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Crossing numbers of complete bipartite graphs

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Abstract

The long standing Zarankiewicz's conjecture states that the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph is $Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$. Using flag algebras we show that $\mathrm{cr}(K_{n,n}) \geq 0.9118 \cdot Z(n,n) + o(n^4)$. We also show that the rectilinear crossing number $\overline{cr}(K_{n,n})$ of $K_{n,n}$ is at least $0.987 \cdot Z(n,n) + o(n^4)$. Finally, we show that if a drawing of $K_{n,n}$ has no $K_{3,4}$ that has exactly two crossings, and these crossings share exactly one vertex, then it has at least $Z(n,n) + o(n^4)$ crossings. This is a local restriction inspired by Turán type problems that gives an asymptotically tight result.

Keywords:
Zarankiewicz; crossing number; complete bipartite graph; flag algebras
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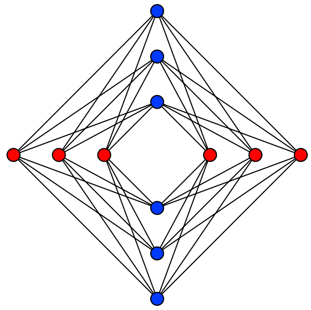

Fig. 1. Zarankiewicz's drawing of $K_{6,6}$

1. Introduction

We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane.

The systematic study of crossing numbers of graphs seems to date back to a question by Paul Turán in 1944. While working in a brick factory outside Budapest, where his work consisted of pushing wagon loads of bricks from kilns to storage sites, Turán noted that rail wagons tended to derail at crossings of tracks, and he wondered what was the best way to design a brickyard, in the sense that the number of track crossings is minimized. In modern graph theory terminology, this question is equivalent to: what is the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$? As pointed out by Beineke and Wilson in their lively survey [1], this problem has a curious history. Zarankiewicz thought he had solved the problem [2]. Around a decade later, his main argument was found to have an unpluggable gap [3]. In any case, Zarankiewicz did exhibit a natural way to draw $K_{m,n}$ with exactly

$$Z(m,n) := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

crossings, see Figure 1 for an example. To this day, no drawing of $K_{m,n}$ with fewer than $Z(m,n)$ crossings has been found, and $\mathrm{cr}(K_{m,n}) = Z(m,n)$ has become known as *Zarankiewicz's Conjecture*. This conjecture has been settled only when $\min\{m,n\} \leq 6$ [4], and in a few additional small cases [5].

There seems to be a consensus among researchers interested in the conjecture that this holds, but that attempting to settle it exactly is a hopeless task. One of the reasons may be that there are many constructions that have asymptotically $Z(m,n)$ crossings. For example, every geodesic drawing of $K_{2m,2n}$ on the sphere such that for each vertex there is another vertex in the same class drawn in its antipodal location has exactly $Z(2m,2n)$ crossings. For this reason, most efforts on this problem are nowadays focused on its asymptotic version. The asymptotic version of Zarankiewicz's Conjecture reads $\mathrm{cr}(K_{m,n}) = m^2n^2/16 + o(m^2n^2)$. This weaker version also remains open. We focus on the diagonal case which reads as follows.

Conjecture 1. $\mathrm{cr}(K_{n,n}) = n^4/16 + o(n^4)$.

To compare the asymptotic results, we let

$$L := \lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)},$$

so that Conjecture 1 may be equivalently paraphrased by saying that $L = 1$.

