GLOBAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN $L^2(\mathbb{R})$

BENJAMIN HARROP-GRIFFITHS, ROWAN KILLIP, MARIA NTEKOUME,
AND MONICA VIŞAN

ABSTRACT. We prove that the derivative nonlinear Schrödinger equation in one space dimension is globally well-posed on the line in $L^2(\mathbb{R})$, which is the scaling-critical space for this equation.

CONTENTS

1. INTRODUCTION

The derivative nonlinear Schrödinger equation

$$\text{(DNLS)} \qquad i\partial_t q + q'' + i\big(|q|^2 q\big)' = 0$$

describes the evolution of a complex-valued field q defined on the line $\mathbb{R}$. Here and below, primes indicate spatial derivatives.

Physical applications of (DNLS) are reviewed briefly in subsection 1.4 below. There, we also discuss certain well-documented changes of variables that convert (DNLS) to other evolutions of interest in the physical sciences.

One of the most basic questions we should ask of any model is whether it is well-posed: Do solutions exist? Are they unique? Do they depend continuously on the initial data? Without such properties, it is unclear whether the model is capable of making experimentally falsifiable predictions. The well-posedness question also forms an important benchmark in our understanding of an equation. Gaps between well- and ill-posedness results leave open the possibility that there are basic physical processes — instabilities and/or stabilizing mechanisms — that remain undiscovered.

The principal goal of this paper is to show that (DNLS) is globally well-posed in $L^2(\mathbb{R})$:

Theorem 1.1. *The* (DNLS) *evolution is globally well-posed in $L^2(\mathbb{R})$. Precisely, there is a jointly continuous map $\Phi : \mathbb{R} \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that agrees with the data-to-solution map when restricted to Schwartz-class initial data.*

It is not wanton abstraction to define the data-to-solution map as an extension from Schwartz-class initial data; indeed, this is the textbook approach to defining the Fourier transform on $L^2(\mathbb{R})$ and is widespread in nonlinear PDE. The heart of the matter is to prove key metric properties that allow one to extend the mapping to general elements of L^2 and then to ensure that the extension retains the many good properties of its Schwartz-class restriction.

A relatively small fraction of this paper would suffice to show that L^2-precompact sets of Schwartz initial data are mapped under the flow to $C_t([-T, T]; L^2(\mathbb{R}))$-precompact sets of orbits. (Here $T > 0$ must be finite, but is otherwise arbitrary.) This would be a new result and it trivially yields the existence of solutions; however, it goes no way to justifying uniqueness, nor continuous dependence on the initial data. This thinking helps us appreciate the uniqueness statement embedded in Theorem 1.1: no matter how we approximate an L^2 initial data by a sequence of Schwartz initial data, the corresponding trajectories will converge and they will converge to the same limit! In particular, our solutions have the group property.

Theorem 1.1 implicitly asserts that Schwartz initial data lead to global unique solutions. This is true. While uniqueness of smooth solutions is easily verified via the Grönwall inequality, the existence of global solutions for large Schwartz-class initial data is, in fact, a very recent result! (See the discussion in subsection 1.1.) Although the existence of such solutions is not a prerequisite for our methods, building on this result leads to a much clearer exposition. Moreover, without the triumphs of these authors, which we celebrate in subsection 1.1, we would not have had the courage to pursue the results of this paper.

Next, we wish to discuss why Theorem 1.1 considers initial data in $L^2(\mathbb{R})$. There are several reasons that make $L^2(\mathbb{R})$ a natural space in which to study (DNLS). First, the L^2-norm is conserved by the flow; indeed, we have the microscopic conservation law

$$(1.1) \qquad \partial_t |q|^2 + \partial_x \left[2 \operatorname{Im}(q' \bar{q}) + \tfrac{3}{2} |q|^4 \right] = 0.$$

Second, it is a scale-invariant space: if $q(t, x)$ is a smooth solution to (DNLS), then so too is

$$(1.2) \qquad q_\lambda(t, x) := \sqrt{\lambda} \, q(\lambda^2 t, \lambda x)$$

for every $\lambda > 0$. Notice that this transformation does not affect the L^2 norm of the initial data (nor indeed at any later time).

Long-standing physical intuition dictates that dispersive equations will be ill-posed below the scaling-critical regularity. For the case of (DNLS), this is justified by the self-similar solutions constructed in [10, 31]. Beyond proving that well-posedness fails in $H^s(\mathbb{R})$ with $s < 0$, these solutions even show that it fails in weak-L^2, which is a scale-invariant space!

It is a matter of some pride for us that we are able to treat (DNLS) in the most natural scale-invariant space. It is only quite recently that global-in-time solutions could be constructed for large data in scaling-critical spaces for any kind of dispersive PDE. Moreover, we are dealing with a focusing nonlinearity (e.g. soliton solutions abound). Many focusing dispersive equations do not admit large-data global solutions; it is typical for wave collapse to occur above a certain threshold size (as measured in scaling-critical spaces).

There is an obvious scapegoat here: (DNLS) is completely integrable, [25]. However scaling-critical well-posedness does not seem to be the norm for such models: it fails for KdV, NLS, and mKdV! The phenomenology of (DNLS) becomes even more curious when we endeavor to find a quantitative expression of the continuous dependence of the solution on its initial data. As discussed below, we know that when $s < \frac{1}{2}$, the data-to-solution map cannot be uniformly continuous on *any* neighborhood of the origin in $H^s(\mathbb{R})$. This is quite different from the behavior of the mass- or energy-critical NLS, for example, where the data-to-solution map is real analytic (cf. [28]).

It is perhaps better to compare (DNLS) to other models with derivative nonlinearity. For the notoriously difficult two-dimensional wave maps equation, for example, Tataru [49] proved that the data-to-solution map (defined on scaling-critical balls) is Lipschitz in lower regularity norms. We will show in Proposition 1.12 that this fails for (DNLS) — again (DNLS) appears *less* continuous! Complete integrability, it seems, is not a stern parent that keeps its flows safe and orderly; rather it is permissive and allows its solutions to become quite wild before issuing the rebuke of ill-posedness.

Earlier we singled out the $H^s(\mathbb{R})$ family of spaces in our discussion of well- and ill-posedness. Already the natural prerequisite that the *linear* Schrödinger equation be well-posed is quite restrictive; this precludes the consideration of L^p-based Sobolev spaces with $p \neq 2$. As we will see below, $H^s(\mathbb{R})$ spaces are both the most natural and most studied classes of initial data; indeed, they arise from the consideration of conservation laws for (DNLS). Building on Theorem 1.1, we will prove

Corollary 1.2. (DNLS) *is globally well-posed in* $H^s(\mathbb{R})$ *for every* $s \geq 0$.

Prior work in this direction is discussed at length in the next subsection. It is evident that Theorem 1.1 guarantees the existence and uniqueness of solutions for data in H^s with $s \geq 0$. That such solutions remain bounded in H^s is known as *persistence of regularity* and may be deduced as a consequence of conservation laws. To complete the proof of well-posedness, one must upgrade continuous dependence from the L^2 metric to the H^s metric. As demonstrated in several prior works of the authors [5, 18, 27, 29], this is easily done if one can verify that H^s-equicontinuous sets of initial data lead to H^s-equicontinuous ensembles of orbits.

Definition 1.3. A subset Q of $H^s(\mathbb{R})$ is $H^s(\mathbb{R})$-*equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\sup_{q \in Q} \sup_{|y| < \delta} \left\| q(x+y) - q(x) \right\|_{H_x^s} < \varepsilon.$$

This definition extends naturally to any translation-invariant Banach space of functions on any group. For bounded continuous functions on $\mathbb{R}^d$, one recovers the notion of equicontinuity familiar from the Arzelà–Ascoli Theorem. Indeed, this more general notion of equicontinuity was introduced precisely to formulate the analogous compactness theorem in $L^p(\mathbb{R}^d)$ spaces; see [42].

We will also need the second key requirement for compactness, albeit only in the L^2 setting:

Definition 1.4. We say that $Q \subseteq L^2(\mathbb{R})$ is *tight* if

$$\limsup_{R \to \infty} \sup_{q \in Q} \int_{|x| \geq R} |q(x)|^2 \, dx = 0.$$

The transportation of L^2 norm is expressed by (1.1). As is characteristic of dispersive equations, we see that the flux of the conserved quantity involves more derivatives than the conserved quantity itself. While this is an obstacle in our path to proving tightness, it is also the key property of microscopic conservation laws that provides for local smoothing estimates.

To formulate local smoothing estimates, we must first agree on how to localize the solution in space. We will do this through the Schwartz-class function

$$(1.3) \qquad \psi(x) := \sqrt{\operatorname{sech}(\tfrac{x}{99})} \quad \text{and its translates} \quad \psi_\mu(x) := \psi(x - \mu).$$

There is nothing terribly special about this choice. The fact that it has slow exponential decay (relative to unit scale) is quite convenient; beyond this, it is merely the case that this choice has served us well in the prior work [18].

Theorem 1.5 (Local smoothing). *Let $Q \subset \mathcal{S}(\mathbb{R})$ be both L^2-bounded and equicontinuous. For each $T > 0$, solutions $q(t)$ to (DNLS) with initial data $q(0) \in Q$ satisfy*

$$(1.4) \qquad \sup_{\mu \in \mathbb{R}} \int_{-T}^{T} \|\psi_\mu^{12} q(t)\|_{H^{\frac{1}{2}}}^2 \, dt \lesssim_{T,Q} \|q(0)\|_{L^2}^2.$$

Corollary 1.6. *The solutions constructed in Theorem 1.1 are distributional solutions; indeed, the data-to-solution map is continuous as a mapping of $L^2(\mathbb{R})$ into $L^3_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R})$.*

The first striking thing about Theorem 1.5 is the fact that the estimate is only claimed for equicontinuous sets, not balls. This is of necessity, as we will show in Proposition 1.11, and reiterates the non-perturbative nature of our analysis.

Other than proving Corollary 1.6, Theorem 1.5 will play no role in the analysis. It is not strong enough! For example, it is not sufficient to prove tightness. For that purpose, we will need the stronger estimate (4.2) expressed in terms of our local smoothing spaces $X_\kappa^{1/2}$ introduced in subsection 2.3. In essence, these spaces capture the local smoothing norm living at frequencies $|\xi| \geq \kappa$. In this way, (4.2) expresses that there is little local smoothing norm at high frequencies and consequently, little transportation of the L^2 norm by the high frequencies.

1.1. **Prior work.** Local well-posedness of (DNLS) was first proved in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ via energy methods in [50, 51]. Subsequently, this was improved to $s \geq \frac{1}{2}$ by Takaoka in [45] via contraction mapping in $X^{s,b}$ spaces. The solution so constructed is a real-analytic function of the initial data.

As explained in [46, §7], the results of [45] show that the data-to-solution map cannot be real-analytic (or even C^3) on $H^s(\mathbb{R})$ for any $s < \frac{1}{2}$. Indeed, by analyzing the family of solitons reviewed in subsection 1.3, it was shown in [4] that the data-to-solution map cannot even be uniformly continuous (on bounded sets) in $H^s(\mathbb{R})$ for $s < \frac{1}{2}$.

Contraction mapping arguments have also been applied in other function spaces. Local well-posedness of (DNLS) in certain Fourier–Lebesgue and modulation spaces was shown in [13] and [14], respectively. In both cases, the spaces are based on $s \geq \frac{1}{2}$ number of derivatives. It is noted in [14] that if fewer derivatives are used, the data-to-solution map cannot be smooth.

We discussed earlier how this irregularity of the data-to-solution map challenges the naivest notions of complete integrability. It also has profound implications in terms of methods. For a generation now, work on well-posedness problems for dispersive PDE has been dominated by contraction mapping arguments in increasingly sophisticated spaces, employing ever subtler harmonic analysis tools. By their very nature, solutions built by contraction mapping will be real-analytic functions of their initial data. The poor regularity of the data-to-solution map in the setting of Theorem 1.1 is a strong signal that very different methods will be needed.

Let us turn now to the question of *global* well-posedness. The standing paradigm here is to extend local-in-time results by employing exact (or approximate) conservation laws. As a completely integrable system, (DNLS) has a multitude of exact conservation laws. The most basic three are

$$(1.5) \qquad M(q) := \int |q(x)|^2 \, dx,$$

$$(1.6) \qquad H(q) := -\frac{1}{2} \int i(q\bar{q}' - \bar{q}q') + |q|^4 \, dx,$$

$$(1.7) \qquad H_2(q) := \int |q'|^2 + \frac{3}{4} i |q|^2 (q\bar{q}' - \bar{q}q') + \frac{1}{2} |q|^6 \, dx.$$

The functional $H(q)$ will be called the Hamiltonian since it generates the (DNLS) dynamics in concert with the Poisson structure

$$(1.8) \qquad \{F, G\} := \int \frac{\delta F}{\delta q} \left(\frac{\delta G}{\delta \bar{q}} \right)' + \frac{\delta F}{\delta \bar{q}} \left(\frac{\delta G}{\delta q} \right)' \, dx.$$

The problem with these conservation laws is that they are *not* coercive for large initial data, specifically, when $M(q)$ is large. It was observed in [19] that coercivity does hold if $M(q) < 2\pi$, which was used to obtain global well-posedness in $H^1(\mathbb{R})$ under this restriction. The subsequent works [7, 8, 37, 46] ultimately led to H^s-well-posedness for $s \geq \frac{1}{2}$ under the $M(q) < 2\pi$ restriction.

Later, Wu showed that the 2π barrier was illusory and that a priori bounds could be obtained under the weaker restriction $M(q) < 4\pi$; see [55, 56] and [11]. Global H^s-well-posedness for $s \geq \frac{1}{2}$ and $M(q) < 4\pi$ was then shown in [15].

The 4π barrier is certainly not illusory: Algebraic solitons (see (1.26)) are explicit solutions of (DNLS) with $M(q) = 4\pi$, but for which all other polynomial conservation laws vanish. Applying the symmetry (1.2) to these algebraic solitons, we see

that the polynomial conservation laws alone cannot provide the kind of control that is needed; see the discussion surrounding (1.27). Because of such obstructions, the behavior of large-data solutions to (DNLS) was for a long time a terra incognita.

The first definitive evidence that large data do not blow up was provided via the inverse scattering approach; see [20, 22, 23, 34, 35, 40, 41, 44]. Among these works, we wish to single out [23] as not only constructing solutions (without any spectral hypotheses), but also for proving continuous dependence on the initial data. Concretely, they proved that (DNLS) is globally well-posed in $H^{2,2}(\mathbb{R}) = \{f \in H^2 : x^2 f \in L^2\}$. Combined with the local-in-time arguments in [19], this result shows that (DNLS) is globally well-posed in Schwartz space.

Strong spatial decay requirements are a prerequisite for the inverse scattering approach as we understand it today. Currently, there is no satisfactory theory of forward nor inverse scattering in any $H^s(\mathbb{R})$ space (not only for (DNLS), but also for KdV, NLS, and mKdV). On the other hand, one of the major strengths of the inverse scattering method is its ability to describe the long-time behavior of solutions. Indeed, a soliton resolution result for generic data in $H^{2,2}(\mathbb{R})$ was proved in [21]; see also [22, 36].

The large-data impasse in Sobolev spaces was dramatically broken by Bahouri and Perelman in the recent paper [3]. By synthesizing the existing well-posedness theory with an in-depth analysis of the transmission coefficient, they proved that (DNLS) is globally well-posed in $H^{1/2}(\mathbb{R})$.

For what follows, it is more convenient to discuss the reciprocal of the transmission coefficient and to define this quantity, $a(k; q)$, via a Fredholm determinant. For $\kappa > 0$, we first define

$$(1.9) \quad \Lambda(\kappa; q) := (\kappa - \partial)^{-\frac{1}{2}} q (\kappa + \partial)^{-\frac{1}{2}} \quad \text{and} \quad \Gamma(\kappa; q) := (\kappa + \partial)^{-\frac{1}{2}} \bar{q} (\kappa - \partial)^{-\frac{1}{2}},$$

which extend to $\kappa < 0$ via $\Lambda(-\kappa; q) = -\Gamma(\kappa; \bar{q})$. By Lemma 2.5 below (reproduced from [30]), both Λ and Γ are Hilbert–Schmidt operators; thus we may define

$$(1.10) \qquad\qquad a(i\kappa; q) := \det\left[1 - i\kappa\Lambda\Gamma\right].$$

This expression originates from a perturbation determinant based on the Lax pair for (DNLS) discovered in [25]. In particular, it is conserved under the (DNLS) flow; see [32] for a proof of this.

It follows from (1.10) that $k \mapsto a(k; q)$ extends to a holomorphic function in both the upper and lower half-planes. While this extension was essential for [3] and the paper [17] that we will discuss shortly, we will not need this here and so restrict our attention to the case where $k = i\kappa$ is purely imaginary.

The central problem overcome by [3] was the ineffectual nature of the conservation laws attendant to (DNLS); however, this solution did not provide new conservation laws with which to fill the void. In particular, [3] does not provide a priori control on lower regularity norms, nor the means to address the question of equicontinuity in such spaces.

Using the ideas of [3] as a jumping-off point, the paper [17] shows that (DNLS) does preserve L^2-equicontinuity. Note that this assertion takes the form of an a priori bound on Schwartz-class solutions since solutions were not known to exist for merely L^2 initial data. In fact (and this will be important for us), the paper [17] shows that this equicontinuity property is enjoyed by any flow preserving the perturbation determinant (1.10) and so by the *entire* (DNLS) hierarchy:

Theorem 1.7 ([17]). *Let $Q \subseteq \mathcal{S}(\mathbb{R})$ be L^2-bounded and equicontinuous. Then*

$$(1.11) \qquad Q_* = \big\{ q \in \mathcal{S}(\mathbb{R}) : a(k;q) \equiv a(k;\tilde{q}) \text{ for some } \tilde{q} \in Q \big\}$$

is also L^2-bounded and equicontinuous.

A key motivation for addressing the equicontinuity question in [17] is that it unlocks a large number of tools in the study of (DNLS); it was this realization that lead [27] to the explicit formulation of this equicontinuity problem.

The first tools unlocked by the equicontinuity property are low-regularity conservation laws, specifically conservation laws at the level of H^s for $0 < s < \frac{1}{2}$. Such laws were first derived in [32] following the approach of [30]; however, they were only applicable to small solutions. The realm of applicability was first raised to $M(q) < 4\pi$ in [27] by proving equicontinuity in that regime and then to arbitrarily large solutions in [17].

Equicontinuity also unlocks higher regularity conservation laws for large data. In [27], L^2-equicontinuity and the conservation of $H_2(q)$ are shown to provide global $H^1(\mathbb{R})$ bounds. In [2], the result of [17] is used as the base step of an inductive argument to cover $H^s(\mathbb{R})$ spaces for all $s \geq \frac{1}{2}$. This brings closure to the question of coercive conservation laws: we now know that H^s-bounded sets of Schwartz-class initial data lead to H^s-bounded solutions for all $s \geq 0$.

To prove local smoothing, we need *microscopic* conservation laws such as (1.1), rather than mere conserved quantities. Note that (1.1) itself is useless for this purpose because the current is not coercive. Already for the proof of (1.4), we need scaling-critical coercive microscopic conservation laws and the full proof of well-posedness will require even more subtle estimates.

Just such microscopic conservation laws were worked out in [47] and will be recapitulated in Proposition 3.5. The structure of these laws closely resembles those of the NLS/mKdV hierarchy presented in [18]. There is a good reason for this: the Kaup–Newell Lax operator for (DNLS) can be written as

$$(1.12) \qquad L(\kappa;q) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \kappa - \partial & \sqrt{\kappa}q \\ i\sqrt{\kappa}\bar{q} & \kappa + \partial \end{bmatrix},$$

which closely resembles the AKNS–ZS Lax operator of the NLS/mKdV hierarchy. We will only discuss this operator for $\kappa \in \mathbb{R}$. Throughout this paper,

$$\sqrt{\kappa} = i\sqrt{|\kappa|} \quad \text{when} \quad \kappa < 0.$$

There is one more prior result that we wish to discuss, namely, global well-posedness in $H^{1/6}(\mathbb{R})$. This was first shown in [27] for initial data satisfying $M(q) < 4\pi$, a restriction that was removed in [17]. This result was shown using the first-generation method of commuting flows introduced in [29] and reviewed below.

1.2. **Description of the method.** The principal problem we must address in order to prove Theorem 1.1 is this: given $T > 0$ and an L^2-convergent sequence $q_n(0)$ of Schwartz initial data, show that the corresponding (Schwartz-class) solutions $q_n(t)$ converge in $L^2(\mathbb{R})$ uniformly for $|t| \leq T$.

Given the breakdown of uniform continuity of the data-to-solution map (on bounded sets) and the further instabilities highlighted in Proposition 1.12, it is difficult to conceive of a method of controlling differences of solutions in terms of their initial data. It was to address this specific challenge that the method of commuting flows was introduced in [29].

To explain the method of commuting flows, let us imagine that we wish to prove well-posedness of the flow generated by a Hamiltonian H in L^2; in our case, H is given in (1.6). Central to the method is the construction of a one-parameter family of Hamiltonians H_κ whose flows satisfy the following three properties:

(1) they commute with the H flow,

(2) they are well-posed in the target well-posedness space L^2, and

(3) they converge to the H flow as $\kappa \to \infty$.

Let us temporarily take for granted the existence of the family of Hamiltonians H_κ satisfying these properties. Their construction for (DNLS) is quite involved and will be discussed shortly. Demonstrating that they satisfy the three properties requires almost the entire bulk of this paper.

Property (1), namely commutativity of the flows, can be expressed as

$$e^{tJ\nabla H} \circ e^{sJ\nabla H_\kappa} = e^{sJ\nabla H_\kappa} \circ e^{tJ\nabla H} = e^{J\nabla[tH+sH_\kappa]} \quad \text{for any } s, t \in \mathbb{R},$$

where we adopt the exponential notation for the flow of a vector field and write $J\nabla$ for the symplectic gradient. The relevance of this relation to demonstrating that a sequence of (Schwartz) solutions $q_n(t)$ is Cauchy in $C([-T, T]; L^2)$ may be best understood via the following identity:

$$q_n(t) - q_m(t) = \left[e^{tJ\nabla H_\kappa}q_n(0) - e^{tJ\nabla H_\kappa}q_m(0)\right] + \left[e^{tJ\nabla(H-H_\kappa)} - \mathrm{Id}\right] \circ e^{tJ\nabla H_\kappa}q_n(0)$$

$$(1.13) \qquad\qquad - \left[e^{tJ\nabla(H-H_\kappa)} - \mathrm{Id}\right] \circ e^{tJ\nabla H_\kappa}q_m(0).$$

Property (2) is well-posedness of the H_κ flows. This implies that the first term on the right-hand side of (1.13) converges to zero as $n, m \to \infty$ for each fixed κ. In order to prove that the sequence $q_n(t)$ is Cauchy in $C([-T, T]; L^2)$, it remains to show that

$$(1.14) \qquad \limsup_{\kappa\to\infty} \sup_n \sup_{|t|\leq T} \left\| \left[e^{tJ\nabla(H-H_\kappa)} - \mathrm{Id}\right] \circ e^{tJ\nabla H_\kappa}q_n(0)\right\|_{L^2} = 0.$$

We will refer to the flow generated by the Hamiltonian $H - H_\kappa$ as the *difference flow*. Relation (1.14) embodies the statement that as $\kappa \to \infty$, the difference flow converges to the identity. This is a quantitative interpretation of property (3).

In implementing the method of commuting flows, we have come to regard properties (1) and (2) as selection criteria for the H_κ Hamiltonians, leaving property (3) as the key analytical difficulty that must be faced.

In our experience, (1.14) has always proved to be a very difficult problem. First, we must acknowledge that the difference flow inherits all the strong instabilities of the original flow. As the H_κ flow is typically a diffeomorphism, it cannot undo these problems. The one big advance, however, is that we no longer need to control *differences* of solutions: q_n and q_m are now completely decoupled. This is partially off-set by the fact that the initial data for the difference flow is not $q_n(0)$, but rather $e^{tJ\nabla H_\kappa}q_n(0)$ where t varies over $[-T, T]$. As a result, we will need to show that the difference flow converges to the identity uniformly across sets of initial data about which we know very little.

As in previous works, we will exploit that $\{e^{tJ\nabla H_\kappa}q_n(0) : n \in \mathbb{N}, |t| \leq T\}$ inherits equicontinuity from the precompact set $\{q_n(0)\}$. This follows from Theorem 1.7 and the fact that the H_κ flows constructed below conserve $a(i\kappa; q)$. Refracted through this perspective, property (3) becomes the following assertion:

$$(1.15) \qquad\qquad \limsup_{\kappa\to\infty} \sup_{q\in Q} \sup_{|t|\leq T} \left\| \left[e^{tJ\nabla(H-H_\kappa)} - \mathrm{Id}\right](q)\right\|_{L^2} = 0$$

for any L^2-bounded and equicontinuous set $Q \subseteq \mathcal{S}(\mathbb{R})$.

It is natural to seek to prove (1.15) by estimating the difference-flow vector field, that is, the time derivative under this flow. As a prerequisite, one needs to be able to make sense of the nonlinearity appearing therein — this includes the nonlinearity under the H flow, which for (DNLS) is $(|q|^2 q)'$.

The method of commuting flows was applied to (DNLS) in [27], treating initial data in $H^{1/6}$. Conservation laws were used to bound the resulting solution in $L_t^\infty H^{1/6}$, which allowed the authors to prove (1.15) with L^2 replaced by H^{-4}. The lost derivatives were then recovered using equicontinuity, specifically, the general statement that if a sequence q_n converges in H^σ and is equicontinuous in H^s for $s > \sigma$, then it converges in H^s. The relevance of $H^{1/6}$ is that it embeds into L^3 and this allows us to interpret the nonlinearity $(|q|^2 q)'$ as an element of $L_t^\infty H^\sigma$ for any $\sigma < -\frac{3}{2}$.

Already in the first application of the method of commuting flows in [29], which was for KdV in H^{-1}, it was not possible to estimate the difference-flow vector field directly. To address this problem, the authors introduced a gauge transformation (a diffeomorphic change of unknown), whose difference-flow dynamics they could estimate pointwise in time (albeit with a sizable loss of derivatives, which were then recovered using equicontinuity).

One advantage of this *first-generation* method of commuting flows, where the difference flow (with or without a gauge) is estimated pointwise in time, is that it works equally well for problems posed both on the line and on the circle. However, there are models (such as NLS and mKdV) where the threshold regularities for well-posedness are different in the two geometries. The treatment of these equations in [18] necessitated the introduction of a *second-generation* method of commuting flows, based on new *local smoothing* and *tightness* estimates.

The job of local smoothing estimates is to make sense of the vector field as a spacetime distribution in instances where this cannot be done pointwise in time. As the central problem is to control the *difference flow* by estimating the size of the corresponding vector field, one must develop local smoothing estimates for *this* flow. This is almost paradoxical: local smoothing is an expression of high-frequency transport; however, our ultimate goal is to demonstrate that the difference flow converges to the identity. The demonstration of sufficiently strong smoothing estimates for the (DNLS) difference flow (see Proposition 6.1) requires a vast amount of work; we will return to this topic after the Hamiltonians H_κ have been introduced.

Using local smoothing, we will only be able to prove convergence of the difference flow to the identity locally in space (cf. Theorem 7.1). The role of the second new ingredient, tightness, is to overcome this limitation. As both radiation and solitons move under the (DNLS) flow, establishing tightness is challenging. For (DNLS) this is accomplished in Proposition 5.1 below and relies on the subtle control of the high-frequency transportation provided by (4.2).

The H_κ flows and their properties. To introduce the Hamiltonians H_κ for (DNLS), we return to the perturbation determinant (1.10), or rather, to the closely related quantity

$$(1.16) \qquad A(\kappa; q) := -\operatorname{sgn}(\kappa) \log[a(i\kappa; q)] = -\operatorname{sgn}(\kappa) \log \det\left[1 - i\kappa \Lambda \Gamma\right].$$

As discussed above, $a(i\kappa; q)$ is conserved by the (DNLS) flow. This guarantees that both the real and imaginary parts of the (complex) functional $A(\kappa; q)$ Poisson commute with the (DNLS) Hamiltonian $H(q)$.

The inclusion of $\mathrm{sgn}(\kappa)$ in (1.16) ensures that $A(\kappa; q)$ has the same asymptotic expansion as $\kappa \to \pm\infty$. This expansion shows that $A(\kappa; q)$ encodes all the polynomial conservation laws of (DNLS); it begins

$$(1.17) \qquad A(\kappa; q) = \tfrac{i}{2}M(q) + \tfrac{1}{4\kappa}H(q) - \tfrac{i}{8\kappa^2}H_2(q) + O\big(\tfrac{1}{\kappa^3}\big)$$

for $q \in \mathcal{S}(\mathbb{R})$. Rearranging this formula leads one to believe that

$$(1.18) \qquad\qquad H_\kappa(q) := 4\kappa \, \mathrm{Re}\, A(\kappa; q)$$

is a good approximation for the (DNLS) Hamiltonian $H(q)$, at least as $\kappa \to \infty$. Moreover, the Poisson commutativity of $\mathrm{Re}\, A(\kappa; q)$ and $H(q)$ noted above guarantees that H_κ and H also commute. This is the sought-after property (1) from our overview of the method of commuting flows.

The preceding discussion has been predicated on the non-vanishing of $a(i\kappa; q)$, so that one may safely take the logarithm in (1.16). This issue is discussed in [27], where it is shown that $A(\kappa; q)$ is well-defined provided κ is sufficiently large; however, (by necessity) the restriction on κ is not dictated solely by the size of q, but also by its frequency distribution.

Informed by the many computations ahead of us, in this paper we adopt the expedient of using (1.2) to rescale solutions q so that we may impose a single restriction on κ, namely, $|\kappa| \geq 1$. The goal of the rescaling is to make $\widehat{q}(\xi)$ small at frequencies $|\xi| \geq 1$; such smallness is conveniently expressed through the following notion:

Definition 1.8. Fix $0 < \sigma < \tfrac{1}{2}$. Given $\delta > 0$, we say that $Q \subseteq \mathcal{S}(\mathbb{R})$ is δ-*good* if it is L^2-bounded, L^2-equicontinuous, and it satisfies

$$(1.19) \qquad\qquad \sup_{q \in Q} \int \frac{|\xi|^{2\sigma}|\hat{q}(\xi)|^2}{(4 + \xi^2)^\sigma}\, d\xi \leq \delta^2.$$

Although the parameter σ could be frozen once and for all, say $\sigma = \tfrac{1}{4}$, we believe that retaining the symbol σ makes it easier to check our computations.

As reviewed in Section 3, the series (1.16) converges uniformly on all δ-good sets (once δ is small enough). Local well-posedness of the H_κ flow for δ-good sets of initial data follows from Picard's Theorem because the corresponding vector field is Lipschitz. Moreover, these solutions remain Schwartz-class and conserve $A(\varkappa; q)$ for all $|\varkappa| \geq 1$. These assertions were shown in [27, §5].

In order to construct a global-in-time H_κ flow, we must ensure that orbits remain δ-good as time progresses. This is accomplished by combining the fact that the H_κ flow preserves $a(i\varkappa; q)$ together with the following consequence of Theorem 1.7:

Corollary 1.9. *Let* $Q \subseteq \mathcal{S}(\mathbb{R})$ *be an* L^2-*bounded and equicontinuous set. Given* $\delta > 0$, *there exists* $\lambda = \lambda(Q, \delta)$ *so that the set*

$$Q_*^\lambda = \big\{ \sqrt{\lambda}\, q(\lambda x) \in \mathcal{S}(\mathbb{R}) : a(i\varkappa; q) \equiv a(i\varkappa; \tilde{q}) \text{ for some } \tilde{q} \in Q \big\} \quad \text{is } \delta\text{-good.}$$

Before turning to the difficult topic of analyzing the difference flow, let us pause to summarize the preceding discussion as a theorem. In particular, this theorem encapsulates properties (1) and (2) of the H_κ flows.

Theorem 1.10. *There exists $\delta_0 > 0$ sufficiently small so that for any $0 < \delta \leq \delta_0$ and L^2-bounded and equicontinuous set $Q \subseteq \mathcal{S}(\mathbb{R})$, the H_κ flow is globally well-posed on the set Q_*^λ in the L^2 topology, where $\lambda(Q, \delta)$ is chosen according to Corollary 1.9. In particular, solutions remain of Schwartz-class and remain in the set Q_*^λ. Moreover, the (DNLS) flow also preserves the set Q_*^λ and commutes with the H_κ flow.*

All the assertions made here about the H_κ flow were proved in [27] contingent on the question of equicontinuity that was subsequently resolved in [17]. As discussed earlier, the existence of global Schwartz-class solutions to (DNLS) follows from [19, 23]. That such (DNLS) solutions conserve the transmission coefficients is a classical result.

Analysis of the difference flow. This occupies the bulk of this paper and requires many new insights.

To understand the difference flow, we must first give the explicit form of this evolution. This relies on the functional derivatives of H_κ, which are easily deduced from those of $A(\kappa; q)$ given in (3.5).

The functions $g_{12}(x)$ and $g_{21}(x)$ appearing in (3.5) are the two off-diagonal entries of the Green's function corresponding to the Lax operator (1.12) evaluated on the diagonal $x = y$. Together with a third component $\gamma(x)$, these functions will be recurrent characters in our story and Section 3 is devoted to a detailed elaboration of their algebraic and analytic properties.

Combining the functional derivatives with the Poisson structure (1.8), we find an explicit formula for the difference flow evolution:

$$(1.20) \qquad i\tfrac{d}{dt}q = -q'' - i(|q|^2 q)' + 2\kappa\left[\sqrt{\kappa}g'_{12}(\kappa) - \sqrt{-\kappa}g'_{12}(-\kappa)\right].$$

The goal of this section is to explain how to prove (1.15) for the difference flow given by (1.20).

Unlike all other terms, the nonlinearity $(|q|^2 q)'$ appearing on the right-hand side of (1.20) does not make sense pointwise in time for $q \in C_t L^2$ and so we are immediately tasked with finding a remedy. Despite strenuous efforts, we were unable to find a gauge transformation that would allow us to estimate the resulting difference-flow vector field pointwise in time. Based on previous successes with the diagonal Green's function [18, 29], it is natural to imagine that this might be a satisfactory gauge. This idea is refuted by (3.28), as explained there. Therefore, we are forced to adopt the second-generation method of commuting flows.

As discussed earlier, the characteristic features of the second-generation method are the use of local smoothing estimates to control the difference flow and the resulting necessity of showing that compact sets of initial data lead to tight ensembles of orbits under the (DNLS) flow, over bounded time intervals. The proof of the tightness statement relies on refined local smoothing estimates for the (DNLS) flow.

Local smoothing estimates are a direct expression of the dispersive nature of an equation: high frequencies travel rapidly and so spend little time in any fixed spacetime region. As such, they originate from the linear/dispersive part of the equation.

As discussed in [48], there are two standard ways for proving local smoothing estimates for a linear equation: via spacetime Fourier transforms and via monotonicity identities. When the nonlinearity may be treated perturbatively, local smoothing estimates can be transferred directly from the underlying linear flow to the full nonlinear equation. Correspondingly, it matters little what method one uses

for establishing the linear estimates. In the non-perturbative regime considered in this paper, we have no choice but to pursue an approach based on monotonicity identities for the full nonlinear flow.

All monotonicity-type identities we know originate from microscopic conservation laws of the form

$$\partial_t \rho + \nabla_x \cdot \vec{\jmath} = 0. \tag{1.21}$$

In one spatial dimension for example, this implies

$$\partial_t \int \tanh(x)\rho(t,x)\,dx = \int \operatorname{sech}^2(x)j(t,x)\,dx. \tag{1.22}$$

In the rare event that one can find such a law with $j \geq 0$, (1.21) constitutes true monotonicity. It is more reflective of actual practice however to find a coercive term j_1 in the current and then integrate (1.22) to obtain

$$\int_{-T}^{T}\int \operatorname{sech}^2(x)j_1(t,x)\,dx\,dt \leq 2 \sup_{|t|\leq T}\left|\int \tanh(x)\rho(t,x)\,dx\right|$$
$$+ \int_{-T}^{T}\int \operatorname{sech}^2(x)[j_1 - j](t,x)\,dx\,dt. \tag{1.23}$$

The utility of this inequality rests on finding a suitable microscopic conservation law. First, one must find a density ρ whose integral can be controlled uniformly in time. Second, one must be able to identify a coercive part j_1 of the current that controls the sought-after local smoothing norm. Third, one must be able to control the contribution of $j_1 - j$.

For our analysis we need two one-parameter families of microscopic conservation laws, one for (DNLS) and one for the difference flow (1.20). These can be found in Proposition 3.5, with the density given in (3.27). Local smoothing for (DNLS) is proved in Proposition 4.1 and for the difference flow in Proposition 6.1.

The first task is to estimate the integral of the density ρ in (3.27) uniformly in time. This is achieved in Lemma 4.2. The complicated structure of this density makes this a nontrivial task. Moreover, to prove tightness of orbits under the (DNLS) flow we need the refined local smoothing estimate (4.2), which requires us to prove that the contribution of ρ converges to zero in the high-frequency regime. The analysis of ρ in Lemma 4.2 relies on the detailed study of the diagonal Green's functions carried out in Section 3.

Our second task is to identify a coercive part in the currents appearing in Proposition 3.5. In our analysis, the quadratic terms $j_{\mathrm{DNLS}}^{[2]}$ and $j_{\mathrm{diff}}^{[2]}$ of the currents will play the role of j_1 in (1.23). Although these are not sign definite, we are able to demonstrate the requisite coercivity up to acceptable errors. For the treatment of $j_{\mathrm{DNLS}}^{[2]}$, see the discussion surrounding (4.32). Extracting coercivity from $j_{\mathrm{diff}}^{[2]}$ requires considerable regrouping and the estimation of many error terms and commutators; see the treatment of (6.40).

The third and most difficult part of obtaining local smoothing estimates is controlling the remainder of the current $j_1 - j$. In defocusing problems, the most dangerous parts of $j_1 - j$ typically have a favorable sign. When the problem treated is subcritical, the second term on RHS(1.23) can be controlled by interpolating between the LHS(1.23) and a priori conservation laws. A small data hypothesis can also provide the smallness needed to bound the contribution of $j_1 - j$ by a

small fraction of LHS(1.23). The problem studied here has none of these favorable features. In fact, any of these features would yield local smoothing estimates that depend only on the norm of the initial data; this is ruled out by Proposition 1.11.

