8\kappa+5}}$ so that

$$\rho^{-1} = \gamma^{-\mu_1}\rho^{8\kappa+4}.$$

We denote

$$\alpha_1 := 4N + 6\kappa + 12, \quad \alpha_2 := 2N + 2\kappa + 2, \quad \mu := \frac{\mu_1}{8\kappa+5}.$$

Then from (4.23), it is not hard to see

$$|\widehat{\beta\theta_h^3}(\tau,\xi)| \leq C\langle\tau,\xi\rangle^{\alpha_2}h^{-\alpha_1}\Big(\gamma^{-\mu} + e^{m_2\gamma}(\varepsilon+\varepsilon^{-2}\delta)\Big), \tag{4.24}$$

with some index $m_2 > m_1 > 0$. For a fixed $M > 1$, by (4.24) and Plancherel theorem, we deduce

$$\begin{aligned}
\|\beta\theta_h^3\|^2_{H^{-1}(\mathbb{R}^{n+1})} &= \int_{|(\tau,\xi)|\le M} \langle\tau,\xi\rangle^{-2}|\widehat{\beta\theta_h^3}(\tau,\xi)|^2 d\tau d\xi + \int_{|(\tau,\xi)|> M} \langle\tau,\xi\rangle^{-2}|\widehat{\beta\theta_h^3}(\tau,\xi)|^2 d\tau d\xi \\
&\le C\left(\int_{|(\tau,\xi)|\le M} \langle\tau,\xi\rangle^{2\alpha_2} h^{-2\alpha_1}\left(\gamma^{-2\mu} + e^{2m_2\gamma}(\varepsilon^2+\varepsilon^{-4}\delta^2)\right) d\tau d\xi + M^{-2}\|\beta\theta_h^3\|^2_{L^2(\mathbb{R}^{n+1})}\right) \\
&\le CM^{2\alpha_2+n+1}h^{-2\alpha_1}\left(\gamma^{-2\mu} + e^{2m_2\gamma}(\varepsilon^2+\varepsilon^{-4}\delta^2)\right) + CM^{-2}m_0^2,
\end{aligned}$$

by recalling that $|\beta| \le m_0$. Thus,

$$\|\beta\theta_h^3\|_{H^{-1}(\mathbb{R}^{n+1})} \le CM^{\alpha_2+\frac{n+1}{2}}h^{-\alpha_1}\left(\gamma^{-\mu} + e^{m_2\gamma}(\varepsilon+\varepsilon^{-2}\delta)\right) + CM^{-1}.$$

By interpolating and (4.4),

$$\begin{aligned}
\|\beta\theta_h^3\|^2_{L^2(Q^*)} &\le \|\beta\theta_h^3\|_{H^{-1}(Q^*)}\|\beta\theta_h^3\|_{H^1(Q^*)} \le C\|\beta\theta_h^3\|_{H^{-1}(Q^*)}h^{-1} \\
&\le CM^{\alpha_2+\frac{n+1}{2}}h^{-\alpha_1-1}\left(\gamma^{-\mu} + e^{m_2\gamma}(\varepsilon+\varepsilon^{-2}\delta)\right) + CM^{-1}h^{-1}.
\end{aligned}$$

In addition, we write

$$\beta = \beta\theta_h^3 + \beta(1-\theta_h^3).$$

Note that $1-\theta_h^3 = 0$ in $[2h, T^*-2h]$, which leads to

$$\|1-\theta_h^3\|^2_{L^2(0,T^*)} \le \int_0^{2h}(1-\theta_h^3)^2 dt + \int_{T^*-2h}^{T^*}(1-\theta_h^3)^2 dt \le 4h.$$

Hence,

$$\begin{aligned}
\|\beta\|^2_{L^2(Q^*)} &\le C(\|\beta\theta_h^3\|^2_{L^2(Q^*)} + \|\beta(1-\theta_h^3)\|^2_{L^2(Q^*)}) \\
&\le CM^{\alpha_2+\frac{n+1}{2}}h^{-\alpha_1-1}\left(\gamma^{-\mu} + e^{m_2\gamma}(\varepsilon+\varepsilon^{-2}\delta)\right) + CM^{-1}h^{-1} + Ch.
\end{aligned}$$

