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Well-posedness of the governing equations for a quasi-linear viscoelastic model with pressure-dependent moduli in which both stress and strain appear linearly

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Abstract. The response of a body described by a quasi-linear viscoelastic constitutive relation, whose material moduli depend on the mechanical pressure (that is one-third the trace of stress) is studied. The constitutive relation stems from a class of implicit relations between the histories of the stress and the relative deformation gradient. A-priori thresholding is enforced through the pressure that ensures that the displacement gradient remains small. The resulting mixed variational problem consists of an evolutionary equation with the Volterra convolution operator; this equation is studied for well-posedness within the theory of maximal monotone graphs. For isotropic extension or compression, a semi-analytic solution of the quasi-linear viscoelastic problem is constructed under stress control. The equations are studied numerically with respect to monotone loading both with and without thresholding. In the example, the thresholding procedure ensures that the solution does not blow-up in finite time.

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1. Introduction

Implicit constitutive relations were introduced by Rajagopal [32] to describe elastic response that could not be described by classical constitutive relations for the same. In that study, he also considered implicit relations for viscoelastic response between the stress and its various time derivatives, and kinematical variables and their various time derivatives. Implicit constitutive relations for viscoelastic response had been introduced much earlier by Burgers [2] and Oldroyd [28]. While the constitutive relation introduced by Maxwell [24] relates the stress and the time rate of stress to the symmetric part of the velocity gradient, it is not an implicit relation as the symmetric part of the velocity gradient can be expressed in terms of the stress and the time rate of stress. Later, Průša and Rajagopal [31] introduced implicit relations between the history of the stress and the history of the relative deformation gradient. Recently, Murru et al. [27] derived a subclass of constitutive relations which are appropriate to describe the elastic response of materials such as rocks, bone, ceramics, concrete, intermetallic alloys and other porous bodies. Rajagopal and Wineman [35] put into place implicit constitutive relations for viscoelastic porous bodies.

Let us consider the implicit response of elastic bodies introduced in [32]. The challenge consists in the fact that the implicit constitutive relation between the Cauchy stress $\boldsymbol{\sigma}$ and the deformation gradient $\mathbf{F}$ is given in the form

$$\mathfrak{F}(\boldsymbol{\sigma}, \mathbf{F}) = \mathbf{0}$$

and cannot be inverted to express the stress as a function of the strain, and vice versa.

A particular subclass of these constitutive relations was introduced by Rajagopal and co-authors [34, 35] in which both the stress and the linearized strain $\boldsymbol{\varepsilon}$ appear linearly. Such material response is nonlinear as it contains the mutual product of stress and strain, allowing one to have linearly inhomogeneous material moduli. The material moduli depend on the density in contrast to the classical linearized elastic model with constant coefficients. In virtue of the balance of mass, the density can be replaced by $\rho = \rho_R/(1+\mathrm{tr}\boldsymbol{\varepsilon})$, where ρ_R is the density in the reference configuration. Well-posedness for the corresponding problems was established within the variational theory by Itou et al. [14–16] by thresholding the moduli, thus preventing them from becoming unbounded.

The first integral constitutive relation to describe viscoelastic response was developed by Boltzmann [1], and this has been followed by integral constitutive relations due to Green and Rivlin [8], Lockett [23], Pipkin and Rogers [29] and others. Fung [7] developed a one-dimensional approximation, referred to as a quasi-linear constitutive relation, to describe the viscoelastic behavior of biological tissues. Muliana et al. [26] developed a quasi-linear viscoelastic model in three-dimensions, wherein the relationship between the stress and strain is nonlinear. These constitutive relations have been extended to implicit constitutive relations, in the context of strain-limiting approach by Bulíček et al. [6] and Itou et al. [10,11]. The quasi-linear viscoelastic model has been used to study the Boussinesq problem by Itou et al. [12,13] for the indentation of half-space by a rigid punch with unknown contact zones. We cite the references [18–21,30] for nonlinear and unilateral boundary conditions which is appropriate to problems concerning the response of viscoelastic materials.