Kleitman [4] showed that $\mathrm{cr}(K_{5,n}) = Z(5,n)$, and an easy counting argument shows that this implies that $L \geq 0.8$. De Klerk, Maharry, Pasechnik, Richter, and Salazar [6] gave a lower bound on $\mathrm{cr}(K_{7,n})$ using semidefinite program-

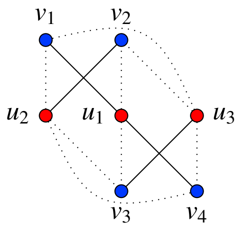

Fig. 2. We let F denote a drawing of $K_{3,4}$ with two crossings sharing exactly one vertex, such as the one illustrated above. Here the solid edges are crossed, while the dashed are uncrossed.

ming (SDP) implying $L > 0.83$. A further improvement to $L > 0.859$ follows from De Klerk, Pasechnik, and Schrijver [7] who also used SDP to give a lower bound on $\mathrm{cr}(K_{9,n})$. For a fixed $m \geq 3$, deciding whether or not Zarankiewicz's conjecture holds for $K_{m,n}$ is a finite problem for every $n \geq m$, see [6]. Recently, Brosch and Polak [8] used SDP to provide lower bounds on $\mathrm{cr}(K_{10,n})$, $\mathrm{cr}(K_{11,n})$, $\mathrm{cr}(K_{12,n})$, and $\mathrm{cr}(K_{13,n})$ giving $L \geq 0.8878$. Using flag algebras, Norin and Zwols [9] obtained the following.

Theorem 2 (Norin and Zwols [9])**.** $L > 0.905$. *In other words* $\mathrm{cr}(K_{n,n}) > 0.905n^4/16 + o(n^4)$.

Their result was presented in a talk but never published in a paper. Norin and Zwols were the first to apply flag algebras to crossing numbers. Their result inspired other applications of flag algebras in graph drawing and geometry [10, 11, 12]. Using their method we obtained a slight improvement of the result.

Theorem 3. $L > 0.9118$. *In other words* $\mathrm{cr}(K_{n,n}) > 0.9118n^4/16 + o(n^4)$.

It is common to look at restrictions of the drawings to try and get improved bounds. We recall that in a *rectilinear drawing* of a graph each edge is drawn as a straight segment, and the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a rectilinear drawing of G. Note that Zarankiewicz's original drawings of $K_{m,n}$ are actually rectilinear. For the rectilinear version of Zarankiewicz's conjecture, we can significantly improve the bound from Theorem 3.

Theorem 4. $\overline{cr}(K_{n,n}) > 0.987n^4/16 + o(n^4)$.

Another restriction one can make is that some given subdrawing does not appear. This sort of condition is reminscent of Turán type problems from extremal graph theory, which go as follows. Let F be a graph. What is the maximum number of edges in an n-vertex graph not containing F as a (possibly induced) subgraph? In our case, let F be a drawing of $K_{3,4}$ with two crossings sharing exactly one vertex, see Figure 2. For example for a vertex set $\{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ the pairs of crossing edges may be (u_1v_1, u_2v_2) and (u_1v_4, u_3v_3). This creates a local combinatorial constraint rather than a geometrical constraint. Observe that F does not appear in Zarankiewicz's drawing of $K_{n,n}$. This restriction allows us to give an asymptotically correct bound.

Theorem 5. *If* $K_{n,n}^D$ *is a drawing of* $K_{n,n}$ *where no* $K_{3,4}$ *induces exactly two crossings sharing one vertex, then* $K_{n,n}^D$ *has at least* $n^4/16 + o(n^4)$ *crossings.*

In the next section we give a brief introduction to flag algebras, the main method used in this paper. Then we describe the more particular setup we use in this paper. Sections 3, 4, and 5 deal with the proofs of Theorems 3, 4, and 5, respectively.

1.1. Notation

We distinguish a graph G from its drawing by denoting a drawing of G by G^D. We denote the crossing number of a graph G by $\mathrm{cr}(G)$. For a drawing G^D, $\mathrm{cr}(G^D)$ is the number of crossings in G^D. For a set of vertices X, we denote the induced subgraph of G by $G[X]$ and $G^D[X]$ denotes the subdrawing of G^D induced by X.