In our case, the remainder $j_1 - j$ comprises the quartic and higher order terms $j_{\mathrm{DNLS}}^{[\geq 4]}$ and $j_{\mathrm{diff}}^{[\geq 4]}$. There is an enormous number of contributions that need to be controlled. Moreover, these cannot be estimated directly using the L^2 norm of q since they involve both derivatives and higher powers of q. Instead, we endeavor to control these contributions using local smoothing and a bootstrap argument.

As we are dealing with a large-data scaling-critical problem, there is no easy source of smallness for closing the bootstrap. This is one of the key analytical challenges we must overcome in this article. The subcriticality of the models treated in [18] expressed itself through the appearance of negative powers of the large parameter κ, which provided the requisite smallness. For (DNLS), we are forced to simultaneously exhibit two copies of the local smoothing norm (to be bootstrapped) and a third factor encoding equicontinuity (the source of smallness) for every single error term. To achieve this, we must identify and exploit many subtle hidden cancellations in the flow — see, for example, the carefully curated decompositions of $j_{\mathrm{DNLS}}^{[\geq 4]}$ and $j_{\mathrm{diff}}^{[\geq 4]}$ appearing in (4.27) and in the proof of Lemma 6.3, respectively.

The analysis of these error terms relies on a large body of work built up in the preceding sections of the paper. In Section 2 we introduce the norms used to quantify both equicontinuity and local smoothing. We also need to introduce and analyze a Banach algebra B of bounded multiplication operators on our equicontinuity spaces. This section also contains a suite of basic nonlinear estimates used later in the paper.

Much of Section 3 is devoted to proving estimates on the diagonal Green's functions. These arise in several places in our analysis: not only are they an integral part of the microscopic conservation laws, but they also appear in (1.20) because they encode the functional derivatives of $A(\kappa; q)$. The culmination of Section 3 is the estimation of the diagonal Green's functions and key nonlinear combinations thereof in the equicontinuity and the local smoothing spaces. We also elucidate the structure of these functions in terms of a new class of paraproducts introduced in this section; see, for example, Lemma 3.11 and Proposition 3.12.

A second class of paraproducts which incorporates the localizing weights intrinsic to local smoothing estimates is introduced in Section 4. A key feature of our analysis is demonstrating that one may distribute these localizing weights to all entries in these paraproducts. This is important since any one of the input functions in a paraproduct may be the highest frequency term and so will need to be estimated in the local smoothing norm. The culmination of Section 4 is the proof of the local smoothing estimates for the (DNLS) flow stated in Proposition 4.1.

Tightness of orbits for solutions to (DNLS) is proved in Section 5. This argument is quite short because of the strength of the estimate (4.2) proved in Section 4.

Section 6 contains a proof of local smoothing for the difference flow (1.20). It is the most demanding part of the paper and relies on all the analysis that precedes it. Indeed, the needs of this section dictated much of the prior development.

Section 7 combines all that precedes it to prove convergence of the difference flow to the identity, locally in space. Finally, a short Section 8 deduces all the main theorems from these prerequisites.

1.3. **The soliton menace.** In this subsection we present the family of soliton solutions to (DNLS) and use them to exhibit some of the instabilities of this equation.

For each value of $\theta \in (0, \frac{\pi}{2})$, the function

$$q_0(x; \theta) := \sqrt{2 \sin(2\theta)} \frac{[\cos(\theta) \cosh(x) - i \sin(\theta) \sinh(x)]^3}{[\cos^2(\theta) \cosh^2(x) + \sin^2(\theta) \sinh^2(x)]^2} e^{-ix \cot(2\theta)}$$

provides initial data for a soliton solution to (DNLS). In understanding the shape of this function, it is useful to note that the central factor can be written as $Z^3/|Z|^4$ with $Z = \cos(\theta) \cosh(x) - i \sin(\theta) \sinh(x) = \cosh(x - i\theta)$. The soliton with this initial data takes the form

$$(1.24) \qquad q(t, x; \theta) = q_0\big(x + 2 \cot(2\theta)t; \theta\big) e^{it \csc^2(2\theta)}.$$

Further solitons can be obtained by translation, phase rotation, and scaling.

The $\theta \to 0$ limit of this solution exists and is identically zero. Indeed

$$(1.25) \qquad \qquad \qquad \|q_0\|_{L^2}^2 = 8\theta.$$

In the form we have presented, the $\theta \to \frac{\pi}{2}$ limit does not exist. However by rescaling in accordance with (1.2), a limit can be recovered, namely, the algebraic soliton:

$$(1.26) \qquad q(t, x) = q_a(x - t)e^{it/4} \quad \text{with initial data} \quad q_a(x) = \frac{2(1 - ix)}{(1 + ix)^2} e^{ix/2}.$$

This solution embodies a key obstruction to coercivity of the polynomial conservation laws. Indeed, while $M(q_a) = 4\pi$ all other polynomial conserved quantities vanish. These properties also hold for all rescalings (1.2) of q_a. However,

$$(1.27) \qquad \qquad \qquad \|q_{a,\lambda}\|_{H^s} \to \infty \quad \text{as} \quad \lambda \to \infty$$

for any $s > 0$.

To see that q_a also witnesses an obstruction to using the perturbation determinant (1.10) to prove equicontinuity we note that $a(i\kappa; q_{a,\lambda}) \equiv 1$ for all $\lambda > 0$. However, $\{q_{a,\lambda} : \lambda > 0\}$ is *not* an equicontinuous family.

Let us now turn our attention to the soliton with $\theta = \frac{\pi}{4}$, which simplifies to

$$(1.28) \qquad q_s(t, x) = \frac{2e^{it}[\cosh(x) - i \sinh(x)]^3}{\left[\cosh^2(x) + \sinh^2(x)\right]^2}.$$

The subscript s appearing here emphasizes that this is a *stationary* soliton. (While it does oscillate in time, it does not propagate through space.) This property makes it the archenemy of local smoothing and Strichartz estimate. In particular, our next proposition shows that the inequality (1.4) does not hold for sets Q that are merely L^2-bounded. This further emphasizes the non-perturbative nature of the (DNLS) flow in the L^2 topology.

Proposition 1.11. *Local smoothing and Strichartz norms cannot be controlled solely by the L^2 norm of the initial data. Concretely, there is a sequence of solutions q_n to (DNLS) satisfying $M(q_n) \equiv 2\pi$ but*

$$\int_{-1}^{1} \| \operatorname{sech}^{12}(x) q_n(t, x)\|_{H_x^{1/2}}^2 \, dt \longrightarrow \infty \quad \text{and} \quad \int_{-1}^{1} \int_{\mathbb{R}} |q_n(t, x)|^6 \, dx \, dt \longrightarrow \infty$$

as $n \to \infty$.

Proof. We choose the q_n to be rescalings of q_s according to (1.2) and observe that $M(q_{s,\lambda}) \equiv 2\pi$ but

$$\int_{-1}^{1} \|\operatorname{sech}^{12}(x) q_{s,\lambda}(t,x)\|_{H_x^{1/2}}^2 \, dt \approx \lambda \quad \text{and} \quad \int_{-1}^{1} \int_{\mathbb{R}} |q_{s,\lambda}(t,x)|^6 \, dx \, dt \approx \lambda^2. \quad \square$$

As the last topic of this section, we demonstrate another instability inherent to (DNLS). Concretely, we will show that it is not possible to prove uniqueness of solutions via Grönwall's inequality in lower regularity spaces. This is a widely successful uniqueness technique and a key ingredient in constructing solutions via compactness/uniqueness arguments; the Grönwall inequality yields Lipschitz dependence in H^s of the data-to-solution map. However, our next proposition shows failure of Lipschitz dependence, no matter how negative one chooses s.

Proposition 1.12. *Fix $s \leq 0$. There are times $t_n \to 0$ and pairs of solutions q_n and $\tilde{q}_n$ to (DNLS) so that*

$$(1.29) \qquad \|q_n\|_{L^2} + \|\tilde{q}_n\|_{L^2} \to 0 \quad \text{but} \quad \frac{\|q_n(t_n) - \tilde{q}_n(t_n)\|_{H^s}}{\|q_n(0) - \tilde{q}_n(0)\|_{H^s}} \to \infty.$$

Proof. We choose q_n and $\tilde{q}_n$ to be distinct rescalings of the soliton solution (1.24) with parameter θ_n. We first choose $\theta_n \to 0$ to ensure that their L^2 norms converge to zero; see (1.25).

The key idea to exploit is the fact that q_n and $\tilde{q}_n$ travel at different speeds. To ensure that their separation at time t_n diverges, we require that

$$(1.30) \qquad |\lambda_n - \tilde{\lambda}_n| \cot(2\theta_n) t_n \to \infty \quad \text{yet} \quad t_n \to 0 \quad \text{as} \quad n \to \infty.$$

In order to compute the overall size of the norms at the times 0 and t_n, it is convenient to compute the Fourier transform of a soliton exactly. The key identity is this:

$$(1.31) \qquad \int \frac{\cosh(x - i\theta)}{\cosh^2(x + i\theta)} e^{-i\xi x} \, dx = \frac{\pi e^{-\theta\xi}}{\cosh(\frac{\pi}{2}\xi)} \big[\cos(2\theta) - \xi \sin(2\theta)\big],$$

which follows by a simple residue computation. It follows from this that $\hat{q}_0(\xi)$ has a simple zero at the origin, yielding three cases: $s < -\frac{3}{2}$, $s = -\frac{3}{2}$, and $s > -\frac{3}{2}$.

In the regime where (1.30) and $|\lambda_n - \tilde{\lambda}_n| \ll \lambda_n$ both hold, elementary (but lengthy) computations show

$$\frac{\lambda_n}{|\lambda_n - \tilde{\lambda}_n|} \|q_n(0) - \tilde{q}_n(0)\|_{H^s(\mathbb{R})} \lesssim \|q_n(0)\|_{H^s(\mathbb{R})} \approx \|q_n(t_n) - \tilde{q}_n(t_n)\|_{H^s}.$$

With this information it is not difficult to choose the necessary parameters. $\square$

1.4. **Equivalent models and their physical origins.** Let us begin by noting that (DNLS) does not admit a focusing/defocusing dichotomy: the sign of the nonlinearity can be reversed by simply replacing $x \mapsto -x$. Likewise, the relative coupling of the three terms in (DNLS) can be freely adjusted by rescaling the space and time variables.

To the best of our knowledge, (DNLS) first appears in the literature as a model for the propagation of large-wavelength Alfvén waves in plasma. For a further discussion of this scenario, including how this effective model informs our understanding of the stability of such Alfvén waves, see [26, 39, 43, 53].

It is easily seen that (DNLS) does not inherit the Galilean symmetry of the linear Schrödinger equation. Indeed, if q solves (DNLS), then

$$(1.32) \qquad v(t,x) = e^{ikx-ik^2t}q(t, x-2kt)$$

solves

$$(1.33) \qquad i\partial_t v + v'' + i(|v|^2 v)' + k|v|^2 v = 0.$$

Here $k \in \mathbb{R}$ is fixed but arbitrary.

This computation indicates that the traditional cubic nonlinear Schrödinger equations (both focusing and defocusing) are 'embedded' inside (DNLS) in the limit of large modulation. While we know of no mathematical work on this embedding, we will describe two physical systems which speak to this phenomenology.

The combined nonlinearities of (1.33) arise naturally in nonlinear optics. While negligible in many experimental scenarios, the derivative nonlinearity becomes physically important in the propagation of short pulses (cf. [1, 52]).

One early application of the cubic nonlinear Schrödinger equation was to modeling amplitude modulations of Alfvén waves, with the unknown function describing deviations from a plane wave. (In our earlier discussion of (DNLS) as model of Alfvén waves, q describes the entire amplitude of the wave, not fluctuations.) One of the key assumptions in deriving this model is that the characteristic length of the modulations far exceeds the carrier wavelength. As argued in [38], the combined nonlinearities of (1.33) allow one to extend the realm of applicability of this effective model to include cases where these two length scales are almost comparable.

As part of a search for completely integrable PDE, a different form of derivative nonlinear Schrödinger equation was uncovered in [6], namely,

$$(1.34) \qquad i\partial_t q + q'' + i|q|^2 q' = 0.$$

It was subsequently discovered (see, e.g. [54]) that this model can be obtained from (DNLS) via a change of variables. The change of variables in question takes the following form:

$$(1.35) \qquad w(t,x) = q(t,x)e^{i\nu\Phi} \quad \text{with} \quad \Phi(t,x) = \int_{-\infty}^{x} |q(t,y)|^2 \, dy$$

and $\nu \in \mathbb{R}$ fixed. With a little work, we find that w satisfies

$$(1.36) \qquad iw_t + w'' = 2i(\nu-1)|w|^2 w' + i(2\nu-1)w^2\bar{w}' - \tfrac{1}{2}\nu(2\nu-1)|w|^4 w.$$

When $\nu = \frac{1}{2}$, we recover (1.34). When $\nu = 1$, we obtain the Gerdjikov–Ivanov form of derivative NLS; see [12].

The more general nonlinearity presented in (1.36), as well as that arising by further incorporating the Galilei transformation (1.32), appears naturally in the study of the Benjamin–Feir instability in the theory of water waves; see [9, 24, 33].

It is easy to see that both changes of variables, (1.33) and (1.35), are real-analytic diffeomorphisms on $L^2(\mathbb{R})$; indeed, they are diffeomorphisms on $H^s(\mathbb{R})$ for every $s \geq 0$. Thus Theorem 1.1 guarantees the following:

Corollary 1.13. *The evolutions (1.33) and (1.36) are globally well-posed in $L^2(\mathbb{R})$.*

When we look at (1.36), it seems all the more surprising that large data GWP holds, since it fails for the focusing quintic nonlinear Schrödinger equation!

2. Preliminaries

Throughout, we will use scaling-homogeneous Littlewood–Paley decompositions with frequency parameters $N \in 2^{\mathbb{Z}}$. Concretely, choosing a smooth, non-negative function φ supported on $|\xi| \leq 2$ with $\varphi(\xi) = 1$ for $|\xi| \leq 1$, we define $P_{\leq N}$ as the Fourier multiplier with symbol $\varphi(\xi/N)$ and then $P_N = P_{\leq N} - P_{\leq N/2}$. Observe that

$$1 = \sum_{N \in 2^{\mathbb{Z}}} P_N.$$

Such decompositions will be ubiquitous and we often adopt the more compact notations $f_N = P_N f$, $f_{\leq N} = P_{\leq N} f$, and $f_{>N} = [1 - P_{\leq N}] f$.

As a similar expedient, we often write Fourier multipliers under their arguments. For example, for $\kappa > 0$,

$$\frac{q}{\sqrt{4\kappa^2 - \partial^2}} := (4\kappa^2 - \partial^2)^{-1/2} q \quad \text{and} \quad \frac{q}{2\kappa \pm \partial} := (2\kappa \pm \partial)^{-1} q.$$

For $s \in \mathbb{R}$ and $|\kappa| \geq 1$ we define the Sobolev space H_κ^s as the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|q\|_{H_\kappa^s}^2 := \int (4\kappa^2 + \xi^2)^s |\hat{q}(\xi)|^2 \, d\xi,$$

and write $H^s = H_1^s$.

Associated to the localizing function ψ_μ defined in (1.3), we have

$$(2.1) \qquad \int_{\mathbb{R}} \psi(x - \mu)^{24} \, d\mu = \tfrac{512}{7} \quad \text{and so} \quad f(x) = \tfrac{7}{512} \int_{\mathbb{R}} f(x) \psi_\mu^{24}(x) \, d\mu.$$

2.1. **Equicontinuity spaces.** To quantify the equicontinuity, for $\sigma, s \in \mathbb{R}$ and $|\kappa| \geq 1$ we define

$$(2.2) \qquad \|q\|_{E_{s,\kappa}^\sigma}^2 := |\kappa|^{2(s-\sigma)} \||\partial|^\sigma q\|_{H_\kappa^{-s}}^2 = \int \frac{|\kappa|^{2(s-\sigma)} |\xi|^{2\sigma}}{(4\kappa^2 + \xi^2)^s} |\hat{q}(\xi)|^2 \, d\xi,$$

and take $E_s^\sigma = E_{s,1}^\sigma$.

We write B for the space of bounded functions that belong to the homogeneous Besov space $\dot{B}_{2,\infty}^{\frac{1}{2}}$. We equip this space with the norm

$$\|f\|_B := \|f\|_{L^\infty} + \sup_{N \in 2^{\mathbb{Z}}} N^{\frac{1}{2}} \|f_N\|_{L^2}.$$

This space is an algebra; see Lemma 2.3. Moreover, by Lemma 2.4, multiplication by functions in B defines a bounded operator on our equicontinuity spaces.

Our next lemma shows how the spaces $E_{\sigma,\kappa}^\sigma$ allow us to track the equicontinuity properties of orbits.

Lemma 2.1. *Let $Q \subset L^2$ be bounded and equicontinuous. For $\sigma > 0$ we have*

$$(2.3) \qquad \lim_{\kappa \to \infty} \sup_{q \in Q} \|q\|_{E_{\sigma,\kappa}^\sigma} = 0.$$

Proof. The claim follows from the following estimate

$$\|q\|_{E_{\sigma,\kappa}^\sigma} \lesssim \|q_{\leq N}\|_{E_{\sigma,\kappa}^\sigma} + \|q_{>N}\|_{E_{\sigma,\kappa}^\sigma} \lesssim \left(\tfrac{N}{\kappa}\right)^\sigma \|q\|_{L^2} + \|q_{>N}\|_{L^2},$$

by choosing the frequency $N \in 2^{\mathbb{Z}}$ appropriately. $\qquad \square$

Lemma 2.2. *For $s > \sigma > 0$, $\beta \in \mathbb{R}$, and $\kappa \geq 1$,*

$$(2.4) \qquad \int_\kappa^\infty \|q\|_{E_{s,\varkappa}^{\sigma+\beta}}^2 \, \varkappa^{2\beta} \, \frac{d\varkappa}{\varkappa} \sim \kappa^{2\beta} \|q\|_{E_{\sigma,\kappa}^{\sigma+\beta}}^2.$$

Further, if $\kappa \geq 2$ and $I_\kappa = [1, \frac{\kappa}{2}] \cup [2\kappa, \infty)$ then

$$(2.5) \qquad \int_{I_\kappa} \|q\|_{E_{s,\varkappa}^{\sigma+\beta}}^2 \, \varkappa^{2\beta} \, \frac{d\varkappa}{\varkappa} \sim \|q\|_{E_\sigma^{\sigma+\beta}}^2.$$

Proof. Decomposing into Littlewood–Paley pieces,

$$\int_\kappa^\infty \|q\|_{E_{s,\varkappa}^{\sigma+\beta}}^2 \, \varkappa^{2\beta} \, \frac{d\varkappa}{\varkappa} \sim \sum_N \int_\kappa^\infty \frac{N^{2(\sigma+\beta)} \varkappa^{2(s-\sigma)}}{(\varkappa+N)^{2s}} \, \frac{d\varkappa}{\varkappa} \|q_N\|_{L^2}^2$$

$$\sim \sum_N \frac{N^{2(\sigma+\beta)}}{(\kappa+N)^{2\sigma}} \|q_N\|_{L^2}^2 \sim \kappa^{2\beta} \|q\|_{E_{\sigma,\kappa}^{\sigma+\beta}}^2.$$

In order to integrate in $\varkappa$ one considers separately the cases $N \leq \kappa$ and $N > \kappa$, breaking the integral into the regions $[\kappa, N]$ and $[N, \infty)$ in the latter case. The estimate (2.5) is proved similarly. $\qquad\square$

Lemma 2.3. *For $0 \leq \sigma < \frac{1}{2}$ and $\varkappa \geq 1$ we have the estimates*

$$(2.6) \qquad \|fg\|_B \lesssim \|f\|_B \|g\|_B,$$

$$(2.7) \qquad \|f\|_B \lesssim \varkappa^{-\frac{1}{2}} \big[\varkappa \|f\|_{E_{2\sigma,\varkappa}^\sigma} + \|f'\|_{E_{2\sigma,\varkappa}^\sigma} \big],$$

$$(2.8) \qquad \|f\|_B \lesssim \|f\|_{L^\infty} + \|f'\|_{L^1}.$$

Proof. To verify (2.6), we employ the basic decomposition

$$(2.9) \qquad P_N(fg) = P_N \Big[f_{\sim N} g_{\lesssim N} + f_{\lesssim N} g_{\sim N} + \sum_{M > 2N} f_M g_M \Big].$$

From Bernstein's and Hölder's inequalities, we see that

$$\|(fg)_N\|_{L^2} \lesssim \|f_{\sim N}\|_{L^2} \|g_{\lesssim N}\|_{L^\infty} + \|f_{\lesssim N}\|_{L^\infty} \|g_{\sim N}\|_{L^2} + \sum_{M > 2N} N^{\frac{1}{2}} \|f_M\|_{L^2} \|g_M\|_{L^2}$$

$$\lesssim N^{-\frac{1}{2}} \|f\|_B \|g\|_B,$$

from which (2.6) then follows easily.

The estimate (2.7) follows from Bernstein's inequality:

$$\|f\|_B \lesssim \varkappa^\sigma \sum_{N \leq |\varkappa|} N^{\frac{1}{2}-\sigma} \|f_N\|_{E_{2\sigma,\varkappa}^\sigma} + \varkappa^{-\sigma} \sum_{N > |\varkappa|} N^{\sigma-\frac{1}{2}} \|f_N'\|_{E_{2\sigma,\varkappa}^\sigma} \lesssim \text{RHS}(2.7).$$

Turning now to (2.8), we recall that the convolution kernel associated to P_N is a Schwartz function which integrates to zero. Indeed, it can be written in the form $NK'(Nx)$ for some Schwartz function K. Integrating by parts, we find

$$N^{\frac{1}{2}} \|f_N\|_{L^2} \leq N^{\frac{1}{2}} \Big\| \int K\big(N(x-y)\big) f'(y) \, dy \Big\|_{L_x^2} \leq \|K\|_{L^2} \|f'\|_{L^1} \lesssim \|f'\|_{L^1}. \qquad \square$$

Lemma 2.4. *Let $0 \leq \sigma < \frac{1}{2}$, $0 \leq s < \sigma + \frac{1}{2}$, and $|\varkappa|, \kappa \geq 1$. Then we have the estimate*

$$(2.10) \qquad |\varkappa| \|f\|_{E_{s,\varkappa}^\sigma} + \|f'\|_{E_{s,\varkappa}^\sigma} \sim \|(2\varkappa - \partial) f\|_{E_{s,\varkappa}^\sigma}$$

and the product estimates

$$(2.11) \qquad \|fg\|_{E_{s,\kappa}^\sigma} \lesssim \|f\|_{E_{s,\kappa}^\sigma} \|g\|_B,$$

(2.12) $$\|fg\|_{E^\sigma_{s,\kappa}} \lesssim |\varkappa|^{-\frac{1}{2}} \|f\|_{E^\sigma_{s,\kappa}} \left[|\varkappa| \|g\|_{E^\sigma_{2\sigma,\varkappa}} + \|g'\|_{E^\sigma_{2\sigma,\varkappa}} \right].$$

In particular, if $\phi' \in \mathcal{S}$ we have the localization estimate

(2.13) $$\|\phi f\|_{E^\sigma_{s,\kappa}} \lesssim_\phi \|f\|_{E^\sigma_{s,\kappa}}.$$

Proof. The estimate (2.10) follows from

$$\|(2\varkappa - \partial)f\|^2_{E^\sigma_{s,\varkappa}} = 4|\varkappa|^2 \|f\|^2_{E^\sigma_{s,\varkappa}} + \|f'\|^2_{E^\sigma_{s,\varkappa}}.$$

For the product estimate (2.11), we have

$$\|fg\|^2_{E^\sigma_{s,\kappa}} \sim \sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \|P_N(fg)\|^2_{L^2}.$$

Decomposing as in (2.9) and using the Hölder and Bernstein inequalities, we estimate

$$\|fg\|^2_{E^\sigma_{s,\kappa}} \lesssim \sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \|f_N\|^2_{L^2} \|g_{\ll N}\|^2_{L^\infty}$$

$$+ \sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \left[\sum_{M \ll N} M^{\frac{1}{2}} \|f_M\|_{L^2} \right]^2 \|g_N\|^2_{L^2}$$

$$+ \sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \left[\sum_{M > 2N} N^{\frac{1}{2}} \|f_M\|_{L^2} \|g_{\sim M}\|_{L^2} \right]^2.$$

The first summand is easily seen to be acceptable. To estimate the second summand, we first sum in N and then apply Schur's test:

$$\sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \left[\sum_{M \ll N} M^{\frac{1}{2}} \|f_M\|_{L^2} \right]^2 \|g_N\|^2_{L^2}$$

$$\lesssim \|g\|^2_B \sum_{M_1 \le M_2 \ll N} \frac{\kappa^{2(s-\sigma)} N^{2\sigma} M_1^{\frac{1}{2}} M_2^{\frac{1}{2}}}{N(\kappa+N)^{2s}} \|f_{M_1}\|_{L^2} \|f_{M_2}\|_{L^2}$$

$$\lesssim \|g\|^2_B \sum_{M_1 \le M_2} \frac{M_1^{\frac{1}{2}-\sigma} (\kappa+M_1)^s}{M_2^{\frac{1}{2}-\sigma} (\kappa+M_2)^s} \|f_{M_1}\|_{E^\sigma_{s,\kappa}} \|f_{M_2}\|_{E^\sigma_{s,\kappa}}$$

$$\lesssim \|f\|^2_{E^\sigma_{s,\kappa}} \|g\|^2_B.$$

Arguing similarly, we estimate the remaining summand by

$$\sum_N \frac{\kappa^{2(s-\sigma)} N^{2\sigma}}{(\kappa+N)^{2s}} \left[\sum_{M > 2N} N^{\frac{1}{2}} \|f_M\|_{L^2} \|g_{\sim M}\|_{L^2} \right]^2$$

$$\lesssim \|g\|^2_B \sum_{N < M_1 \le M_2} \frac{\kappa^{2(s-\sigma)} N^{2\sigma+1} M_1^{-\frac{1}{2}} M_2^{-\frac{1}{2}}}{(\kappa+N)^{2s}} \|f_{M_1}\|_{L^2} \|f_{M_2}\|_{L^2}$$

$$\lesssim \|g\|^2_B \sum_{M_1 \le M_2} \frac{M_1^{\sigma+\frac{1}{2}} (\kappa+M_2)^s}{M_2^{\sigma+\frac{1}{2}} (\kappa+M_1)^s} \|f_{M_1}\|_{E^\sigma_{s,\kappa}} \|f_{M_2}\|_{E^\sigma_{s,\kappa}}$$

$$\lesssim \|f\|^2_{E^\sigma_{s,\kappa}} \|g\|^2_B.$$

The estimate (2.12) follows as a corollary of (2.11) and (2.7). The localization estimate (2.13) follows from (2.11) and (2.8). $\qquad \square$

2.2. **Operator estimates.** We begin with the basic Hilbert-Schmidt bound for the operators $\Lambda(q)$ and $\Gamma(q)$ introduced in (1.9). This estimate appeared already in [30, Lemma 4.1]:

Lemma 2.5 ([30]). *For $q \in L^2$ and $\kappa > 0$ we have*

$$(2.14) \qquad \|\Lambda\|_{\mathfrak{J}_2}^2 = \|\Gamma\|_{\mathfrak{J}_2}^2 \approx \int_{\mathbb{R}} \log(4 + \tfrac{\xi^2}{\kappa^2}) \frac{|\hat{q}(\xi)|^2}{\sqrt{4\kappa^2 + \xi^2}} \, d\xi \lesssim \kappa^{-1} \|q\|_{L^2}^2,$$

Using this lemma as our basic tool, we obtain the following basic estimates when $q \in L^2$ is frequency localized.

Lemma 2.6 (Operator estimates). *For $|\kappa| \geq 1$, $0 \leq \sigma < \frac{1}{2}$, and $0 \leq s < \sigma + \frac{1}{2}$ we have*

$$(2.15) \qquad \|\Lambda(q_N)\|_{\mathfrak{J}_2} = \|\Gamma(q_N)\|_{\mathfrak{J}_2} \approx \sqrt{\tfrac{1}{|\kappa|+N} \log\left(4 + \tfrac{N^2}{\kappa^2}\right)} \|q_N\|_{L^2},$$

$$(2.16) \qquad \|\Lambda(q_N)\|_{\mathrm{op}} = \|\Gamma(q_N)\|_{\mathrm{op}} \lesssim \tfrac{\sqrt{N}}{|\kappa|+N} \sqrt{\log\left(4 + \tfrac{N^2}{\kappa^2}\right)} \|q_N\|_{L^2}$$

$$(2.17) \qquad \sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \tfrac{N}{|\kappa|+N} \log^{\frac{3}{2}}\left(4 + \tfrac{N^2}{\kappa^2}\right) \sup_{M \in 2^{\mathbb{Z}}} M^{-\frac{1}{2}} \|f_M\|_{L^2},$$

$$(2.18) \qquad \sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \tfrac{|\kappa|^{-\frac{1}{2}} N^{\frac{1}{2}-\sigma}}{(|\kappa|+N)^{\frac{1}{2}-\sigma}} \|f\|_{E_{s,\kappa}^\sigma},$$

$$(2.19) \qquad \sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \tfrac{N^{\frac{1}{2}-\sigma}(1+N)^s}{|\kappa|^{\frac{1}{2}+\sigma-s}(|\kappa|+N)^{\frac{1}{2}+s-\sigma}} \|f\|_{E_s^\sigma}.$$

Proof. The estimate (2.15) follows immediately from Lemma 2.5. The estimate (2.16) follows from (2.15) for $N > \kappa$ and the Bernstein inequality:

$$(2.20) \qquad \|\Lambda(q_N)\|_{\mathrm{op}} \lesssim \tfrac{1}{|\kappa|} \|q_N\|_{L^\infty} \lesssim \tfrac{\sqrt{N}}{|\kappa|} \|q_N\|_{L^2},$$

for $N \leq \kappa$.

Claim (2.17) follows from (2.16):

$$\sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \sum_{M \leq N} \tfrac{M}{|\kappa|+M} \log^{\frac{1}{2}}\left(4 + \tfrac{M^2}{\kappa^2}\right) M^{-\frac{1}{2}} \|f_M\|_{L^2}$$

$$\lesssim \tfrac{N}{|\kappa|+N} \log^{\frac{3}{2}}\left(4 + \tfrac{N^2}{\kappa^2}\right) \sup_{M \in 2^{\mathbb{Z}}} M^{-\frac{1}{2}} \|f_M\|_{L^2},$$

as does (2.18):

$$\sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \sum_{M \leq N} \tfrac{M^{\frac{1}{2}}}{|\kappa|+M} \log^{\frac{1}{2}}\left(4 + \tfrac{M^2}{\kappa^2}\right) \|f_M\|_{L^2} \lesssim \tfrac{|\kappa|^{-\frac{1}{2}} N^{\frac{1}{2}-\sigma}}{(|\kappa|+N)^{\frac{1}{2}-\sigma}} \|f\|_{E_{s,\kappa}^\sigma}$$

and (2.19):

$$\sum_{M \leq N} \|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \sum_{M \leq N} \tfrac{M^{\frac{1}{2}-\sigma}(1+M)^s}{|\kappa|+M} \log^{\frac{1}{2}}\left(4 + \tfrac{M^2}{\kappa^2}\right) \|f_M\|_{E_s^\sigma} \lesssim \mathrm{RHS}(2.19). \qquad \square$$

Corollary 2.7. *For $q \in L^2$ and $0 \leq \sigma < \frac{1}{2}$,*

$$(2.21) \qquad \sqrt{\kappa} \big\|\Lambda(q)\big\|_{\mathrm{op}} \lesssim \|q\|_{E_{\sigma,\kappa}^\sigma} \quad \text{uniformly for } \kappa \geq 1.$$

In particular, if Q is a bounded and equicontinuous subset of L^2,

$$\lim_{\kappa \to \infty} \sup_{q \in Q} \sqrt{\kappa} \|\Lambda(q)\|_{\mathrm{op}} = 0. \tag{2.22}$$

Proof. Using (2.18) and Lemma 2.5, we may bound

$$\sqrt{\kappa}\|\Lambda(q)\|_{\mathrm{op}} \lesssim \sqrt{\kappa} \sum_{M \leq \kappa} \|\Lambda(q_M)\|_{\mathrm{op}} + \sqrt{\kappa}\,\|\Lambda(q_{>\kappa})\|_{\mathrm{op}}$$

$$\lesssim \|q\|_{E^\sigma_{\sigma,\kappa}} + \|q_{>\kappa}\|_{L^2} \lesssim \|q\|_{E^\sigma_{\sigma,\kappa}},$$

which settles (2.21). Lemma 2.1 then yields (2.22). $\qquad\square$

Lemma 2.8. *For $0 \leq \sigma < \frac{1}{2}$ and $\kappa \geq 1$ we have*

$$\|\Lambda(\psi_\mu^3 q; \kappa)\|_{\mathfrak{I}_8}^8 \lesssim \kappa^{-5} \left\| \frac{\psi_\mu^3 q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E^\sigma_\sigma}^6. \tag{2.23}$$

Proof. Decomposing into Littlewood–Paley pieces, applying (2.19) and (2.13) at low frequency, and (2.15) at high frequency, we have

$$\mathrm{LHS}(2.23) = \left| \mathrm{tr}\left\{ \Lambda(\psi_\mu^3 q; \kappa)^8 \right\} \right|$$

$$\lesssim \sum_{N_1 \sim N_2 \geq \cdots \geq N_8} \left\|\Lambda\big(P_{N_1}(\psi_\mu^3 q)\big)\right\|_{\mathfrak{I}_2} \left\|\Lambda\big(P_{N_2}(\psi_\mu^3 q)\big)\right\|_{\mathfrak{I}_2} \prod_{j=3}^8 \left\|\Lambda\big(P_{N_j}(\psi_\mu^3 q)\big)\right\|_{\mathrm{op}}$$

$$\lesssim \sum_{N_1 \sim N_2} \frac{N_1^{3-6\sigma}(1+N_1)^{6\sigma}}{\kappa^3(1+N_1)^3(\kappa+N_1)^2} \log\left(4 + \frac{N_1^2}{\kappa^2}\right) \left\| \frac{P_{N_1}(\psi_\mu^3 q)}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}} \left\| \frac{P_{N_2}(\psi_\mu^3 q)}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}} \|q\|_{E^\sigma_\sigma}^6$$

$$\lesssim \mathrm{RHS}(2.23).$$

The fact that $N_1 \sim N_2$ must hold is most evident by computing the trace (which is unitarily invariant) in Fourier variables. $\qquad\square$

2.3. **Local smoothing spaces.**

To control the local smoothing property, for $s, h \in \mathbb{R}$ and $|\kappa| \geq 1$ we define

$$\|q\|_{F^s_\kappa(h)}^2 := \int \left\| \frac{\psi_\mu^{12} q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{s+1}}^2 e^{-\frac{1}{200}|h-\mu|} \, d\mu, \tag{2.24}$$

$$\|q\|_{X^s_\kappa}^2 := \sup_{h \in \mathbb{R}} \int_{-1}^1 \|q(t)\|_{F^s_\kappa(h)}^2 \, dt. \tag{2.25}$$

As before, when $\kappa = 1$ we denote $F^s(h) = F^s_1(h)$ and $X^s = X^s_1$.

We note that as a consequence of Lemma 2.10 below, we obtain an alternative characterization of X^s_κ that is closer to that used in [18]; see Remark 2.11. The additional complexity apparent in (2.24) is necessitated by the scaling criticality of the problem.

Multiplicative commutators are an essential tool for repositioning the spatial localization factors within the paraproducts appearing in our analysis of local smoothing estimates. The next lemma, which extends [18, Lemma 2.8], is our basic workhorse in this task:

Lemma 2.9. *For $|\varkappa|, |\kappa| \geq 1$, $s, \sigma \in \mathbb{R}$, $1 < p < \infty$, $r \in \mathbb{Z}$, and integer $|\ell| \leq 24$, all fixed, we have the following uniformly for $\mu \in \mathbb{R}$ and $q \in \mathcal{S}(\mathbb{R})$:*

$$\|(2 \pm \partial)^s(2\kappa + \partial)^\sigma \psi_\mu^\ell (\varkappa - \partial)^r q\|_{L^p} \sim \|(2 \pm \partial)^s(2\kappa + \partial)^\sigma(\varkappa - \partial)^r \psi_\mu^\ell q\|_{L^p}. \tag{2.26}$$

If both $s, \sigma \in \mathbb{Z}$ then (2.26) also holds for $p \in \{1, \infty\}$.

Proof. The Mikhlin multiplier theorem shows that the choice of $\pm$ signs is immaterial and we shall restrict attention to the $+$ case.

Both inequalities can be treated simultaneously through a slightly larger family of estimates involving two parameters $\theta, \nu \in \mathbb{Z}$. Specifically, adopting the notation

$$W = (2 + \partial)^s (2\kappa + \partial)^\sigma (\varkappa - \partial)^\nu$$

it suffices to show that for each pair $\theta, \nu \in \mathbb{Z}$,

$$\|W\psi^\ell(\varkappa - \partial)^\theta \psi^{-\ell}(\varkappa - \partial)^{-\theta} W^{-1} q\|_{L^p} \lesssim \|q\|_{L^p}.$$

(In fact, just the two cases $(\theta, \nu) = (r, 0)$ and $(\theta, \nu) = (-r, r)$ are truly needed.)

When $1 < p < \infty$, complex interpolation allows us to restrict attention to the case where $s, \sigma \in \mathbb{Z}$, which we do in what follows.

The next step is to perform additive commutations, moving each positive power of a differential operator toward its inverse, one factor at a time. Proceeding in this fashion until all positive powers of said differential operators are exhausted leaves a very concrete (but combinatorially very messy) finite linear combination of products of operators from the following list:

$$\psi^{-\ell}(\partial^m \psi^\ell), \ \psi^\ell(\partial^m \psi^{-\ell}), \ (2 + \partial)^{-1}, \ (2\kappa + \partial)^{-1}, \ (\varkappa - \partial)^{-1}, \ \text{and } \psi^\ell(\varkappa - \partial)^{-1}\psi^{-\ell},$$

where m is any integer satisfying $0 \le m \le |\sigma| + |s| + |\nu| + |\theta|$. In this way, we see that the proof will be complete if we can show that any operator on the list is L^p bounded for every $1 \le p \le \infty$.