In order to introduce the quasi-linear constitutive relation, we begin with a consideration of the implicit constitutive relation for elastic response due to Rajagopal [34] in which both the stress $\boldsymbol{\sigma}$ and the linearized strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$ appear linearly:

$$(1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma})\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1(1+\lambda_1 \mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})\boldsymbol{\sigma} + E_2(1+\lambda_2 \mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \tag{1.1}$$

where $\mathbf{I}$ is the identity transformation, E_1, E_2 and $\lambda_1, \lambda_2, \lambda_3$ are constants. Equation (1.1) is nonlinear since it involves the products of variables $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{\mathrm{e}}$. When all $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the above constitutive relation reduces to that for classical linearized elasticity. Let $\lambda_1 = \lambda_2 = 0$ such that (1.1) can be inverted as an explicit constitutive expression for the strain in terms of the stress

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1 \frac{1}{1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma}}\boldsymbol{\sigma} + E_2 \frac{1}{1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma}}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \tag{1.2}$$

where the parameters $E_1 > 0$ and $E_2 > 0$, and $\lambda_3 \in \mathbb{R}$ is a material moduli. Notice that the linearized strain is a nonlinear function of the stress, but in (1.1), the linearized strain and stress are such that they appear linearly. Below we make the assumptions in order to get from (1.2) a specific model suitable for mathematical analysis.

Decomposing the stress and strain tensors into its deviatoric and spherical parts

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \quad \boldsymbol{\varepsilon}^{\mathrm{e}} = (\boldsymbol{\varepsilon}^{\mathrm{e}})^* + \frac{1}{3}(\mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}})\mathbf{I}, \tag{1.3}$$

we assume that the nonlinearity in the deviatoric part of (1.2) is negligible, for example, for isotropic extension or compression in Sect. 4, such that

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1\boldsymbol{\sigma}^* + E_3 \frac{1}{1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma}}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \tag{1.4}$$

where $E_3 := E_1/3 + E_2$. This assumption eliminates the mixed term $\boldsymbol{\sigma}^*/(1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma})$ from our consideration. It is worth noting that $\mathrm{tr}\boldsymbol{\sigma}$ determines the mechanical pressure

$$p = -\frac{\mathrm{tr}\boldsymbol{\sigma}}{3}, \tag{1.5}$$

thus implying the pressure-dependent factor in front of $\mathrm{tr}\boldsymbol{\sigma}$ (see Rajagopal [33] for a discussion of the concept of pressure).

The next assumption allows us to guarantee properties of ellipticity and boundedness for the constitutive relation. It can be observed that small $|1+\lambda_3 \mathrm{tr}\boldsymbol{\sigma}|$ leads to the equation (1.4), wherein the strain

$\boldsymbol{\varepsilon}^{\mathrm{e}}$ becomes unbounded. For consistency with the assumption of small displacement, according to [14–16], we prescribe the lower and upper thresholds

$$0 < \underline{M} \leq 1 \leq \overline{M}, \tag{1.6}$$

where $\underline{M}$ may be small, and the cut-off function

$$B(\mathrm{tr}\boldsymbol{\sigma}) := \begin{cases} \dfrac{\mathrm{tr}\boldsymbol{\sigma}}{\underline{M}}, & \text{if } 1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} < \underline{M} \\ \dfrac{\mathrm{tr}\boldsymbol{\sigma}}{1+\lambda_3\mathrm{tr}\boldsymbol{\sigma}}, & \text{if } \underline{M} \leq 1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} \leq \overline{M} \\ \dfrac{\mathrm{tr}\boldsymbol{\sigma}}{\overline{M}}, & \text{if } 1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} > \overline{M} \end{cases}. \tag{1.7}$$

Using (1.6) and (1.7), the thresholding equation is introduced as

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1\boldsymbol{\sigma}^* + E_3 B(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} =: \mathcal{F}[\boldsymbol{\sigma}]. \tag{1.8}$$

This includes equation (1.4) if $\underline{M} \leq 1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} \leq \overline{M}$ in (1.7). Otherwise, if $1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} < \underline{M}$, then (1.8) turns into the linearized relation

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1\boldsymbol{\sigma}^* + \frac{E_3}{\underline{M}}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}, \tag{1.9}$$

on the other hand, if $1+\lambda_3\mathrm{tr}\boldsymbol{\sigma} > \overline{M}$, then it is linearized as well