2. Flag Algebras

2.1. General Bipartite Setup

Flag algebras is a tool developed by Razborov [13] for solving problems in extremal combinatorics. It has a variety of applications due to its generality. It can be applied to graphs, hypergraphs, oriented graphs, and other combinatorial structures [14, 15, 16, 17]. Here we give a very brief introduction somewhat tweaked for drawings of complete bipartite graphs by stating the basic definitions and claims. Proofs of the claims made in this section can be found in [13].

We call a drawing of a graph *good* if (a) adjacent edges intersect only at their common endvertex; (b) any two edges have at most one intersection; and (c) intersections of two nonadjacent edges are crossings, not tangential. Crossing-minimal drawings are necessarily good. Since we investigate crossing-minimal drawings of graphs and their induced subdrawings, we assume that all drawings are good.

To streamline the exposition, we assume that complete bipartite graphs are vertex 2-colored, where one part is red and the other part is blue. Notation $K^D_{r,b}$ means a drawing of $K_{r,b}$ where the red part has r vertices and the blue part has b vertices. We use $V_r(K^D_{r,b})$ and $V_b(K^D_{r,b})$ to denote the set of red and blue vertices, respectively. The following text uses the notion of isomorphims of two drawings of graphs that we specify later. The choice of isomorphism is not important for the theory but it is crucial for applications. For now we require that there are only finitely many non-isomorphic drawings of $K_{r,b}$ for each $r, b \geq 0$.

Let $K^D_{r_1,b_1}$ and $K^D_{r_2,b_2}$ be two drawings, where $r_1 \leq r_2$ and $b_1 \leq b_2$. Let $X_r \subseteq V_r(K^D_{r_2,b_2})$ and $X_b \subseteq V_b(K^D_{r_2,b_2})$ be subsets of red and blue vertices respectively picked uniformly at random, where $|X_r| = r_1$ and $|X_b| = b_1$. Denote by $P(K^D_{r_1,b_1}, K^D_{r_2,b_2})$ the probability that $K^D_{r_2,b_2}[X_r \cup X_b]$ is isomorphic to $K^D_{r_1,b_1}$. This is also a density of $K^D_{r_1,b_1}$ in $K^D_{r_2,b_2}$.

Let $\mathcal{F}$ denote the collection of all drawings of all complete bipartite graphs up to isomorphism. Let $\mathcal{F}_{r,b}$ denote those drawings in $\mathcal{F}$ with red and blue parts of size r and b, respectively. Let $\mathbb{R}\mathcal{F}$ denote the set of all formal finite linear combinations of drawings in $\mathcal{F}$ with real coefficients. Let $\mathcal{K}$ be a linear subspace generated by all linear combinations of

$$K^D_{r,b} - \sum_{K^D \in \mathcal{F}_{r',b'}} P(K^D_{r,b}, K^D) K^D,$$

where $r' \geq r$ and $b' \geq b$.

The algebra $\mathcal{A}$ is the space $\mathbb{R}\mathcal{F}$ factorized by $\mathcal{K}$. Observe that addition and multiplication by a real number are naturally defined in $\mathcal{A}$. Next we introduce a multiplication of two drawings in $\mathcal{F}$. Let $K^D_{r_1,b_1}, K^D_{r_2,b_2} \in \mathcal{F}$. Fix one $K^D \in \mathcal{F}_{r_1+r_2,b_1+b_2}$. Let $X_{r_1} \subseteq V_r(K^D)$ and $X_{b_1} \subseteq V_b(K^D)$ be subsets picked uniformly at random, where $|X_{r_1}| = r_1$ and $|X_{b_1}| = b_1$. Let $X_{r_2} = V_r(K^D) \setminus X_{r_1}$ and $X_{b_2} = V_b(K^D) \setminus X_{b_1}$. Denote by $P(K^D_{r_1,b_1}, K^D_{r_2,b_2}; K^D)$ the probability that $K^D[X_{r_1} \cup X_{b_1}]$ is isomorphic to $K^D_{r_1,b_1}$ and $K^D[X_{r_2} \cup X_{b_2}]$ is isomorphic to $K^D_{r_2,b_2}$. Define

$$K^D_{r_1,b_1} \cdot K^D_{r_2,b_2} = \sum_{K^D \in \mathcal{F}_{r_1+r_2,b_1+b_2}} P(K^D_{r_1,b_1}, K^D_{r_2,b_2}; K^D) K^D. \tag{1}$$