Boundedness of the first two operators in the list is trivial given our choice of ψ. Boundedness of the remaining operators can be deduced from their explicit kernels. Indeed, $\psi^\ell(\varkappa - \partial)^{-1}\psi^{-\ell}$ has kernel

$$K(x, y) = \psi(x)^\ell \psi(y)^{-\ell} e^{\varkappa(x-y)} \mathbb{1}_{x<y}, \quad \text{which satisfies} \quad |K(x,y)| \lesssim_\ell e^{-\frac{1}{2}\varkappa|x-y|}.$$

Thus L^p boundedness follows from Schur's test. $\qquad \square$

As we are dealing with a nonlinear equation, one needs to understand how to estimate products in our local smoothing spaces. Due to the low regularity of the objects we are treating in this paper, each term in the product must itself satisfy local smoothing estimates in order for the product to be bounded. This dictates the structure of our basic product estimates below.

Lemma 2.10. *If* $0 \le \sigma < \frac{1}{2}$, $|\varkappa|, \kappa \ge 1$, $h \in \mathbb{R}$, $1 \le \ell \le 12$ *is an integer, and* $\phi' \in \mathcal{S}$ *then*

$$(2.27) \qquad |\varkappa|\|q\|_{F_\kappa^{\frac{1}{2}}(h)} + \|q\|_{F_\kappa^{\frac{3}{2}}(h)} \sim \|(2\varkappa - \partial)q\|_{F_\kappa^{\frac{1}{2}}(h)}$$

and we have the product estimates

$$(2.28)$$
$$\|fg\|_{F_\kappa^{\frac{1}{2}}(h)} \lesssim |\varkappa|^{-\frac{1}{2}}\left[\|f\|_{F_\kappa^{\frac{1}{2}}(h)}\|(2\varkappa - \partial)g\|_{E_{2\sigma,\varkappa}} + \|f\|_{E_{2\sigma,\varkappa}}\|(2\varkappa - \partial)g\|_{F_\kappa^{\frac{1}{2}}(h)}\right],$$
$$(2.29)$$
$$\|fg\|_{F_\kappa^{\frac{3}{2}}(h)} \lesssim |\varkappa|^{-\frac{1}{2}}\left[\|f\|_{F_\kappa^{\frac{3}{2}}(h)}\|(2\varkappa - \partial)g\|_{E_{2\sigma,\varkappa}} + \|(2\varkappa - \partial)f\|_{E_{2\sigma,\varkappa}}\|g\|_{F_\kappa^{\frac{3}{2}}(h)}\right].$$

We also have the localization estimates

$$(2.30) \qquad \int \Big\|\frac{\psi_\mu^\ell q}{\sqrt{4\kappa^2 - \partial^2}}\Big\|_{H^{\frac{3}{2}}}^2 e^{-\frac{1}{200}|h-\mu|} \, d\mu \lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2,$$

$$(2.31) \qquad \|\phi q\|_{F_\kappa^{\frac{1}{2}}(h)} \lesssim_\phi \|q\|_{F_\kappa^{\frac{1}{2}}(h)}.$$

Proof. For (2.27), we first use Lemma 2.9 to obtain

$$\|(2\varkappa - \partial)q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \sim \int \big\|(2\varkappa - \partial)\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}^2 \, e^{-\frac{1}{200}|h-\mu|}\, d\mu$$

$$\sim 4|\varkappa|^2 \int \big\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}^2 \, e^{-\frac{1}{200}|h-\mu|}\, d\mu + \int \big\|\big(\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big)'\big\|_{H^{\frac{3}{2}}}^2 \, e^{-\frac{1}{200}|h-\mu|}\, d\mu$$

$$\sim |\varkappa|^2\|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 + \|q\|_{F_\kappa^{\frac{3}{2}}(h)}.$$

Using space-translation invariance, it suffices to prove (2.30) for $h = 0$. If $\ell = 12$, the claim follows from the definition. Otherwise, define $T_{\mu,\nu}\colon L^2 \to L^2$ to have integral kernel

$$\frac{(4+\xi^2)^{\frac{3}{4}}}{(4\kappa^2+\xi^2)^{\frac{1}{2}}}\widehat{\psi_\mu^\ell\psi_\nu^{12}}(\xi-\eta)\frac{(4\kappa^2+\eta^2)^{\frac{1}{2}}}{(4+\eta^2)^{\frac{3}{4}}},$$

and apply Schur's test to bound

$$\|T_{\mu,\nu}\|_{\mathrm{op}} \lesssim \|\langle\xi\rangle^{\frac{5}{2}}\widehat{\psi_\mu^\ell\psi_\nu^{12}}(\xi)\|_{L_\xi^1} \lesssim \|\psi_\mu^\ell\psi_\nu^{12}\|_{H^{\frac{7}{2}}} \lesssim e^{-\frac{1}{200}|\mu-\nu|}.$$

We then apply (2.1) to obtain

$$\int\big\|\tfrac{\psi_\mu^\ell q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}^2 \, e^{-\frac{1}{200}|\mu|}\, d\mu$$

$$\lesssim \iiint \big\|\tfrac{\psi_\mu^\ell\psi_{\nu_1}^{24}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}\big\|\tfrac{\psi_\mu^\ell\psi_{\nu_2}^{24}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}} \, e^{-\frac{1}{200}|\mu|}\, d\nu_1\, d\nu_2\, d\mu$$

$$\lesssim \iiint \|T_{\mu,\nu_1}\|_{\mathrm{op}}\|T_{\mu,\nu_2}\|_{\mathrm{op}}\big\|\tfrac{\psi_{\nu_1}^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}\big\|\tfrac{\psi_{\nu_2}^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}} \, e^{-\frac{1}{200}|\mu|}\, d\nu_1\, d\nu_2\, d\mu$$

$$\lesssim \iiint e^{-\frac{|\nu_1|+|\nu_2|}{400}} e^{-\frac{|\mu-\nu_1|+|\mu-\nu_2|}{400}}\big\|\tfrac{\psi_{\nu_1}^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}}\big\|\tfrac{\psi_{\nu_2}^{12}q}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{\frac{3}{2}}} \, d\nu_1\, d\nu_2\, d\mu$$

$$\lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(0)}^2.$$

In the last step we first integrated in μ and then used L^2 boundedness of the resulting convolution operator.

Turning next to the product estimates (2.28) and (2.29), we take $s \in \{\frac{1}{2}, \frac{3}{2}\}$, $\mu \in \mathbb{R}$, and use (2.9) to estimate

$$\big\|\tfrac{1}{\sqrt{4\kappa^2-\partial^2}}(\psi_\mu^{12}fg)\big\|_{H^{s+1}}^2$$

$$\lesssim \sum_N \frac{(1+N)^{2s+2}}{(\kappa+N)^2}\|P_{\sim N}(\psi_\mu^6 f)\|_{L^2}^2\|P_{\lesssim N}(\psi_\mu^6 g)\|_{L^\infty}^2$$

$$+ \sum_N \frac{(1+N)^{2s+2}}{(\kappa+N)^2}\|P_{\lesssim N}(\psi_\mu^6 f)\|_{L^\infty}^2\|P_{\sim N}(\psi_\mu^6 g)\|_{L^2}^2$$

$$+ \sum_N \frac{(1+N)^{2s+2}}{(\kappa+N)^2}\Big[\sum_{M>2N} N^{\frac{1}{2}}\|P_M(\psi_\mu^6 f)\|_{L^2}\|P_{\sim M}(\psi_\mu^6 g)\|_{L^2}\Big]^2.$$

Recalling (2.7), (2.10), and (2.13), we bound the first summand by

$$\sum_N \frac{(1+N)^{2s+2}}{(\kappa+N)^2}\|P_N(\psi_\mu^6 f)\|_{L^2}^2\|P_{\lesssim N}(\psi_\mu^6 g)\|_{L^\infty}^2$$

$$\lesssim |\varkappa|^{-1}\big\|\tfrac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}}\big\|_{H^{s+1}}^2\Big[|\varkappa|\|\psi_\mu^6 g\|_{E_{2\sigma,\varkappa}} + \|(\psi_\mu^6 g)'\|_{E_{2\sigma,\varkappa}}\Big]^2$$

$$\lesssim |\varkappa|^{-1} \big\| \tfrac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{s+1}}^2 \big\| (2\varkappa - \partial) g \big\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

For (2.29), we bound the second summand in a symmetric fashion, reversing the roles of f, g. For (2.28), we instead apply Schur's test with (2.10), (2.13), and (2.26) to bound

$$\sum_N \frac{(1+N)^3}{(\kappa+N)^2} \Big[\sum_{M \lesssim N} M^{\frac{1}{2}} \| P_M(\psi_\mu^6 f) \|_{L^2} \Big]^2 \| P_N(\psi_\mu^6 g) \|_{L^2}^2$$

$$\lesssim \big\| \tfrac{(2\varkappa-\partial)(\psi_\mu^6 g)}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{\frac{3}{2}}}^2 \sum_{M_1 \le M_2} \frac{(\varkappa+M_1)^{2\sigma} M_1^{\frac{1}{2}-\sigma} M_2^{\frac{1}{2}-\sigma}}{|\varkappa|^{2\sigma}(\varkappa+M_2)^{2-2\sigma}} \big\| P_{M_1}(\psi_\mu^6 f) \big\|_{E_{2\sigma,\varkappa}^\sigma} \big\| P_{M_2}(\psi_\mu^6 f) \big\|_{E_{2\sigma,\varkappa}^\sigma}$$

$$\lesssim |\varkappa|^{-1} \big\| \tfrac{\psi_\mu^6 (2\varkappa-\partial) g}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{\frac{3}{2}}}^2 \| f \|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

The third summand is again bounded using Schur's test, (2.7), (2.10), and (2.13):

$$\sum_N \frac{(1+N)^{2s+2}}{(\kappa+N)^2} \Big[\sum_{M>2N} N^{\frac{1}{2}} \| P_M(\psi_\mu^6 f) \|_{L^2} \big\| P_{\sim M} g \big\|_{L^2} \Big]^2$$

$$\lesssim \| g \|_B^2 \sum_{M_1 \le M_2} \frac{(1+M_1)^{s+1}(\kappa+M_2) M_1^{\frac{1}{2}}}{(1+M_2)^{s+1}(\kappa+M_1) M_2^{\frac{1}{2}}} \big\| \tfrac{P_{M_1}(\psi_\mu^6 f)}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{s+1}} \big\| \tfrac{P_{M_2}(\psi_\mu^6 f)}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{s+1}}$$

$$\lesssim |\varkappa|^{-1} \big\| \tfrac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{s+1}}^2 \| (2\varkappa-\partial) g \|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

The estimates (2.28), (2.29) then follow from (2.30).

Applying Schur's test in Fourier variables, we find

$$\sup_\mu \Big\| \tfrac{\langle\partial\rangle^{3/2}}{\sqrt{4\kappa^2-\partial^2}} \phi \psi_\mu^6 \tfrac{\sqrt{4\kappa^2-\partial^2}}{\langle\partial\rangle^{3/2}} \Big\|_{\mathrm{op}} \lesssim_\phi 1.$$

In this way, we see that

$$\big\| \tfrac{\psi_\mu^{12} \phi q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{\frac{3}{2}}} \lesssim_\phi \big\| \tfrac{\psi_\mu^6 q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{H^{\frac{3}{2}}},$$

and consequently, (2.31) follows from (2.30). □

Remark 2.11. As a consequence of the proof of Lemma 2.10, we note that

$$(2.32) \qquad \| q \|_{X_\kappa^{\frac{1}{2}}}^2 \sim \sup_{\mu \in \mathbb{R}} \big\| \tfrac{\psi_\mu^{12} q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{L_t^2 H^{\frac{3}{2}}}^2.$$

Indeed, the inequality

$$\| q \|_{X_\kappa^{\frac{1}{2}}}^2 \lesssim \sup_{\mu \in \mathbb{R}} \big\| \tfrac{\psi_\mu^{12} q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{L_t^2 H^{\frac{3}{2}}}^2$$

follows immediately from the definition. For the converse inequality, we argue as in (2.30): in view of (2.1),

$$\big\| \tfrac{\psi_\mu^{12} q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{L_t^2 H^{\frac{3}{2}}}^2 \lesssim \iint \| T_{\mu,\nu_1} \|_{\mathrm{op}} \| T_{\mu,\nu_2} \|_{\mathrm{op}} \big\| \tfrac{\psi_{\nu_1}^{12} q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{L_t^2 H^{\frac{3}{2}}} \big\| \tfrac{\psi_{\nu_2}^{12} q}{\sqrt{4\kappa^2-\partial^2}} \big\|_{L_t^2 H^{\frac{3}{2}}} \, d\nu_1 \, d\nu_2$$

$$\lesssim \| q \|_{L_t^2 F_\kappa^{\frac{1}{2}}(h)}^2.$$

3. Green's functions and microscopic conservation laws

By the Green's function, we mean the integral kernel associated to the inverse of the Lax operator presented in (1.12). It is not a given that this operator is invertible; we will rely on the subtle interplay between the spectral parameter κ and the equicontinuity properties of q. This same issue was discussed in the introduction in connection with making sense of $A(\kappa; q)$. Indeed, it formed the central rationale for introducing the notion of a δ-good subset of $L^2(\mathbb{R})$; see Definition 1.8. Let us begin our discussion by revisiting the construction of $A(\kappa; q)$.

We subsequently take $0 < \sigma < \frac{1}{2}$ to be fixed. If Q is a δ-good subset of $L^2(\mathbb{R})$, then Corollary 2.7 shows that

$$(3.1) \qquad |\kappa|^{\frac{1}{2}} \big\| \Lambda(q; \kappa) \big\|_{\mathrm{op}} \lesssim \|q\|_{E_\sigma^\sigma} \leq \delta \quad \text{uniformly for } |\kappa| \geq 1 \text{ and } q \in Q.$$

As shown in [27, Lemma 5.1], it follows that if δ is sufficiently small then

$$A(\kappa; q) = -\operatorname{sgn}(\kappa) \log \det \big[1 - i\kappa(\kappa - \partial)^{-1} q (\kappa + \partial)^{-1} \bar{q} \big]$$

$$(3.2) \qquad\qquad = \operatorname{sgn}(\kappa) \sum_{\ell \geq 1} \tfrac{1}{\ell} \operatorname{tr}\big\{ (i\kappa \Lambda \Gamma)^\ell \big\}$$

defines a real-analytic function of κ and q. Moreover, the domain of this function includes all $|\kappa| \geq 1$ and an L^2-neigborhood of Q.

It is important to define $A(\kappa; q)$ in such a neighborhood of Q (rather than just on Q) to ensure that the functional derivatives are well defined. We find that

$$(3.3) \quad \int \tfrac{\delta A}{\delta q} f + \tfrac{\delta A}{\delta \bar{q}} \bar{f} \, dx = \operatorname{sgn}(\kappa) \sum_{m \geq 0} \operatorname{tr}\Big\{ \big(i\kappa \Lambda(q) \Gamma(q) \big)^m i\kappa [\Lambda(f)\Gamma(q) + \Lambda(q)\Gamma(f)] \Big\}.$$

As Lemma 2.5 and (3.1) show, this series defines a bounded linear functional on $f \in L^2$ and correspondingly the functional derivatives exist as L^2 functions.

Duality also gives an efficient way to introduce the functions $\gamma(\kappa; q)$, $g_{12}(\kappa; q)$, and $g_{21}(\kappa; q)$ that will be of central importance in what follows: for $a, b, c \in L^2$,

$$(3.4) \qquad \int g_{21} b + g_{12} c + \gamma a \, dx = \operatorname{sgn}(\kappa) \operatorname{tr}\Big\{ \big[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix} \big] \big[L^{-1} - L_0^{-1} \big] \Big\}.$$

Here L_0 denotes the Lax operator (1.12) with $q \equiv 0$.

Lemma 2.5 and (3.1) guarantee that a Neumann expansion of the right-hand side of (3.4) yields a convergent series for all $a, b, c \in L^2(\mathbb{R})$. On comparing these series with those of (3.3), we find

$$(3.5) \qquad\qquad \tfrac{\delta A}{\delta \bar{q}} = i\sqrt{\kappa}\, g_{12} \quad \text{and} \quad \tfrac{\delta A}{\delta q} = -\sqrt{\kappa}\, g_{21}.$$

It is evident from (3.4) that g_{12}, g_{21}, and γ are closely connected with the matrix Green's function evaluated on the diagonal (i.e. at the coincidence of the two spatial points). For the continuity needed to make sense of this directly, see see [18, Prop. 3.1].

For simplicity of exposition, the discussion above only constructed g_{12}, g_{21}, and γ as L^2 functions. By estimating more carefully (as was done in [27]), one finds that the series defining g_{12}, g_{21}, and γ converge in H^1 to real analytic functions of q; moreover, these functions are Schwartz whenever $q \in \mathcal{S}$.

Direct computations also reveal certain basic identities among these functions; see [18] or [47]. Concretely, we have

$$(3.6) \qquad g_{12}(\kappa) = -\overline{g_{21}(-\kappa)}, \quad \gamma(\kappa) = \overline{\gamma(-\kappa)}, \quad \text{and} \quad A(\kappa) = -\overline{A(-\kappa)}$$

as well as

$$(3.7) \qquad g'_{12} = 2\kappa g_{12} - \kappa^{\frac{1}{2}} q(\gamma + 1)$$

$$(3.8) \qquad g'_{21} = -2\kappa g_{21} - i\kappa^{\frac{1}{2}} \bar{q}(\gamma + 1)$$

$$(3.9) \qquad \gamma' = 2\kappa^{\frac{1}{2}}(qg_{21} + i\bar{q}g_{12}).$$

Lastly, we have the quadratic identity

$$(3.10) \qquad 2g_{12}g_{21} + \tfrac{1}{2}\gamma^2 + \gamma = 0,$$

which can be proved by differentiating the left-hand side and applying (3.7)–(3.9).

Using these relations, we may write

$$(3.11) \quad g_{12} = \frac{\sqrt{\kappa}}{2\kappa-\partial}\big[q(\gamma+1)\big], \qquad g_{21} = \frac{-i\sqrt{\kappa}}{2\kappa+\partial}\big[\bar{q}(\gamma+1)\big], \qquad \gamma = -\frac{4g_{12}g_{21}}{2+\gamma},$$

$$\frac{g_{12}}{2+\gamma} = \frac{\sqrt{\kappa}}{2(2\kappa-\partial)}\big[q + 4i\bar{q}\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big], \quad \text{and} \quad \frac{g_{21}}{2+\gamma} = \frac{-i\sqrt{\kappa}}{2(2\kappa+\partial)}\big[\bar{q} - 4iq\big(\tfrac{g_{21}}{2+\gamma}\big)^2\big].$$

While it is very elementary to check these last two identities, it is much less obvious that they are key to efficiently treating a lot of what follows.

To prove our theorems, we require bounds for these functions in the B, E, and F spaces introduced in Section 2. We start with the following:

Lemma 3.1. *For δ sufficiently small and $Q \subset \mathcal{S}$ a δ-good set, the following estimates hold uniformly for $q \in Q$ and $|\kappa| \geq 1$:*

$$(3.12) \qquad \|\gamma(\kappa;q)\|_B \lesssim \|q\|^2_{E^\sigma_{2\sigma,\kappa}} \leq \delta^2,$$

$$(3.13) \qquad \|g_{12}(\kappa;q)\|_B + \|g_{21}(\kappa;q)\|_B \lesssim \|q\|_{E^\sigma_{2\sigma,\kappa}},$$

$$(3.14) \qquad \big\|\tfrac{g_{12}(\kappa;q)}{2+\gamma(\kappa;q)}\big\|_B + \big\|\tfrac{g_{21}(\kappa;q)}{2+\gamma(\kappa;q)}\big\|_B \lesssim \|q\|_{E^\sigma_{2\sigma,\kappa}}.$$

Proof. To prove (3.12), we argue by duality. To this end, we test against functions f satisfying $\sup_{M \in 2^{\mathbb{Z}}} M^{-\frac{1}{2}}\|f_M\|_{L^2} \leq 1$ and employ

$$\langle f, \gamma(\kappa;q)\rangle = \mathrm{sgn}(\kappa)\sum_{\ell \geq 1}(i\kappa)^\ell \,\mathrm{tr}\Big\{\big[(\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\bar{q}\big]^\ell(\kappa-\partial)^{-1}\bar{f}\Big\}$$

$$+ \mathrm{sgn}(\kappa)\sum_{\ell \geq 1}(i\kappa)^\ell \,\mathrm{tr}\Big\{\big[(\kappa+\partial)^{-1}\bar{q}(\kappa-\partial)^{-1}q\big]^\ell(\kappa+\partial)^{-1}\bar{f}\Big\}.$$

Let us first observe that by (2.18) and (1.19), there exists $C \geq 1$ so that

$$\sup_{N \in 2^{\mathbb{Z}}} |\kappa|^{\frac{1}{2}}\sum_{M \leq N}\|\Lambda(q_M)\|_{\mathrm{op}} \leq C\delta,$$

uniformly in $|\kappa| \geq 1$. Note also that by Lemma 2.6 and our assumptions on f,

$$\sum_{M \leq N}\|\Lambda(f_M)\|_{\mathrm{op}} \lesssim \tfrac{N}{|\kappa|+N}\log^{\frac{3}{2}}\big(4 + \tfrac{N^2}{\kappa^2}\big) \quad \text{and} \quad \|\Lambda(f_N)\|_{\mathfrak{I}_2} \lesssim \sqrt{\tfrac{N}{|\kappa|+N}\log\big(4+\tfrac{N^2}{\kappa^2}\big)}.$$

Decomposing into Littlewood–Paley pieces, using that the two highest frequencies must be comparable, together with Lemma 2.6 and the preceding estimates, we find that

$$\Big|(i\kappa)^\ell \,\mathrm{tr}\Big\{\big[(\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\bar{q}\big]^\ell(\kappa-\partial)^{-1}\bar{f}\Big\}\Big|$$

$$\lesssim (C\delta)^{2(\ell-1)}\sum_{N_1 \sim N_2}\tfrac{|\kappa|N_1}{|\kappa|+N_1}\log^{\frac{3}{2}}\big(4+\tfrac{N_1^2}{\kappa^2}\big)\|\Lambda(q_{N_1})\|_{\mathfrak{I}_2}\|\Lambda(q_{N_2})\|_{\mathfrak{I}_2}$$

$$+ (C\delta)^{2(\ell-1)} \sum_{N_1 \sim N_2 \geq N_3} |\kappa| \|\Lambda(q_{N_1})\|_{\mathfrak{I}_2} \|\Lambda(f_{N_2})\|_{\mathfrak{I}_2} \|\Lambda(q_{N_3})\|_{\mathrm{op}}$$

$$\lesssim (C\delta)^{2(\ell-1)} \sum_{N_1 \sim N_2} \frac{|\kappa|N_1}{(|\kappa|+N_1)^2} \log^{\frac{5}{2}}\big(4 + \tfrac{N_1^2}{|\kappa|^2}\big) \|q_{N_1}\|_{L^2} \|q_{N_2}\|_{L^2}$$

$$+ (C\delta)^{2(\ell-1)} \sum_{N_1 \sim N_2} \frac{|\kappa|^{\frac{1}{2}} N_1^{1-\sigma}}{(|\kappa|+N_1)^{\frac{3}{2}-\sigma}} \log\big(4 + \tfrac{N_1^2}{|\kappa|^2}\big) \|q_{N_1}\|_{L^2} \|q\|_{E_{2\sigma,\kappa}^\sigma}$$

$$\lesssim (C\delta)^{2(\ell-1)} \|q\|_{E_{2\sigma,\kappa}^\sigma}^2.$$

Choosing δ sufficiently small, we may sum in $\ell \geq 1$ and so deduce (3.12).

Further reducing δ, if necessary, the estimates (3.12) and (2.6) allow us to sum the geometric series and so obtain

$$(3.15) \qquad \qquad \big\|\tfrac{1}{2+\gamma}\big\|_B \lesssim 1.$$

Regarding (3.13), we use (2.7), (2.10), (3.11), and (2.11) to obtain

$$\|g_{12}\|_B \lesssim |\kappa|^{-\frac{1}{2}} \|(2\kappa - \partial)g_{12}\|_{E_{2\sigma,\kappa}^\sigma} \lesssim \big[1 + \|\gamma\|_B\big] \|q\|_{E_{2\sigma,\kappa}^\sigma}.$$

The bound (3.13) then follows from (3.12) and (3.6).

The estimate (3.14) follows directly from (3.13), (3.15), and (2.6). $\qquad \square$

Lemma 3.2. *For δ sufficiently small and $Q \subset \mathcal{S}$ a δ-good set, the following estimates hold uniformly for $q \in Q$ and $|\varkappa|, \kappa \geq 1$:*

$$(3.16) \qquad |\varkappa| \|g_{12}(\varkappa)\|_{E_{s,\kappa}^\sigma} + \|g_{12}'(\varkappa)\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma},$$

$$(3.17) \qquad |\varkappa| \|\gamma(\varkappa)\|_{E_{s,\kappa}^\sigma} + \|\gamma'(\varkappa)\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma} \|q\|_{E_{2\sigma,\varkappa}^\sigma},$$

$$(3.18) \qquad |\varkappa| \big\|\tfrac{g_{12}(\varkappa)}{2+\gamma(\varkappa)}\big\|_{E_{s,\kappa}^\sigma} + \big\|\big(\tfrac{g_{12}}{2+\gamma}\big)'(\varkappa)\big\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma}.$$

Moreover, in view of (3.6), g_{21} also satisfies the estimates (3.16) and (3.18).

Proof. In view of (3.7), (2.11) and (3.12),

$$\|(2\varkappa - \partial)g_{12}\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q(1+\gamma)\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma} \big[1 + \|\gamma\|_B\big] \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma}.$$

The estimate (3.16) then follows from (2.10).

Recalling $-\gamma = \frac{4g_{12}g_{21}}{2+\gamma}$ from (3.11), the estimates (3.16) and (3.14) show that

$$|\varkappa| \|\gamma(\varkappa)\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa| \|g_{12}(\varkappa)\|_{E_{s,\kappa}^\sigma} \big\|\tfrac{g_{21}(\varkappa)}{2+\gamma(\varkappa)}\big\|_B \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma} \|q\|_{E_{2\sigma,\varkappa}^\sigma}.$$

To complete the proof of (3.17), we complement this with the estimate

$$\|\gamma'(\varkappa)\|_{E_{s,\kappa}^\sigma} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{2\sigma,\varkappa}^\sigma} \|q\|_{E_{s,\kappa}^\sigma},$$

for which we employed (3.9), (2.12), and (3.16).

Using (2.11), the fact that B is an algebra, and the estimates (3.14), (3.15), (3.16), (3.17), and (1.19), we obtain

$$\big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{E_{s,\kappa}^\sigma} \lesssim \big\|(2\varkappa - \partial)g_{12}\big\|_{E_{s,\kappa}^\sigma} \big\|\tfrac{1}{2+\gamma}\big\|_B + \|\gamma'\|_{E_{s,\kappa}^\sigma} \big\|\tfrac{g_{12}}{(2+\gamma)^2}\big\|_B$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma}\big(1 + \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2\big) \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{s,\kappa}^\sigma}.$$

The estimate (3.18) now follows from (2.10). $\qquad \square$

Lemma 3.3. *For δ sufficiently small and $Q \subset \mathcal{S}$ a δ-good set, the following estimates hold uniformly for $q \in Q$, $|\varkappa|, \kappa \geq 1$, and $h \in \mathbb{R}$:*

$$(3.19) \qquad |\varkappa| \|g_{12}(\varkappa)\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}(\varkappa)\|_{F_\kappa^{\frac{3}{2}}(h)} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}(h)},$$

$$(3.20) \qquad |\varkappa| \|\gamma(\varkappa)\|_{F_\kappa^{\frac{1}{2}}(h)} + \|\gamma(\varkappa)\|_{F_\kappa^{\frac{3}{2}}(h)} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}(h)} \|q\|_{E_{2\sigma,\varkappa}},$$

$$(3.21) \qquad |\varkappa| \big\| \tfrac{g_{12}(\varkappa)}{2+\gamma(\varkappa)} \big\|_{F_\kappa^{\frac{1}{2}}(h)} + \big\| \tfrac{g_{12}(\varkappa)}{2+\gamma(\varkappa)} \big\|_{F_\kappa^{\frac{3}{2}}(h)} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}.$$

Proof. Using the quadratic identity (3.10) together with (2.27), (2.28), and (2.29), followed by (3.16) and (3.17) we get

$$|\varkappa| \|\gamma(\varkappa)\|_{F_\kappa^{\frac{1}{2}}(h)} + \|\gamma(\varkappa)\|_{F_\kappa^{\frac{3}{2}}(h)}$$

$$\lesssim |\varkappa|^{-\frac{1}{2}} \|(2\varkappa - \partial)\gamma\|_{E_{2\sigma,\varkappa}} \|(2\varkappa - \partial)\gamma\|_{F_\kappa^{\frac{1}{2}}(h)}$$

$$+ |\varkappa|^{-\frac{1}{2}} \max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{E_{2\sigma,\kappa}} + \|g_{12}'\|_{E_{2\sigma,\kappa}} \Big] \max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}\|_{F_\kappa^{\frac{3}{2}}(h)} \Big]$$

$$\lesssim \|q\|_{E_{2\sigma,\varkappa}}^2 \|(2\varkappa - \partial)\gamma\|_{F_\kappa^{\frac{1}{2}}(h)} + \|q\|_{E_{2\sigma,\varkappa}} \max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}\|_{F_\kappa^{\frac{3}{2}}(h)} \Big].$$

Using (1.19) we deduce

$$(3.22) \qquad \|(2\varkappa - \partial)\gamma\|_{F_\kappa^{\frac{1}{2}}(h)} \lesssim \|q\|_{E_{2\sigma,\varkappa}} \max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}\|_{F_\kappa^{\frac{3}{2}}(h)} \Big]$$

and so (3.20) will follow from (3.19).

From (3.22) and (3.7) together with (2.27), (2.28), and (3.17), we obtain

$$\max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}\|_{F_\kappa^{\frac{3}{2}}(h)} \Big]$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q(1+\gamma)\|_{F_\kappa^{\frac{1}{2}}(h)}$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}(h)} + \|q\|_{F_\kappa^{\frac{1}{2}}(h)} \|(2\varkappa - \partial)\gamma\|_{E_{2\sigma,\varkappa}} + \|q\|_{E_{2\sigma,\varkappa}} \|(2\varkappa - \partial)\gamma\|_{F_\kappa^{\frac{1}{2}}(h)}$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}(h)} \big(1 + \|q\|_{E_{2\sigma,\varkappa}}^2\big) + \|q\|_{E_{2\sigma,\varkappa}}^2 \max_{\pm \varkappa} \Big[|\varkappa| \|g_{12}\|_{F_\kappa^{\frac{1}{2}}(h)} + \|g_{12}\|_{F_\kappa^{\frac{3}{2}}(h)} \Big].$$

The estimate (3.19) then follows from (1.19).

It remains to prove (3.21). As a preliminary, let us pause to observe that

$$(2\varkappa - \partial)\big(\tfrac{g_{12}}{2+\gamma}\big)^2 = \tfrac{2g_{12}}{2+\gamma} \cdot (\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}.$$

Using (2.11), (3.14), and (3.18), we get

$$(3.23) \qquad \big\|(2\varkappa - \partial)\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big\|_{E_{2\sigma,\varkappa}} \lesssim \big\|\tfrac{g_{12}}{2+\gamma}\big\|_B \big\|(\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{E_{2\sigma,\varkappa}} \lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{E_{2\sigma,\varkappa}}^2,$$

while by (2.28) and (3.18),

$$\big\|(2\varkappa - \partial)\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big\|_{F_\kappa^{\frac{1}{2}}} \lesssim |\varkappa|^{-\frac{1}{2}} \big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{E_{2\sigma,\varkappa}} \big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{F_\kappa^{\frac{1}{2}}}$$

$$(3.24) \qquad \qquad \lesssim \|q\|_{E_{2\sigma,\varkappa}} \big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{F_\kappa^{\frac{1}{2}}}.$$

Using (2.27) together with (3.11) followed by (2.28), (3.23), and (3.24) we obtain

$$\mathrm{LHS}(3.21) \lesssim \big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{F_\kappa^{\frac{1}{2}}}$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}} + \|q\|_{F_\kappa^{\frac{1}{2}}} \big\|(2\varkappa - \partial)\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big\|_{E_{2\sigma,\varkappa}} + \|q\|_{E_{2\sigma,\varkappa}} \big\|(2\varkappa - \partial)\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big\|_{F_\kappa^{\frac{1}{2}}}$$

$$\lesssim |\varkappa|^{\frac{1}{2}} \|q\|_{F_\kappa^{\frac{1}{2}}} \big[1 + \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2\big] + \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2 \big\|(2\varkappa - \partial)\tfrac{g_{12}}{2+\gamma}\big\|_{F_\kappa^{\frac{1}{2}}}.$$

Using (1.19), we may absorb the second term on the right-hand side above into the left-hand side, thus settling (3.21). □

A consequence of Lemmas 3.2 and 3.3 is the following:

Corollary 3.4. *Fix δ sufficiently small and let $Q \subset \mathcal{S}$ be a δ-good set. All functions f from the following list (and so finite linear combinations thereof):*

$$q, \quad \bar{q},$$

$$(3.25) \qquad \tfrac{2\varkappa \pm \partial}{\sqrt{\varkappa}} g_{12}(\varkappa), \quad \tfrac{2\varkappa \pm \partial}{\sqrt{\varkappa}} g_{21}(\varkappa), \quad \tfrac{2\varkappa \pm \partial}{\sqrt{\varkappa}} \tfrac{g_{12}(\varkappa)}{2+\gamma(\varkappa)}, \quad \tfrac{2\varkappa \pm \partial}{\sqrt{\varkappa}} \tfrac{g_{21}(\varkappa)}{2+\gamma(\varkappa)},$$

$$\tfrac{2\kappa \pm \partial}{\sqrt{\kappa}} g_{12}(\kappa), \quad \tfrac{2\kappa \pm \partial}{\sqrt{\kappa}} g_{21}(\kappa), \quad \tfrac{2\kappa \pm \partial}{\sqrt{\kappa}} \tfrac{g_{12}(\kappa)}{2+\gamma(\kappa)}, \quad \tfrac{2\kappa \pm \partial}{\sqrt{\kappa}} \tfrac{g_{21}(\kappa)}{2+\gamma(\kappa)},$$

satisfy the estimates

$$(3.26) \qquad \|f\|_{L^2} \lesssim \|q\|_{L^2}, \quad \|f\|_{E_{2\sigma,\varkappa}^\sigma} \lesssim \|q\|_{E_{2\sigma,\varkappa}^\sigma}, \quad \|f\|_{E_\sigma^\sigma} \lesssim \|q\|_{E_\sigma^\sigma},$$
$$\|f\|_{F^{\frac{1}{2}}(h)} \lesssim \|q\|_{F^{\frac{1}{2}}(h)}, \qquad \|f\|_{F_\kappa^{\frac{1}{2}}(h)} \lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(h)}$$

uniformly for $q \in Q$, $|\varkappa|, |\kappa| \geq 1$, and $h \in \mathbb{R}$.

As discussed in the introduction, multi-parameter local smoothing estimates are essential for our analysis. As we are in the non-perturbative regime, the only reasonable approach to proving such estimates is via monotonicity identities. In all examples that we are aware of, such monotonicity identities stem from a proper understanding of conserved densities and their corresponding currents. This line of thinking leads inevitably to the problem of finding a microscopic representation for the conservation of $A(\varkappa; q)$.

It is invariably easy to find microscopic representations for conserved quantities that are polynomial in the underlying field (and its derivatives), such as the mass or energy. However, even in these simple cases there is no universal algorithm for finding such microscopic laws; indeed, this is an ill-posed problem — the corresponding cohomology class does not have a unique representative.

When the conserved quantity in question is more complex, discovering a microscopic representation becomes truly challenging. All the more so when we need our representative to be coercive, if it is to be useful. This is the case for $A(\varkappa; q)$, which is defined as the logarithm of a Fredholm determinant or as an infinite series of traces; see (3.2). Structurally, each of these traces is a paraproduct in q. Nevertheless, just such a microscopic representation was presented in [47] based on the density

$$(3.27) \qquad \rho(\varkappa; q) := q \cdot \frac{i g_{21}(\varkappa; q)}{\sqrt{\varkappa}(2 + \gamma(\varkappa; q))} + \bar{q} \cdot \frac{g_{12}(\varkappa; q)}{\sqrt{\varkappa}(2 + \gamma(\varkappa; q))}.$$

In finding (3.27), the authors of [47] were very much guided by the analogous form for the AKNS–ZS hierarchy discovered in [18]. The analogue for KdV was found in [29], although this is of little assistance. Indeed, these few examples would lead one to believe that the answer will always be a rational function of matrix elements of the diagonal Green's function; this notion is refuted in [16].

Once these densities have been discovered, it is not fundamentally difficult to derive the corresponding current (though it may require considerable labour) because the time derivative of the Green's function may be deduced from the Lax

pair representation of the flow. For example, under the (DNLS) flow,

$$
\begin{aligned}
(3.28) \quad i\tfrac{d}{dt}g_{12} &= -\big[4\varkappa^2 - 2i\varkappa|q|^2\big]g_{12} + \big[2\varkappa^{\frac{3}{2}}q + i\varkappa^{\frac{1}{2}}|q|^2 q + \varkappa^{\frac{1}{2}}q'\big](1+\gamma) \\
&= -g_{12}'' - i\big(2|q|^2 g_{12}' + iq^2 g_{21}'\big).
\end{aligned}
$$

We include this to illustrate the point made in the introduction that (unlike for NLS and mKdV) this change of variables alone does not allow us to treat $q \in L^2$. Concretely, we note that for $q \in L^2$ one cannot make sense of the term $|q|^2 q$ appearing in the former expression as a distribution. This cannot be remedied by the other terms because they are distributions!