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1\boldsymbol{\sigma}^* + \frac{E_3}{\overline{M}}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}. \tag{1.10}$$

Furthermore, if $\lambda_3 = 0$, then $B(\mathrm{tr}\boldsymbol{\sigma}) = \mathrm{tr}\boldsymbol{\sigma}$ in (1.7) due to (1.6), and we recover the equation of classical linearized elasticity

$$\boldsymbol{\varepsilon}^{\mathrm{e}} = E_1\boldsymbol{\sigma}^* + E_3(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I} = E_1\boldsymbol{\sigma} + E_2(\mathrm{tr}\boldsymbol{\sigma})\mathbf{I}. \tag{1.11}$$

From (1.11), the material moduli are identified as

$$E_1 = \frac{1+\nu}{E} = \frac{1}{2\mu} > 0, \quad E_2 = -\frac{\nu}{E}, \quad E_3 = \frac{1-2\nu}{3E} = \frac{1}{9K} > 0, \tag{1.12}$$

where $E > 0$ and $\nu \in (0, 1/2)$ are the Young's modulus and Poisson's ratio, which determine the Lame parameter μ and the bulk modulus K, whereas λ_3 in (1.4) and the thresholds $\underline{M}, \overline{M}$ in (1.8) are fitting parameters.

For time $t \geq 0$, following Rajagopal and co-authors [26,35], we introduce the viscoelastic constitutive relation corresponding to that in the thresholding equation (1.8):

$$\boldsymbol{\varepsilon}^{\mathrm{v}}(t) = \int_0^t J'(t-s)\mathcal{F}[\boldsymbol{\sigma}(s)]\,\mathrm{d}s =: \mathcal{I}\big[\mathcal{F}[\boldsymbol{\sigma}]\big](t). \tag{1.13}$$

It corresponds to the constitutive equation (1.4) when

$$\boldsymbol{\varepsilon}^{\mathrm{v}} = \mathcal{I}\Big[E_1\boldsymbol{\sigma}^* + E_3\frac{\mathrm{tr}\boldsymbol{\sigma}}{1+\lambda_3\mathrm{tr}\boldsymbol{\sigma}}\mathbf{I}\Big]. \tag{1.14}$$

The kernel $J \geq 0$ that appears in the Volterra convolution equation (1.13) is typically given by the exponential sum

$$J(t) = \sum_{n=1}^{N} J_n\left[1 - \exp\left(-\frac{t}{\tau_n}\right)\right], \quad J'(t) = \sum_{n=1}^{N} \frac{J_n}{\tau_n}\exp\left(-\frac{t}{\tau_n}\right) \tag{1.15}$$

with creep parameters $J_1, \dots, J_N$, $\tau_1, \dots \tau_N \geq 0$. It is worth noting that the integral operator $\mathcal{I}$ cannot be inverted in general, even if $\mathcal{F}$ would be linear in $\boldsymbol{\sigma}$. In the particular case of $N = 1$ in (1.15), equation (1.13) reads

$$\boldsymbol{\varepsilon}^{\mathrm{v}}(t) = \frac{J_1}{\tau_1} \int_0^t \exp\left(\frac{s-t}{\tau_1}\right) \mathcal{F}[\boldsymbol{\sigma}(s)]\, \mathrm{d}s. \tag{1.16}$$

On differentiating (1.16) with respect to t, we calculate $J''(t) = -J'(t)/\tau_1$ and can invert the equation, thus arriving at the generalized Kelvin–Voigt model:

$$\boldsymbol{\varepsilon}^{\mathrm{v}} + \tau_1 \dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} = J_1 \mathcal{F}[\boldsymbol{\sigma}], \tag{1.17}$$

where dot denotes the time derivative. The classical linearized Kelvin–Voigt model $\boldsymbol{\varepsilon}^{\mathrm{v}} + \tau_1 \dot{\boldsymbol{\varepsilon}}^{\mathrm{v}} = E_1 \boldsymbol{\sigma}$ is recovered from equation (1.17) with the parameters chosen as $J_1 = 1$ and $\tau_1 = \alpha E_1$, where $\alpha > 0$ is the viscosity.