This multiplication has a unique linear extension to $\mathbb{R}\mathcal{F}$ which yields a well-defined multiplication in $\mathcal{A}$. This is proved formally in [13, Lemma 2.4]. Notice that a multiplication of drawings is a linear combination of some other drawings on more vertices.

For each σ in $\mathcal{F}$, define all drawings with a fixed labeled copy of σ by $\mathcal{F}^\sigma$. It is possible to define an algebra A^σ built from $\mathcal{F}^\sigma$ in a similar way as $\mathcal{A}$ is built from $\mathcal{F}$. See [11] or [13] for more details.

A sequence of drawings of graphs $(K^D_{n,n})_{n\geq 1}$ is said to be *convergent* if for every drawing $K^D \in \mathcal{F}$ the following limit converges

$$\lim_{n\to\infty} P(K^D, K^D_{n,n}).$$

A compactness argument using Tychonoff's theorem yields that every infinite sequence has a convergent subsequence. Let $(K^D_{n,n})_{n\geq 1}$ be a convergent sequence. Let $\phi(K^D) := \lim_{n\to\infty} P(K^D, K^D_{n,n})$ for every $K^D \in \mathcal{F}$ and extend ϕ to $\mathcal{A}$ by linearity. The mapping ϕ can be seen as the *limit of the sequence* $(K^D_{n,n})_{n\geq 1}$. The resulting mapping $\phi : \mathcal{A} \to \mathbb{R}$ is actually a homomorphism satisfying $\phi(K^D) \in [0, 1]$ for every $K^D \in \mathcal{F}$.

Let $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ be the set of all homomorphisms ψ from the algebra $\mathcal{A}$ to $\mathbb{R}$ such that $\psi(K^D) \geq 0$ for every $K^D \in \mathcal{F}$. This set of homomorphisms is exactly the set of all limits of convergent sequences in $\mathcal{F}$, see [13, Theorem 3.3].

Similarly to $\mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, it is possible to define $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ as limits of convergent sequences where each drawing has a fixed labeled subdrawing of σ. Finally, there is a linear *unlabeling operator* $[\![.]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}$ such that if $\phi^\sigma(A^\sigma) \geq 0$ for some $A \in \mathcal{A}$, then $\phi([\![A]\!]_\sigma) \geq 0$.

The proofs in the paper are constructed as follows. For fixed r and b we want to find a constant C such that for all $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\phi\left(\sum_{F\in\mathcal{F}_{r,b}} \mathrm{cr}(F)\, F\right) \geq C. \tag{2}$$

Letting $C := \min_{F\in\mathcal{F}_{r,b}} \mathrm{cr}(F)$ would work but it may be improved. We do it using sums of squares obtained in $\mathcal{A}^\sigma$ for various σ. Denote by T a set of types, some labeled drawings from $\mathcal{F}$. We do the following for each $\sigma \in T$. Pick $r', b' \geq 1$ such that $2r' - |V_r(\sigma)| = r$ and $2b' - |V_b(\sigma)| = b$. This is necessary since we will perform multiplications and we need the result as a linear combination in $\mathcal{F}_{r,b}$; recall (1). Let $n_\sigma = |\mathcal{F}^\sigma_{r',b'}|$ and let F^σ be a vector with an arbitrary ordering of elements from $\mathcal{F}^\sigma_{r',b'}$. Then for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ and a positive semidefinite matrix $M_\sigma \in \mathbb{R}^{n_\sigma\times n_\sigma}$ the following holds