The following proposition gives the currents associated to the density (3.27) and was proved in [47]:

Proposition 3.5. *Let $Q \subset \mathcal{S}$ be δ-good for δ sufficiently small. Under the (DNLS) flow, we have that $\partial_t \rho(\varkappa) + \partial_x j_{\mathrm{DNLS}}(\varkappa) = 0$ for all $|\varkappa| \geq 1$, where*

$$
\begin{aligned}
j_{\mathrm{DNLS}} &= (|q|^2 - 2i\varkappa)\rho + \frac{q'g_{21} + i\bar{q}'g_{12}}{\sqrt{\varkappa}(2+\gamma)} + i|q|^2 \\
(3.29) \quad &= \tfrac{1}{\sqrt{\varkappa}}\tfrac{g_{21}}{2+\gamma} \cdot (2\varkappa + \partial + i|q|^2)q - \tfrac{i}{\sqrt{\varkappa}}\tfrac{g_{12}}{2+\gamma} \cdot (2\varkappa - \partial + i|q|^2)\bar{q} + i|q|^2.
\end{aligned}
$$

Likewise, for $\kappa \geq 1$, $\partial_t \rho(\varkappa) + \partial_x j_{\mathrm{diff}}(\varkappa, \kappa) = 0$ under the $H - H_\kappa$ flow. Here,

$$
\begin{aligned}
(3.30) \quad j_{\mathrm{diff}}(\varkappa, \kappa) &= \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa)\Big[(2\varkappa + \partial + i|q|^2)q - 2\kappa^{\frac{5}{2}}\big(\tfrac{g_{12}(\kappa)}{\kappa-\varkappa} - \tfrac{ig_{12}(-\kappa)}{\kappa+\varkappa}\big)\Big] \\
&\quad - \tfrac{i}{\sqrt{\varkappa}}\big(\tfrac{g_{12}}{2+\gamma}\big)(\varkappa)\Big[(2\varkappa - \partial + i|q|^2)\bar{q} - 2\kappa^{\frac{5}{2}}\big(\tfrac{g_{21}(-\kappa)}{\kappa+\varkappa} + \tfrac{ig_{21}(\kappa)}{\kappa-\varkappa}\big)\Big] \\
&\quad + i|q|^2 - \tfrac{\kappa^2}{\kappa-\varkappa}\gamma(\kappa) + \tfrac{\kappa^2}{\kappa+\varkappa}\gamma(-\kappa).
\end{aligned}
$$

In our application of the microscopic conservation laws to the proof of local smoothing, we require a detailed understanding of the structure of the lower order terms (in powers of q) of the currents. To this end, we adopt the notation from [18] of using square brackets to identify specific terms in power series expansions:

$$
g_{12}^{[2\ell+1]}(\kappa) = \mathrm{sgn}(\kappa)\kappa^{\frac{1}{2}}(i\kappa)^\ell \big\langle \delta_x, \big[(\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\bar{q}\big]^\ell (\kappa-\partial)^{-1}q(\kappa+\partial)^{-1}\delta_x \big\rangle
$$

so that

$$
(3.31) \quad g_{12}^{[\geq 2\ell+1]}(\kappa) = \sum_{m=\ell}^{\infty} g_{12}^{[2m+1]}(\kappa) \quad \text{and} \quad g_{12}(\kappa) = \sum_{\ell=0}^{\infty} g_{12}^{[2\ell+1]}(\kappa).
$$

The terms $g_{21}^{[2\ell+1]}$ and $\gamma^{[2\ell]}$ admit correspondingly simple definitions; however, we will also use this notation on more complicated analytic functions of q such as $\big(\tfrac{g_{12}}{2+\gamma}\big)^{[2\ell+1]}$, $\rho^{[2\ell]}$, and $j_{\mathrm{DNLS}}^{[2\ell]}$.

Using (3.11), we can derive the following explicit expressions:

$$
(3.32) \quad g_{12}^{[1]} = \tfrac{\sqrt{\kappa}q}{2\kappa-\partial}, \qquad g_{21}^{[1]} = \tfrac{-i\sqrt{\kappa}\bar{q}}{2\kappa+\partial}, \qquad \gamma^{[2]} = 2i\kappa\tfrac{q}{2\kappa-\partial} \cdot \tfrac{\bar{q}}{2\kappa+\partial},
$$

$$
(3.33) \quad g_{12}^{[3]} = \tfrac{2i\kappa^{\frac{3}{2}}}{2\kappa-\partial}\big[q \cdot \tfrac{q}{2\kappa-\partial} \cdot \tfrac{\bar{q}}{2\kappa+\partial}\big] \quad \text{and} \quad g_{21}^{[3]} = \tfrac{2\kappa^{\frac{3}{2}}}{2\kappa+\partial}\big[\bar{q} \cdot \tfrac{q}{2\kappa-\partial} \cdot \tfrac{\bar{q}}{2\kappa+\partial}\big],
$$

as well as

$$
(3.34) \quad \tfrac{1}{\sqrt{\kappa}}\big(\tfrac{g_{12}}{2+\gamma}\big)^{[\geq 3]} = \tfrac{2i}{2\kappa-\partial}\big[\bar{q}\big(\tfrac{g_{12}}{2+\gamma}\big)^2\big],
$$

$$
(3.35) \quad \tfrac{1}{\sqrt{\kappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]} = \tfrac{-2}{2\kappa+\partial}\big[q\big(\tfrac{g_{21}}{2+\gamma}\big)^2\big].
$$

To represent our expansions for higher order terms in a concise form, we introduce a space of paraproducts. We begin by introducing a generating set of operators:

$$(3.36) \qquad \mathcal{G} := \left\{ \mathrm{Id} \right\} \cup \left\{ \tfrac{2\varkappa}{2\varkappa \pm \partial} : |\varkappa| \geq 1 \right\}.$$

One may regard the elements of $\mathcal{G}$ as letters in an alphabet. We then define the set $\mathcal{G}^\star$ to be the set of (finite) words built from the alphabet $\mathcal{G}$. Specifically, elements in $\mathcal{G}^\star$ are finite products of the form

$$(3.37) \qquad L_1 L_2 \ldots L_n \quad \text{where } L_i \in \mathcal{G}.$$

Note that each factor L_i may have a different parameter $\varkappa_i$.

Example 3.6. For all $|\varkappa| \geq 1$, the operator $\frac{4\varkappa^2}{4\varkappa^2 - \partial^2} = \frac{2\varkappa}{2\varkappa - \partial} \frac{2\varkappa}{2\varkappa + \partial} \in \mathcal{G}^\star$ is a word over the alphabet $\mathcal{G}$.

We say that the paraproduct m belongs to the class $S(1)$ if it admits a representation as a finite linear combination of paraproducts m_i satisfying

$$m_i[f] = T_i f \quad \text{where } T_i \in \mathcal{G}^\star.$$

We will frequently consider families of paraproducts depending continuously on one or more parameters. For example, all the paraproducts in Proposition 3.12 depend continuously on the parameter $\varkappa$. When this is the case, we will say such a family is in $S(1)$ only if the coefficients in this linear combination are uniformly bounded in the parameter(s).

Example 3.7. For all $|\varkappa| \geq 1$, the operators

$$\tfrac{\partial}{2\varkappa + \partial} = \mathrm{Id} - \tfrac{2\varkappa}{2\varkappa + \partial} \quad \text{and} \quad \tfrac{2\varkappa - \partial}{2\varkappa + \partial} = 2 \tfrac{2\varkappa}{2\varkappa + \partial} - \mathrm{Id}$$

are elements of $S(1)$.

For $n \geq 2$, we inductively define $S(n)$ as linear combinations of paraproducts that admit the representation

$$(3.38) \qquad m[f_1, \ldots, f_n] = T \left[\prod_{j=1}^{J} m_j \left[f_{\sigma(n_{j-1}+1)}, \ldots, f_{\sigma(n_j)} \right] \right],$$

where $0 = n_0 < \cdots < n_J = n$ are integers, $m_j \in S(n_j - n_{j-1})$, $T \in \mathcal{G}^\star$, and $\sigma \in \mathfrak{S}_n$ is a permutation. The product appearing in (3.38) is a pointwise product of functions (and not a composition of operators). As in the case $n = 1$, we require the coefficients in the linear combination to be uniformly bounded in any parameters.

This definition is clearly symmetric, in the sense that whenever $m \in S(n)$ and $\sigma \in \mathfrak{S}_n$ we have

$$(3.39) \qquad m\big[f_{\sigma(1)}, \ldots, f_{\sigma(n)}\big] \in S(n).$$

Further, by induction on n, our definition is consistent with interior products, in the sense that if $2 \leq k \leq n$, $m_1 \in S(k)$, and $m_2 \in S(n+1-k)$ then

$$(3.40) \qquad m_1\big[f_1, \ldots, f_{k-1}, m_2[f_k, \ldots, f_n]\big] \in S(n).$$

We illustrate our paraproduct classes with an example motivated by (3.33):

Example 3.8. For all $|\varkappa| \geq 1$, the paraproduct

$$m[f_1, f_2, f_3] = \tfrac{8\varkappa^3}{2\varkappa - \partial}\left[f_1 \tfrac{f_2}{2\varkappa - \partial} \tfrac{f_3}{2\varkappa + \partial}\right]$$

is an element of $S(3)$ that can be expressed in the form (3.38) with $\sigma = \mathrm{Id}$,

$$T = \tfrac{2\varkappa}{2\varkappa - \partial}, \qquad m_1[f_1] = f_1, \qquad m_2[f_2] = \tfrac{2\varkappa}{2\varkappa - \partial} f_2, \qquad m_3[f_3] = \tfrac{2\varkappa}{2\varkappa + \partial} f_3.$$

While paraproducts are often regarded as multilinear objects, our use of them here is closer to that of a polynomial in a single variable, namely, q. More accurately, our paraproducts will solely be populated by the objects appearing in the list presented in Corollary 3.4. Moreover, in estimating these paraproducts, we will only be employing the information (3.26) about these objects. With these considerations in mind, we will frequently employ the expedient of writing paraproducts as $m[f, \ldots, f]$. Similarly, if an expression involves paraproducts $m_1, \ldots, m_k \in S(n)$, we simply denote each paraproduct by m as in, e.g., (3.47) below.

We demonstrate this notation with two examples from the proof of Proposition 3.12 below:

Example 3.9. If $|\varkappa| \geq 1$ the paraproduct

$$m[f_1, f_2, f_3] = 4\varkappa^2 \tfrac{f_1}{2\varkappa + \partial} \tfrac{f_2}{2\varkappa + \partial} f_3 = \tfrac{2\varkappa}{2\varkappa + \partial} f_1 \cdot \tfrac{2\varkappa}{2\varkappa + \partial} f_2 \cdot f_3$$

is an element of $S(3)$. We may then write

$$(3.41) \qquad \tfrac{8i\varkappa^3}{2\varkappa - \partial}\left[\tfrac{q}{4\varkappa^2 - \partial^2}\left| \tfrac{q'}{4\varkappa^2 - \partial^2}\right|^2\right] = \tfrac{2i\varkappa}{2\varkappa - \partial} m\left[\tfrac{q}{2\varkappa - \partial}, \tfrac{q'}{2\varkappa - \partial}, \tfrac{\bar{q}'}{4\varkappa^2 - \partial^2}\right]$$

$$= \tfrac{2i\varkappa}{2\varkappa - \partial} m\left[\tfrac{f}{2\varkappa - \partial}, \tfrac{f'}{2\varkappa - \partial}, \tfrac{f'}{4\varkappa^2 - \partial^2}\right],$$

where each f represents either q or $\bar{q}$, which are both elements of the list (3.25).

One should observe that using the definition of $S(3)$, the expression (3.41) could be further simplified to read

$$m_1\left[\tfrac{f}{2\varkappa - \partial}, f, \tfrac{f'}{4\varkappa^2 - \partial^2}\right] \quad \text{or even} \quad (2\varkappa)^{-2} m_2[f, f, f]$$

with $m_1, m_2 \in S(3)$. However, we will need the additional structure appearing in (3.41) when we apply Proposition 3.12.

Example 3.10. If $|\varkappa| \geq 1$ then, recalling Example 3.7, the paraproduct

$$m[f_1, f_2, f_3] = 2i\left(\tfrac{2\varkappa - \partial}{2\varkappa + \partial} f_1\right) f_2 f_3$$

is again an element of $S(3)$. From (3.32), we then have

$$(3.42) \qquad \tfrac{1}{2\varkappa - \partial}\left[\gamma^{[2]}(\varkappa) \tfrac{q''}{4\varkappa^2 - \partial^2}\right] = \tfrac{2i\varkappa}{2\varkappa - \partial}\left[\tfrac{\bar{q}}{2\varkappa + \partial} \tfrac{q}{2\varkappa - \partial} \tfrac{q''}{4\varkappa^2 - \partial^2}\right]$$

$$= \tfrac{\varkappa}{2\varkappa - \partial} m\left[\tfrac{\bar{q}}{2\varkappa - \partial}, \tfrac{q}{2\varkappa - \partial}, \tfrac{q''}{4\varkappa^2 - \partial^2}\right]$$

$$= \tfrac{\varkappa}{2\varkappa - \partial} m\left[\tfrac{f}{2\varkappa - \partial}, \tfrac{f}{2\varkappa - \partial}, \tfrac{f''}{4\varkappa^2 - \partial^2}\right],$$

where each f represents an element of the list (3.25).

Just as in the case of (3.41), one could further simplify (3.42) to read

$$m_1\left[\tfrac{f}{2\varkappa - \partial}, \tfrac{f}{2\varkappa - \partial}, f\right] \quad \text{or} \quad (2\varkappa)^{-2} m_2[f, f, f]$$

with $m_1, m_2 \in S(3)$. The additional structure in (3.42) will be exploited later.

As a first application of our paraproducts we have the following:

Lemma 3.11. *For $\ell \geq 1$ and $|\varkappa| \geq 1$ we have the representations*

$$(3.43) \qquad \frac{1}{\sqrt{\varkappa}} g_{12}^{[2\ell+1]}(\varkappa) = \frac{\varkappa}{2\varkappa-\partial} m\Big[\underbrace{f,\ldots,f}_{\ell}, \underbrace{\frac{f}{2\varkappa-\partial},\ldots,\frac{f}{2\varkappa-\partial}}_{\ell+1}\Big],$$

$$(3.44) \qquad \frac{1}{\sqrt{\varkappa}} g_{12}^{[\geq 2\ell+1]}(\varkappa) = \frac{\varkappa}{2\varkappa-\partial} m\Big[\underbrace{f,\ldots,f}_{\ell}, \underbrace{\frac{f}{2\varkappa-\partial},\ldots,\frac{f}{2\varkappa-\partial}}_{\ell+1}\Big],$$

$$(3.45) \qquad \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[2\ell+1]}(\varkappa) = \frac{\varkappa}{2\varkappa+\partial} m\Big[\underbrace{f,\ldots,f}_{\ell}, \underbrace{\frac{f}{2\varkappa-\partial},\ldots,\frac{f}{2\varkappa-\partial}}_{\ell+1}\Big],$$

$$(3.46) \qquad \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 2\ell+1]}(\varkappa) = \frac{\varkappa}{2\varkappa+\partial} m\Big[\underbrace{f,\ldots,f}_{\ell}, \underbrace{\frac{f}{2\varkappa-\partial},\ldots,\frac{f}{2\varkappa-\partial}}_{\ell+1}\Big],$$

where $m \in S(2\ell+1)$ and each f represents an element from the list (3.25). Similar representations hold for g_{21} and $\frac{g_{12}}{2+\gamma}$.

Proof. We prove all four identities simultaneously by strong induction on ℓ.

When $\ell = 1$, the identity (3.43) follows from (3.33). Using (3.11) and (3.32), we may write

$$\frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[3]}(\varkappa) = \frac{\varkappa}{2(2\varkappa+\partial)}\Big[q \cdot \frac{\bar{q}}{2\varkappa+\partial} \cdot \frac{\bar{q}}{2\varkappa+\partial}\Big],$$

which gives (3.45) when $\ell = 1$. Using (3.11) we may also write

$$\frac{1}{\sqrt{\varkappa}} g_{12}^{[\geq 3]}(\varkappa) = -\frac{4\varkappa}{2\varkappa-\partial}\Big[q \cdot \frac{1}{2\varkappa-\partial}\Big(\frac{2\varkappa-\partial}{\sqrt{\varkappa}} g_{12}\Big) \cdot \frac{1}{2\varkappa+\partial}\Big(\frac{2\varkappa+\partial}{\sqrt{\varkappa}} \frac{g_{21}}{2+\gamma}\Big)\Big],$$

$$\frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 3]}(\varkappa) = -\frac{2\varkappa}{2\varkappa+\partial}\Big[q \cdot \frac{1}{2\varkappa+\partial}\Big(\frac{2\varkappa+\partial}{\sqrt{\varkappa}} \frac{g_{21}}{2+\gamma}\Big) \cdot \frac{1}{2\varkappa+\partial}\Big(\frac{2\varkappa+\partial}{\sqrt{\varkappa}} \frac{g_{21}}{2+\gamma}\Big)\Big];$$

the identities (3.44) and (3.46) for $\ell = 1$ then follow from Corollary 3.4.

Now assume that (3.43)–(3.46) are true for all $0 \leq m \leq \ell - 1$. Using (3.11) we see that

$$\frac{1}{\sqrt{\varkappa}} g_{12}^{[2\ell+1]}(\varkappa) = -\sum_{n=0}^{\ell-1} \frac{4\varkappa}{2\varkappa-\partial}\Big[q \frac{1}{\sqrt{\varkappa}} g_{12}^{[2n+1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[2(\ell-1-n)+1]}\Big],$$

$$\frac{1}{\sqrt{\varkappa}} g_{12}^{[\geq 2\ell+1]}(\varkappa) = -\sum_{n=0}^{\ell-2} \frac{4\varkappa}{2\varkappa-\partial}\Big[q \frac{1}{\sqrt{\varkappa}} g_{12}^{[2n+1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 2(\ell-1-n)+1]}\Big]$$
$$\qquad\qquad - \frac{4\varkappa}{2\varkappa-\partial}\Big[q \frac{1}{\sqrt{\varkappa}} g_{12}^{[\geq 2\ell-1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)\Big],$$

$$\frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[2\ell+1]}(\varkappa) = -\sum_{n=0}^{\ell-1} \frac{2\varkappa}{2\varkappa+\partial}\Big[q \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[2n+1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[2(\ell-1-n)+1]}\Big],$$

$$\frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 2\ell+1]}(\varkappa) = -\sum_{n=0}^{\ell-2} \frac{2\varkappa}{2\varkappa+\partial}\Big[q \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{12}}{2+\gamma}\Big)^{[2n+1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 2(\ell-1-n)+1]}\Big]$$
$$\qquad\qquad - \frac{2\varkappa}{2\varkappa-\partial}\Big[q \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)^{[\geq 2\ell-1]} \frac{1}{\sqrt{\varkappa}}\Big(\frac{g_{21}}{2+\gamma}\Big)\Big],$$

and hence (3.43) through (3.46) follow from the inductive hypothesis, (3.39), and (3.40). $\qquad\square$

As a second application, we have the following:

Proposition 3.12. *Denote* $u = \frac{q}{4\varkappa^2 - \partial^2}$. *Then we have the representations*

$$\frac{1}{\sqrt{\varkappa}} g_{12}^{[3]}(\varkappa) = 16i\varkappa^4 |u|^2 u + 24i\varkappa^3 |u|^2 u' + \frac{\varkappa}{2\varkappa - \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right]$$

$$+ \frac{\varkappa}{2\varkappa - \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{4\varkappa^2 - \partial^2}\right]$$

$$(3.47) \qquad = 16i\varkappa^4 |u|^2 u + 24i\varkappa^3 |u|^2 u' + 8i\varkappa^2 |u|^2 u''$$

$$+ 8i\varkappa^2 |u'|^2 u + 12i\varkappa^2 (u')^2 \bar{u} + \frac{1}{2\varkappa - \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right]$$

$$+ \frac{\partial}{2\varkappa - \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right] + \frac{\partial}{2\varkappa - \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{4\varkappa^2 - \partial^2}\right],$$

$$(3.48) \quad \gamma^{[4]}(\varkappa) = -96\varkappa^6 |u|^4 - 96\varkappa^5 |u|^2 \left(u'\bar{u} - u\bar{u}'\right)$$

$$+ m\left[\frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right] + m\left[f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right],$$

$$\frac{1}{\sqrt{\varkappa}} g_{12}^{[5]}(\varkappa) = -192\varkappa^7 |u|^4 u - 480\varkappa^6 |u|^4 u' + \frac{1}{2\varkappa - \partial} m\left[f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right]$$

$$(3.49) \qquad + \frac{1}{2\varkappa - \partial} m\left[f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right],$$

$$(3.50) \quad \gamma^{[6]}(\varkappa) = -1280i\varkappa^9 |u|^6 + m\left[f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right],$$

$$(3.51) \quad \frac{1}{\sqrt{\varkappa}} g_{12}^{[7]}(\varkappa) = -2560i\varkappa^{10} |u|^6 u + \frac{1}{2\varkappa - \partial} m\left[f, f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right].$$

Further, we have the representations

$$\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2 + \gamma}\right)^{[3]}(\varkappa) = 4\varkappa^4 |u|^2 \bar{u} + \frac{\varkappa}{2\varkappa + \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right]$$

$$(3.52) \qquad = 4\varkappa^4 |u|^2 \bar{u} - 8\varkappa^3 |u|^2 \bar{u}' - 2\varkappa^3 \bar{u}^2 u' + \frac{1}{2\varkappa + \partial} m\left[f, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right]$$

$$+ \frac{1}{2\varkappa + \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right]$$

$$= 4\varkappa^4 |u|^2 \bar{u} - 8\varkappa^3 |u|^2 \bar{u}' - 2\varkappa^3 \bar{u}^2 u' + 4\varkappa^2 |u|^2 \bar{u}''$$

$$+ 6\varkappa^2 |u'|^2 \bar{u} + 5\varkappa^2 (\bar{u}')^2 u + \frac{1}{2\varkappa + \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right]$$

$$+ \frac{\partial}{2\varkappa + \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right] + \frac{\partial}{2\varkappa + \partial} m\left[\frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{4\varkappa^2 - \partial^2}\right],$$

$$\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2 + \gamma}\right)^{[5]}(\varkappa) = 32i\varkappa^7 |u|^4 \bar{u} + \frac{\varkappa}{2\varkappa + \partial} m\left[f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right]$$

$$(3.53) \qquad = 32i\varkappa^7 |u|^4 \bar{u} - 128i\varkappa^6 |u|^4 \bar{u}' - 48i\varkappa^6 |u|^2 \bar{u}^2 u'$$

$$+ \frac{1}{2\varkappa + \partial} m\left[f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\varkappa^2 - \partial^2}\right]$$

$$+ \frac{1}{2\varkappa + \partial} m\left[f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right],$$

$$(3.54)$$

$$\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2 + \gamma}\right)^{[7]}(\varkappa) = -320\varkappa^{10} |u|^6 \bar{u} + \frac{1}{2\varkappa + \partial} m\left[f, f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'}{2\varkappa - \partial}\right].$$

Throughout, the paraproduct m *lies in* $S(n)$ *for an appropriate integer* n, *each* f *represents an element from the list* (3.25), *and identical expressions hold with* $\varkappa$ *replaced by* $-\varkappa$.

Proof. Using (3.32), we derive the identity

$$(3.55) \qquad \gamma^{[2]} = 8i\varkappa^3|u|^2 + 4i\varkappa^2(u'\bar{u} - u\bar{u}') - 2i\varkappa|u'|^2.$$

To obtain the first expansion for $\frac{1}{\sqrt{\varkappa}}g^{[3]}_{12}(\varkappa)$ in (3.47), we use (3.11) to write

$$\frac{1}{\sqrt{\varkappa}}g^{[3]}_{12} = \frac{4\varkappa^2}{2\varkappa-\partial}\big[u\gamma^{[2]}\big] - \frac{1}{2\varkappa-\partial}\big[u''\gamma^{[2]}\big]$$
$$= \frac{32i\varkappa^5}{2\varkappa-\partial}\big[|u|^2u\big] + \frac{16i\varkappa^4}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big] - \frac{8i\varkappa^3}{2\varkappa-\partial}\big[|u'|^2u\big] - \frac{1}{2\varkappa-\partial}\big[\gamma^{[2]}u''\big].$$

Recalling Examples 3.9 and 3.10, the last two summands are of the form

$$(3.56) \qquad \frac{2\varkappa i}{2\varkappa-\partial}m\Big[\frac{f}{2\varkappa-\partial}, \frac{f'}{2\varkappa-\partial}, \frac{f'}{4\varkappa^2-\partial^2}\Big] + \frac{\varkappa}{2\varkappa-\partial}m\Big[\frac{f}{2\varkappa-\partial}, \frac{f}{2\varkappa-\partial}, \frac{f''}{4\varkappa^2-\partial^2}\Big]$$

and so acceptable.

For the remaining summands we first use that

$$(3.57) \qquad \frac{1}{2\varkappa-\partial} = \frac{1}{2\varkappa} + \frac{1}{2\varkappa}\frac{\partial}{2\varkappa-\partial}$$

to write

$$\frac{16i\varkappa^4}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big] = 8i\varkappa^3(u'\bar{u}-u\bar{u}')u + \frac{8i\varkappa^3\partial}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big],$$

where we note that the second summand is of the form (3.56) and so acceptable. Expanding further, we have

$$(3.58) \qquad \frac{1}{2\varkappa-\partial} = \frac{1}{2\varkappa} + \frac{\partial}{4\varkappa^2} + \frac{1}{4\varkappa^2}\frac{\partial^2}{2\varkappa-\partial},$$

which we apply to get

$$\frac{32i\varkappa^5}{2\varkappa-\partial}\big[|u|^2u\big] = 16i\varkappa^4|u|^2u + 16i\varkappa^3|u|^2u' + 8i\varkappa^3u^2\bar{u}' + \frac{8i\varkappa^3\partial^2}{2\varkappa-\partial}\big[|u|^2u\big],$$

where the final summand is again of the form (3.56). This completes the proof of the first expansion recorded in (3.47).

We now turn to the second expression for $\frac{1}{\sqrt{\varkappa}}g^{[3]}_{12}(\varkappa)$ in (3.47). We again use (3.11) to write

$$\frac{1}{\sqrt{\varkappa}}g^{[3]}_{12} = \frac{32i\varkappa^5}{2\varkappa-\partial}\big[|u|^2u\big] + \frac{16i\varkappa^4}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big] - \frac{8i\varkappa^3}{2\varkappa-\partial}\big[|u'|^2u + |u|^2u''\big]$$
$$- \frac{1}{2\varkappa-\partial}\big[(\gamma^{[2]} - 8i\varkappa^3|u|^2)u''\big].$$

By (3.55), the final summand is seen to be of the form $\frac{1}{2\varkappa-\partial}m[\frac{f}{2\varkappa-\partial}, \frac{f'}{2\varkappa-\partial}, \frac{f''}{4\varkappa^2-\partial^2}]$, and so acceptable. We then use (3.57) to write

$$-\frac{8i\varkappa^3}{2\varkappa-\partial}\big[|u'|^2u + |u|^2u''\big] = -4i\varkappa^2\big[|u'|^2u + |u|^2u''\big] - \frac{4i\varkappa^2\partial}{2\varkappa-\partial}\big[|u'|^2u + |u|^2u''\big]$$

and note that the final summand is of the form

$$(3.59) \qquad \frac{\partial}{2\varkappa-\partial}m\Big[\frac{f}{2\varkappa-\partial}, \frac{f}{2\varkappa-\partial}, \frac{f''}{4\varkappa^2-\partial^2}\Big] + \frac{\partial}{2\varkappa-\partial}m\Big[\frac{f}{2\varkappa-\partial}, \frac{f'}{2\varkappa-\partial}, \frac{f'}{4\varkappa^2-\partial^2}\Big],$$

and so acceptable. Applying (3.58), we similarly have

$$\frac{16i\varkappa^4}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big] = 8i\varkappa^3\big[(u'\bar{u}-u\bar{u}')u\big] + 4i\varkappa^2\big[(u'\bar{u}-u\bar{u}')u\big]' + \frac{4i\varkappa^2\partial^2}{2\varkappa-\partial}\big[(u'\bar{u}-u\bar{u}')u\big],$$

where the final summand is again of the form (3.59). Expanding even further, we have

$$\frac{1}{2\varkappa-\partial} = \frac{1}{2\varkappa} + \frac{\partial}{4\varkappa^2} + \frac{\partial^2}{8\varkappa^3} + \frac{1}{8\varkappa^3}\frac{\partial^3}{2\varkappa-\partial},$$

which we apply to the remaining term to get

$$\frac{32i\varkappa^5}{2\varkappa-\partial}\big[|u|^2u\big] = 16i\varkappa^4|u|^2u + 8i\varkappa^3\big[|u|^2u\big]' + 4i\varkappa^2\big[|u|^2u\big]'' + \frac{4i\varkappa^2\partial^3}{2\varkappa-\partial}\big[|u|^2u\big].$$

The final term is once again of the form (3.59) and so acceptable. Combining these expressions, one obtains the second expansion recorded in (3.47).

Turning next to $\gamma^{[4]}$, we use (3.10) to write

$$\gamma^{[4]} = -\tfrac{1}{2}\big[\gamma^{[2]}\big]^2 - 2g_{12}^{[3]}g_{21}^{[1]} - 2g_{12}^{[1]}g_{21}^{[3]}.$$

For the first summand, we use (3.55) to express

$$-\tfrac{1}{2}\big[\gamma^{[2]}\big]^2 = 32\varkappa^6|u|^4 + 32\varkappa^5|u|^2(u'\bar{u} - u\bar{u}') + m\Big[\tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\Big].$$

For the second summand, we use (3.47) and (3.32) to write

$$-2g_{12}^{[3]}\,g_{21}^{[1]} = -32\varkappa^5|u|^2u\,\tfrac{\bar{q}}{2\varkappa+\partial} - 48\varkappa^4|u|^2u'\,\tfrac{\bar{q}}{2\varkappa+\partial}$$
$$+ m\Big[\tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\Big] + m\Big[f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big].$$

Writing

$$(3.60) \qquad\qquad\qquad \tfrac{\bar{q}}{2\varkappa+\partial} = 2\varkappa\bar{u} - \bar{u}',$$

in the first two terms, we obtain

$$-2g_{12}^{[3]}\,g_{21}^{[1]} = -64\varkappa^6|u|^4 + 32\varkappa^5|u|^2u\bar{u}' - 96\varkappa^5|u|^2u'\bar{u}$$
$$+ m\Big[\tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\Big] + m\Big[f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big].$$

Thus, using (3.6) we also have

$$-2g_{12}^{[1]}\,g_{21}^{[3]} = -64\varkappa^6|u|^4 - 32\varkappa^5|u|^2\bar{u}u' + 96\varkappa^5|u|^2\bar{u}'u$$
$$+ m\Big[\tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\Big] + m\Big[f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big].$$

Combining these expressions gives us (3.48).

Next, consider (3.49). Using (3.11) and then applying (3.48), we find

$$\tfrac{1}{\sqrt{\varkappa}}g_{12}^{[5]} = \tfrac{1}{2\varkappa-\partial}\Big[\gamma^{[4]}\,q\Big]$$
$$= \tfrac{1}{2\varkappa-\partial}\Big[-96\varkappa^6|u|^4q - 96\varkappa^5|u|^2(u'\bar{u} - u\bar{u}')q\Big]$$
$$+ \tfrac{1}{2\varkappa-\partial}m\Big[f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\Big]$$
$$+ \tfrac{1}{2\varkappa-\partial}m\Big[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big].$$

For the first term, we employ (3.58) to write

$$-\tfrac{96\varkappa^6}{2\varkappa-\partial}\Big[|u|^4\,q\Big] = -192\varkappa^7|u|^4u - 96\varkappa^6(|u|^4u)' - \tfrac{96\varkappa^6\partial^2}{2\varkappa-\partial}\Big[|u|^4u\Big] + \tfrac{96\varkappa^6}{2\varkappa-\partial}\Big[|u|^4\,u''\Big],$$

where the third and fourth terms are seen to be acceptable. Similarly, for the remaining term we apply (3.57) to get

$$-\tfrac{96\varkappa^5}{2\varkappa-\partial}\Big[|u|^2(u'\bar{u} - u\bar{u}')\,q\Big] = -192\varkappa^6|u|^2(u'\bar{u} - u\bar{u}')u - \tfrac{192\varkappa^6\partial}{2\varkappa-\partial}\Big[|u|^2(u'\bar{u} - u\bar{u}')u\Big]$$
$$+ \tfrac{96\varkappa^5}{2\varkappa-\partial}\Big[|u|^2(u'\bar{u} - u\bar{u}')u''\Big],$$

where the second and third summands are similarly acceptable. This completes the proof of (3.49).

For $\gamma^{[6]}$, we again use the quadratic identity (3.10) to write

$$\gamma^{[6]} = -\gamma^{[2]}\gamma^{[4]} - 2g_{12}^{[5]}g_{21}^{[1]} - 2g_{12}^{[3]}g_{21}^{[3]} - 2g_{12}^{[1]}g_{21}^{[5]}.$$

For the first term, we use (3.48) and (3.32) followed by (3.55) to obtain

$$-\gamma^{[2]}\gamma^{[4]} = 96\varkappa^6|u|^4\gamma^{[2]} + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right]$$

$$= 768i\varkappa^9|u|^6 + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right].$$

Next, applying (3.49) and (3.32) followed by (3.60), we have

$$-2g_{12}^{[5]}g_{21}^{[1]} = -384i\varkappa^8|u|^4u\tfrac{\bar{q}}{2\varkappa+\partial} + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right]$$

$$= -768i\varkappa^9|u|^6 + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right].$$

Another application of (3.6) then gives us

$$-2g_{12}^{[1]}g_{21}^{[5]} = -768i\varkappa^9|u|^6 + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right].$$

For the remaining term, we use the first expansion in (3.47), (3.6), and (3.33) to express

$$-2g_{12}^{[3]}g_{21}^{[3]} = -16i\varkappa^{\frac{9}{2}}|u|^2ug_{21}^{[3]} - 16\varkappa^{\frac{9}{2}}|u|^2\bar{u}g_{12}^{[3]} + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right]$$

$$= -512i\varkappa^9|u|^6 + m\left[f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right].$$

Combining all of these expressions gives us (3.50).

Using (3.11) and (3.50), we may write

$$\tfrac{1}{\sqrt{\varkappa}}g_{12}^{[7]} = \tfrac{1}{2\varkappa-\partial}\left[\gamma^{[6]}q\right] = \tfrac{1}{2\varkappa-\partial}[-1280i\varkappa^9|u|^6q] + \tfrac{1}{2\varkappa-\partial}m\left[f, f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right]$$

$$= -2560i\varkappa^{10}|u|^6u + \tfrac{1}{2\varkappa-\partial}m\left[f, f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}\right],$$

which settles (3.51).

Using (3.11), (3.32), and (3.60) we may write

$$\tfrac{1}{\sqrt{\varkappa}}\left(\tfrac{g_{21}}{2+\gamma}\right)^{[3]} = \tfrac{\varkappa}{2(2\varkappa+\partial)}\left[q\left(\tfrac{\bar{q}}{2\varkappa+\partial}\right)^2\right]$$

$$= \tfrac{2\varkappa^3}{2\varkappa+\partial}\left[u\left(\tfrac{\bar{q}}{2\varkappa+\partial}\right)^2\right] - \tfrac{\varkappa}{2(2\varkappa+\partial)}\left[u''\left(\tfrac{\bar{q}}{2\varkappa+\partial}\right)^2\right]$$

$$= \tfrac{8\varkappa^5}{2\varkappa+\partial}\left[|u|^2\bar{u}\right] - \tfrac{8\varkappa^4}{2\varkappa+\partial}\left[|u|^2\bar{u}'\right] + \tfrac{2\varkappa^3}{2\varkappa+\partial}\left[u(\bar{u}')^2\right] - \tfrac{\varkappa}{2(2\varkappa+\partial)}\left[u''\left(\tfrac{\bar{q}}{2\varkappa+\partial}\right)^2\right],$$

from which the first expansion in (3.52) follows easily using that

$$\tfrac{1}{2\varkappa+\partial} = \tfrac{1}{2\varkappa} - \tfrac{\partial}{2\varkappa(2\varkappa+\partial)}.$$

To obtain the second expression in (3.52), we expand even further

$$\tfrac{1}{\sqrt{\varkappa}}\left(\tfrac{g_{21}}{2+\gamma}\right)^{[3]} = \tfrac{8\varkappa^5}{2\varkappa+\partial}\left[|u|^2\bar{u}\right] - \tfrac{8\varkappa^4}{2\varkappa+\partial}\left[|u|^2\bar{u}'\right] + \tfrac{2\varkappa^3}{2\varkappa+\partial}\left[u(\bar{u}')^2\right] - \tfrac{2\varkappa^3}{2\varkappa+\partial}\left[\bar{u}^2u''\right]$$

$$+ \tfrac{1}{2\varkappa+\partial}m\left[\tfrac{f}{2\varkappa-\partial}, \tfrac{f'}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\right]$$

and use that

$$\tfrac{1}{2\varkappa+\partial} = \tfrac{1}{2\varkappa} - \tfrac{\partial}{4\varkappa^2} + \tfrac{\partial^2}{4\varkappa^2(2\varkappa+\partial)}$$

for the first summand and that

$$\tfrac{1}{2\varkappa+\partial} = \tfrac{1}{2\varkappa} - \tfrac{\partial}{2\varkappa(2\varkappa+\partial)}$$

for the second summand. The last expression in (3.52) is derived by expanding $\tfrac{1}{2\varkappa+\partial}$ one further degree for each summand.

For (3.53) we again use (3.11), (3.32), and (3.60) to obtain

$$
\begin{aligned}
\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]} &= \tfrac{2i\varkappa}{2\varkappa+\partial}\Big[q\tfrac{\bar{q}}{2\varkappa+\partial}\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big] \\
&= \tfrac{8i\varkappa^3}{2\varkappa+\partial}\Big[u\tfrac{\bar{q}}{2\varkappa+\partial}\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big] - \tfrac{2i\varkappa}{2\varkappa+\partial}\Big[u''\tfrac{\bar{q}}{2\varkappa+\partial}\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big] \\
&= \tfrac{16i\varkappa^4}{2\varkappa+\partial}\Big[|u|^2\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big] - \tfrac{8i\varkappa^3}{2\varkappa+\partial}\Big[u\bar{u}'\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big] \\
&\quad - \tfrac{2i\varkappa}{2\varkappa+\partial}\Big[u''\tfrac{\bar{q}}{2\varkappa+\partial}\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big].
\end{aligned}
$$

We then expand $\frac{1}{2\varkappa+\partial}$ and use (3.52) and to obtain (3.53).