From the mathematical point of view, the principal difficulty of quasi-linear viscoelastic equations (1.13) and (1.14) concerns the fact that the Volterra convolution operator $\mathcal{I}$ is not monotone or coercive. Therefore, the monotone operator theory is inapplicable to guarantee solvability for the corresponding variational problems that will be stated. To remedy this difficulty, following [11–13], we introduce an auxiliary relation between the strains $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and $\boldsymbol{\varepsilon}^{\mathrm{v}}$ by the integral formula

$$\boldsymbol{\varepsilon}^{\mathrm{v}} = \mathcal{I}(\boldsymbol{\varepsilon}^{\mathrm{e}}). \tag{1.18}$$

Indeed, inserting into (1.18) the strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$ satisfying the constitutive relation for elastic response (1.8), we find $\boldsymbol{\varepsilon}^{\mathrm{v}} = \mathcal{I}\big[\mathcal{F}[\boldsymbol{\sigma}]\big]$ which reduces to the constitutive equation for viscoelastic response (1.13).

The nonlinear equation (1.8) can be described by a graph $\mathfrak{G}$ between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{\mathrm{e}}$. The theory of graphs is well-suited to study implicit and multi-valued functions. Following Bulíček et al. [4,5], we will prove that $\mathfrak{G}$ is maximal monotone and coercive on the appropriate selection $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon}(\mathbf{w})$, in this manner justifying well-posedness for the underlying variational problem. The concept of maximal monotony for nonlinear operators was established by Browder [3] and Minty [25]. We also cite relevant results using the pseudo-monotone operators in [9,17] and hemi-variational inequalities in [22,36].

The structure of the paper is as follows. In Sect. 2, we document a boundary-value problem for the quasi-linear viscoelastic model for unknown $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^{\mathrm{v}} = \boldsymbol{\varepsilon}(\mathbf{u})$ within the context of the thresholding equation (1.13). Well-posedness for the corresponding mixed variational problem is proved in Sect. 3 based on the representation (1.18). In Sect. 4, we apply the theory to the semi-analytic example of isotropic extension or compression when deformation is controlled by the pressure. The numerical simulation tests are presented first applying monotone load, and second maintaining and then removing the load. The thresholding model (1.13) demonstrates an interesting feature when compared to the unthresholded model (1.14), whereas in the latter, a solution may blow-up under finite pressure when subject to monotone loading.

2. The quasi-linear viscoelastic problem

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^d$, where the spatial dimensions $d = 2$ and $d = 3$ are physically relevant. Let its boundary $\partial\Omega$ be Lipschitz continuous, and the unit normal vector $\mathbf{n} = (n_1, \dots, n_d)$ at $\partial\Omega$ be dire cted outward Ω. We assume that the boundary is comprised of two mutually disjoint sets $\partial\Omega = \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{D}}}$ corresponding to the Neumann Γ_{N} and Dirichlet $\Gamma_{\mathrm{D}} \neq \emptyset$ boundary conditions. For spatial points $\mathbf{x} = (x_1, \dots, x_d)$ in the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ and times $t \in [0, T]$ with some final time $T > 0$ fixed, we denote the right time-space cylinder by $\Omega^T = (0, T) \times \Omega$ with the side consisting of two parts $\Gamma_{\mathrm{N}}^T = (0, T) \times \Gamma_{\mathrm{N}}$ and $\Gamma_{\mathrm{D}}^T = (0, T) \times \Gamma_{\mathrm{D}}$.

We look for the displacement $\mathbf{u} = (u_1, \dots, u_d)(t, \mathbf{x})$ in $\overline{\Omega^T}$. It determines the linearized strain $\boldsymbol{\varepsilon}(\mathbf{u})$ valued in the space of second-order symmetric tensors $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ as the symmetric gradient with the entries

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \dots, d. \tag{2.1}$$

For the given body force $\mathbf{f} = (f_1, \dots, f_d)(t, \mathbf{x})$, $\mathbf{f} \in C([0,T]; L^2(\Omega; \mathbb{R}^d))$, we look also for the stress tensor $\boldsymbol{\sigma}(t, \mathbf{x}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfying the equilibrium equation omitting the inertia terms:

$$-\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad i = 1, \dots, d, \quad \text{in } \Omega^T. \tag{2.2}$$