$$0 \leq \phi\left([\![(F^\sigma)^T M_\sigma F^\sigma]\!]_\sigma\right) = \phi\left(\sum_{F\in\mathcal{F}_{r,b}} c_F(M_\sigma, \sigma)\, F\right), \tag{3}$$

where $c_F(M_\sigma, \sigma)$ are real numbers depending on F, M_σ and σ. Since some entries in M_σ may be negative, also some $c_F(M_\sigma, \sigma)$ may be negative. This is what makes the approach work. Equations (2) and (3) can be combined since they are linear combinations in $\mathcal{F}_{r,b}$. Using linearity of ϕ we get

$$\begin{aligned}\phi\left(\sum_{F\in\mathcal{F}_{r,b}} \mathrm{cr}(F)\, F\right) &\geq \phi\left(\sum_{F\in\mathcal{F}_{r,b}} \mathrm{cr}(F)\, F\right) - \underbrace{\sum_{\sigma\in T} \phi\left([\![(F^\sigma)^T M_\sigma F^\sigma]\!]_\sigma\right)}_{\geq 0} = \phi\left(\sum_{F\in\mathcal{F}_{r,b}} c_F F\right)\\ &\geq \min_{F\in\mathcal{F}_{r,b}} (c_F) \cdot \underbrace{\phi\left(\sum_{F\in\mathcal{F}_{r,b}} F\right)}_{=1} = \min_{F\in\mathcal{F}_{r,b}} (c_F) = C,\end{aligned}$$

where the coefficients c_F are functions of the M_σs since they are obtained as $c_F := \mathrm{cr}(F) - \sum_\sigma c_F(M_\sigma, \sigma)$. The goal is to obtain the largest possible C, which means solving

$$\max_{M_\sigma, \sigma \in T} \min_{F \in \mathcal{F}_{r,b}} (c_F). \tag{4}$$

This is a semidefinite program that can be numerically solved. We used solvers CSDP [18] and Mosek [19]. Notice that if one proves for some $A \in \mathcal{A}$ that $\phi(A) \geq 0$ for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$, then $-y \cdot \phi(A)$ for a parameter $y \geq 0$ can be added to the calculation.

Finally, in order to obtain a rigorous bound, the numerical solution needs to be rounded to a rational solution. It can be done using software we developed in SageMath [20]. For more details about ideas behind the rounding see [21, 22]. The proofs are certified by the positive semidefinite the matrices M_σs that can be obtained at `http://lidicky.name/pub/knn/`. There is no need to solve a semidefinite program in order to verify the solution.

The machinery of flag algebras works with limits of convergent sequences. The same machinery can be applied also in $K^D_{n,n}$ but multiplication (1) will come with an additive error $O(1/n)$. Since the number of multiplications in our calculations is finite, the limit result can be transferred to a result about a (sufficiently large) graph at the cost of $O(1/n)$ error.

2.2. Isomorphism

In order to apply the method of flag algebras to crossing numbers, we need to define when two drawings are isomorphic. Let G^D be a drawing of a graph. The following types of information can be extracted from G^D.

(C1) For each pair of edges e_1 and e_2, whether or not they cross each other.
(C2) For each quadruple $Q \subset V(G^D)$, whether $G^D[Q]$ induces a crossing or not.

In case of a rectilinear drawing, the following information can be also extracted from G^D.

(C3) For vertices u_1, u_2, u_3, u_4 whether or not line u_1u_2 crosses the segment u_3u_4.
(C4) For vertices u_1, u_2, u_3 if u_3 is on the left or right of the line u_1u_2.
(C5) For each vertex the cyclic clockwise order of its neighbors (that is, the *rotation* at the vertex).