Similarly, for (3.54) we use (3.52) and (3.53) to write

$$
\begin{aligned}
\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[7]} &= \tfrac{2i\varkappa}{2\varkappa+\partial}\Big[q\tfrac{\bar{q}}{2\varkappa+\partial}\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]}\Big] - \tfrac{2\varkappa}{2\varkappa+\partial}\Big[q\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}\Big]^2\Big] \\
&= -\tfrac{64\varkappa^8}{2\varkappa+\partial}\Big[q\tfrac{\bar{q}}{2\varkappa+\partial}|u|^4\bar{u}\Big] - \tfrac{32\varkappa^9}{2\varkappa+\partial}\Big[q|u|^4\bar{u}^2\Big] \\
&\quad + \tfrac{1}{2\varkappa+\partial}m\Big[f,f,f,\tfrac{f}{2\varkappa-\partial},\tfrac{f}{2\varkappa-\partial},\tfrac{f}{2\varkappa-\partial},\tfrac{f'}{2\varkappa-\partial}\Big],
\end{aligned}
$$

to which we apply (3.60).   □

4. LOCAL SMOOTHING FOR THE DNLS

In this section we prove local smoothing for Schwartz solutions of (DNLS):

Proposition 4.1 (Local smoothing for the DNLS). *Let $Q \subset \mathcal{S}$ be an L^2 bounded and equicontinuous set such that*

$$Q_* = \big\{e^{tJ\nabla H}q : |t| \leq 1 \text{ and } q \in Q\big\}$$

is a δ-good set for a sufficiently small δ. Then the local smoothing estimate

$$(4.1) \qquad \|q\|_{X^{\frac{1}{2}}} \lesssim \|q(0)\|_{L^2}$$

holds uniformly for $q(0) \in Q$.

Further, equicontinuity holds in the local smoothing topology, in the sense that

$$(4.2) \qquad \lim_{\kappa \to \infty} \sup_{q(0) \in Q} \|q\|_{X_\kappa^{\frac{1}{2}}} = 0.$$

We remind the reader that Corollary 1.9 guarantees that for any L^2 bounded and equicontinuous set Q, there is a uniform rescaling so that the corresponding Q_* is δ-good. For the remainder of this section, we fix $Q_* \subset \mathcal{S}$ satisfying the hypotheses of Proposition 4.1.

Our proof of Proposition 4.1 rests on the microscopic conservation law presented in Proposition 3.5. Taking

$$(4.3) \qquad \phi_\mu(x) = \int_0^x \psi_\mu(y)^{24}\,dy,$$

and integrating by parts, we obtain

$$(4.4) \qquad \operatorname{Im}\int_{-1}^1 \int j_{\mathrm{DNLS}}(\varkappa; q(t))\,\psi_\mu^{24}\,dx\,dt = \operatorname{Im}\int \rho(\varkappa; q(t))\phi_\mu\,dx\Big|_{t=-1}^{t=1}.$$

To bound the right-hand side of this expression, we use the following:

Lemma 4.2 (Estimate for ρ). *The following estimates hold uniformly for $q \in Q_*$, $\kappa, \varkappa \geq 1$, and $\mu \in \mathbb{R}$:*

$$(4.5) \qquad \left| \int \operatorname{Im} \rho(\varkappa; q)\, \phi_\mu \, dx \right| \lesssim \varkappa^{-1} \|q\|_{E^\sigma_{2\sigma,\varkappa}}^2 \left[1 + \|q\|_{L^2}^2 \right],$$

$$(4.6) \qquad \int_\kappa^\infty \int \left| \operatorname{Im} \int \rho(\varkappa; q)\phi_\mu \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu \, d\varkappa \lesssim \|q\|_{E^\sigma_{\sigma,\kappa}}^2 \left[1 + \|q\|_{L^2}^2 \right].$$

Proof. A computation yields

$$\operatorname{Im} \int \rho^{[2]}(\varkappa; q)\, \phi_\mu \, dx = \frac{1}{2i} \int \bar{q}\, \phi_\mu \, \frac{q'}{4\varkappa^2 - \partial^2} \, dx - \frac{1}{2i} \int q\, \phi_\mu \, \frac{\bar{q}'}{4\varkappa^2 - \partial^2} \, dx,$$

so using (2.13) we get

$$\left| \operatorname{Im} \int \rho^{[2]}(\varkappa; q)\, \phi_\mu \, dx \right| \leq \varkappa^{-1} \|\phi_\mu q\|_{E^{1/2}_{1,\varkappa}} \|q\|_{E^{1/2}_{1,\varkappa}} \lesssim \varkappa^{-1} \|q\|_{E^\sigma_{2\sigma,\varkappa}}^2.$$

Turning to the higher order terms, we use (3.34) and (3.35) to write

$$\rho^{[\geq 4]}(\varkappa; q) = q \cdot \frac{-2i}{2\varkappa + \partial} \left[q\left(\frac{g_{21}}{2+\gamma} \right)^2 \right] + \bar{q} \cdot \frac{2i}{2\varkappa - \partial} \left[\bar{q}\left(\frac{g_{12}}{2+\gamma} \right)^2 \right].$$

Thus, by (3.14) and the fact that $\|\frac{1}{2\varkappa \pm \partial}\|_{\mathrm{op}} \lesssim \varkappa^{-1}$, we obtain

$$\left| \operatorname{Im} \int \rho^{[\geq 4]}(\varkappa; q)\, \phi_\mu \, dx \right| \lesssim \varkappa^{-1} \|\phi_\mu q\|_{L^2} \|q\|_{L^2} \left[\left\| \tfrac{g_{12}}{2+\gamma} \right\|_{L^\infty}^2 + \left\| \tfrac{g_{21}}{2+\gamma} \right\|_{L^\infty}^2 \right]$$

$$\lesssim \varkappa^{-1} \|q\|_{L^2}^2 \|q\|_{E^\sigma_{2\sigma,\varkappa}}^2,$$

which completes the proof of (4.5).

The estimate (4.6) follows from (4.5) and Lemma 2.2. $\qquad \square$

Turning to the left-hand side of (4.4), our main challenge will be to control the remainder terms $j_{\mathrm{DNLS}}^{[\geq 24]}$. To do so, we need to distribute the exponential weight ψ_μ^{24} across the arguments of paraproducts in $S(n)$. To accomplish this, we introduce a modified space of paraproducts, $S_{\mathrm{loc}}(n)$, which involve a spatial parameter $\mu \in \mathbb{R}$. While this construction will aid our proof of Proposition 4.1, its true value will only become clear when we turn to the significantly more involved problem of obtaining local smoothing estimates for the difference flow in Section 6.

We define an extended set of generators

$$(4.7)$$

$$\mathcal{G}_{\mathrm{loc}} = \mathcal{G} \cup \left\{ S^{\pm 1} \psi_\mu^\ell S^{\mp 1} \psi_\mu^{-\ell}, \ \psi_\mu^\ell S^{\pm 1} \psi_\mu^{-\ell} S^{\mp 1} : |\ell| \leq 24 \text{ is an integer and } S \in \mathcal{G}^\star \right\},$$

which are μ-dependent operators. Recall that the set $\mathcal{G}$ was defined in (3.36) and generates the set $\mathcal{G}^\star$ of finite words over the alphabet $\mathcal{G}$, as in (3.37). We similarly take $\mathcal{G}_{\mathrm{loc}}^\star$ to be the set of words over the alphabet $\mathcal{G}_{\mathrm{loc}}$.

Example 4.3. If $|\varkappa| \geq 1$ and $|\ell| \leq 24$ is an integer then

$$\psi_\mu^\ell \frac{2\varkappa}{2\varkappa + \partial} \psi_\mu^{-\ell} = \left(\psi_\mu^\ell \frac{2\varkappa}{2\varkappa + \partial} \psi_\mu^{-\ell} \frac{2\varkappa + \partial}{2\varkappa} \right) \frac{2\varkappa}{2\varkappa + \partial} \in \mathcal{G}_{\mathrm{loc}}^\star.$$

Paralleling the construction of $S(n)$, we say that a paraproduct $m \in S_{\mathrm{loc}}(1)$ if it admits a representation as a finite linear combination of elements of $\mathcal{G}_{\mathrm{loc}}^\star$.

For $n \geq 2$, we inductively define $S_{\mathrm{loc}}(n)$ as finite linear combinations of paraproducts that admit the representation

$$(4.8) \qquad m[f_1, \ldots, f_n] = T\left[\prod_{j=1}^{J} m_j[f_{\sigma(n_{j-1}+1)}, \ldots, f_{\sigma(n_j)}]\right],$$

where $0 = n_0 < \cdots < n_J = n$ are integers, $m_j \in S_{\mathrm{loc}}(n_j - n_{j-1})$, $T \in \mathcal{G}_{\mathrm{loc}}^{\star}$, and $\sigma \in \mathfrak{S}_n$. On both sides of (4.8), all paraproducts are evaluated at a common value of μ. This will be the standing convention whenever we combine paraproducts in $S_{\mathrm{loc}}(n)$.

As in the case of $S(n)$, we require all coefficients in these linear combinations to be uniformly bounded in any parameters.

Example 4.4. Recall from Example 3.8 that for $|\varkappa| \geq 1$, the paraproduct

$$m[f_1, f_2, f_3] = \tfrac{8\varkappa^3}{2\varkappa - \partial}\left[f_1 \tfrac{f_2}{2\varkappa - \partial} \tfrac{f_3}{2\varkappa + \partial}\right]$$

is in $S(3)$. We may write

$$\psi_\mu^{24} m[f_1, f_2, f_3] = \widetilde{m}[\psi_\mu^8 f_1, \psi_\mu^8 f_2, \psi_\mu^8 f_3],$$

where $\widetilde{m} \in S_{\mathrm{loc}}(3)$ has representation

$$\widetilde{m}[f_1, f_2, f_3] = \psi_\mu^{24} \tfrac{8\varkappa^3}{2\varkappa - \partial}\left[\psi_\mu^{-8} f_1 \cdot \tfrac{\psi_\mu^{-8} f_2}{2\varkappa - \partial} \cdot \tfrac{\psi_\mu^{-8} f_3}{2\varkappa + \partial}\right],$$

which can be expressed as in (4.8) with $\sigma = \mathrm{Id}$, $T = \psi_\mu^{24} \tfrac{2\varkappa}{2\varkappa - \partial} \psi_\mu^{-24}$,

$$m_1[f_1] = f_1, \qquad m_2[f_2] = \psi_\mu^8 \tfrac{2\varkappa}{2\varkappa - \partial}(\psi_\mu^{-8} f_2), \qquad m_3[f_3] = \psi_\mu^8 \tfrac{2\varkappa}{2\varkappa + \partial}(\psi_\mu^{-8} f_3).$$

Example 4.4 demonstrates one of the key motivations for the introduction of this class of paraproducts; this is codified in property (i) of Lemma 4.6. It is mandated by the necessity of employing local smoothing estimates on each and every argument of our paraproducts.

We first record a result which will be used in the proof of Lemma 4.6.

Lemma 4.5. If $T \in \mathcal{G}_{\mathrm{loc}}^{\star}$, then the conjugated operators

$$(4.9) \qquad \tfrac{2+\partial}{2} T \tfrac{2}{2+\partial}, \qquad \tfrac{2-\partial}{2} T \tfrac{2}{2-\partial}, \qquad \tfrac{2}{2+\partial} T \tfrac{2+\partial}{2} \quad \text{and} \quad \tfrac{2}{2-\partial} T \tfrac{2-\partial}{2}$$

belong to $S_{\mathrm{loc}}(1)$.

Proof. By definition, any $T \in \mathcal{G}_{\mathrm{loc}}^{\star}$ may be written as a finite product of the generators $\mathcal{G}_{\mathrm{loc}}$. As conjugating a product is equivalent to conjugating each factor, it suffices to verify the claim in the case $T \in \mathcal{G}_{\mathrm{loc}}$.

The elements of $\mathcal{G}_{\mathrm{loc}}$ come in five kinds. The easiest case to deal with is $T \in \mathcal{G}$ because then T commutes with $2 \pm \partial$ and all four operators in (4.9) are all equal to T. In what follows, we will treat the first two operators in (4.9) for each of the four remaining kinds of generators in $\mathcal{G}_{\mathrm{loc}}$. The remaining two operators in (4.9) may be treated in a parallel manner.

To unify our treatment of the first two operators in (4.9), we will show that

$$(4.10) \qquad \tfrac{2\kappa + \partial}{2\kappa} T \tfrac{2\kappa}{2\kappa + \partial} \quad \text{is a linear combination of words in } \mathcal{G}_{\mathrm{loc}}^{\star}$$

for any $|\kappa| \geq 1$, whenever $T = S^{\pm 1} \psi_\mu^\ell S^{\mp 1} \psi_\mu^{-\ell}$ or $T = \psi_\mu^\ell S^{\pm 1} \psi_\mu^{-\ell} S^{\mp 1}$ with $S \in \mathcal{G}^{\star} \backslash \{\mathrm{Id}\}$ and $|\ell| \leq 24$.

If $T = S^{-1}\psi_\mu^\ell S\psi_\mu^{-\ell}$, we write

$$\tfrac{2\kappa+\partial}{2\kappa}T\tfrac{2\kappa}{2\kappa+\partial} = \left[\left(S\tfrac{2\kappa}{2\kappa+\partial}\right)^{-1}\psi_\mu^\ell\left(S\tfrac{2\kappa}{2\kappa+\partial}\right)\psi_\mu^{-\ell}\right]\left[\psi_\mu^\ell\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^{-\ell}\tfrac{2\kappa}{2\kappa+\partial}\right] \in \mathcal{G}_{\mathrm{loc}}^*.$$

Similarly, if $T = \psi_\mu^\ell S^{-1}\psi_\mu^{-\ell}S$ we may write

$$\tfrac{2\kappa+\partial}{2\kappa}T\tfrac{2\kappa}{2\kappa+\partial} = \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\kappa}{2\kappa+\partial}\psi_\mu^{-\ell}\right]\left[\psi_\mu^\ell\left(S\tfrac{2\kappa}{2\kappa+\partial}\right)^{-1}\psi_\mu^{-\ell}\left(S\tfrac{2\kappa}{2\kappa+\partial}\right)\right] \in \mathcal{G}_{\mathrm{loc}}^*.$$

Next we consider the case $T = S\psi_\mu^\ell S^{-1}\psi_\mu^{-\ell}$. By the definition of $\mathcal{G}^\star$, we may write $S = \tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{S}$ or $S = \tfrac{2\varkappa}{2\varkappa-\partial}\widetilde{S}$ for some $|\varkappa| \geq 1$ and $\widetilde{S} \in \mathcal{G}^\star$. We present the details in the case $S = \tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{S}$; the remaining case can be treated analogously.

Using the identity

$$\tfrac{2\kappa+\partial}{2\kappa}\tfrac{2\varkappa}{2\varkappa+\partial} = \tfrac{2\varkappa}{2\varkappa+\partial} + \tfrac{\varkappa}{\kappa}\tfrac{\partial}{2\varkappa+\partial}$$

and the symmetric identity with $\varkappa \leftrightarrow \kappa$, we may express

$$\begin{aligned}
\tfrac{2\kappa+\partial}{2\kappa}S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial} &= S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{\varkappa}{\kappa}\tfrac{\partial}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{2\varkappa+\partial}{2\varkappa}\tfrac{2\kappa}{2\kappa+\partial} \\
&= S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{\varkappa}{\kappa}\tfrac{\partial}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\left[\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{\kappa}{\varkappa}\tfrac{\partial}{2\kappa+\partial}\right] \\
&= S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial} + \left[\tfrac{2\kappa+\partial}{2\kappa}\tfrac{2\varkappa}{2\varkappa+\partial} - \tfrac{2\varkappa}{2\varkappa+\partial}\right]\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{2\kappa}{2\kappa+\partial} \\
&\quad + \tfrac{\partial}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{\partial}{2\kappa+\partial} \\
&= S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{2\kappa+\partial}{2\kappa}\tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{2\kappa}{2\kappa+\partial} \\
&\quad - \tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{\partial}{2\varkappa+\partial}\widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\tfrac{\partial}{2\kappa+\partial}.
\end{aligned} \tag{4.11}$$

We then apply (4.11) to get

$$\begin{aligned}
\tfrac{2\kappa+\partial}{2\kappa}T\tfrac{2\kappa}{2\kappa+\partial} &= \left[\tfrac{2\kappa+\partial}{2\kappa}S\psi_\mu^\ell S^{-1}\tfrac{2\kappa}{2\kappa+\partial}\right]\left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^{-\ell}\tfrac{2\kappa}{2\kappa+\partial}\right] \\
&= T\tfrac{2\kappa}{2\kappa+\partial} + \tfrac{2\varkappa}{2\varkappa+\partial}\left[\tfrac{2\kappa+\partial}{2\kappa}\widetilde{T}\tfrac{2\kappa}{2\kappa+\partial}\right] - \tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{T}\tfrac{2\kappa}{2\kappa+\partial} \\
&\quad + \tfrac{\partial}{2\varkappa+\partial}\widetilde{T}\left[\psi_\mu^\ell\tfrac{\partial}{2\varkappa+\partial}\psi_\mu^{-\ell}\right]\left[\psi_\mu^\ell\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^{-\ell}\tfrac{2\kappa}{2\kappa+\partial}\right],
\end{aligned} \tag{4.12}$$

where $\widetilde{T} = \widetilde{S}\psi_\mu^\ell\widetilde{S}^{-1}\psi_\mu^{-\ell} \in \mathcal{G}_{\mathrm{loc}}^\star$. By using Examples 3.7 and 4.3, we see that

$$(4.13) \quad \tfrac{\partial}{2\varkappa+\partial} = \mathrm{Id} - \tfrac{2\varkappa}{2\varkappa+\partial} \quad \text{and} \quad \psi_\mu^\ell\tfrac{\partial}{2\varkappa+\partial}\psi_\mu^{-\ell} = \mathrm{Id} - \left[\psi_\mu^\ell\tfrac{2\varkappa}{2\varkappa+\partial}\psi_\mu^{-\ell}\tfrac{2\varkappa+\partial}{2\varkappa}\right]\tfrac{2\varkappa}{2\varkappa+\partial}$$

are linear combinations of words in $\mathcal{G}_{\mathrm{loc}}^\star$. As a consequence, (4.12) shows that if $\tfrac{2\kappa+\partial}{2\kappa}\widetilde{T}\tfrac{2\kappa}{2\kappa+\partial}$ is a linear combination of words in $\mathcal{G}_{\mathrm{loc}}^\star$ then so is $\tfrac{2\kappa+\partial}{2\kappa}T\tfrac{2\kappa}{2\kappa+\partial}$. In this case, (4.10) follows by induction on the minimal number of letters required to spell S as in the sense (3.37). The base case corresponds to taking $\widetilde{T} = \mathrm{Id}$.

Finally, we consider the case $T = \psi_\mu^\ell S\psi_\mu^{-\ell}S^{-1}$. Arguing as in the previous case and using (4.11) (with ℓ replaced by $-\ell$), we get

$$\begin{aligned}
\tfrac{2\kappa+\partial}{2\kappa}T\tfrac{2\kappa}{2\kappa+\partial} &= \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\kappa}{2\kappa+\partial}\right]\left[\tfrac{2\kappa+\partial}{2\kappa}S\psi_\mu^{-\ell}S^{-1}\tfrac{2\kappa}{2\kappa+\partial}\right] \\
&= \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\kappa}{2\kappa+\partial}\psi_\mu^{-\ell}\right]T\tfrac{2\kappa}{2\kappa+\partial} + \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\varkappa}{2\varkappa+\partial}\psi_\mu^{-\ell}\tfrac{2\kappa}{2\kappa+\partial}\right]\tfrac{2\kappa+\partial}{2\kappa}\widetilde{T}\tfrac{2\kappa}{2\kappa+\partial} \\
&\quad - \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\kappa}{2\kappa+\partial}\psi_\mu^{-\ell}\right]\left[\psi_\mu^\ell\tfrac{2\varkappa}{2\varkappa+\partial}\psi_\mu^{-\ell}\tfrac{2\varkappa+\partial}{2\varkappa}\right]\tfrac{2\varkappa}{2\varkappa+\partial}\widetilde{T}\tfrac{2\kappa}{2\kappa+\partial} \\
&\quad + \left[\tfrac{2\kappa+\partial}{2\kappa}\psi_\mu^\ell\tfrac{2\kappa}{2\kappa+\partial}\psi_\mu^{-\ell}\right]\left[\psi_\mu^\ell\tfrac{\partial}{2\varkappa+\partial}\psi_\mu^{-\ell}\right]\widetilde{T}\tfrac{\partial}{2\kappa+\partial},
\end{aligned}$$

where $\widetilde{T} = \psi_\mu^\ell \widetilde{S} \psi_\mu^{-\ell} \widetilde{S}^{-1} \in \mathcal{G}_{\mathrm{loc}}^\star$. Using (4.13) and observing that

$$\tfrac{2\kappa+\partial}{2\kappa} \psi_\mu^\ell \tfrac{2\varkappa}{2\varkappa+\partial} \psi_\mu^{-\ell} \tfrac{2\kappa}{2\kappa+\partial} = \tfrac{2\varkappa}{2\varkappa+\partial} \left[\tfrac{2\kappa+\partial}{2\kappa} \tfrac{2\varkappa+\partial}{2\varkappa} \psi_\mu^\ell \tfrac{2\varkappa}{2\varkappa+\partial} \tfrac{2\kappa}{2\kappa+\partial} \psi_\mu^{-\ell} \right] \left[\psi_\mu^\ell \tfrac{2\kappa+\partial}{2\kappa} \psi_\mu^{-\ell} \tfrac{2\kappa}{2\kappa+\partial} \right]$$

is a word over the alphabet $\mathcal{G}_{\mathrm{loc}}$, we again see that whenever $\tfrac{2\kappa+\partial}{2\kappa} \widetilde{T} \tfrac{2\kappa}{2\kappa+\partial}$ is a linear combination of words in $\mathcal{G}_{\mathrm{loc}}^\star$, so is $\tfrac{2\kappa+\partial}{2\kappa} T \tfrac{2\kappa}{2\kappa+\partial}$. The proof of (4.10) in this case is completed by inducting on the minimal number of letters required to spell S.   □

Lemma 4.6 (Properties of $S_{\mathrm{loc}}(n)$).

(i) (Distribution of exponential weights) If $m \in S(n)$ then for any non-negative $\ell_0 + \cdots + \ell_n = 24$ we have

$$m[f_1, \ldots, f_n] \psi_\mu^{24} = \psi_\mu^{\ell_0} m_1[\psi_\mu^{\ell_1} f_1, \ldots, \psi_\mu^{\ell_n} f_n],$$
$$m[f_1, \ldots, f_n \psi_\mu^{24}] = \psi_\mu^{\ell_0} m_2[\psi_\mu^{\ell_1} f_1, \ldots, \psi_\mu^{\ell_n} f_n],$$

where $m_j \in S_{\mathrm{loc}}(n)$.

(ii) (Symmetry) If $m \in S_{\mathrm{loc}}(n)$ and $\sigma \in \mathfrak{S}_n$ then

$$m[f_{\sigma(1)}, \ldots, f_{\sigma(n)}] \in S_{\mathrm{loc}}(n).$$

(iii) (Interior products) If $2 \le k \le n$, $m_1 \in S_{\mathrm{loc}}(k)$, and $m_2 \in S_{\mathrm{loc}}(n+1-k)$ then

$$m_1[f_1, \ldots, f_{k-1}, m_2[f_k, \ldots, f_n]] \in S_{\mathrm{loc}}(n).$$

(iv) (Leibniz rule) If $n \ge 2$ and $m \in S_{\mathrm{loc}}(n)$ then

$$(4.14) \qquad m[f_1', \ldots, f_{n-1}, f_n] = \partial m_1[f_1, \ldots, f_n] + m_2[f_1, (2-\partial)f_2, f_3, \ldots, f_n]$$
$$+ \cdots + m_n[f_1, \ldots, f_{n-1}, (2-\partial)f_n],$$

where $m_1, \ldots, m_n \in S_{\mathrm{loc}}(n)$.

(v) (Hölder's inequality) If $m \in S_{\mathrm{loc}}(n)$ and $1 \le p, p_j \le \infty$ are so that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$ then

$$(4.15) \qquad \|m[f_1, \ldots, f_n]\|_{L^p} \lesssim \prod_{j=1}^{n} \|f_j\|_{L^{p_j}},$$

uniformly in μ.

Proof. Part (ii) is an immediate consequence of the definition. Part (iii) follows from an easy induction in k.

We turn now to the proof of part (i), which we prove by induction on n. We will prove the very slightly stronger statement that for all $m \in S(n)$, and non-negative integers $\ell_0 + \cdots + \ell_n = \ell \le 24$ we may find $\widetilde{m} \in S_{\mathrm{loc}}(n)$ so that

$$(4.16) \qquad m[f_1, \ldots, f_n] \psi_\mu^\ell = \psi_\mu^{\ell_0} \widetilde{m}[\psi_\mu^{\ell_1} f_1, \ldots, \psi_\mu^{\ell_n} f_n].$$

For the base case of (4.16) we write $m \in S(1)$ as a linear combination

$$m[f] = \sum_{i=1}^{n} c_i T_i f,$$

where $c_i \in \mathbb{C}$ and $T_i \in \mathcal{G}^\star$. In this case, (4.16) follows from writing

$$m[f] \psi_\mu^\ell = \psi_\mu^{\ell_0} \sum_{i=1}^{n} c_i (\psi_\mu^{\ell_1} T_i \psi_\mu^{-\ell_1}) [\psi_\mu^{\ell_1} f] = \psi_\mu^{\ell_0} \widetilde{m} [\psi_\mu^{\ell_1} f]$$

and noting that $\psi_\mu^{\ell_1} T_i \psi_\mu^{-\ell_1} = [\psi_\mu^{\ell_1} T_i \psi_\mu^{-\ell_1} T_i^{-1}] T_i \in \mathcal{G}_{\mathrm{loc}}^\star$.

For the inductive step, we fix $N \geq 2$ and assume that (4.16) is true for all $1 \leq n \leq N-1$. We recall that elements of $S(N)$ are linear combinations of paraproducts with the representation

$$m[f_1, \ldots, f_N] = T\left[\prod_{j=1}^J m_j[f_{\sigma(n_{j-1}+1)}, \ldots, f_{\sigma(n_j)}]\right],$$

where $0 = n_0 < \cdots < n_J = N$, $m_j \in S(n_j - n_{j-1})$, $T \in \mathcal{G}^\star$, and $\sigma \in \mathfrak{S}_N$. Without loss of generality, we assume $\sigma = \mathrm{Id}$. Applying the inductive hypothesis we write

$$m[f_1, \ldots, f_N]\,\psi_\mu^\ell = \psi_\mu^{\ell_0}(\psi_\mu^{\ell-\ell_0} T_i \psi_\mu^{\ell_0-\ell})\left[\psi_\mu^{\ell-\ell_0}\prod_{j=1}^J m_j[f_{n_{j-1}+1}, \ldots, f_{n_j}]\right]$$

$$= \psi_\mu^{\ell_0}\widetilde{T}\left[\prod_{j=1}^J \widetilde{m}_j\left[\psi_\mu^{\ell_{n_{j-1}+1}} f_{n_{j-1}+1}, \ldots, \psi_\mu^{\ell_{n_j}} f_{n_j}\right]\right]$$

where $\widetilde{T} = \psi_\mu^{\ell-\ell_0} T \psi_\mu^{\ell_0-\ell} \in \mathcal{G}_{\mathrm{loc}}^\star$ and $\widetilde{m}_j \in S_{\mathrm{loc}}(n_j - n_{j-1})$. From (4.8), it is then clear that $\widetilde{m} \in S_{\mathrm{loc}}(N)$. The proof of the inductive step is completed by considering linear combinations of paraproducts of this form.

We now turn to part (iv), which is also proved by induction on n. All the requisite ideas can be understood most transparently from the treatment of the base case $n = 2$.

Given a word $T \in \mathcal{G}_{\mathrm{loc}}^\star$, we express

$$(4.17) \qquad \frac{2}{2-\partial} T\partial = \frac{2}{2+\partial} T \frac{2+\partial}{2}\frac{2+\partial}{2-\partial} - \frac{2}{2-\partial} T\frac{2-\partial}{2},$$

$$(4.18) \qquad \partial T \frac{2}{2-\partial} = \frac{2+\partial}{2} T \frac{2}{2+\partial}\frac{2+\partial}{2-\partial} - \frac{2-\partial}{2} T \frac{2}{2-\partial}.$$

By Example 3.7 (with $\varkappa = -1$) the operator $\frac{2+\partial}{2-\partial}$ is a linear combination of words in $\mathcal{G}^\star$. Combining this with Lemma 4.5, we see that both LHS(4.17) and LHS(4.18) are linear combinations of words in $\mathcal{G}_{\mathrm{loc}}^\star$.

For the base step $n = 2$, it suffices to consider $m \in S_{\mathrm{loc}}(2)$ that can be expressed as

$$m[f_1, f_2] = T[h_1[f_1] \cdot h_2[f_2]],$$

where $T \in \mathcal{G}_{\mathrm{loc}}^\star$ and $h_1, h_2 \in S_{\mathrm{loc}}(1)$. We define the paraproducts k_1, k_2 via

$$k_1[f] = \frac{1}{2-\partial} h_1[f'], \qquad k_2[f] = \partial h_2\left[\frac{f}{2-\partial}\right].$$

By (4.17) and (4.18) we see that $k_1, k_2 \in S_{\mathrm{loc}}(1)$. We also define

$$\ell[f] = h_2\left[\frac{2}{2-\partial} f\right]$$

and have $\ell \in S_{\mathrm{loc}}(1)$ by definition.

We then compute

$$m[f_1', f_2] = T\big[(2-\partial)k_1[f_1] \cdot h_2[f_2]\big]$$

$$= T(2-\partial)\big[k_1[f_1] \cdot h_2[f_2]\big] + T\big[k_1[f_1] \cdot \partial h_2[f_2]\big]$$

$$= (2-\partial)S\big[k_1[f_1] \cdot h_2[f_2]\big] + T\big[k_1[f_1] \cdot k_2[(2-\partial)f_2]\big]$$

$$= \underbrace{-\partial S\big[k_1[f_1] \cdot h_2[f_2]\big] + S\big[k_1[f_1] \cdot \ell[(2-\partial)f_2]\big]}_{=\partial m_1[f_1, f_2]} + \underbrace{T\big[k_1[f_1] \cdot k_2[(2-\partial)f_2]\big]}_{=m_2[f_1,(2-\partial)f_2]},$$

where $S = \frac{2}{2-\partial} T \frac{2-\partial}{2} \in \mathcal{G}_{\mathrm{loc}}^\star$ by (4.10) and $m_1, m_2 \in S_{\mathrm{loc}}(2)$ by definition.

To prove part (v), we first apply Lemma 2.9 to see that every element of $\mathcal{G}^\star_{\mathrm{loc}}$ is bounded on L^p whenever $1 \leq p \leq \infty$. The claim follows from a final induction on n. $\qquad\square$

To state our paraproduct estimates for (DNLS), it will once again be convenient to employ the convention that if $m \in S_{\mathrm{loc}}(n)$ and $f_1, \ldots, f_n$ satisfy the estimates (3.26), then we denote the expression $m[f_1, \ldots, f_n]$ by $m[f, \ldots, f]$. Moreover, if an expression involves several paraproducts $m_1, \ldots, m_k \in S_{\mathrm{loc}}(n)$, then we denote each paraproduct by m.

With this convention in hand, we turn to our paraproduct estimates for (DNLS):

Lemma 4.7. *Let $m \in S_{\mathrm{loc}}(4)$ and f satisfy (3.26). Then, the following estimates hold uniformly for $h \in \mathbb{R}$ and $|\varkappa| \geq 1$:*

$$(4.19) \qquad \int \left| \int m\Big[\psi^6_\mu f, \psi^6_\mu f, \tfrac{\psi^6_\mu f}{2\varkappa - \partial}, \tfrac{2-\partial}{2\varkappa - \partial}(\psi^6_\mu f)\Big]\, dx \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu$$
$$+ \int \left| \int m\Big[\psi^6_\mu f, \psi^6_\mu f, \psi^6_\mu f, \tfrac{2-\partial}{4\varkappa^2 - \partial^2}(\psi^6_\mu f)\Big]\, dx \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}(h)} \|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

Further, if $m \in S_{\mathrm{loc}}(6)$ then uniformly for $h \in \mathbb{R}$ and $|\varkappa| \geq 1$:

$$(4.20) \qquad \int \left| \int m\Big[\psi^4_\mu f, \psi^4_\mu f, \psi^4_\mu f, \psi^4_\mu f, \tfrac{\psi^4_\mu f}{2\varkappa - \partial}, \tfrac{\psi^4_\mu f}{2\varkappa - \partial}\Big]\, dx \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}(h)} \|q\|^2_{E^\sigma_\sigma} \|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

Proof. To prove these estimates, we decompose each f into Littlewood–Paley pieces and estimate the two highest frequencies in L^2 with a view to employ (2.30). To estimate the remaining lower frequency pieces, we rely on the following lemma.

Lemma 4.8. *For any f satisfying (3.26), we have*

$$(4.21) \qquad \|(\psi^\ell_\mu f)_N\|_{L^\infty} \lesssim N^{\frac{1}{2} - \sigma}(1 + N)^\sigma \|q\|_{E^\sigma_\sigma},$$

$$(4.22) \qquad \|(\psi^\ell_\mu f)_N\|_{L^\infty} \lesssim |\varkappa|^{-\sigma} N^{\frac{1}{2} - \sigma}(|\varkappa| + N)^{2\sigma} \|q\|_{E^\sigma_{2\sigma,\varkappa}},$$

uniformly for $0 \leq \ell \leq 12$ and $N \in 2^{\mathbb{Z}}$. In particular,

$$(4.23) \qquad \sum_{M \leq N} \tfrac{1}{|\varkappa| + M} \|(\psi^\ell_\mu f)_M\|_{L^\infty} \lesssim \tfrac{N^{\frac{1}{2} - \sigma}(1 + N)^\sigma}{|\varkappa|^{\frac{1}{2}}(|\varkappa| + N)^{\frac{1}{2}}} \|q\|_{E^\sigma_\sigma},$$

$$(4.24) \qquad \sum_{M \leq N} \tfrac{1}{|\varkappa| + M} \|(\psi^\ell_\mu f)_M\|_{L^\infty} \lesssim \tfrac{N^{\frac{1}{2}}}{|\varkappa|^{\frac{1}{2}}(|\varkappa| + N)^{\frac{1}{2}}} \|q\|_{E^\sigma_{2\sigma,\varkappa}}.$$

Proof. The bounds (4.21) and (4.22) follow easily from Bernstein's inequality, (2.13), and (3.26). To obtain the last two bounds, one considers separately the contribution from $M \leq |\varkappa|$ and $M > |\varkappa|$. $\qquad\square$

We start by considering (4.20). Decomposing into Littlewood–Paley pieces and employing Lemma 4.8, we find

$$\int m\Big[\psi^4_\mu f, \psi^4_\mu f, \psi^4_\mu f, \psi^4_\mu f, \tfrac{\psi^4_\mu f}{2\varkappa - \partial}, \tfrac{\psi^4_\mu f}{2\varkappa - \partial}\Big]\, dx$$

$$\lesssim \sum_{N_1 \geq \cdots \geq N_6} \frac{1}{(|\varkappa|+N_5)(|\varkappa|+N_6)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \prod_{j=3}^{6} \|P_{N_j}(\psi_\mu^4 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{2-4\sigma}(1+N_2)^{2\sigma}}{(|\varkappa|+N_2)^{1-2\sigma}} \|P_{N_1}(\psi_\mu^4 f)\|_{L^2} \|P_{N_2}(\psi_\mu^4 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \|\psi_\mu^4 f\|_{H^{\frac{1}{2}}}^2 \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

In view of (2.30) and (3.26), this contribution is acceptable.

Next we consider the first term on LHS(4.19). Decomposing once again into Littlewood–Paley pieces, we have

$$m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}\big(\psi_\mu^6 f\big)\Big]$$
$$= \sum_{M_1,M_2,M_3,M_4} m\Big[P_{M_1}(\psi_\mu^6 f), P_{M_2}(\psi_\mu^6 f), \tfrac{P_{M_3}(\psi_\mu^6 f)}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial} P_{M_4}(\psi_\mu^6 f)\Big].$$

To continue, we split the sum into two pieces: the first where $M_4 < \max\{M_j\}$ and the second where $M_4 = \max\{M_j\}$.

For the first summand, we apply (4.15) to estimate the terms with the two highest frequencies in L^2 and the remaining terms in L^∞. We then apply Bernstein's inequality followed by (4.24) to the N_4-term and (4.22) to the N_3-term to estimate

$$\sum_{\substack{M_1,M_2,M_3,M_4 \\ M_4<\max\{M_j\}}} \left|\int m\Big[P_{M_1}(\psi_\mu^6 f), P_{M_2}(\psi_\mu^6 f), \tfrac{P_{M_3}(\psi_\mu^6 f)}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial} P_{M_4}(\psi_\mu^6 f)\Big] dx\right|$$

$$\lesssim \sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \frac{(1+N_2)}{(|\varkappa|+N_2)(|\varkappa|+N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2 \geq N_3} \frac{(1+N_2)N_3^{1-\sigma}}{|\varkappa|^{\frac{1}{2}+\sigma}(|\varkappa|+N_2)(|\varkappa|+N_3)^{\frac{1}{2}-2\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{(1+N_2)^{2-\sigma}}{|\varkappa|^{\frac{1}{2}+\sigma}(|\varkappa|+N_2)^{\frac{3}{2}-2\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{(1+N_2)^{1-\sigma}}{|\varkappa|^{\frac{1}{2}+\sigma}(|\varkappa|+N_2)^{\frac{3}{2}-2\sigma}} \big(\tfrac{1+N_2}{1+N_1}\big)^{\frac{1}{2}} \|P_{N_1}(\psi_\mu^6 f)\|_{H^{\frac{1}{2}}} \|P_{N_2}(\psi_\mu^6 f)\|_{H^{\frac{1}{2}}} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \|\psi_\mu^6 f\|_{H^{\frac{1}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

Note that the frequency parameters N_j represent a permutation of the parameters M_j so as to account for the largest contribution. Integrating with respect to the measure $e^{-\frac{1}{200}|h-\mu|} d\mu$ and applying (2.30) and (3.26), we obtain an acceptable contribution.