Choose $h < T^*/4$ satisfying $M^{-1}h^{-1} = h$ (i.e., $h = M^{-\frac{1}{2}}$) such that the last two terms above have the same order. This results in

$$\|\beta\|^2_{L^2(Q^*)} \le CM^{\alpha_3}\left(\gamma^{-\mu} + e^{m_2\gamma}(\varepsilon+\varepsilon^{-2}\delta)\right) + CM^{-\frac{1}{2}},$$

where $\alpha_3 := \alpha_2 + \frac{1}{2}(\alpha_1+n+2)$. We also further choose $M = \gamma^{\frac{\mu}{\frac{1}{2}+\alpha_3}}$ such that

$$\gamma^{-\mu}M^{\alpha_3} = M^{-\frac{1}{2}},$$

which implies that there exist constants $0 < \mu' < 1$ and $m_3 > m_2 > 0$ such that

$$\|\beta\|^2_{L^2(Q^*)} \le C\left(e^{m_3\gamma}\varepsilon^{-2}\delta + e^{m_3\gamma}\varepsilon + \gamma^{-\mu'}\right). \tag{4.25}$$

For $\delta \in (0, \min\{1,\, e^{-6m_3\gamma^*},\, \Lambda^{\frac{1}{2}}\})$ with $\Lambda > 1$, we take

$$\varepsilon = \frac{\lambda}{4}\Lambda^{\frac{-1}{6}}\delta^{\frac{1}{3}} \quad \text{and} \quad \gamma = \frac{1}{6m_3}|\log(\delta)|.$$

Then (4.25) becomes

$$\|\beta\|^2_{L^2(Q^*)} \le C\left(\delta^{\frac{1}{6}} + |\log(\delta)|^{-\mu'}\right).$$

where C depends on Ω, T, T^*, m_0, and λ and Λ.

$\square$

Now we verify the small condition in the well-posedness.

Remark 4.2. *From the above proof, the parameters are defined by*

$$\rho = \gamma^{\mu},\ M = \gamma^{\frac{\mu}{\alpha_3+\frac{1}{2}}},\ h = M^{-\frac{1}{2}} = \gamma^{-\frac{\mu}{2\alpha_3+1}},\ \gamma = \frac{1}{6m_3}|\log(\delta)|.$$

From (4.22)*, for* $j = 1, 2$*,*

$$\|f_j\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} \le C\rho^{4\kappa+4}h^{-N-2\kappa-3} \le C\gamma^{(4\kappa+4)\mu+\frac{1}{2\alpha_3+1}(N+2\kappa+3)\mu} \le Ce^{m_3\gamma}.$$

We took $\varepsilon_j = \varepsilon$ *above. Due to* $\delta < \Lambda^{\frac{1}{2}}$*, it follows that*

$$|\varepsilon_j| \le \frac{\lambda}{4}\Lambda^{\frac{-1}{6}}\delta^{\frac{1}{3}} < \frac{\lambda}{4},$$

and

$$\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|_{H^{2\kappa+\frac{3}{2},2\kappa+\frac{3}{2}}(\Sigma)} \le C\frac{\lambda}{2}\Lambda^{\frac{-1}{6}}\delta^{\frac{1}{3}}e^{m_3\gamma} = C\frac{\lambda}{2}\Lambda^{\frac{-1}{6}}\delta^{\frac{1}{6}} < C\frac{\lambda}{2}\Lambda^{\frac{-1}{12}} < \lambda,$$

provided Λ *is sufficiently large. Hence, the Dirichlet data* $\varepsilon_1 f_1 + \varepsilon_2 f_2$ *belongs to* $\mathcal{S}_\lambda(\Sigma)$*. This justifies the well-posedness and our procedures discussed above.*