Prescribing the boundary force $\mathbf{g} = (g_1, \dots, g_d)(t, \mathbf{x})$, $\mathbf{g} \in C([0,T]; L^2(\Gamma_{\mathrm{N}}; \mathbb{R}^d))$, we augment (2.2) with the mixed Dirichlet–Neumann boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma^T_{\mathrm{D}}, \tag{2.3}$$

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma^T_{\mathrm{N}}, \tag{2.4}$$

where $\boldsymbol{\sigma}\mathbf{n} = (\sum_{j=1}^{d} \sigma_{1j} n_j, \dots, \sum_{j=1}^{d} \sigma_{dj} n_j)$. The quasi-static equilibrium problem (2.1)–(2.4) is rendered complete with the constitutive equation for viscoelastic response (1.13) for $\boldsymbol{\varepsilon}^{\mathrm{v}} = \boldsymbol{\varepsilon}(\mathbf{u})$ as

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \mathcal{I}\big[\mathcal{F}[\boldsymbol{\sigma}]\big]. \tag{2.5}$$

Following the argument for the use of (1.18) which was articulated in the Introduction, we look for the displacement $\mathbf{u}$ in the form

$$\mathbf{u} = \mathcal{I}[\mathbf{w}], \tag{2.6}$$

where the selection $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon}(\mathbf{w})$ satisfies equation (1.8). The relation (1.8) between stress and strain can be generalized to an implicit relation on graph $\mathfrak{G} \subset (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$ which we define by the inclusion

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{e}}) \in \mathfrak{G} \Leftrightarrow \boldsymbol{\varepsilon}^{\mathrm{e}} = \mathcal{F}[\boldsymbol{\sigma}]. \tag{2.7}$$

Lemma 2.1. (Selection on the graph) *The quasi-linear equation* (2.5) *for viscoelastic response completing the equilibrium problem* (2.1)–(2.4) *can be generalized to the identity* (2.6) *and the selection* $\boldsymbol{\varepsilon}^{\mathrm{e}} = \boldsymbol{\varepsilon}(\mathbf{w})$ *on the graph* $\mathfrak{G}$ *in* (2.7) *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{w})) \in \mathfrak{G}. \tag{2.8}$$

Proof. If the selection $\boldsymbol{\varepsilon}(\mathbf{w}) = \mathcal{F}[\boldsymbol{\sigma}]$ in (2.8) holds, then the variable $\mathbf{w}$ can be reduced by use of (2.6). Indeed, applying to the both sides of (2.6) the symmetric gradient from (2.1), we can interchange the linear operators $\boldsymbol{\varepsilon}$ and $\mathcal{I}$, thus obtaining

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathcal{I}[\mathbf{w}]) = \mathcal{I}[\boldsymbol{\varepsilon}(\mathbf{w})] = \mathcal{I}\big[\mathcal{F}[\boldsymbol{\sigma}]\big] \tag{2.9}$$

and resulting in (2.5).

Conversely, $\boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varepsilon}(\mathcal{I}[\mathbf{w}])$ follows (2.6) excluding rigid motions, which vanish here due to the homogeneous Dirichlet condition (2.3). Then, (2.5) and (2.6) together with (2.9) justify the selection (2.8). □

In the next lemma, we establish some useful properties of $\mathfrak{G}$.

Lemma 2.2. (Properties of the graph)

(i) The graph $\mathfrak{G}$ *in* (2.7) *includes the origin:*

$$(\mathbf{0}, \mathbf{0}) \in \mathfrak{G}. \tag{2.10}$$

(ii) For all $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{e}}) \in \mathfrak{G}$*, the graph is coercive with the uniform estimate:*

$$\boldsymbol{\varepsilon}^{\mathrm{e}} \cdot \boldsymbol{\sigma} \geq C_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}\|^2 + C_{\varepsilon} \|\boldsymbol{\varepsilon}^{\mathrm{e}}\|^2, \tag{2.11}$$

where the dot stands for the scalar product of tensors $\boldsymbol{\varepsilon}^{\mathrm{e}} \cdot \boldsymbol{\sigma} = \sum_{i,j=1}^{d} \varepsilon^{\mathrm{e}}_{ij} \sigma_{ij}$*, Frobenius norm* $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}$*, and the constant factors are*

$$C_{\boldsymbol{\sigma}} := \frac{1}{2} \min\Big(E_1, \frac{E_3 d}{\overline{M}^2}\Big), \quad C_{\varepsilon} := \frac{1}{2} \min\Big(\frac{1}{E_1}, \frac{\underline{M}^4}{(E_3 d)\overline{M}^2}\Big). \tag{2.12}$$