For the second summand, we first use (4.14) to redistribute the derivative:

$$\sum_{M_4 \geq M_1,M_2,M_3} \left|\int m\Big[P_{M_1}(\psi_\mu^6 f), P_{M_2}(\psi_\mu^6 f), \tfrac{P_{M_3}(\psi_\mu^6 f)}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial} P_{M_4}(\psi_\mu^6 f)\Big] dx\right|$$

$$\leq \sum_{M_4 \geq M_1,M_2,M_3} \left|\int m\Big[(2-\partial)P_{M_1}(\psi_\mu^6 f), P_{M_2}(\psi_\mu^6 f), \tfrac{P_{M_3}(\psi_\mu^6 f)}{2\varkappa-\partial}, \tfrac{P_{M_4}(\psi_\mu^6 f)}{2\varkappa-\partial}\Big] dx\right|$$

$$+ \sum_{M_4 \geq M_1, M_2, M_3} \left| \int m\Big[P_{M_1}(\psi_\mu^6 f), (2-\partial)P_{M_2}(\psi_\mu^6 f), \tfrac{P_{M_3}(\psi_\mu^6 f)}{2\varkappa - \partial}, \tfrac{P_{M_4}(\psi_\mu^6 f)}{2\varkappa - \partial}\Big] dx \right|$$

$$+ \sum_{M_4 \geq M_1, M_2, M_3} \left| \int m\Big[P_{M_1}(\psi_\mu^6 f), P_{M_2}(\psi_\mu^6 f), \tfrac{2-\partial}{2\varkappa - \partial}P_{M_3}(\psi_\mu^6 f), \tfrac{P_{M_4}(\psi_\mu^6 f)}{2\varkappa - \partial}\Big] dx \right|,$$

where each $m_j \in S_{\mathrm{loc}}(4)$. We then proceed as in the first case to estimate each term by

$$\sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \tfrac{(1+N_2)}{(|\varkappa|+N_1)(|\varkappa|+N_4)} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^4 \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \tfrac{(1+N_2)^{2-\sigma}}{|\varkappa|^{\sigma+\frac{1}{2}}(|\varkappa|+N_1)(|\varkappa|+N_2)^{\frac{1}{2}-2\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2}\|P_{N_2}(\psi_\mu^6 f)\|_{L^2}\|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1}\|\psi_\mu^6 f\|_{H^{\frac{1}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is seen to be acceptable after an application of (2.30) and (3.26).

Applying a parallel argument, the second term on LHS(4.19) can be bounded by

$$\sum_{N_1 \geq N_2 \geq N_3 \geq N_4} \tfrac{(1+N_2)}{(|\varkappa|+N_2)(|\varkappa|+N_4)} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^4 \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty},$$

which is acceptable, as demonstrated above. This completes the proof of (4.19). $\square$

Combining Proposition 3.12 and Lemma 4.7, we have the following:

Lemma 4.9. *The following estimate holds uniformly for $q \in Q_*$, $h \in \mathbb{R}$, and $\varkappa \geq 1$:*

$$(4.25) \qquad \int \left| \mathrm{Im} \int j_{\mathrm{DNLS}}^{[\geq 4]}(\varkappa)\, \psi_\mu^{24}\, dx \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu \lesssim |\varkappa|^{-1}\|q\|_{F^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

In particular, in view of Lemma 2.2,

$$(4.26) \qquad \int_\kappa^\infty \int \left| \mathrm{Im} \int_{-1}^1 \int j_{\mathrm{DNLS}}^{[\geq 4]}(\varkappa)\, \psi_\mu^{24}\, dx\, dt \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu\, d\varkappa \lesssim \|q\|_{X^{\frac{1}{2}}}^2 \|q\|_{L_t^\infty E_{\sigma,\kappa}^\sigma}^2.$$

Proof. Using (3.29) and (3.6), we may write

$$\mathrm{Im}\, j_{\mathrm{DNLS}}^{[\geq 4]}(\varkappa) = \mathrm{Im}\Bigg\{ 2\Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]q$$

$$+ \Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]q'$$

$$(4.27) \qquad\qquad + i\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(-\varkappa)\Big]|q|^2 q \Bigg\}.$$

Using (3.46), we may integrate by parts to obtain

$$\int \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa)\, q'\, \psi_\mu^{24}\, dx = \int m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f)\Big] dx,$$

where $m \in S_{\mathrm{loc}}(4)$. We then apply (4.19) to obtain

$$(4.28) \qquad \int \left| \int \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa)\, q'\, \psi_\mu^{24}\, dx \right| e^{-\frac{1}{200}|h-\mu|}\, d\mu \lesssim |\varkappa|^{-1}\|q\|_{F^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

Similarly, from (3.46) we have

$$\int \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa)\,|q|^2 q\,\psi_\mu^{24}\,dx = \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(6)$. Recalling (3.32) and (3.60) we obtain

$$\int \Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(-\varkappa)\Big]\,|q|^2 q\,\psi_\mu^{24}\,dx$$

$$= i\int \bar{u}'|q|^2 q\,\psi_\mu^{24}\,dx = \int m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \psi_\mu^6 f, \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(4)$ and we recall that $u = \tfrac{q}{4\varkappa^2-\partial^2}$. Applying (4.19) and (4.20) then gives us

(4.29) $$\int \left|\int \Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(-\varkappa)\Big]\,|q|^2 q\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu$$

$$\lesssim |\varkappa|^{-1}\|q\|_{F^{\frac{1}{2}}(h)}^2\|q\|_{E^\sigma_{2\sigma,\varkappa}}^2.$$

Another application of (3.46) gives us

$$\int \sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 5]}(\varkappa)\,q\,\psi_\mu^{24}\,dx = \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(6)$. Further, from (3.52) we have

$$\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(\varkappa) - 4\varkappa^5|u|^2\bar{u}\Big]\,q\,\psi_\mu^{24}\,dx = \int m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f)\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(4)$. Once again we use (4.19) and (4.20) to bound

$$\int \left|\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) - 4\varkappa^5|u|^2\bar{u}\Big]\,q\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim |\varkappa|^{-1}\|q\|_{F^{\frac{1}{2}}(h)}^2\|q\|_{E^\sigma_{2\sigma,\varkappa}}^2.$$

In particular,

$$\int \left|\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]\,q\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu$$

(4.30) $$\lesssim |\varkappa|^{-1}\|q\|_{F^{\frac{1}{2}}(h)}^2\|q\|_{E^\sigma_{2\sigma,\varkappa}}^2.$$

The estimate (4.25) then follows from combining (4.28) through (4.30). $\qquad\square$

We are now in a position to complete the:

Proof of Proposition 4.1. Combining the identity (4.4) with Lemmas 4.2 and 4.9 we obtain

$$\int_\kappa^\infty \int \left|\mathrm{Im}\int_{-1}^1 \int j_{\mathrm{DNLS}}^{[2]}(\varkappa)\,\psi_\mu^{24}\,dx\,dt\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu\,d\varkappa$$

(4.31) $$\lesssim \|q\|_{L_t^\infty E^\sigma_{\sigma,\kappa}}^2\Big[1 + \|q\|_{L_t^\infty L^2}^2 + \|q\|_{X^{\frac{1}{2}}}^2\Big].$$

It remains to consider the quadratic terms in j_{DNLS}. A computation yields

(4.32) $$\mathrm{Im}\,j_{\mathrm{DNLS}}^{[2]} = 2\,\mathrm{Re}\Big\{\tfrac{q'}{4\varkappa^2-\partial^2}\bar{q}'\Big\} - \mathrm{Re}\Big\{\tfrac{q'}{4\varkappa^2-\partial^2}\bar{q}\Big\}'.$$

We may then integrate by parts to obtain

$$\operatorname{Im}\int j_{\mathrm{DNLS}}^{[2]}(\varkappa)\,\psi_\mu^{24}\,dx = 2\|(\psi_\mu^{12}q)'\|_{H_\varkappa^{-1}}^2 + 2\operatorname{Re}\int [\psi_\mu^{12},\tfrac{\partial}{4\varkappa^2-\partial^2}]q\,(\psi_\mu^{12}\bar q)'\,dx$$

To estimate the error, we use that

$$[\psi_\mu^{12},\tfrac{\partial}{4\varkappa^2-\partial^2}] = -\tfrac{1}{4\varkappa^2-\partial^2}\big[(\psi_\mu^{12})'' + 2(\psi_\mu^{12})'\partial\big]\tfrac{\partial}{4\varkappa^2-\partial^2} - \tfrac{1}{4\varkappa^2-\partial^2}(\psi_\mu^{12})'$$

together with (2.13) to bound

$$\left|\int [\psi_\mu^{12},\tfrac{\partial}{4\varkappa^2-\partial^2}]q\,(\psi_\mu^{12}\bar q)'\,dx\right| \lesssim \varkappa^{-1}\|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

Thus,

$$\|\psi_\mu^{12}q\|_{E_{1,\varkappa}^1}^2 \lesssim \left|\operatorname{Im}\int j_{\mathrm{DNLS}}^{[2]}(\varkappa)\,\psi_\mu^{24}\,dx\right| + \varkappa^{-1}\|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

Integrating in $\varkappa$ over $[\kappa,\infty)$, using (2.4) and then (4.31) we get

$$\kappa\int \|\psi_\mu^{12}q\|_{L_t^2 E_{\frac{1}{2},\kappa}^1}^2\,e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim \|q\|_{L_t^\infty E_{\sigma,\kappa}^\sigma}^2\left[1 + \|q\|_{L_t^\infty L_x^2}^2 + \|q\|_{X^{\frac{1}{2}}}^2\right].$$

Estimating

$$\left\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\right\|_{L_t^2 H^{\frac{3}{2}}}^2 \lesssim \left\|\tfrac{P_{\le 1}(\psi_\mu^{12}q)}{\sqrt{4\kappa^2-\partial^2}}\right\|_{L_t^2 H^{\frac{3}{2}}}^2 + \left\|\tfrac{P_{>1}(\psi_\mu^{12}q)}{\sqrt{4\kappa^2-\partial^2}}\right\|_{L_t^2 H^{\frac{3}{2}}}^2$$

$$\lesssim \kappa^{-2}\|q\|_{L_t^\infty L_x^2}^2 + \kappa\|\psi_\mu^{12}q\|_{L_t^2 E_{\frac{1}{2},\kappa}^1}^2,$$

we then get

(4.33) $$\|q\|_{X_\kappa^{\frac{1}{2}}}^2 \lesssim \|q\|_{L_t^\infty E_{\sigma,\kappa}^\sigma}^2\|q\|_{X^{\frac{1}{2}}}^2 + \|q\|_{L_t^\infty E_{\sigma,\kappa}^\sigma}^2\left[1 + \|q\|_{L_t^\infty L_x^2}^2\right] + \kappa^{-2}\|q\|_{L_t^\infty L_x^2}^2$$

To prove (4.1) we first take $\kappa = 1$ and use (1.19) to absorb the first term on the right-hand side of (4.33) into the left-hand side, and then invoke the conservation of the L^2 norm. The estimate (4.2) follows from using (4.1) in (4.33) and then applying (2.3).   □

5. Tightness

The goal of this section is to show that the family of orbits emanating from an L^2-precompact set of Schwartz initial data remains tight, at least for times $t \in [-1,1]$. We begin by constructing a suitable function ϕ_R for localizing to the spatial region $|x| \ge R$ for given $R \ge 100$. For such R, we first define

$$\chi_R(x) = \tfrac{1}{R}\int_{R\le|\mu|\le 2R} \operatorname{sgn}(\mu)\psi_\mu^{24}(x)\,d\mu.$$

Note that χ_R is odd and vanishes at $x = 0$. We then define

$$\phi_R(x) = \int_0^x \chi_R(y)\,dy,$$

which is even and everywhere positive. Indeed

$$\phi_R(x) \gtrsim 1 \quad \text{uniformly for} \quad |x| \ge 2R \ge 200.$$

In view of this property of ϕ_R, a bounded subset $Q \subset L^2$ is tight in L^2 if

$$\int \phi_R(x)|q(x)|^2\,dx \to 0 \quad \text{as } R \to \infty, \text{ uniformly for } q \in Q.$$

Proposition 5.1 (Tightness for (DNLS)). *Suppose $Q \subset \mathcal{S}$ is an L^2 bounded and equicontinuous set for which*

$$Q_* = \left\{ e^{tJ\nabla H} q : |t| \leq 1 \text{ and } q \in Q \right\}$$

is δ-good for some sufficiently small δ. If Q is also tight, then so too is Q_*.

Proof. From the microscopic conservation law (1.1) we obtain

$$(5.1) \qquad \frac{d}{dt} \int \phi_R |q|^2 \, dx = \frac{1}{R} \int_{R \leq |\mu| \leq 2R} \operatorname{sgn}(\mu) \int \left[2 \operatorname{Im}(q' \bar{q}) + \tfrac{3}{2} |q|^4 \right] \psi_\mu^{24} \, dx \, d\mu.$$

Our goal is to deduce the tightness of Q_* from that of Q by estimating the right-hand side above in $L^1([-1,1]; dt)$ and showing that it converges to zero as $R \to \infty$.

For the first term on RHS(5.1) we have

$$\int_{-1}^{1} \left| \int q' \bar{q} \, \psi_\mu^{24} \, dx \right| dt \lesssim \|\psi_\mu^{12} q\|_{L_t^2 H^{\frac{1}{2}}}^2 \lesssim \kappa \|\psi_\mu^{12} q\|_{L_{t,x}^2}^2 + \left\| \tfrac{\psi_\mu^{12} q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{L_t^2 H^{\frac{3}{2}}}^2,$$

where the second inequality follows from decomposing into frequencies $\leq \kappa$ and $> \kappa$. For the second term on RHS(5.1), we apply the Gagliardo–Nirenberg inequality to bound

$$\|\psi_\mu^6 q\|_{L_{t,x}^4}^4 \lesssim \|\psi_\mu^6 q\|_{L_t^2 H^{\frac{1}{2}}}^2 \|\psi_\mu^6 q\|_{L_t^\infty L_x^2}^2 \lesssim \left[\kappa \|\psi_\mu^6 q\|_{L_{t,x}^2}^2 + \left\| \tfrac{\psi_\mu^6 q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{L_t^2 H^{\frac{3}{2}}}^2 \right] \|\psi_\mu^6 q\|_{L_t^\infty L_x^2}^2.$$

In this way we deduce that

$$\|\mathrm{RHS}(5.1)\|_{L_t^1} \lesssim \left[\tfrac{\kappa}{R} \|\psi_\mu^{12} q\|_{L_\mu^2 L_t^2 L_x^2}^2 + \left\| \tfrac{\psi_\mu^{12} q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{L_\mu^\infty L_t^2 H^{\frac{3}{2}}}^2 \right] \left[1 + \|q\|_{L_t^\infty L_x^2} \right].$$

By Fubini, (2.1), and (2.32) we then have

$$\|\mathrm{RHS}(5.1)\|_{L_t^1} \lesssim \left[\tfrac{\kappa}{R} \|q\|_{L_t^\infty L_x^2}^2 + \|q\|_{X_\kappa^{\frac{1}{2}}}^2 \right] \left[1 + \|q\|_{L_t^\infty L_x^2} \right],$$

which can be made arbitrarily small by first choosing κ large and applying (4.2), and then choosing R large. $\qquad \square$

6. Local smoothing for the difference flow

In this section we prove local smoothing for Schwartz solutions of the difference flow.

Proposition 6.1 (Local smoothing for the difference flow). *Let $Q \subset \mathcal{S}$ be an L^2 bounded and equicontinuous set such that*

$$Q_* = \left\{ q(t; \kappa) = e^{tJ\nabla(H - H_\kappa)} q : t \in \mathbb{R}, \ q \in Q, \ \text{and} \ \kappa \geq 2 \right\}$$

is a δ-good set for a sufficiently small $\delta > 0$. Then the local smoothing estimate

$$(6.1) \qquad\qquad\qquad \|q\|_{X_\kappa^{\frac{1}{2}}} \lesssim \|q(0)\|_{L^2}.$$

holds uniformly for $\kappa \geq 2$ and $q(0) \in Q$.

Our proof will mirror that of Proposition 4.1. Once again, we take $0 < \sigma < \frac{1}{2}$ and Q_* as in the hypothesis of Proposition 6.1. Taking ϕ_μ to be defined as in (4.3) and using Propostion 3.5, we integrate by parts to obtain

$$(6.2) \qquad \operatorname{Im} \int_{-1}^{1} \int j_{\mathrm{diff}}(q; \varkappa, \kappa) \, \psi_\mu^{24} \, dx \, dt = \operatorname{Im} \int \rho(q; \varkappa) \phi_\mu \, dx \Big|_{t=-1}^{t=1}.$$

Once again, our main challenge will be to estimate the remainder terms $j_{\text{diff}}^{[\geq 4]}$. To this end, we start with the following collection of paraproduct estimates:

Lemma 6.2. *The following paraproduct estimates hold uniformly for $q \in Q_*$, $h \in \mathbb{R}$, $|\varkappa|, |\kappa| \geq 1$, and functions f that are admissible in the sense of* (3.26):
i) Quartic paraproducts. Let $m \in S_{\text{loc}}(4)$. Then we have the estimates

$$(6.3) \quad \int \left| \int m\left[\psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{(2-\partial)^3}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim |\varkappa|^{-1} \tfrac{\kappa^2+\varkappa^2}{\varkappa^2} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

$$(6.4) \quad \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \psi_\mu^6 f, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma^\sigma}^2,$$

(6.5)

$$\int \left| \int m\left[\psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{(2-\partial)^3}{(2\varkappa-\partial)(4\kappa^2-\partial^2)}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^6 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^6 f, \psi_\mu^6 f, \psi_\mu^6 f, \tfrac{(2-\partial)^3}{(4\kappa^2-\partial^2)(4\varkappa^2-\partial^2)}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \kappa^4 \int m\left[\tfrac{\psi_\mu^6 f}{4\kappa^2-\partial^2}, \tfrac{\psi_\mu^6 f}{4\kappa^2-\partial^2}, \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim |\varkappa|^{-1} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

ii) Sextic paraproducts. If $m \in S_{\text{loc}}(6)$ then we have the estimates

(6.6)

$$\int \left| \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^4 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$+ \int \left| \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f)\right] dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim |\varkappa|^{-1} \tfrac{\kappa^2+\varkappa^2}{\varkappa^2} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^2_{E_\sigma} \|q\|^2_{E_{2\sigma,\varkappa}},$$

(6.7)
$$\int \left| \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$\lesssim |\kappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^4_{E_\sigma},$$

(6.8)
$$\int \left| \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \kappa^2 \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{4-\partial^2}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \varkappa^2 \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{4\varkappa^2-\partial^2}, \tfrac{4-\partial^2}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \kappa^8 \int m\left[\tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{\psi_\mu^4 f}{4\kappa^2-\partial^2}, \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^2_{E_\sigma} \|q\|^2_{E_{2\sigma,\varkappa}},$$

(6.9)
$$\int \left| \kappa \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \kappa \int m\left[\psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f), \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^2_{E_\sigma} \|q\|^2_{E_{2\sigma,\varkappa}}.$$

iii) Octic paraproducts. If $m \in S_{\mathrm{loc}}(8)$ *then we have the estimates*

(6.10)
$$\int \left| \varkappa \int m\left[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^3 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$\lesssim |\varkappa|^{-1} \tfrac{\kappa^2+\varkappa^2}{\varkappa^2} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^4_{E_\sigma} \|q\|^2_{E_{2\sigma,\varkappa}},$$

(6.11)
$$\int \left| \kappa \int m\left[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^3 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \kappa^2 \int m\left[\psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^3 f), \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial} \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$+ \int \left| \kappa^2 \int m\left[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^3 f) \right] dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|^2_{F^{\frac{1}{2}}_\kappa(h)} \|q\|^4_{E_\sigma} \|q\|^2_{E_{2\sigma,\varkappa}}.$$

iv) Decic paraproducts. If $m \in S_{\mathrm{loc}}(10)$ then we have the estimates

$$(6.12) \qquad \int \left| \varkappa^2 \int m\left[\underbrace{\psi_\mu^2 f, \ldots, \psi_\mu^2 f}_{4}, \underbrace{\tfrac{\psi_\mu^2 f}{2\varkappa - \partial}, \ldots, \tfrac{\psi_\mu^2 f}{2\varkappa - \partial}}_{6}\right] \psi_\mu^4 \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim |\varkappa|^{-1} \tfrac{\kappa^2 + \varkappa^2}{\varkappa^2} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma^\sigma}^6 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

$$(6.13) \quad \int \left| \kappa^2 \int m\left[\underbrace{\psi_\mu^2 f, \ldots, \psi_\mu^2 f}_{4}, \tfrac{\psi_\mu^2 f}{2\varkappa - \partial}, \tfrac{\psi_\mu^2 f}{2\varkappa - \partial}, \underbrace{\tfrac{\psi_\mu^2 f}{2\kappa - \partial}, \ldots, \tfrac{\psi_\mu^2 f}{2\kappa - \partial}}_{4}\right] \psi_\mu^4 \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim |\varkappa|^{-1} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma^\sigma}^6 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2.$$

Proof. For all of the ensuing estimates, we follow the argument of Lemma 4.7:

(1) Decompose into Littlewood–Paley pieces.
(2) If derivatives fall at high frequency, integrate by parts using (4.14).
(3) Estimate the two highest frequency terms in L^2 and the remaining terms in L^∞ using (4.15).
(4) Estimate the low frequency terms using Lemma 4.8.
(5) Bound the highest two frequency terms using (2.30).

We illustrate this in detail with the first term on LHS(6.3). Decomposing into Littlewood–Paley pieces, we consider

$$\sum_{M_1, M_2, M_3, M_4} \int m\left[P_{M_1}(\psi_\mu^6 f), \tfrac{P_{M_2}(\psi_\mu^6 f)}{2\varkappa - \partial}, \tfrac{2-\partial}{2\varkappa - \partial} P_{M_3}(\psi_\mu^6 f), \tfrac{4 - \partial^2}{4\varkappa^2 - \partial^2} P_{M_4}(\psi_\mu^6 f)\right] dx.$$

We now decompose the sum into three parts.

The first summand is where $M_3, M_4 < \max\{M_j\}$. Here, we apply (4.15), Bernstein's inequality, and Lemma 4.8 to estimate

$$\sum_{\substack{M_1, M_2, M_3, M_4 \\ M_3, M_4 < \max\{M_j\}}} \left| \int m\left[P_{M_1}(\psi_\mu^6 f), \tfrac{P_{M_2}(\psi_\mu^6 f)}{2\varkappa - \partial}, \tfrac{2-\partial}{2\varkappa - \partial} P_{M_3}(\psi_\mu^6 f), \tfrac{4 - \partial^2}{4\varkappa^2 - \partial^2} P_{M_4}(\psi_\mu^6 f)\right] dx \right|$$

$$\lesssim \sum_{N_1 \geq \cdots \geq N_4} \tfrac{(1+N_2)^2 (1+N_3)}{(|\varkappa| + N_2)^2 (|\varkappa| + N_3)(|\varkappa| + N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \tfrac{|\varkappa|^{-1} N_2^{1-2\sigma}(1+N_2)^3}{(|\varkappa| + N_2)^{3-2\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \left\| \tfrac{\psi_\mu^6 f}{\sqrt{4\varkappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2 \lesssim |\varkappa|^{-1} \tfrac{\kappa^2 + \varkappa^2}{\varkappa^2} \left\| \tfrac{\psi_\mu^6 f}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

where we note that the M_j have been permuted in the first inequality to account fo the largest contribution. This is acceptable after integrating with respect to $e^{-\frac{1}{200}|h-\mu|} \, d\mu$ and applying (2.30).

The second summand is where $M_3 \leq M_4 = \max\{M_j\}$. Here, we use (4.14) to integrate by parts and then proceed as for the first summand to obtain

$$\sum_{\substack{M_1, M_2, M_3, M_4 \\ M_3 \leq M_4 = \max\{M_j\}}} \left| \int m\left[P_{M_1}(\psi_\mu^6 f), \tfrac{P_{M_2}(\psi_\mu^6 f)}{2\varkappa - \partial}, \tfrac{2-\partial}{2\varkappa - \partial} P_{M_3}(\psi_\mu^6 f), \tfrac{4 - \partial^2}{4\varkappa^2 - \partial^2} P_{M_4}(\psi_\mu^6 f)\right] dx \right|$$

$$\lesssim \sum_{N_1 \geq \cdots \geq N_4} \frac{(1+N_2)^3}{(|\varkappa|+N_1)^2(|\varkappa|+N_2)(|\varkappa|+N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{N_2^{1-2\sigma}(1+N_2)^3}{|\varkappa|^{\frac{1}{2}+\sigma}(|\varkappa|+N_2)^{\frac{7}{2}-3\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \left\| \frac{\psi_\mu^6 f}{\sqrt{4\varkappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2 \lesssim |\varkappa|^{-1} \frac{\kappa^2+\varkappa^2}{\varkappa^2} \left\| \frac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is again acceptable.

The final summand, where $M_4 < M_3 = \max\{M_j\}$, is estimated in a similar way, using (4.14) to obtain a contribution of

$$\sum_{\substack{M_1,M_2,M_3,M_4 \\ M_4 < M_3 = \max\{M_j\}}} \left| \int m\left[P_{M_1}(\psi_\mu^6 f), \frac{P_{M_2}(\psi_\mu^6 f)}{2\varkappa-\partial}, \frac{2-\partial}{2\varkappa-\partial}P_{M_3}(\psi_\mu^6 f), \frac{4-\partial^2}{4\varkappa^2-\partial^2}P_{M_4}(\psi_\mu^6 f) \right] dx \right|$$

$$\lesssim \sum_{N_1 \geq \cdots \geq N_4} \frac{(1+N_2)^3}{(|\varkappa|+N_1)(|\varkappa|+N_2)^2(|\varkappa|+N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty},$$

which is again acceptable, as before.

The remaining terms on LHS(6.3) are estimated similarly. In each case, their contribution is bounded by

$$\sum_{N_1 \geq \cdots \geq N_4} \frac{(1+N_2)^3}{(|\varkappa|+N_2)^2(|\varkappa|+N_3)(|\varkappa|+N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty},$$

which is acceptable after summation.

For (6.4), we proceed similarly, to obtain a bound of

$$\sum_{N_1 \geq \cdots \geq N_4} \frac{(1+N_2)^2}{(|\kappa|+N_2)^2} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{N_2^{1-2\sigma}(1+N_2)^{2+2\sigma}}{(|\kappa|+N_2)^2} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^2$$

$$\lesssim \left\| \frac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma^\sigma}^2,$$

which is acceptable.

For the estimate (6.5) we argue as before to obtain a bound of

$$\sum_{N_1 \geq \cdots \geq N_4} \frac{(1+N_2)^3}{(|\kappa|+N_2)^2(|\varkappa|+N_3)(|\varkappa|+N_4)} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^6 f)\|_{L^2} \prod_{j=3}^{4} \|P_{N_j}(\psi_\mu^6 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1}N_2^{1-2\sigma}(1+N_2)^3}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{1-2\sigma}} \|P_{N_1}(\psi_\mu^6 f)\|_{L^2} \|P_{N_2}(\psi_\mu^6 f)\|_{L^2} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \left\| \frac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is once again acceptable.

The sextic terms are estimated similarly. For (6.6), after integrating by parts we obtain a bound of

$$\sum_{N_1 \geq \cdots \geq N_6} \frac{(1+N_2)^2}{(|\varkappa|+N_2)^2(|\varkappa|+N_5)(|\varkappa|+N_6)} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \prod_{j=3}^6 \|P_{N_j}(\psi_\mu^4 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{2-4\sigma}(1+N_2)^{2+2\sigma}}{(|\varkappa|+N_2)^{3-2\sigma}} \|P_{N_1}(\psi_\mu^4 f)\|_{L^2} \|P_{N_2}(\psi_\mu^4 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \frac{\kappa^2+\varkappa^2}{\varkappa^2} \left\| \frac{\psi_\mu^4 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is acceptable.

Similarly, we may bound LHS(6.7) by

$$\sum_{N_1 \geq \cdots \geq N_6} \frac{(1+N_2)}{(|\kappa|+N_2)(|\kappa|+N_5)(|\kappa|+N_6)} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \prod_{j=3}^6 \|P_{N_j}(\psi_\mu^4 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{N_2^{2-4\sigma}(1+N_2)^{1+4\sigma}}{|\kappa|(|\kappa|+N_2)^2} \|P_{N_1}(\psi_\mu^4 f)\|_{L^2} \|P_{N_2}(\psi_\mu^4 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^4$$

$$\lesssim |\kappa|^{-1} \left\| \frac{\psi_\mu^4 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma^\sigma}^4,$$

which is acceptable.

For (6.8), after integrating by parts we may bound each term by

$$\sum_{N_1 \geq \cdots \geq N_6} \frac{(1+N_2)^2}{(|\kappa|+N_2)^2(|\varkappa|+N_5)(|\varkappa|+N_6)} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \prod_{j=3}^6 \|P_{N_j}(\psi_\mu^4 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{2-4\sigma}(1+N_2)^{2+2\sigma}}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{1-2\sigma}} \|P_{N_1}(\psi_\mu^4 f)\|_{L^2} \|P_{N_2}(\psi_\mu^4 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2$$

$$\lesssim |\varkappa|^{-1} \left\| \frac{\psi_\mu^6 f}{\sqrt{4\kappa^2-\partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is again acceptable.

Turning to (6.9), our basic technique gives us a bound of

$$\sum_{\tau \in \mathfrak{S}} \sum_{N_1 \geq \cdots \geq N_6} \frac{|\kappa|(1+N_2)^2}{(|\kappa|+N_2)(|\kappa|+N_{\tau(3)})(|\kappa|+N_{\tau(4)})(|\varkappa|+N_{\tau(5)})(|\varkappa|+N_{\tau(6)})}$$

$$\prod_{j=1}^2 \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \prod_{j=3}^6 \|P_{N_j}(\psi_\mu^4 f)\|_{L^\infty},$$

where $\mathfrak{S}$ is the set of permutations of $\{3,4,5,6\}$. Estimating the $N_{\tau(3)}, N_{\tau(4)}$ terms in E_σ^σ using (4.21) and the $N_{\tau(5)}, N_{\tau(6)}$ terms in $E_{2\sigma,\varkappa}^\sigma$ using (4.22), we obtain a bound of

$$\sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{2-4\sigma}(1+N_2)^{2+2\sigma}}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{1-2\sigma}} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^4 f)\|_{L^2} \|q\|_{E_\sigma^\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

which is acceptable.

Turning next to the octic estimates, we may bound LHS(6.10) by

$$\sum_{N_1 \geq \cdots \geq N_8} \frac{|\varkappa|(1+N_2)}{(|\varkappa|+N_2)} \prod_{j=5}^8 \frac{1}{|\varkappa|+N_j} \prod_{j=1}^2 \|P_{N_j}(\psi_\mu^3 f)\|_{L^2} \prod_{j=3}^8 \|P_{N_j}(\psi_\mu^3 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{3-6\sigma}(1+N_2)^{1+4\sigma}}{(|\kappa|+N_2)^{3-2\sigma}} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^3 f)\|_{L^2} \|q\|_{E_\sigma}^4 \|q\|_{E_{2\sigma,\varkappa}}^2$$

$$\lesssim |\varkappa|^{-1} \frac{\kappa^2+\varkappa^2}{\varkappa^2} \left\| \frac{\psi_\mu^3 f}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma}^4 \|q\|_{E_{2\sigma,\varkappa}}^2,$$

which is acceptable.

For (6.11), we argue as in (6.9) so that after estimating the low frequency terms in E_σ^σ or $E_{2\sigma,\varkappa}^\sigma$ (depending on the associated denominator) we obtain a bound of

$$\sum_{N_1 \geq N_2} \left\{ \frac{|\varkappa|^{-1} N_2^{3-6\sigma}(1+N_2)^{1+4\sigma}}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{1-2\sigma}} + \frac{|\varkappa|^{-\frac{1}{2}-\sigma} N_2^{3-2\sigma}(1+N_2)^{1+\sigma}}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{\frac{3}{2}-2\sigma}} \right\} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^3 f)\|_{L^2} \|q\|_{E_\sigma}^4 \|q\|_{E_{2\sigma,\varkappa}}^2$$

which is acceptable.

Again applying our basic technique to (6.12), we obtain a bound of

$$\sum_{N_1 \geq \cdots \geq N_{10}} |\varkappa|^2 \prod_{j=5}^{10} \frac{1}{|\varkappa|+N_j} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^2 f)\|_{L^2} \prod_{j=3}^{10} \|P_{N_j}(\psi_\mu^2 f)\|_{L^\infty}$$

$$\lesssim \sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{4-8\sigma}(1+N_2)^{6\sigma}}{(|\kappa|+N_2)^{3-2\sigma}} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^2 f)\|_{L^2} \|q\|_{E_\sigma}^6 \|q\|_{E_{2\sigma,\varkappa}}^2$$

$$\lesssim |\varkappa|^{-1} \frac{\kappa^2+\varkappa^2}{\varkappa^2} \left\| \frac{\psi_\mu^2 f}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma}^6 \|q\|_{E_{2\sigma,\varkappa}}^2,$$

which is acceptable.