4.5. **Proof of Theorem 1.3.**

Proof of Theorem 1.3. From Theorem 1.1, From Theorem 1.1 and the strict convexity assumption, in $\mathbb{R}^n$ ($n \ge 3$), for $p \in \Omega$ close enough to Γ, we could find $\omega_1, \omega_2 \in \mathbb{S}^{n-1}$ satisfying $\omega_1 \perp \omega_2$ and $((\gamma_{p,\omega_1} \cup \gamma_{p,\omega_2} \cup \gamma_{p,\omega_1+\omega_2}) \cap \partial\Omega) \subset \Gamma$. Consequently, we obtain that $\beta_1 = \beta_2$ in a neighborhood of Γ. Combining with the hypothesis that $\beta_1 - \beta_2 = 0$ on $(0, T) \times \mathcal{O}'$ yields that $\beta_1 - \beta_2 = 0$ near the boundary $\partial\Omega$. Thus one can assume that $\beta = 0$ in some open neighborhood $\mathcal{O}$ of $\partial\Omega$. Applying the result in Theorem 1.2, for any $T^* \in (0, T)$, we derive that $\beta_1 = \beta_2$ in $(0, T^*) \times \Omega$ by letting $\delta \to 0$, which completes the proof. ☐

Remark 4.3. *Theorem 1.1 and Theorem 1.2 hold true for more general nonlinearity, such as* $\beta(t, x)u^m$ *or* $\beta(t, x)|u|^{2m}u$*. For the former case, the integral identity becomes* $\int \beta U_1 U_2 \ldots U_m \overline{U}_0\, dxdt = 0$*, where* U_j *is the solution to the linear equation. Like the setting* $m = 2$ *discussed above, the vectors* ω_j *in the phase functions of GO solutions are chosen to satisfy*

$$\omega_0 = \omega_1 + \ldots + \omega_m \quad \text{and} \quad |\omega_0|^2 = |\omega_1|^2 + \ldots + |\omega_m|^2$$

so that the leading complex phase functions vanish eventually in the integral identity. Once the phase functions are determined, following similar arguments in the proof of theorems lead to the unique and stable determination of β*.*

For the case of Gross-Pitaevskii equation with nonlinearity $\beta|u|^2u$ *and the generalized* $\beta|u|^{2m}u$*, we can treat similarly to obtain the integral identity*

$$\int \beta U_1\overline{U_2}U_3\overline{U_4}\ldots U_{2m-1}\overline{U_{2m}}U_{2m+1}\overline{U_0}\ dxdt = 0$$

and choose

$$\omega_1 - \omega_2 + \omega_3 - \omega_4 + \ldots + \omega_{2m+1} - \omega_0 = 0$$
$$|\omega_1|^2 - |\omega_2|^2 + |\omega_3|^2 - |\omega_4|^2 + \ldots + |\omega_{2m+1}|^2 - |\omega_0|^2 = 0.$$

We can choose $U_1, \overline{U_2}, U_{2m+1}$ *and* $\overline{U}_0$ *to be GO-solutions supported near four straight lines* γ_{p,ω_1}*,* γ_{p,ω_2}*,* $\gamma_{p,\omega_{2m+1}}$*, and* γ_{p,ω_0}*, respectively, and let* U_j *and* U_{j+1} *be GO-solutions supported near* γ_{p,ω_1} *for*

$j = 3, 5, \ldots, 2m-1$ *so that their complex phases will cancel the other in pairs. Hence,* $\omega_1, \omega_2, \omega_{2m+1}, \omega_0$ *should satisfy*

$$\omega_0 + \omega_2 = \omega_1 + \omega_{2m+1},$$
$$|\omega_0|^2 + |\omega_2|^2 = |\omega_1|^2 + |\omega_{2m+1}|^2,$$

which can be achieved, for instance, by choosing

$$\omega_0 = (1, -1, \ldots, 0), \quad \omega_1 = (\sqrt{1-r^2}, -1, \ldots, r),$$
$$\omega_2 = (\sqrt{1-r^2}, \sqrt{1-r^2}, 0, \ldots, r), \quad \omega_{2m+1} = (1, \sqrt{1-r^2}, \ldots, 0), \quad 0 < r < 1.$$

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