The lists above are not exhaustive and there may be other possibilities of what information to extract from G^D. Notice that distinct drawings can result in identical configurations in $\mathcal{F}$. Typically preserving more information about the drawings gives stronger bounds. On the other hand, flag algebras needs to work with a list of all drawings in $\mathcal{F}_{r,b}$ for some fixed r, b, and larger r, b usually give better bounds. For computational purposes, $|\mathcal{F}_{r,b}|$ needs to be under 200,000. It is not clear if it is better to have smaller r, b with configurations keeping more information from the drawing or larger r, b and configurations with less information.

3. General Drawings

In this section we describe the proof of Theorem 3. The main step is the following lemma whose proof is using flag algebras.

Lemma 6. *For every sufficiently large n, in every drawing of $K_{n,n}$ the average number of crossings over all subgraphs isomorphic to $K_{3,4}$ is greater than 4.1.*

Proof. We used isomorphism (C1). We obtained all drawings of $K_{3,4}$ up to isomorphisms under (C1), denoted here by $\mathcal{F}_{3,4}$ from a program developed by Norin and Zwols [23]. In this setting, $|\mathcal{F}_{3,4}| = 6393$. Using flag algebras, we obtained that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$:

$$\phi\left(\sum_{K^D \in \mathcal{F}_{3,4}} \mathrm{cr}(K^D) K^D\right) \geq \frac{16413782150067826979825224800019758560899432}{4000000000000000000000000000000000000000000} \approx 4.103445537516957.$$

In addition to using (3), we included the following constraints. Let $K^D \in \mathcal{F}_{3,3}$ and K_s^D be obtained from K^D by swapping red and blue colors. The included constraints are $K^D = K_s^D$. Without these symmetry constraints, the lower bound would be 4.0929. □

Proof of Theorem 3. Let $K_{n,n}^D$ be a drawing of $K_{n,n}$. We proceed by double-counting the pairs of crossings and copies of $K_{3,4}$ in $K_{n,n}^D$ containing them. Each copy of $K_{2,2}$ in $K_{n,n}^D$ is in $2(n-2)\binom{n-2}{2} = n^3 + o(n^3)$ different copies of $K_{3,4}$ in $K_{n,n}^D$. On the other hand, there are $2\binom{n}{3}\binom{n}{4}$ copies of $K_{3,4}$ in $K_{n,n}^D$ and by Lemma 6 the average number of crossings per copy is at least 4.1. This gives

$$\begin{aligned}(n^3 + o(n^3))\mathrm{cr}(K_{n,n}^D) &> 4.1/72n^7 + o(n^7)\\ \mathrm{cr}(K_{n,n}^D) &> 8.2/9 \cdot n^4/16 + o(n^4) = 0.91\overline{1} \cdot n^4/16 + o(n^4).\end{aligned}$$

□

In the proof of Lemma 6 we also tried isomorphism using (C2). Under (C2), $|\mathcal{F}_{3,4}| = 355$, which greatly increases the speed of the calculation but the resulting bound 4.044 is weaker.

4. Rectilinear Drawings

In this section we describe the proof of Theorem 4.

Lemma 7. *For every sufficiently large n, in every rectilinear drawing of $K_{n,n}$ the expected number of crossings in a $K_{4,4}$ chosen uniformly at random is more than 8.8837.*

Proof. We used isomorphism (C4). To obtain rectilinear drawings of $K_{4,4}$, we start with all rectilinear drawings of K_8 found by Aichholzer [24] and try all possible splits into $K_{4,4}$. This eventually lead to a program using 112401 configurations. Using flag algebras, we obtain

$$\phi\left(\sum_{K^D \in \mathcal{F}_{4,4}} \overline{cr}(K^D) K^D\right) \geq \frac{8883777481283804668150619490360970766373630}{1000000000000000000000000000000000000000000} \approx 8.883777481283804.$$

In order to make the problem feasible, we included the following constraints. Let $K^D \in \mathcal{F}_{4,4}$ and K_s^D be obtained from K^D by swapping red and blue colors. Instead of including the constraints $K^D = K_s^D$, we reduced the number of variables. Notice that we used 112401 configurations while $|\mathcal{F}_{4,4}| = 231922$. Unlike in Lemma 6, these constraints did not improve the result but they made it computable. □