Finally, (6.13) again follows the argument of (6.9), (6.11), estimating the low frequency terms in E_σ^σ or $E_{2\sigma,\varkappa}^\sigma$ to obtain an acceptable bound of

$$\sum_{N_1 \geq N_2} \frac{|\varkappa|^{-1} N_2^{4-8\sigma}(1+N_2)^{6\sigma}}{(|\kappa|+N_2)^2(|\varkappa|+N_2)^{1-2\sigma}} \prod_{j=1}^{2} \|P_{N_j}(\psi_\mu^2 f)\|_{L^2} \|q\|_{E_\sigma}^6 \|q\|_{E_{2\sigma,\varkappa}}^2.$$

This completes the proof of the lemma. $\qquad\square$

Combining Propositions 3.12 and Lemma 6.2, we obtain the following:

Lemma 6.3. *Let* $q \in Q_*$, $h \in \mathbb{R}$, $\kappa \geq 1$, *and* $\varkappa \in I_\kappa = [1, \frac{\kappa}{2}] \cup [2\kappa, \infty)$. *Then we have the estimate*

$$(6.14) \qquad \int \left| \mathrm{Im} \int j_{\mathrm{diff}}^{[\geq 4]} \psi_\mu^{24} dx \right| e^{-\frac{1}{200}|h-\mu|} d\mu$$

$$\lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \left[\varkappa^{-1} \|q\|_{E_{2\sigma,\varkappa}}^2 + \frac{\kappa}{\kappa^2+\varkappa^2} \|q\|_{E_\sigma}^2 \right].$$

Proof. From (3.32), we have

$$\frac{1}{\kappa} q - \left[\frac{1}{\sqrt{\kappa}} g_{12}^{[1]}(\kappa) - \frac{1}{\sqrt{-\kappa}} g_{12}^{[1]}(-\kappa) \right] = -\frac{1}{\kappa} u(\kappa)'',$$

$$\frac{1}{2\kappa} q' - \left[\sqrt{\kappa} g_{12}^{[1]}(\kappa) - \sqrt{-\kappa} g_{12}^{[1]}(-\kappa) \right] = -\frac{1}{2\kappa} u(\kappa)'''.$$

We may then use (3.30) and (3.6) to decompose

$$\mathrm{Im}\, j_{\mathrm{diff}}^{[\geq 4]} = \mathrm{Im} \Bigg\{$$

$$(6.15) \qquad\qquad - \frac{2\varkappa^2}{\kappa^2-\varkappa^2} \left[\sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 3]}(\varkappa) + \sqrt{-\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 3]}(-\varkappa) \right] q$$

$$(6.16) \qquad -\tfrac{\varkappa^2}{\kappa^2-\varkappa^2}\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]q'$$

$$(6.17) \qquad -\tfrac{i\varkappa^2}{\kappa^2-\varkappa^2}\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(-\varkappa)\Big]|q|^2 q$$

$$(6.18) \qquad -\tfrac{2\kappa^2}{\kappa^2-\varkappa^2}\Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]u(\kappa)''$$

$$(6.19) \qquad -\tfrac{\kappa^2}{\kappa^2-\varkappa^2}\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]u(\kappa)'''$$

$$(6.20) \qquad -\tfrac{2\kappa^3}{\kappa^2-\varkappa^2}\Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)(-\varkappa)\Big]$$

$$\times \Big[\tfrac{1}{\sqrt{\kappa}}g_{12}^{[\geq 3]}(\kappa) - \tfrac{1}{\sqrt{-\kappa}}g_{12}^{[\geq 3]}(-\kappa)\Big]$$

$$(6.21) \qquad -\tfrac{2\kappa^3}{\kappa^2-\varkappa^2}\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)(-\varkappa)\Big]$$

$$\times \Big[\sqrt{\kappa}g_{12}^{[\geq 3]}(\kappa) - \sqrt{-\kappa}g_{12}^{[\geq 3]}(-\kappa) - \tfrac{i}{2\kappa}|q|^2 q\Big]$$

$$(6.22) \qquad -\tfrac{\kappa^3}{\kappa^2-\varkappa^2}\Big[\gamma^{[\geq 4]}(\kappa) - \gamma^{[\geq 4]}(-\kappa)\Big]$$

$$\Bigg\}.$$

We now proceed to use Proposition 3.12 and Lemma 6.2 to remove the leading order terms from each line as follows:

For (6.15), we first use (3.46) to write

$$\int \sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 9]}(\varkappa)\, q\, \psi_\mu^{24}\, dx = \varkappa^2 \int m\Big[\underbrace{\psi_\mu^2 f, \dots, \psi_\mu^2 f}_{4}, \underbrace{\tfrac{\psi_\mu^2 f}{2\varkappa-\partial}, \dots, \tfrac{\psi_\mu^2 f}{2\varkappa-\partial}}_{6}\Big] \psi_\mu^4\, dx,$$

where $m \in S_{\mathrm{loc}}(10)$, to which we apply (6.12). Next, we use (3.54) to write

$$\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[7]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[7]}(-\varkappa)\Big] q\, \psi_\mu^{24}\, dx$$

$$= \varkappa \int m\Big[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^3 f)\Big]\, dx,$$

where $m \in S_{\mathrm{loc}}(8)$, which we bound using (6.10). Similarly, we use (3.53) to write

$$\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]}(-\varkappa) - 64i\varkappa^8|u(\varkappa)|^4\bar{u}(\varkappa)\Big] q\, \psi_\mu^{24}\, dx$$

$$= \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^4 f)\Big]\, dx$$

$$+ \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f)\Big]\, dx,$$

where each $m \in S_{\mathrm{loc}}(6)$, which can be estimated using (6.6). Finally, we use the second expression on RHS(3.52) to obtain

$$\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(-\varkappa)$$

$$+ 16\varkappa^4|u(\varkappa)|^2\bar{u}(\varkappa)' + 4\varkappa^4\bar{u}(\varkappa)^2 u(\varkappa)'\Big] q\, \psi_\mu^{24}\, dx$$

$$= \int m\Big[\psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\Big]\, dx$$

$$+ \int m\left[\frac{\psi_\mu^6 f}{2\varkappa - \partial}, \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f)\right] dx,$$

where each $m \in S_{\mathrm{loc}}(4)$ and then apply (6.3). Combining these estimates and using (1.19) gives us

$$(6.23) \quad \int\left|\int\left[(6.15) - \frac{2\varkappa^2}{\kappa^2-\varkappa^2}\left[16\varkappa^4|u(\varkappa)|^2\bar{u}(\varkappa)' + 4\varkappa^4\bar{u}(\varkappa)^2 u(\varkappa)'\right.\right.\right.$$
$$\left.\left.\left. - 64i\varkappa^8|u(\varkappa)|^4\bar{u}(\varkappa)\right]q\right]\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim |\varkappa|^{-1}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma}.$$

We estimate the contribution of (6.16) similarly. From (3.46), we have

$$\int \frac{1}{\sqrt{\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[\geq 7]}(\varkappa)\,q'\,\psi_\mu^{24}\,dx$$
$$= \varkappa\int m\left[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \frac{\psi_\mu^3 f}{2\varkappa-\partial}, \frac{\psi_\mu^3 f}{2\varkappa-\partial}, \frac{\psi_\mu^3 f}{2\varkappa-\partial}, \frac{\psi_\mu^3 f}{2\varkappa-\partial}, \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^3 f)\right]dx,$$

where $m \in S_{\mathrm{loc}}(8)$, whereas from (3.53) we have

$$\int\left[\frac{1}{\sqrt{\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[5]}(\varkappa) + \frac{1}{\sqrt{-\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[5]}(-\varkappa)\right]q'\,\psi_\mu^{24}\,dx$$
$$= \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \frac{\psi_\mu^4 f}{2\varkappa-\partial}, \frac{\psi_\mu^4 f}{2\varkappa-\partial}, \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f), \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f)\right]dx,$$

where each $m \in S_{\mathrm{loc}}(6)$. Using the second expression on RHS(3.52) we get

$$\int\left[\frac{1}{\sqrt{\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[3]}(\varkappa) + \frac{1}{\sqrt{-\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[3]}(-\varkappa) - 8\varkappa^4|u(\varkappa)|^2\bar{u}(\varkappa)\right]q'\,\psi_\mu^{24}\,dx$$
$$= \int m\left[\psi_\mu^6 f, \frac{\psi_\mu^6 f}{2\varkappa-\partial}, \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \frac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\right]dx$$
$$+ \int m\left[\frac{\psi_\mu^6 f}{2\varkappa-\partial}, \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \frac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f)\right]dx,$$

where each $m \in S_{\mathrm{loc}}(4)$. Again applying (6.10), (6.6), (6.3), respectively, and using (1.19) we have the estimate

$$(6.24) \quad \int\left|\int\left[(6.16) - \frac{\varkappa^2}{\kappa^2-\varkappa^2}\left[-8\varkappa^4|u(\varkappa)|^2\bar{u}(\varkappa)\right]q'\right]\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu$$
$$\lesssim |\varkappa|^{-1}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma}.$$

For (6.17) we first write

$$|q|^2 q = 16\varkappa^4|u(\varkappa)|^2 q + m\left[f, f, \frac{f''}{4\varkappa^2-\partial^2}\right],$$

where $m \in S(3)$. From (3.46), we then have

$$\int \frac{1}{\sqrt{\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[\geq 3]}(\varkappa)\left[|q|^2 - 16\varkappa^4|u(\varkappa)|^2\right]q\,\psi_\mu^{24}\,dx$$
$$= \int m\left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \frac{\psi_\mu^4 f}{2\varkappa-\partial}, \frac{\psi_\mu^4 f}{2\varkappa-\partial}, \frac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^4 f)\right]dx,$$

where $m \in S_{\mathrm{loc}}(6)$, to which we can apply (6.6). Next, we use (3.46) to write

$$\int \frac{1}{\sqrt{\varkappa}}\left(\frac{g_{21}}{2+\gamma}\right)^{[\geq 7]}(\varkappa)\,16\varkappa^4|u(\varkappa)|^2 q\,\psi_\mu^{24}\,dx$$

$$= \varkappa^2 \int m\Big[\underbrace{\psi_\mu^2 f, \dots, \psi_\mu^2 f}_{4}, \underbrace{\tfrac{\psi_\mu^2 f}{2\varkappa - \partial}, \dots, \tfrac{\psi_\mu^2 f}{2\varkappa - \partial}}_{6}\Big]\, \psi_\mu^4\, dx,$$

where $m \in S_{\mathrm{loc}}(10)$, which can be estimated using (6.12). Similarly, using (3.53) we have

$$\int \Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[5]}(-\varkappa)\Big]\, 16\varkappa^4 |u(\varkappa)|^2 q\, \psi_\mu^{24}\, dx$$

$$= \varkappa \int m\Big[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^3 f)\Big]\, dx,$$

where $m \in S_{\mathrm{loc}}(6)$, which we estimate using (6.6). From (3.52) we have

$$\int \Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(-\varkappa) - 8\varkappa^4 |u(\varkappa)|^2 \bar{u}(\varkappa)\Big]\, 16\varkappa^4 |u(\varkappa)|^2 q\, \psi_\mu^{24}\, dx$$

$$= \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\varkappa^2-\partial^2}(\psi_\mu^4 f)\Big]\, dx$$

$$+ \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^4 f)\Big]\, dx,$$

where $m \in S_{\mathrm{loc}}(4)$, and can then apply (6.6). Finally, using that

$$\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(-\varkappa) = i\bar{u}(\varkappa)'$$

and (1.19), we obtain the estimate

$$\int \Big|\int \Big[(6.17) - \tfrac{i\varkappa^2}{\kappa^2-\varkappa^2}\big[-i\bar{u}(\varkappa)' - 8\varkappa^4 |u(\varkappa)|^2 \bar{u}(\varkappa)\big] 16\varkappa^4 |u(\varkappa)|^2 q\Big]\, \psi_\mu^{24}\, dx\Big|\, e^{-\frac{1}{200}|h-\mu|}\, d\mu$$

$$\tag{6.25} \lesssim |\varkappa|^{-1}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

For (6.18), we use (3.46) to write

$$\int \sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 5]}(\kappa)\, u(\kappa)''\, \psi_\mu^{24}\, dx = \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^4 f)\Big]\, dx,$$

where $m \in S_{\mathrm{loc}}(6)$, and (3.52) to write

$$\int \Big[\sqrt{\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(\varkappa) + \sqrt{-\varkappa}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(-\varkappa)\Big] u(\kappa)''\, \psi_\mu^{24}\, dx$$

$$= \int m\Big[\psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\varkappa-\partial}(\psi_\mu^6 f), \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^6 f)\Big]\, dx,$$

where $m \in S_{\mathrm{loc}}(4)$. Similarly, for (6.19) we use (3.46) to write

$$\int \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa)\, u(\kappa)'''\, \psi_\mu^{24}\, dx = \int m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{\psi_\mu^6 f}{2\varkappa-\partial}, \tfrac{(2-\partial)^3}{(2\varkappa-\partial)(4\kappa^2-\partial^2)}(\psi_\mu^6 f)\Big]\, dx.$$

Applying (6.5) and (6.8) gives us

$$\tag{6.26} \int \Big|\int \big[(6.18) + (6.19)\big]\, \psi_\mu^{24}\, dx\Big|\, e^{-\frac{1}{200}|h-\mu|}\, d\mu \lesssim |\varkappa|^{-1}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

Turning to (6.22), we first take a test function $w \in L^\infty$ and use (2.26), (2.23), (2.21), and (1.19) to bound

$$\Big|\int w\, \gamma^{[\geq 8]}(\pm\kappa)\, \psi_\mu^{24}\, dx\Big| \lesssim \sum_{\ell=0}^\infty \kappa^4 \|\Lambda(w)\|_{\mathrm{op}} \|\Lambda(q\psi_\mu^3)\|^8_{\mathfrak{J}_8} \big(C\sqrt{\kappa}\|\Lambda(q)\|_{\mathrm{op}}\big)^{2\ell}$$

$$\lesssim \kappa^{-2} \left\| \frac{\psi_\mu^3 q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma}^6 \|w\|_{L^\infty}.$$

By duality, this yields

$$\|\gamma^{[\geq 8]}(\pm\kappa)\,\psi_\mu^{24}\|_{L^1} \lesssim \kappa^{-2} \left\| \frac{\psi_\mu^3 q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma}^6.$$

For the remaining terms, first use (3.50) to write

$$\int \left[\gamma^{[6]}(\kappa) - \gamma^{[6]}(-\kappa) + 2560i\kappa^9 |u(\kappa)|^6 \right] \psi_\mu^{24}\, dx$$

$$= \kappa^{-1} \int m \left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\kappa - \partial}, \tfrac{\psi_\mu^4 f}{2\kappa - \partial}, \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^4 f) \right] dx,$$

which we bound using (6.7). Second, apply (3.48) to obtain

$$\int \left[\gamma^{[4]}(\kappa) - \gamma^{[4]}(-\kappa) + 192\kappa^5 |u(\kappa)|^2 \left[u(\kappa)' \bar{u}(\kappa) - u(\kappa)\bar{u}(\kappa)' \right] \right] \psi_\mu^{24}\, dx$$

$$= \kappa^{-2} \int m \left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^6 f), \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^6 f) \right] dx$$

$$+ \kappa^{-2} \int m \left[\psi_\mu^6 f, \psi_\mu^6 f, \psi_\mu^6 f, \tfrac{4 - \partial^2}{4\kappa^2 - \partial^2}(\psi_\mu^6 f) \right] dx,$$

which we estimate by (6.4). Together, these give us

$$(6.27) \quad \int \left| \int \left[(6.22) - \tfrac{\kappa^3}{\kappa^2 - \varkappa^2} \left[192\kappa^5 |u(\kappa)|^2 \left[u(\kappa)'\bar{u}(\kappa) - u(\kappa)\bar{u}(\kappa)' \right] \right. \right. \right.$$

$$\left. \left. \left. + 2560i\kappa^9 |u(\kappa)|^6 \right] \right] \psi_\mu^{24}\, dx \right| e^{-\frac{1}{200}|h - \mu|}\, d\mu$$

$$\lesssim \tfrac{\kappa}{\kappa^2 + \varkappa^2} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma}^2.$$

For (6.20), we take $h = \sqrt{\pm\varkappa}\left(\tfrac{g_{21}}{2 + \gamma}\right)(\pm\varkappa)$ and again apply (2.26), (2.23), (2.21), (1.19), and Corollary 3.4 to estimate

$$\left| \int h \tfrac{1}{\sqrt{\pm\kappa}} g_{12}^{[\geq 7]}(\pm\kappa)\, \psi_\mu^{24}\, dx \right| \lesssim \sum_{\ell=0}^\infty \kappa^3 \|\Lambda(h\psi_\mu^3)\|_{\mathfrak{I}_8} \|\Lambda(q\psi_\mu^3)\|_{\mathfrak{I}_8}^7 \left(C\sqrt{\kappa}\|\Lambda(q)\|_{\mathrm{op}} \right)^{2\ell}$$

$$\lesssim \kappa^{-2} \left\| \tfrac{\psi_\mu^3 q}{\sqrt{4\kappa^2 - \partial^2}} \right\|_{H^{\frac{3}{2}}}^2 \|q\|_{E_\sigma}^6,$$

which is acceptable. For the lower order terms, we use (3.47) and (3.49) to write

$$\int h \left[\tfrac{1}{\sqrt{\kappa}} g_{12}^{[3]}(\kappa) - \tfrac{1}{\sqrt{-\kappa}} g_{12}^{[3]}(-\kappa) - 48i\kappa^3 |u(\kappa)|^2 u(\kappa)' \right] \psi_\mu^{24}\, dx$$

$$= \kappa^{-2} \int m \left[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^6 f), \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^6 f) \right] dx$$

$$+ \kappa^{-2} \int m \left[\psi_\mu^6 f, \psi_\mu^6 f, \psi_\mu^6 f, \tfrac{4 - \partial^2}{4\kappa^2 - \partial^2}(\psi_\mu^6 f) \right] dx,$$

where each $m \in S_{\mathrm{loc}}(4)$, and

$$\int h \left[\tfrac{1}{\sqrt{\kappa}} g_{12}^{[5]}(\kappa) - \tfrac{1}{\sqrt{-\kappa}} g_{12}^{[5]}(-\kappa) + 384\kappa^7 |u(\kappa)|^4 u(\kappa) \right] \psi_\mu^{24}\, dx$$

$$= \kappa^{-1} \int m \left[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\kappa - \partial}, \tfrac{\psi_\mu^4 f}{2\kappa - \partial}, \tfrac{2 - \partial}{2\kappa - \partial}(\psi_\mu^4 f) \right] dx,$$

where each $m \in S_{\mathrm{loc}}(6)$. As a consequence, if we introduce

(6.28)
$$= \frac{2\kappa^3}{\kappa^2 - \varkappa^2} \left[\sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)(\varkappa) + \sqrt{-\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)(-\varkappa) \right]$$
$$\times \left[-48 i \kappa^3 |u(\kappa)|^2 u(\kappa)' + 384 \kappa^7 |u(\kappa)|^4 u(\kappa) \right],$$

we may apply (6.4) and (6.7) to obtain

(6.29)
$$\int \left| \int \left[(6.20) - (6.28) \right] \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu \lesssim \frac{\kappa}{\kappa^2 + \varkappa^2} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma}^2.$$

To estimate (6.28), we first use (3.46) to write

$$\int \sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 5]}(\varkappa) \, \kappa^3 |u(\kappa)|^2 u(\kappa)' \, \psi_\mu^{24} \, dx$$
$$= \int m \left[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \frac{\psi_\mu^3 f}{2\varkappa - \partial}, \frac{\psi_\mu^3 f}{2\varkappa - \partial}, \frac{\psi_\mu^3 f}{2\kappa - \partial}, \frac{\psi_\mu^3 f}{2\kappa - \partial}, \frac{2-\partial}{2\kappa - \partial} (\psi_\mu^3 f) \right] dx,$$

where $m \in S_{\mathrm{loc}}(8)$, to which we apply (6.11). Similarly,

$$\int \sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 5]}(\varkappa) \, \kappa^7 |u(\kappa)|^4 u(\kappa) \, \psi_\mu^{24} \, dx$$
$$= \kappa \int m \left[\underbrace{\psi_\mu^2 f, \dots, \psi_\mu^2 f}_{4}, \frac{\psi_\mu^2 f}{2\varkappa - \partial}, \frac{\psi_\mu^2 f}{2\varkappa - \partial}, \underbrace{\frac{\psi_\mu^2 f}{2\kappa - \partial}, \dots, \frac{\psi_\mu^2 f}{2\kappa - \partial}}_{4} \right] \psi_\mu^4 \, dx,$$

where $m \in S_{\mathrm{loc}}(10)$, which can be bounded using (6.13). Further, from (3.52) we have

$$\int \left[\sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[3]}(\varkappa) + \sqrt{-\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[3]}(-\varkappa) \right] \kappa^3 |u(\kappa)|^2 u(\kappa)' \, \psi_\mu^{24} \, dx$$
$$= \int m \left[\psi_\mu^4 f, \frac{\psi_\mu^4 f}{2\varkappa - \partial}, \frac{2-\partial}{2\varkappa - \partial} (\psi_\mu^4 f), \frac{\psi_\mu^4 f}{2\kappa - \partial}, \frac{\psi_\mu^4 f}{2\kappa - \partial}, \frac{2-\partial}{2\kappa - \partial} (\psi_\mu^4 f) \right] dx,$$

where $m \in S_{\mathrm{loc}}(6)$ to which we apply (6.9), and

$$\int \left[\sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[3]}(\varkappa) + \sqrt{-\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[3]}(-\varkappa) \right] \kappa^7 |u(\kappa)|^4 u(\kappa) \, \psi_\mu^{24} \, dx$$
$$= \kappa \int m \left[\psi_\mu^3 f, \psi_\mu^3 f, \frac{\psi_\mu^3 f}{2\varkappa - \partial}, \frac{2-\partial}{2\varkappa - \partial} (\psi_\mu^3 f), \frac{\psi_\mu^3 f}{2\kappa - \partial}, \frac{\psi_\mu^3 f}{2\kappa - \partial}, \frac{\psi_\mu^3 f}{2\kappa - \partial}, \frac{\psi_\mu^3 f}{2\kappa - \partial} \right] dx,$$

which we estimate with (6.11). Observing that

$$\sqrt{\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[1]}(\varkappa) + \sqrt{-\varkappa} \left(\frac{g_{21}}{2+\gamma} \right)^{[1]}(-\varkappa) = -2i\varkappa^2 \bar{u}(\varkappa),$$

and once again using (1.19), these bounds combine to give us the estimate

(6.30)
$$\int \left| \int \left[(6.28) + \frac{2\kappa^3}{\kappa^2 - \varkappa^2} \left[-2i\varkappa^2 \bar{u}(\varkappa) \right] \right. \right.$$
$$\left. \left. \times \left[48 i \kappa^3 |u(\kappa)|^2 u(\kappa)' - 384 \kappa^7 |u(\kappa)|^4 u(\kappa) \right] \right] \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$
$$\lesssim |\varkappa|^{-1} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_\sigma}^2 \|q\|_{E_{2\sigma,\varkappa}}^2.$$

It remains to extract the leading order terms from (6.21). We first use (3.11) and (3.46) to write

$$\text{(6.31)} \qquad \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[1]}(-\varkappa) = i\bar{u}(\varkappa)'$$

$$\text{(6.32)} \qquad \tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[3]}(-\varkappa) = \tfrac{\varkappa}{2\varkappa+\partial}m\Big[f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}\Big],$$

where $m \in S(3)$. From (3.44), we then have

$$\int i\bar{u}(\varkappa)'\sqrt{\pm\kappa}\,g_{12}^{[\geq 7]}(\pm\kappa)\,\psi_\mu^{24}\,dx$$
$$= \kappa\int m\Big[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^3 f)\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(8)$ and

$$\int\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa)\Big]\sqrt{\pm\kappa}\,g_{12}^{[\geq 7]}(\pm\kappa)\,\psi_\mu^{24}\,dx$$
$$= \kappa\int m\Big[\underbrace{\psi_\mu^2 f, \ldots, \psi_\mu^2 f}_{4}, \tfrac{\psi_\mu^2 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^2 f}{2\varkappa-\partial}, \underbrace{\tfrac{\psi_\mu^2 f}{2\kappa-\partial}, \ldots, \tfrac{\psi_\mu^2 f}{2\kappa-\partial}}_{4}\Big]\psi_\mu^4\,dx,$$

where $m \in S_{\mathrm{loc}}(10)$. These terms can be estimated using (6.11) and (6.13), respectively. Applying (3.49) we have

$$\int i\bar{u}(\varkappa)'\Big[\sqrt{\kappa}\,g_{12}^{[5]}(\kappa) - \sqrt{-\kappa}\,g_{12}^{[5]}(-\kappa)\Big]\psi_\mu^{24}\,dx$$
$$= \int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{\psi_\mu^4 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^4 f)\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(6)$, which we bound using (6.9). Similarly, we have

$$\int\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]\Big[\sqrt{\kappa}\,g_{12}^{[5]}(\kappa) - \sqrt{-\kappa}\,g_{12}^{[5]}(-\kappa)\Big]\psi_\mu^{24}\,dx$$
$$= \int m\Big[\psi_\mu^3 f, \psi_\mu^3 f, \psi_\mu^3 f, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{\psi_\mu^3 f}{2\kappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^3 f)\Big]\,dx,$$

where $m \in S_{\mathrm{loc}}(8)$, which is bounded using (6.11). Using (3.47) we have

$$\int i\bar{u}(\varkappa)'\Big[\sqrt{\kappa}\,g_{12}^{[3]}(\kappa) - \sqrt{-\kappa}\,g_{12}^{[3]}(-\kappa) - 32i\kappa^5|u(\kappa)|^2 u(\kappa)\Big]\psi_\mu^{24}\,dx$$
$$= \kappa^{-1}\int m\Big[\psi_\mu^6 f, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^6 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\Big]\,dx$$
$$+ \kappa^{-1}\int m\Big[\psi_\mu^6 f, \psi_\mu^6 f, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^6 f), \tfrac{2-\partial}{4\varkappa^2-\partial^2}(\psi_\mu^6 f)\Big]\,dx$$

where $m \in S_{\mathrm{loc}}(4)$. This is estimated using (6.5). Also using (3.47) we have

$$\int\Big[\tfrac{1}{\sqrt{\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(\varkappa) + \tfrac{1}{\sqrt{-\varkappa}}\big(\tfrac{g_{21}}{2+\gamma}\big)^{[\geq 3]}(-\varkappa)\Big]$$
$$\Big[\sqrt{\kappa}\,g_{12}^{[3]}(\kappa) - \sqrt{-\kappa}\,g_{12}^{[3]}(-\kappa) - 32i\kappa^5|u(\kappa)|^2 u(\kappa)\Big]\psi_\mu^{24}\,dx$$
$$= \kappa^{-1}\int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f), \tfrac{2-\partial}{2\kappa-\partial}(\psi_\mu^4 f)\Big]\,dx$$
$$+ \kappa^{-1}\int m\Big[\psi_\mu^4 f, \psi_\mu^4 f, \psi_\mu^4 f, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{\psi_\mu^4 f}{2\varkappa-\partial}, \tfrac{4-\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^4 f)\Big]\,dx,$$

where $m \in m_\mu(6)$. A final application of (6.8) and (1.19) gives us the estimate

$$(6.33) \qquad \int \left| \int \left[(6.21) - (6.34) \right] \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu \lesssim |\varkappa|^{-1} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

where we define

$$(6.34) \quad = \frac{2\kappa^3}{\kappa^2 - \varkappa^2} \left[\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)(\varkappa) + \frac{1}{\sqrt{-\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)(-\varkappa) \right] \left[\frac{i}{2\kappa} |q|^2 q - 32 i\kappa^5 |u(\kappa)|^2 u(\kappa) \right].$$

However, by writing

$$\frac{i}{2\kappa} |q|^2 q - 32 i\kappa^5 |u(\kappa)|^2 u(\kappa) = -\frac{i}{2\kappa} |q|^2 u(\kappa)'' - 2i\kappa u(\kappa) \bar{u}(\kappa)'' q - 8i\kappa^3 |u(\kappa)|^2 u(\kappa)'',$$

and using (6.31), (6.32), we obtain

$$\left[\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)^{[1]}(\varkappa) + \frac{1}{\sqrt{-\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)^{[1]}(-\varkappa) \right] \left[\frac{i}{2\kappa} |q|^2 q - 32 i\kappa^5 |u(\kappa)|^2 u(\kappa) \right]$$

$$= \kappa^{-1} m \left[f, f, \frac{f'}{4\varkappa^2 - \partial^2}, \frac{f''}{4\kappa^2 - \partial^2} \right],$$

$$\left[\frac{1}{\sqrt{\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 3]}(\varkappa) + \frac{1}{\sqrt{-\varkappa}} \left(\frac{g_{21}}{2+\gamma} \right)^{[\geq 3]}(-\varkappa) \right] \left[\frac{i}{2\kappa} |q|^2 q - 32 i\kappa^5 |u(\kappa)|^2 u(\kappa) \right]$$

$$= \kappa^{-1} m \left[f, f, f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f''}{4\kappa^2 - \partial^2} \right],$$

for $m \in S(4)$ and $S(6)$, respectively. As a consequence, we may apply (6.5) and (6.8) to obtain

$$(6.35) \qquad \int \left| \int (6.34) \, \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu \lesssim |\varkappa|^{-1} \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2,$$

We now collect the leading order terms from our above estimates into quartic and sextic contributions, as folllows

$$J_1 = \frac{1}{\kappa^2 - \varkappa^2} \Big\{ -48\varkappa^6 |u(\varkappa)|^2 \bar{q} u(\varkappa)' + 8\varkappa^6 q \bar{u}(\varkappa)^2 u(\varkappa)' - 8\varkappa^6 |u(\varkappa)|^2 \bar{u}(\varkappa) q'$$

$$- 192\kappa^6 \varkappa^2 |u(\kappa)|^2 \bar{u}(\varkappa) u(\kappa)' + 384\kappa^8 |u(\kappa)|^2 \bar{u}(\kappa) u(\kappa)' \Big\},$$

$$J_2 = \frac{1}{\kappa^2 - \varkappa^2} \Big\{ -256 i\varkappa^{10} |u(\varkappa)|^4 \bar{u}(\varkappa) q - 1536 i\kappa^{10} \varkappa^2 |u(\kappa)|^4 u(\kappa) \bar{u}(\varkappa)$$

$$+ 2560 i\kappa^{12} |u(\kappa)|^6 \Big\}.$$

Combining the estimates (6.23)–(6.27), (6.29), (6.30), (6.33), (6.35) gives us

$$(6.36) \qquad \int \left| \mathrm{Im} \int \left[j_{\mathrm{diff}}^{[\geq 4]} - J_1 - J_2 \right] \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu$$

$$\lesssim \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \left[|\varkappa|^{-1} \|q\|_{E_{2\sigma,\varkappa}^\sigma}^2 + \frac{\kappa}{\kappa^2 + \varkappa^2} \|q\|_{E_\sigma^\sigma}^2 \right].$$

To estimate J_1, we write

$$-48\varkappa^6 |u(\varkappa)|^2 \bar{q} u(\varkappa)' = -3\varkappa^2 |q|^2 \bar{q} u(\varkappa)' - 3\varkappa^2 |q|^2 \bar{u}(\varkappa)'' u(\varkappa)' - 12\varkappa^4 u(\varkappa)'' \bar{q} u(\varkappa)'$$

$$= -3\varkappa^2 |q|^2 \bar{q} u(\varkappa)' + \varkappa^2 m \left[f, f, \frac{f'}{4\varkappa^2 - \partial^2}, \frac{f''}{4\varkappa^2 - \partial^2} \right],$$

where $m \in S(4)$. Similarly, we have

$$8\varkappa^6 q \bar{u}(\varkappa)^2 u(\varkappa)' = \frac{1}{2} \varkappa^2 |q|^2 \bar{q} u(\varkappa)' + \varkappa^2 m \left[f, f, \frac{f'}{4\varkappa^2 - \partial^2}, \frac{f''}{4\varkappa^2 - \partial^2} \right],$$

$$-8\varkappa^6 |u(\varkappa)|^2 \bar{u}(\varkappa) q' = -\frac{1}{2} \varkappa^2 |q|^2 \bar{q} u(\varkappa)' + \varkappa^2 m \left[f, \frac{f}{2\varkappa - \partial}, \frac{f}{2\varkappa - \partial}, \frac{f'''}{4\varkappa^2 - \partial^2} \right]$$

$$+ \varkappa^2 m\Big[f, f, \tfrac{f'}{4\varkappa^2-\partial^2}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big],$$

$$-192\kappa^6\varkappa^2|u(\kappa)|^2\bar{u}(\varkappa)u(\kappa)' = -48\kappa^6|u(\kappa)|^2\bar{q}u(\kappa)' - 48\kappa^6|u(\kappa)|^2\bar{u}(\varkappa)''u(\kappa)'$$

$$= -3\kappa^2|q|^2\bar{q}u(\kappa)' + \kappa m\Big[f, f, f, \tfrac{f''}{4\kappa^2-\partial^2}\Big]$$

$$+ \kappa^6 m\Big[\tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f'}{4\kappa^2-\partial^2}, \tfrac{f''}{4\kappa^2-\partial^2}\Big],$$

$$384\kappa^8|u(\kappa)|^2\bar{u}(\kappa)u(\kappa)' = 6\kappa^2|q|^2\bar{q}u(\kappa)' + \kappa m\Big[f, f, f, \tfrac{f''}{4\kappa^2-\partial^2}\Big],$$

where each $m \in S(4)$. Applying (6.3), (6.4), and (6.5) we then have

$$\int\left|\int\Big[J_1 - \tfrac{3[\kappa^2 u(\kappa)'-\varkappa^2 u(\varkappa)']}{\kappa^2-\varkappa^2}|q|^2\bar{q}\Big]\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu$$

$$\lesssim \|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\Big[|\varkappa|^{-1}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma} + \tfrac{\kappa}{\kappa^2+\varkappa^2}\|q\|^2_{E_\sigma^\sigma}\Big].$$

For the remaining term in J_1, we compute that

$$\tfrac{\kappa^2 u(\kappa)'-\varkappa^2 u(\varkappa)'}{\kappa^2-\varkappa^2} = -\tfrac{q'''}{(4\kappa^2-\partial^2)(4\varkappa^2-\partial^2)},$$

and applying (6.5) gives us

$$\int\left|\int\tfrac{\kappa^2 u(\kappa)'-\varkappa^2 u(\varkappa)'}{\kappa^2-\varkappa^2}|q|^2\bar{q}\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim |\varkappa|^{-1}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma}.$$

Collecting these bounds gives us

$$(6.37) \quad \int\left|\int J_1\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim \|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\Big[|\varkappa|^{-1}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma} + \tfrac{\kappa}{\kappa^2+\varkappa^2}\|q\|^2_{E_\sigma^\sigma}\Big].$$

Finally, we consider J_2. Arguing as for J_1 we have

$$-256i\varkappa^{10}|u(\varkappa)|^4\bar{u}(\varkappa)q = -4i\varkappa^4|q|^4|u(\varkappa)|^2 + \varkappa^2 m\Big[f, f, f, \tfrac{f}{2\varkappa-\partial}, \tfrac{f}{2\varkappa-\partial}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big],$$

$$-1536i\kappa^{10}\varkappa^2|u(\kappa)|^4 u(\kappa)\bar{u}(\varkappa) = -6i\kappa^4|q|^4|u(\kappa)|^2 + \kappa^2 m\Big[f, f, f, \tfrac{f}{2\kappa-\partial}, \tfrac{f}{2\kappa-\partial}, \tfrac{f'}{2\kappa-\partial}\Big]$$

$$+ \kappa^{10} m\Big[\tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f}{4\kappa^2-\partial^2}, \tfrac{f''}{4\varkappa^2-\partial^2}\Big],$$

$$2560i\kappa^{12}|u(\kappa)|^6 = 10i\kappa^4|q|^4|u(\kappa)|^2 + \kappa^2 m\Big[f, f, f, \tfrac{f}{2\kappa-\partial}, \tfrac{f}{2\kappa-\partial}, \tfrac{f'}{2\kappa-\partial}\Big],$$

where each $m \in S(6)$. Applying (6.6), (6.7), and (6.8) gives us

$$\int\left|\int\Big[J_2 - \tfrac{4i[\kappa^4|u(\kappa)|^2-\varkappa^4|u(\varkappa)|^2]}{\kappa^2-\varkappa^2}|q|^4\Big]\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu$$

$$\lesssim \|q\|^2_{E_\sigma^\sigma}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\Big[|\varkappa|^{-1}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma} + \tfrac{\kappa}{\kappa^2+\varkappa^2}\|q\|^2_{E_\sigma^\sigma}\Big].$$

For the remaining term we compute that

$$\tfrac{\kappa^4|u(\kappa)|^2-\varkappa^4|u(\varkappa)|^4}{\kappa^2-\varkappa^2} = -\kappa^2\bar{u}(\kappa)\tfrac{q''}{(4\kappa^2-\partial^2)(4\varkappa^2-\partial^2)} - \varkappa^2 u(\varkappa)\tfrac{\bar{q}''}{(4\kappa^2-\partial^2)(4\varkappa^2-\partial^2)},$$

and can then apply (6.8) to conclude that

$$\int\left|\int\tfrac{4i[\kappa^4|u(\kappa)|^2-\varkappa^4|u(\varkappa)|^2]}{\kappa^2-\varkappa^2}|q|^4\,\psi_\mu^{24}\,dx\right|e^{-\frac{1}{200}|h-\mu|}\,d\mu \lesssim \tfrac{1}{|\varkappa|}\|q\|^2_{F_\kappa^{\frac{1}{2}}(h)}\|q\|^2_{E_\sigma^\sigma}\|q\|^2_{E_{2\sigma,\varkappa}^\sigma}.$$

Together these, yield

(6.38)
$$\int \left| \int J_2 \, \psi_\mu^{24} \, dx \right| e^{-\frac{1}{200}|h-\mu|} \, d\mu \lesssim \|q\|_{E_\sigma}^2 \|q\|_{F_\kappa^{\frac{1}{2}}(h)}^2 \left[|\varkappa|^{-1} \|q\|_{E_{2\sigma,\varkappa}}^2 + \frac{\kappa}{\kappa^2+\varkappa^2} \|q\|_{E_\sigma}^2 \right].$$

The estimate (6.14) now follows from combining (6.36), (6.37), and (6.38). $\quad\square$

Proof of Proposition 6.1. Recall (6.2). Using Lemma 4.2 and (2.4), we estimate

(6.39)
$$\left| \int_{I_\kappa} \int \operatorname{Im} \int \rho(\varkappa) \phi_\mu \, dx \Big|_{t=-1}^{t=1} e^{-\frac{1}{200}|h-\mu|} \, d\mu \, d\varkappa \right| \lesssim \|q\|_{L_t^\infty E_\sigma}^2 \left[1 + \|q\|_{L_t^\infty L^2}^2 \right].$$

Turning to LHS(6.2) and recalling that $u(\kappa) = \frac{q}{4\kappa^2-\partial^2}$, we use (3.30) and (3.32) to write

$$\operatorname{Im} j_{\mathrm{diff}}^{[2]}(\varkappa, \kappa) = \operatorname{Re} \left\{ 16\kappa^2 \frac{u(\kappa)''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)'' + 2\frac{u(\kappa)'''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)''' \right\}$$
$$- \operatorname{Re} \left\{ 8\kappa^2 \frac{u(\kappa)''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)' + \frac{u(\kappa)'''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)'' \right\}'$$
(6.40)
$$- 8\kappa^2 \operatorname{Re} \left\{ \frac{u(\kappa)'}{4\varkappa^2-\partial^2} \bar{u}(\kappa)' \right\}'' + 4\kappa^2 \operatorname{Re} \left\{ \frac{u(\kappa)'}{4\varkappa^2-\partial^2} \bar{u}(\kappa) \right\}'''.$$

Integrating by parts and then applying (2.13) we may bound the contribution of the last three summands as follows:

$$\left| \int \left\{ 8\kappa^2 \frac{u(\kappa)''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)' + \frac{u(\kappa)'''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)'' \right\}' \psi_\mu^{24} \, dx \right|$$
$$\lesssim \kappa^2 \varkappa^{-1} \|u(\kappa)'\|_{E_{1,\varkappa}^{\frac{1}{2}}} \|(\psi_\mu^{24})' u(\kappa)'\|_{E_{1,\varkappa}^{\frac{1}{2}}} + \varkappa^{-1} \|u(\kappa)''\|_{E_{1,\varkappa}^{\frac{1}{2}}} \|(\psi_\mu^{24})' u(\kappa)''\|_{E_{1,\varkappa}^{\frac{1}{2}}}$$
$$\lesssim \varkappa^{-1} \|q\|_{E_{2\sigma,\varkappa}}^2,$$

$$\left| \int \left\{ 8\kappa^2 \frac{u(\kappa)'}{4\varkappa^2-\partial^2} \bar{u}(\kappa)' \right\}'' \psi_\mu^{24} \, dx \right|$$
$$\lesssim \kappa^2 \varkappa^{-1} \|u(\kappa)\|_{E_{1,\varkappa}^{\frac{1}{2}}} \|(\psi_\mu^{24})'' u(\kappa)'\|_{E_{1,\varkappa}^{\frac{1}{2}}} \lesssim \kappa^{-1} \varkappa^{-1} \|q\|_{E_{2\sigma,\varkappa}}^2,$$

$$\left| \int \left\{ 4\kappa^2 \frac{u(\kappa)'}{4\varkappa^2-\partial^2} \bar{u}(\kappa) \right\}''' \psi_\mu^{24} \, dx \right|$$
$$\lesssim \kappa^2 \varkappa^{-1} \|u(\kappa)\|_{E_{1,\varkappa}^{\frac{1}{2}}} \|(\psi_\mu^{24})''' u(\kappa)\|_{E_{1,\varkappa}^{\frac{1}{2}}} \lesssim \kappa^{-2} \varkappa^{-1} \|q\|_{E_{2\sigma,\varkappa}}^2.$$