Proof of Theorem 4. Let $K_{n,n}^D$ be a rectilinear drawing of $K_{n,n}$. By double-counting the pairs of crossings and copies of $K_{4,4}$ in $K_{n,n}^D$ containing them and using Lemma 7 we get

$$(1/4n^4 + o(n^4))\overline{cr}(K_{n,n}^D) > 8.8837/24^2 n^8 + o(n^8)$$
$$\overline{cr}(D) > 8.8837/9 \cdot n^4/16 + o(n^4) > 0.987 \cdot n^4/16 + o(n^4).$$

□

5. Turán Type Restrictions

In this section we prove Theorem 5. It is enough to work with (C2), translating the problem to a 4-uniform hypergraph setting.

Lemma 8. *For sufficiently large n, in every drawing of $K_{n,n}$ the expected number of crossings in a random $K_{3,4}$ is at least 4.5.*

Proof. We used isomorphism (C2) to work with $\mathcal{F}_{3,4}$. We started with the drawings used in Lemma 6 under (C1) and applied (C2) to them. This gave 355 drawings up to isomorphism. Removing the drawings with two crossing sharing exactly one vertex resulted in $|\mathcal{F}_{3,4}| = 354$. Using flag algebras, we obtained that for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\phi\left(\sum_{K^D \in \mathcal{F}_{3,4}} \mathrm{cr}(K^D) K^D\right) \geq 9/2.$$

□

Proof of Theorem 3. Let $K_{n,n}^D$ be a drawing of $K_{n,n}$. Using exactly the same argument as in Theorem 3 and Lemma 8 we obtain

$$(n^3 + o(n^3))\mathrm{cr}(K_{n,n}^D) \geq 4.5/72 n^7 + o(n^7)$$
$$\mathrm{cr}(K_{n,n}^D) \geq n^4/16 + o(n^4).$$

□

6. Conclusion

Flag algebras are applicable to other variants of crossing number. For rectilinear drawings where the red and blue vertices are separated by a straight line we obtained $L = 0.995$. For 2-page drawings, in which there is a simple closed curve containing all the vertices but not crossing any edge, we obtained $L = 0.976$. Finally, for rectilinear separated 2-page drawings we obtained $L = 0.99998$. This last bound is definitely of interest, since there exist rectilinear separated 2-page drawings with exactly $Z(n, n)$ crossings.