For the remaining term, we write

$$\int \left\{ 16\kappa^2 \frac{u(\kappa)''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)'' + 2\frac{u(\kappa)'''}{4\varkappa^2-\partial^2} \bar{u}(\kappa)''' \right\} \psi_\mu^{24} \, dx$$
$$= \int \left\{ 16\kappa^2 \frac{(\psi_\mu^{12} q)''}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)} \frac{(\psi_\mu^{24} \bar{q})''}{4\kappa^2-\partial^2} + 2\frac{(\psi_\mu^{12} q)'''}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)} \frac{(\psi_\mu^{24} \bar{q})'''}{4\kappa^2-\partial^2} \right\} dx$$
$$+ \int \left\{ 16\kappa^2 \left[\psi_\mu^{12}, \frac{\partial^2}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)} \right] q \cdot \frac{(\psi_\mu^{24} \bar{q})''}{4\kappa^2-\partial^2} \right\} dx$$
$$+ \int \left\{ 2 \left[\psi_\mu^{12}, \frac{\partial^3}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)} \right] \cdot \frac{(\psi_\mu^{12} \bar{q})'''}{4\kappa^2-\partial^2} \right\} dx$$
$$+ \int \left\{ 16\kappa^2 \frac{u(\kappa)''}{4\varkappa^2-\partial^2} q \cdot \left[\psi_\mu^{12}, \frac{\partial^2}{4\kappa^2-\partial^2} \right] \bar{q} + 2\frac{u(\kappa)'''}{4\varkappa^2-\partial^2} \cdot \left[\psi_\mu^{12}, \frac{\partial^3}{4\kappa^2-\partial^2} \right] \bar{q} \right\} \psi_\mu^{12} \, dx.$$

Using that

$$\left[\psi_\mu^{12}, \frac{1}{4\varkappa^2-\partial^2} \right] = -\frac{1}{4\varkappa^2-\partial^2} \left\{ 2(\psi_\mu^{12})' \partial + (\psi_\mu^{12})'' \right\} \frac{1}{4\varkappa^2-\partial^2},$$

$$\left[\psi_\mu^{12}, \tfrac{\partial^2}{4\kappa^2-\partial^2}\right] = -\tfrac{4\kappa^2}{4\kappa^2-\partial^2}\left\{2(\psi_\mu^{12})'\partial + (\psi_\mu^{12})''\right\}\tfrac{1}{4\kappa^2-\partial^2},$$

$$\left[\psi_\mu^{12}, \tfrac{\partial^3}{4\kappa^2-\partial^2}\right] = -\tfrac{4\kappa^2}{4\kappa^2-\partial^2}\left\{2(\psi_\mu^{12})'\partial + (\psi_\mu^{12})''\right\}\tfrac{\partial}{4\kappa^2-\partial^2} - \tfrac{\partial^2}{4\kappa^2-\partial^2}(\psi_\mu^{12})',$$

we may again use (2.13) to bound

$$\left|\int\left\{16\kappa^2\left[\psi_\mu^{12}, \tfrac{\partial^2}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)}\right]q \cdot \tfrac{(\psi_\mu^{24}\bar{q})''}{4\kappa^2-\partial^2}\right\}dx\right|$$

$$\lesssim \kappa^2\left\{\varkappa\left\|\left[\psi_\mu^{12}, \tfrac{1}{4\varkappa^2-\partial^2}\right]u(\kappa)''\right\|_{E^{\frac{1}{2}}_{-1,\varkappa}} + \varkappa^{-1}\left\|\left[\psi_\mu^{12}, \tfrac{\partial^2}{4\kappa^2-\partial^2}\right]q\right\|_{E^{\frac{1}{2}}_{1,\varkappa}}\right\}\left\|\tfrac{(\psi_\mu^{24}\bar{q})'}{4\kappa^2-\partial^2}\right\|_{E^{\frac{1}{2}}_{1,\varkappa}}$$

$$\lesssim \varkappa^{-1}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}},$$

$$\left|\int\left\{2\left[\psi_\mu^{12}, \tfrac{\partial^3}{(4\varkappa^2-\partial^2)(4\kappa^2-\partial^2)}\right]q \cdot \tfrac{(\psi_\mu^{12}\bar{q})'''}{4\kappa^2-\partial^2}\right\}dx\right|$$

$$\lesssim \left\{\varkappa\left\|\left[\psi_\mu^{12}, \tfrac{1}{4\varkappa^2-\partial^2}\right]u(\kappa)'''\right\|_{E^{\frac{1}{2}}_{-1,\varkappa}} + \varkappa^{-1}\left\|\left[\psi_\mu^{12}, \tfrac{\partial^3}{4\kappa^2-\partial^2}\right]q\right\|_{E^{\frac{1}{2}}_{1,\varkappa}}\right\}\left\|\tfrac{(\psi_\mu^{24}\bar{q})''}{4\kappa^2-\partial^2}\right\|_{E^{\frac{1}{2}}_{1,\varkappa}}$$

$$\lesssim \varkappa^{-1}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}},$$

$$\left|\int\left\{16\kappa^2\tfrac{u(\kappa)''}{4\varkappa^2-\partial^2}q \cdot \left[\psi_\mu^{12}, \tfrac{\partial^2}{4\kappa^2-\partial^2}\right]\bar{q}\right\}\psi_\mu^{12}\,dx\right|$$

$$\lesssim \kappa^2\varkappa^{-1}\|u(\kappa)'\|_{E^{\frac{1}{2}}_{1,\varkappa}}\left\|\psi_\mu^{12}\left[\psi_\mu^{12}, \tfrac{\partial^2}{4\kappa^2-\partial^2}\right]q\right\|_{E^{\frac{1}{2}}_{1,\varkappa}} \lesssim \varkappa^{-1}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}},$$

$$\left|\int\left\{2\tfrac{u(\kappa)'''}{4\varkappa^2-\partial^2} \cdot \left[\psi_\mu^{12}, \tfrac{\partial^3}{4\kappa^2-\partial^2}\right]\bar{q}\right\}\psi_\mu^{12}\,dx\right|$$

$$\lesssim \varkappa^{-1}\|u(\kappa)''\|_{E^{\frac{1}{2}}_{1,\varkappa}}\left\|\psi_\mu^{12}\left[\psi_\mu^{12}, \tfrac{\partial^3}{4\kappa^2-\partial^2}\right]q\right\|_{E^{\frac{1}{2}}_{1,\varkappa}} \lesssim \varkappa^{-1}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

Combining these estimates gives us

$$(6.41) \qquad \varkappa^2\left\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{E^2_{1,\varkappa}} \lesssim \left|\operatorname{Im}\int j_{\text{diff}}(\varkappa,\kappa)^{[2]}\,\psi_\mu^{24}\,dx\right| + \varkappa^{-1}\|q\|^2_{E^\sigma_{2\sigma,\varkappa}}.$$

Collecting (6.39), (6.41), (6.14), and using (2.5), we have

$$\int\left\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{L^2_t E^{\frac{1}{2}}_{\frac{1}{2}}}e^{-\frac{1}{200}|h-\mu|}\,d\mu$$

$$\lesssim \left|\int_{-1}^1\int_{I_\kappa}\int \operatorname{Im}j_{\text{diff}}^{[2]}(\varkappa,\kappa)\,\psi_\mu^{12}\,dx\,e^{-\frac{1}{200}|h-\mu|}\,d\mu\,d\varkappa\,dt\right| + \|q\|^2_{L^\infty_t E^\sigma}$$

$$\lesssim \|q\|^2_{L^\infty_t E^\sigma}\left[1 + \|q\|^2_{L^\infty_t L^2_x} + \|q\|^2_{X^{\frac{1}{2}}_\kappa}\right].$$

Estimating

$$\left\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{L^2_t H^{\frac{3}{2}}} \lesssim \left\|\tfrac{P_{\leq 1}(\psi_\mu^{12}q)}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{L^2_t H^{\frac{3}{2}}} + \left\|\tfrac{P_{>1}(\psi_\mu^{12}q)}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{L^2_t H^{\frac{3}{2}}}$$

$$\lesssim \kappa^{-2}\|q\|^2_{L^\infty_t L^2_x} + \left\|\tfrac{\psi_\mu^{12}q}{\sqrt{4\kappa^2-\partial^2}}\right\|^2_{L^2_t E^2_{\frac{1}{2}}},$$

we then get

$$\|q\|^2_{X^{\frac{1}{2}}_\kappa} \lesssim \|q\|^2_{L^\infty_t E^\sigma}\|q\|^2_{X^{\frac{1}{2}}_\kappa} + \left(\|q\|^2_{L^\infty_t E^\sigma} + \kappa^{-2}\right)\left(1 + \|q\|^2_{L^\infty_t L^2_x}\right).$$

Using (1.19) to absorb the first term on the right-hand side into the left-hand side, (6.1) then follows from the conservation of the L^2 norm. $\qquad\square$

7. Convergence of the difference flow

Our main goal in this section is to prove that as $\kappa \to \infty$, the flow determined by the difference of the Hamiltonians $H_\kappa^{\mathrm{diff}} = H - H_\kappa$ converges to the identity, locally in spacetime, uniformly over L^2-bounded and equicontinuous sets.

Theorem 7.1 (Difference flow converges to the identity). *Let $Q \subset \mathcal{S}(\mathbb{R})$ be an L^2-bounded and equicontinuous set such that*

$$Q_* = \left\{ e^{tJ\nabla H_\kappa^{\mathrm{diff}}} q : q \in Q,\ |t| \leq 1,\ and\ \kappa \geq 2 \right\}$$

is a δ-good set for a sufficiently small $\delta > 0$. Then

$$\lim_{\kappa \to \infty} \sup_{q \in Q} \sup_{\mu \in \mathbb{R}} \sup_{|t| \leq 1} \left\| \psi_\mu^{12} e^{tJ\nabla H_\kappa^{\mathrm{diff}}} q - \psi_\mu^{12} q \right\|_{L_x^2} = 0.$$

Proof. The L^2-boundedness and equicontinuity of the set Q_* extends readily to the set

$$\{ \psi_\mu^j q : q \in Q_*,\ 1 \leq |j| \leq 12,\ \mu \in \mathbb{R} \}.$$

In view of this equicontinuity property and employing the fundamental theorem of calculus, the proof of the theorem reduces to showing that

(7.1) $$\lim_{\kappa \to \infty} \sup_{q \in Q} \sup_{\mu \in \mathbb{R}} \left\| \tfrac{d}{dt} \left(\psi_\mu^{12} e^{tJ\nabla H_\kappa^{\mathrm{diff}}} q \right) \right\|_{L_t^1([-1,1]; H_x^{-5})} = 0.$$

A quick computation reveals that

$$i \tfrac{d}{dt} \left(\psi_\mu^{12} e^{tJ\nabla H_\kappa^{\mathrm{diff}}} q \right) = \psi_\mu^{12} \left[F_\kappa \left(e^{tJ\nabla H_\kappa^{\mathrm{diff}}} q \right) \right]'$$

where

$$F_\kappa(q) = -q' - i|q|^2 q + 2\kappa \left[\sqrt{\kappa} g_{12}(\kappa) - \sqrt{-\kappa} g_{12}(-\kappa) \right].$$

Thus, (7.1) will follow from

(7.2) $$\lim_{\kappa \to \infty} \sup_{q \in Q_*} \sup_{\mu \in \mathbb{R}} \left\| \psi_\mu^{12} F_\kappa(q) \right\|_{L_t^1([-1,1]; H_x^{-4})} = 0.$$

Employing (3.32) and (3.47), we decompose

$$F_\kappa(q) = \tfrac{\partial^3}{4\kappa^2 - \partial^2} q + m\left[f, f, \tfrac{f''}{4\kappa^2 - \partial^2} \right] + m\left[f, \tfrac{f'}{2\kappa - \partial}, \tfrac{f'}{2\kappa - \partial} \right]$$
$$+ 2\kappa \left[\sqrt{\kappa} g_{12}^{[\geq 5]}(\kappa) - \sqrt{-\kappa} g_{12}^{[\geq 5]}(-\kappa) \right],$$

where f satisfies (3.26) and each paraproduct m lies in $S(3)$.

The contribution of the linear term is easily estimated via

$$\left\| \psi_\mu^{12} \tfrac{\partial^3}{4\kappa^2 - \partial^2} q \right\|_{L_t^1 H_x^{-4}} \lesssim \kappa^{-2} \| \psi_\mu^{12} \|_{H_x^4} \| q \|_{L_t^\infty L_x^2}$$

which converges to 0 as $\kappa \to \infty$, uniformly for all $q \in Q_*$ and all $\mu \in \mathbb{R}$, in view of the conservation of the L^2 norm.

To estimate the contribution of the cubic and higher order terms in $F_\kappa(q)$, we rely on the consequences of the local smoothing estimates proved in Proposition 6.1. To simplify our bounds, we introduce

$$\| q \|_{\mathrm{DNLS}_\kappa} := \| q \|_{L_t^\infty L_x^2} + \| q \|_{X_\kappa^{\frac{1}{2}}}$$

and note that L^2-conservation and Proposition 6.1 yield

$$(7.3) \qquad \|e^{tJ\nabla H_\kappa^{\text{diff}}}q\|_{\text{DNLS}_\kappa} \lesssim \|q\|_{L^2}$$

uniformly for $\kappa \geq 2$ and $q \in Q$.

Lemma 7.2 (Local smoothing estimates). *For f satisfying* (3.26), *we have*

$$(7.4) \qquad \sup_{\mu\in\mathbb{R}} \left\|\frac{\partial(\psi_\mu^4 f)}{2\kappa\pm\partial}\right\|_{L_t^2 L_x^4} \lesssim \kappa^{-\frac{1}{6}}\|q\|_{\text{DNLS}_\kappa},$$

$$(7.5) \qquad \sup_{\mu\in\mathbb{R}} \left\|\psi_\mu^4 f\right\|_{L_{t,x}^4} \lesssim \kappa^{\frac{1}{6}}\|q\|_{\text{DNLS}_\kappa}.$$

Moreover, the following estimate holds uniformly for $\kappa \geq 2$, $q \in Q_*$, and $\mu \in \mathbb{R}$:

$$(7.6) \qquad \int_{-1}^1 \|\Lambda(\psi_\mu q)\|_{\mathfrak{I}_5}^5 \, dt \lesssim \kappa^{-3}\left[\kappa^{-\frac{1}{4}}\|q\|_{L^2} + \left\|(\psi_\mu q)_{>\kappa^{\frac{1}{3}}}\right\|_{L^2}\right]\|q\|_{\text{DNLS}_\kappa}^4.$$

Proof. Decomposing into low and high frequencies and using Bernstein's inequality for frequencies $\leq \kappa^{\frac{2}{3}}$, and Proposition 6.1 for frequencies $> \kappa^{\frac{2}{3}}$, we may bound

$$(7.7) \qquad \left\|\psi_\mu^4 f\right\|_{L_t^2 H^{\frac{1}{2}}} \lesssim \kappa^{\frac{1}{3}}\|f\|_{L_t^\infty L_x^2} + \kappa^{\frac{1}{3}}\left\|\frac{\psi_\mu f}{\sqrt{4\kappa^2-\partial^2}}\right\|_{L_t^2 H_x^{\frac{3}{2}}} \lesssim \kappa^{\frac{1}{3}}\|q\|_{\text{DNLS}_\kappa}.$$

A parallel argument yields

$$\left\|\frac{(\psi_\mu^4 f)'}{2\kappa\pm\partial}\right\|_{L_{t,x}^2} \lesssim \kappa^{-\frac{1}{3}}\|q\|_{\text{DNLS}_\kappa}.$$

Using this latter bound, Sobolev embedding, and interpolation, we obtain

$$\left\|\frac{\partial(\psi_\mu^4 f)}{2\kappa\pm\partial}\right\|_{L_t^2 L_x^4}^2 \lesssim \left\||\partial|^{\frac{1}{4}}\frac{\partial(\psi_\mu^4 f)}{2\kappa\pm\partial}\right\|_{L_{t,x}^2}^2 \lesssim \left\|\frac{\psi_\mu^4 f}{2\kappa\pm\partial}\right\|_{L_t^2 H_x^{\frac{3}{2}}}\left\|\frac{(\psi_\mu^4 f)'}{2\kappa\pm\partial}\right\|_{L_{t,x}^2} \lesssim \kappa^{-\frac{1}{3}}\|q\|_{\text{DNLS}_\kappa}^2,$$

which settles (7.4).

Arguing similarly and using (7.7), we may bound

$$\int_{-1}^1 \left\|\psi_\mu^4 f\right\|_{L_x^4}^4 \, dt \lesssim \int_{-1}^1 \left\||\partial|^{\frac{1}{4}}(\psi_\mu^4 f)\right\|_{L_x^2}^4 \, dt \lesssim \left\|\psi_\mu^4 f\right\|_{L_t^2 H_x^{\frac{1}{2}}}^2 \left\|\psi_\mu^4 f\right\|_{L_{t,x}^2}^2 \lesssim \kappa^{\frac{2}{3}}\|q\|_{\text{DNLS}_\kappa}^4.$$

We now turn to the proof of (7.6). By the Bernstein inequality and Lemma 2.6,

$$\sum_{N\leq N_0} \|\Lambda_N(q)\|_{\text{op}} \lesssim \begin{cases} \frac{\sqrt{N_0}}{\kappa}\|q\|_{L^2} & \text{for } N_0 \leq \kappa^{\frac{1}{3}} \\ \kappa^{-\frac{5}{6}}\|q\|_{L^2} + \frac{\sqrt{N_0}}{\kappa}\|q_{>\kappa^{\frac{1}{3}}}\|_{L^2} & \text{for } \kappa^{\frac{1}{3}} < N_0 \leq \kappa \\ \frac{1}{\sqrt{\kappa}}\|q\|_{L^2} & \text{for } N_0 > \kappa. \end{cases}$$

Employing this estimate and Lemma 2.5, we may bound

$$\|\Lambda(\psi_\mu q)\|_{\mathfrak{I}_6}^6 = \sum_{N_1\sim N_2} \left\|\Lambda\left[(\psi_\mu q)_{N_1}\right]\right\|_{\mathfrak{I}_2}\left\|\Lambda\left[(\psi_\mu q)_{N_2}\right]\right\|_{\mathfrak{I}_2}\left[\sum_{N_3\leq N_2}\left\|\Lambda\left[(\psi_\mu q)_{N_3}\right]\right\|_{\text{op}}\right]^4$$

$$\lesssim \sum_{\kappa^{\frac{1}{3}}\geq N_1\sim N_2} \frac{N_2^2}{\kappa^5}\|(\psi_\mu q)_{N_1}\|_{L^2}\|(\psi_\mu q)_{N_2}\|_{L^2}\|q\|_{L^2}^4$$

$$+ \sum_{\kappa^{\frac{1}{3}} < N_1\sim N_2\leq \kappa^{\frac{2}{3}}} \frac{1}{\kappa}\|(\psi_\mu q)_{N_1}\|_{L^2}\|(\psi_\mu q)_{N_2}\|_{L^2}\left[\kappa^{-\frac{10}{3}}\|q\|_{L^2}^4 + \frac{N_2^2}{\kappa^4}\|(\psi_\mu q)_{>\kappa^{\frac{1}{3}}}\|_{L^2}^4\right]$$

$$+ \sum_{\kappa^{\frac{2}{3}} < N_1\sim N_2\leq \kappa} \frac{\kappa}{N_2^3}\left\|\frac{(\psi_\mu q)_{N_1}}{\sqrt{4\kappa^2-\partial^2}}\right\|_{H^{\frac{3}{2}}}\left\|\frac{(\psi_\mu q)_{N_2}}{\sqrt{4\kappa^2-\partial^2}}\right\|_{H^{\frac{3}{2}}}\left[\kappa^{-\frac{10}{3}}\|q\|_{L^2}^4 + \frac{N_2^2}{\kappa^4}\|(\psi_\mu q)_{>\kappa^{\frac{1}{3}}}\|_{L^2}^4\right]$$

$$+ \sum_{N_1 \sim N_2 > \kappa} \frac{1}{\kappa^2 N_2^2} \log\left(4 + \frac{N_2^2}{\kappa^2}\right) \left\|\frac{(\psi_\mu q)_{N_1}}{\sqrt{4\kappa^2 - \partial^2}}\right\|_{H^{\frac{3}{2}}} \left\|\frac{(\psi_\mu q)_{N_2}}{\sqrt{4\kappa^2 - \partial^2}}\right\|_{H^{\frac{3}{2}}} \|q\|_{L^2}^4$$

$$\lesssim \left[\kappa^{-\frac{12}{3}} \|q\|_{L^2}^2 + \kappa^{-\frac{11}{3}} \left\|(\psi_\mu q)_{>\kappa^{\frac{1}{3}}}\right\|_{L^2}^2\right] \|q\|_{L^2}^2 \left[\|q\|_{L^2}^2 + \left\|\frac{\psi_\mu q}{\sqrt{4\kappa^2 - \partial^2}}\right\|_{H^{\frac{3}{2}}}^2\right].$$

The estimate (7.6) now follows by interpolation between this bound and

$$\|\Lambda(\psi_\mu q)\|_{\mathfrak{I}_2} \lesssim \kappa^{-\frac{1}{2}} \|q\|_{L^2}.$$

This completes the proof of the lemma. □

Returning to the contribution of the cubic terms in $F_\kappa(q)$ to LHS(7.2), we employ Lemmas 4.6 and 7.2 as well as (2.32) to estimate

$$\left\|\psi_\mu^{12} m\left[f, f, \frac{f''}{4\kappa^2 - \partial^2}\right]\right\|_{L_t^1([-1,1]; H_x^{-4})}$$
$$\lesssim \left\|\psi_\mu^{12} m\left[f, f, \frac{f''}{4\kappa^2 - \partial^2}\right]\right\|_{L_{t,x}^1} \lesssim \|\psi_\mu^4 f\|_{L_{t,x}^4}^2 \left\|\frac{(\psi_\mu f)''}{4\kappa^2 - \partial^2}\right\|_{L_{t,x}^2} \lesssim \kappa^{\frac{1}{3}} \kappa^{-\frac{1}{2}} \|q\|_{\mathrm{DNLS}_\kappa}^3,$$

$$\left\|\psi_\mu^{12} m\left[f, \frac{f'}{2\kappa - \partial}, \frac{f'}{2\kappa - \partial}\right]\right\|_{L_t^1([-1,1]; H_x^{-4})}$$
$$\lesssim \left\|\psi_\mu^{12} m\left[f, \frac{f'}{2\kappa - \partial}, \frac{f'}{2\kappa - \partial}\right]\right\|_{L_{t,x}^1} \lesssim \|\psi_\mu^4 f\|_{L_t^\infty L_x^2} \left\|\frac{(\psi_\mu f)'}{2\kappa - \partial}\right\|_{L_t^2 L_x^4}^2 \lesssim \kappa^{-\frac{1}{3}} \|q\|_{\mathrm{DNLS}_\kappa}^3.$$

In view of (7.3), the contribution of these cubic terms is acceptable.

To estimate the contribution of the quintic and higher order terms in $F_\kappa(q)$, we argue by duality. To this end, fix $w \in H_x^4$. Using Lemma 2.9 and (7.6), we estimate

$$\int_{-1}^1 \left|\int \psi_\mu^{12} \kappa^{\frac{3}{2}} g_{12}^{[\geq 5]}(\kappa) \, w \, dx\right| dt \lesssim \sum_{\ell \geq 2} \kappa^4 \|\Lambda(\psi_\mu q)\|_{\mathfrak{I}_5}^5 \left(\sqrt{\kappa} \|\Lambda(q)\|_{\mathrm{op}}\right)^{2\ell - 4} \|\Lambda(w\psi_\mu^7)\|_{\mathrm{op}}$$

$$\lesssim \|w\|_{L^\infty} \left[\kappa^{-\frac{1}{4}} \|q\|_{L^2} + \left\|(\psi_\mu q)_{>\kappa^{\frac{1}{3}}}\right\|_{L^2}\right] \|q\|_{\mathrm{DNLS}_\kappa}^4,$$

where we used Corollary 2.7 combined with the fact that Q_* is δ-good in order to sum in $\ell \geq 2$. By (7.3) and equicontinuity, the contribution of these terms to LHS(7.2) is also acceptable. □

8. Proofs of the main theorems

All the main difficulties have already been addressed in previous sections, albeit under the assumption that the solutions remain in a δ-good set for some universally small δ and only for the time interval $[-1, 1]$. In this section, we put all the pieces together and show how to circumvent these illusory restrictions.

Proof of Theorem 1.1. The fundamental question to settle is this: Given an L^2-Cauchy sequence of initial data $q_n(0) \in \mathcal{S}$ and a Cauchy sequence of times $t_n \in \mathbb{R}$, show that their evolutions $q_n(t_n)$ under (DNLS) form an L^2-Cauchy sequence.

Evidently, the set $\{q_n(0)\}$ is L^2-precompact. Thus by Corollary 1.9, there is a uniform rescaling parameter λ so that not only are the rescaled initial data δ-good, but so are their evolutions under (DNLS) as well as any other dynamics preserving $A(\varkappa; q)$. This rescaling does not meaningfully alter our original ambition — we just replace the original sequences of solutions and times by their rescaled values (for which we reuse the original names).

It suffices to treat the case where $|t_n| \leq 1$, because larger values can be treated by iterating the argument. For example, if $t_n \to \frac{3}{2}$, then we first run the argument

with $t_n \equiv 1$ and then use $q_n(1)$, which we now know to be convergent, as initial data to extend up to the chosen $t_n \to \frac{3}{2}$.

Assuming now that $|t_n| \leq 1$, Theorem 1.7 guarantees that $\{q_n(t_n)\}$ is equicontinuous and Proposition 5.1 guarantees that it is tight. Thus every subsequence has an L^2-convergent subsequence; we just need to verify that all such subsequential limits agree. For this purpose, it suffices to test against some fixed $w \in C_c^\infty(\mathbb{R})$.

It is at this moment that we employ the commutativity of (DNLS) and the H_κ flows. Using (1.13), our task is reduced to verifying the following two claims:

$$(8.1) \qquad \sup_{\kappa \geq 1} \limsup_{m,n \to \infty} \left| \left\langle w, e^{t_n J \nabla H_\kappa} q_n(0) - e^{t_m J \nabla H_\kappa} q_m(0) \right\rangle \right| = 0$$

$$(8.2) \qquad \limsup_{\kappa \to \infty} \sup_{q_0 \in Q_*} \sup_{|t| \leq 1} \left| \left\langle w, \left[e^{t J \nabla (H - H_\kappa)} - \mathrm{Id} \right](q_0) \right\rangle \right| = 0$$

where Q_* is defined via (1.11) with the choice $Q = \{q_n(0) : n \in \mathbb{N}\}$.

The first of these two claims follows from the L^2-wellposedness of the H_κ flow shown already in [27, Cor. 5.4]. The second was addressed by Theorem 7.1. $\square$

Proof of Corollary 1.2. Recall that local well-posedness for $s \geq \frac{1}{2}$ was proved already by Takaoka in [45]. This result is rendered global by the a priori bounds shown in [2, 3].

Consider now $0 \leq s < \frac{1}{2}$. Evidently, the existence of solutions follows immediately from Theorem 1.1, as does continuous dependence in the L^2 metric. Continuous dependence in the H^s metric follows from this and H^s-equicontinuity, which was shown in [27, Th. 5.6] contingent on the equicontinuity conjecture that was subsequently resolved in [17]. $\square$

Proof of Theorem 1.5. By Corollary 1.9, there is a uniform rescaling of the set of initial data Q so that not only are the rescaled initial data δ-good, but so are their entire (DNLS) evolutions. This allows us to invoke (4.1) from Proposition 4.1 to obtain the local smoothing estimate (over any unit time interval) for the rescaled solutions. The estimate (1.4) for the unrescaled solutions follows by a simple covering argument. Naturally the resulting constant depends on the rescaling parameter; however, this is dictated solely by Q. $\square$

Proof of Corollary 1.6. Given initial data $q(0) \in L^2$, we choose Schwartz-class initial data $q_n(0)$ that converge to it in L^2. By Theorem 1.1, the solutions $q_n(t)$ converge to $q(t)$ in $C([-T, T]; L^2(\mathbb{R}))$. To deduce that $q(t)$ is a distributional solution, we need another form of convergence to handle the nonlinearity. This can be obtained with the aid of Theorem 1.5 and a Gagliardo–Nirenberg inequality:

$$\|\psi^8[q_n - q]\|_{L^3_{t,x}}^3 \lesssim \|q_n - q\|_{L^\infty_t L^2_x} \left\{ \|\psi^{12} q_n\|_{L^2_t H^{1/2}_x}^2 + \|\psi^{12} q\|_{L^2_t H^{1/2}_x}^2 \right\} \to 0 \ \text{ as } n \to \infty.$$

Here all norms are taken over the spacetime slab $[-T, T] \times \mathbb{R}$. $\square$

REFERENCES

[1] D. Anderson and M. Lisak. Nonlinear asymmetric self-phase modulation and self-steepening of pulses in long optical waveguides. *Phys. Rev. A*, 27:1393–1398, Mar 1983.

[2] H. Bahouri, T. M. Leslie, and G. Perelman. H^s bounds for the derivative nonlinear Schrödinger equation. *Preprint*, 2021. `arXiv:2107.12297`.

[3] H. Bahouri and G. Perelman. Global well-posedness for the derivative nonlinear Schrödinger equation. *Invent. Math.*, 229(2):639–688, 2022.

[4] H. A. Biagioni and F. Linares. Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations. *Trans. Amer. Math. Soc.*, 353(9):3649–3659, 2001.

[5] B. Bringmann, R. Killip, and M. Visan. Global well-posedness for the fifth-order KdV equation in $H^{-1}(\mathbb{R})$. *Ann. PDE*, 7(2):Paper No. 21, 46, 2021.

[6] H. H. Chen, Y. C. Lee, and C. S. Liu. Integrability of nonlinear Hamiltonian systems by inverse scattering method. *Phys. Scripta*, 20(3-4):490–492, 1979. Special issue on solitons in physics.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global well-posedness for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 33(3):649–669, 2001.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. A refined global well-posedness result for Schrödinger equations with derivative. *SIAM J. Math. Anal.*, 34(1):64–86, 2002.

[9] K. Dysthe, H. E. Krogstad, and P. Müller. Oceanic rogue waves. *Ann. Rev. Fluid Mech.*, 40(1):287–310, 2008.

[10] K. Fujiwara, V. Georgiev, and T. Ozawa. Self-similar solutions to the derivative nonlinear Schrödinger equation. *J. Differential Equations*, 268(12):7940–7961, 2020.

[11] N. Fukaya, M. Hayashi, and T. Inui. A sufficient condition for global existence of solutions to a generalized derivative nonlinear Schrödinger equation. *Anal. PDE*, 10(5):1149–1167, 2017.

[12] V. S. Gerdjikov and M. I. Ivanov. A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures. *Bulgar. J. Phys.*, 10:130–143, 1983.

[13] A. Grünrock. Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. *Int. Math. Res. Not.*, (41):2525–2558, 2005.

[14] S. Guo, X. Ren, and B. Wang. Local well-posedness for the derivative nonlinear Schrödinger equation with L^2-subcritical data. *Discrete Contin. Dyn. Syst.*, 41(9):4207–4253, 2021.

[15] Z. Guo and Y. Wu. Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$. *Discrete Contin. Dyn. Syst.*, 37(1):257–264, 2017.

[16] B. Harrop-Griffiths, R. Killip, and M. Vişan. Microscopic conservation laws for integrable lattice models. *Monatsh. Math.*, 196(3):477–504, 2021.

[17] B. Harrop-Griffiths, R. Killip, and M. Vişan. Large-data equicontinuity for the derivative NLS. *Int. Math. Res. Not. IMRN*, (6):4601–4642, 2023.

[18] B. Harrop-Griffiths, R. Killip, and M. Visan. Sharp well-posedness for the cubic NLS and mKdV in $H^s(\mathbb{R})$. *Preprint*, 2020. `arXiv:2003.05011`.

[19] N. Hayashi and T. Ozawa. On the derivative nonlinear Schrödinger equation. *Phys. D*, 55(1-2):14–36, 1992.

[20] R. Jenkins, J. Liu, P. Perry, and C. Sulem. Global well-posedness for the derivative non-linear Schrödinger equation. *Comm. Partial Differential Equations*, 43(8):1151–1195, 2018.

[21] R. Jenkins, J. Liu, P. Perry, and C. Sulem. Soliton resolution for the derivative nonlinear Schrödinger equation. *Comm. Math. Phys.*, 363(3):1003–1049, 2018.

[22] R. Jenkins, J. Liu, P. Perry, and C. Sulem. The derivative nonlinear Schrödinger equation: global well-posedness and soliton resolution. *Quart. Appl. Math.*, 78(1):33–73, 2020.

[23] R. Jenkins, J. Liu, P. Perry, and C. Sulem. Global existence for the derivative nonlinear Schrödinger equation with arbitrary spectral singularities. *Anal. PDE*, 13(5):1539–1578, 2020.

[24] R. S. Johnson. On the modulation of water waves in the neighbourhood of $kh \approx 1.363$. *Proc. Roy. Soc. London Ser. A*, 357(1689):131–141, 1977.

[25] D. J. Kaup and A. C. Newell. An exact solution for a derivative nonlinear Schrödinger equation. *J. Mathematical Phys.*, 19(4):798–801, 1978.

[26] T. Kawahara. Weak non-linear magnetohydrodynamic waves traveling parallel to an external magnetic field in a cold collisionless plasma. In *Singular perturbation and its application in hydrodynamic equations*, volume 187 of *RIMS Kôkyûroku*, pages 1–14. 1973.

[27] R. Killip, M. Ntekoume, and M. Vişan. On the well-posedness problem for the derivative nonlinear Schrödinger equation. *Anal. PDE*, 16(5):1245–1270, 2023.

[28] R. Killip and M. Vişan. Nonlinear Schrödinger equations at critical regularity. In *Evolution equations*, volume 17 of *Clay Math. Proc.*, pages 325–437. Amer. Math. Soc., Providence, RI, 2013.

[29] R. Killip and M. Vişan. KdV is well-posed in H^{-1}. *Ann. of Math. (2)*, 190(1):249–305, 2019.

[30] R. Killip, M. Vişan, and X. Zhang. Low regularity conservation laws for integrable PDE. *Geom. Funct. Anal.*, 28(4):1062–1090, 2018.

[31] A. V. Kitaev. Self-similar solutions of a modified nonlinear Schrödinger equation. *Teoret. Mat. Fiz.*, 64(3):347–369, 1985.

[32] F. Klaus and R. Schippa. A priori estimates for the derivative nonlinear Schrödinger equation. *Funkcial. Ekvac.*, 65(3):329–346, 2022.

[33] A. Kundu. Exact solutions to higher-order nonlinear equations through gauge transformation. *Phys. D*, 25(1-3):399–406, 1987.

[34] J. Liu. *Global Well-posedness for the Derivative Nonlinear Schrödinger Equation Through Inverse Scattering.* ProQuest LLC, Ann Arbor, MI, 2017. Thesis (Ph.D.)–University of Kentucky.

[35] J. Liu, P. A. Perry, and C. Sulem. Global existence for the derivative nonlinear Schrödinger equation by the method of inverse scattering. *Comm. Partial Differential Equations*, 41(11):1692–1760, 2016.

[36] J. Liu, P. A. Perry, and C. Sulem. Long-time behavior of solutions to the derivative nonlinear Schrödinger equation for soliton-free initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(1):217–265, 2018.

[37] C. Miao, Y. Wu, and G. Xu. Global well-posedness for Schrödinger equation with derivative in $H^{\frac{1}{2}}(\mathbb{R})$. *J. Differential Equations*, 251(8):2164–2195, 2011.

[38] K. Mio, T. Ogino, K. Minami, and S. Takeda. Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Jpn.*, 41:265–271, Jul 1976.

[39] E. Mjølhus. On the modulational instability of hydromagnetic waves parallel to the magnetic field. *J. Plasma Phys.*, 16(3):321–334, 1976.

[40] D. E. Pelinovsky, A. Saalmann, and Y. Shimabukuro. The derivative NLS equation: global existence with solitons. *Dyn. Partial Differ. Equ.*, 14(3):271–294, 2017.

[41] D. E. Pelinovsky and Y. Shimabukuro. Existence of global solutions to the derivative NLS equation with the inverse scattering transform method. *Int. Math. Res. Not. IMRN*, (18):5663–5728, 2018.

[42] M. Riesz. Sur les ensembles compacts de fonctions sommables. *Acta Sci. Math. (Szeged)*, 6:136–142, 1933.

[43] A. Rogister. Parallel propagation of nonlinear low-frequency waves in high-β plasma. *Physics of Fluids*, 14(12):2733–2739, 1971.

[44] A. Saalmann. Global existence for the derivative NLS equation in the presence of solitons. *Preprint*, 2017. arXiv:1704.00071.

[45] H. Takaoka. Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity. *Adv. Differential Equations*, 4(4):561–580, 1999.

[46] H. Takaoka. Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces. *Electron. J. Differential Equations*, pages No. 42, 23, 2001.

[47] X. Tang and G. Xu. Microscopic conservation laws for the derivative nonlinear Schrödinger equation. *Lett. Math. Phys.*, 111(6):Paper No. 138, 25, 2021.

[48] T. Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[49] D. Tataru. Rough solutions for the wave maps equation. *Amer. J. Math.*, 127(2):293–377, 2005.

[50] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem. *Funkcial. Ekvac.*, 23(3):259–277, 1980.

[51] M. Tsutsumi and I. Fukuda. On solutions of the derivative nonlinear Schrödinger equation. II. *Funkcial. Ekvac.*, 24(1):85–94, 1981.

[52] N. Tzoar and M. Jain. Self-phase modulation in long-geometry optical waveguides. *Phys. Rev. A*, 23:1266–1270, Mar 1981.

[53] M. Wadati, H. Sanuki, K. Konno, and Y.-H. Ichikawa. Circular polarized nonlinear Alfvén waves—a new type of nonlinear evolution equation in plasma physics. *Rocky Mountain J. Math.*, 8(1-2):323–331, 1978.

[54] M. Wadati and K. Sogo. Gauge transformations in soliton theory. *J. Phys. Soc. Japan*, 52(2):394–398, 1983.

[55] Y. Wu. Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space. *Anal. PDE*, 6(8):1989–2002, 2013.

[56] Y. Wu. Global well-posedness on the derivative nonlinear Schrödinger equation. *Anal. PDE*, 8(5):1101–1112, 2015.

Department of Mathematics, University of California, Los Angeles, CA 90095, USA
Email address: harropgriffiths@math.ucla.edu

Department of Mathematics, University of California, Los Angeles, CA 90095, USA
Email address: killip@math.ucla.edu

Department of Mathematics, Rice University, Houston, TX 77005-1892, USA
Email address: maria.ntekoume@rice.edu

Department of Mathematics, University of California, Los Angeles, CA 90095, USA
Email address: visan@math.ucla.edu