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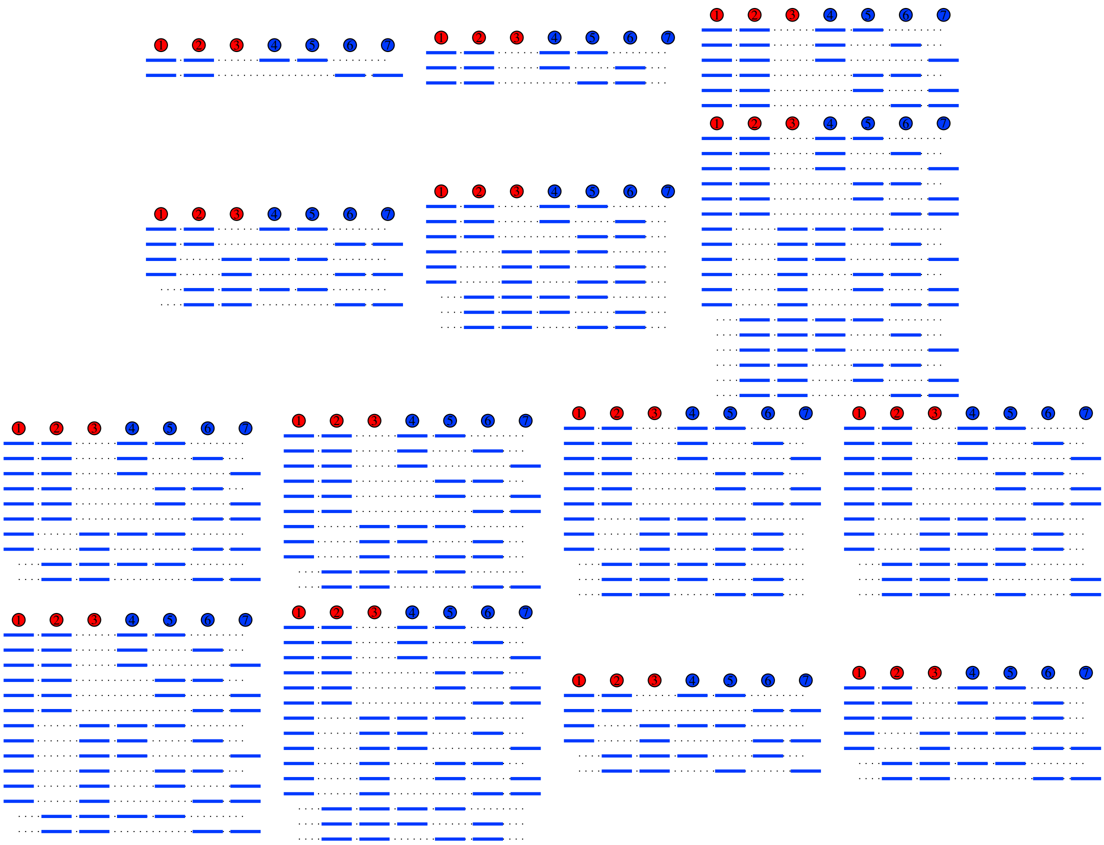
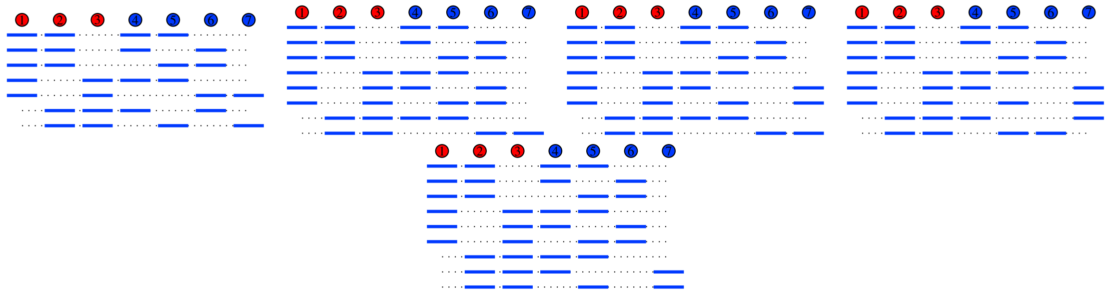
Appendix A. Tight Constructions

Lemma 8 is asymptotically tight. This gives a set $X \subseteq \mathcal{F}_{3,4}$ such that every construction that asymptotically gives 4.5 can contain nontrivial densities only for drawings in X. More precisely, for every $\phi \in \mathrm{Hom}^+(\mathcal{A}, \mathbb{R})$ satisfying

$$\phi\left(\sum_{K^D \in \mathcal{F}_{3,4}} \mathrm{cr}(K^D) K^D\right) = 9/2$$

the following holds: for every $K^D \in \mathcal{F}_{3,4}$ if $\phi(K^D) > 0$ we have that $K^D \in X$. While $|\mathcal{F}_{3,4}| = 354$, we get $|X| = 19$. We depict X below by indicating which vertices are red and blue and then each line corresponds to one 4-edge. This short list suggests there may be a nice description of the extremal construction. However, X is still larger than the list generated by the construction depicted in Figure 1, which has only 6 configurations that are depicted as the first 6 in the following list containing all configurations in X.





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