(iii) For all pairs $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1), (\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}$ *the graph is monotone:*

$$(\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0. \tag{2.13}$$

(iv) For $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$*, the graph is maximal monotone:*

$$\text{if } (\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 0 \text{ for all } (\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}, \quad \text{then } (\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in \mathfrak{G}. \tag{2.14}$$

Proof. For $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, it is straightforward to check that the function $B : \mathbb{R} \mapsto \mathbb{R}$ defined in (1.7) is Lipschitz continuous:

$$\big|B(\mathrm{tr}\boldsymbol{\sigma}^1) - B(\mathrm{tr}\boldsymbol{\sigma}^2)\big| \leq \frac{1}{\underline{M}^2} \big|\mathrm{tr}(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2)\big|, \tag{2.15}$$

and strongly monotone:

$$\big(B(\mathrm{tr}\boldsymbol{\sigma}^1) - B(\mathrm{tr}\boldsymbol{\sigma}^2)\big)\mathrm{tr}(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq \frac{1}{\overline{M}^2} \mathrm{tr}^2(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2), \tag{2.16}$$

hence bounded and coercive, too.

According to (1.3) extended to $\mathbb{R}^d$ and (1.8), the strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$ admits the decomposition

$$(\boldsymbol{\varepsilon}^{\mathrm{e}})^* = E_1 \boldsymbol{\sigma}^*, \quad \frac{1}{d} \mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}} = E_3 B(\mathrm{tr}\boldsymbol{\sigma}). \tag{2.17}$$

Forming the scalar product of (2.17) with $\boldsymbol{\sigma}$, using estimates (2.15) and (2.16) yields the lower bound

$$\begin{aligned}
\boldsymbol{\varepsilon}^{\mathrm{e}} \cdot \boldsymbol{\sigma} &= E_1 \|\boldsymbol{\sigma}^*\|^2 + E_3 B(\mathrm{tr}\boldsymbol{\sigma})\mathrm{tr}\boldsymbol{\sigma} \geq E_1 \|\boldsymbol{\sigma}^*\|^2 + \frac{E_3}{\overline{M}^2} \mathrm{tr}^2 \boldsymbol{\sigma} \\
&\geq \frac{E_1}{2} \Big(\|\boldsymbol{\sigma}^*\|^2 + \Big\|\frac{1}{E_1} (\boldsymbol{\varepsilon}^{\mathrm{e}})^*\Big\|^2\Big) + \frac{E_3}{2\overline{M}^2} \Big(\mathrm{tr}^2 \boldsymbol{\sigma} + \underline{M}^4 \big(\frac{1}{E_3 d} \mathrm{tr}\boldsymbol{\varepsilon}^{\mathrm{e}}\big)^2\Big) \\
&\geq \frac{1}{2} \min\Big(E_1, \frac{E_3 d}{\overline{M}^2}\Big) \big(\|\boldsymbol{\sigma}^*\|^2 + \frac{1}{d} \mathrm{tr}^2 \boldsymbol{\sigma}\big) + \frac{1}{2} \min\Big(\frac{1}{E_1}, \frac{\underline{M}^4}{(E_3 d)\overline{M}^2}\Big) \big(\|(\boldsymbol{\varepsilon}^{\mathrm{e}})^*\|^2 + \frac{1}{d} \mathrm{tr}^2 (\boldsymbol{\varepsilon}^{\mathrm{e}})\big).
\end{aligned}$$

In virtue of the norm identity $\|\boldsymbol{\sigma}\|^2 = \|\boldsymbol{\sigma}^*\|^2 + \mathrm{tr}^2 \boldsymbol{\sigma}/d$ and the notation for factors in (2.12), the lower estimate (2.11) follows.

For points on the graph $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1), (\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}$ the definition (2.7) implies that

$$\boldsymbol{\varepsilon}^n = E_1 (\boldsymbol{\sigma}^n)^* + E_3 B(\mathrm{tr}\boldsymbol{\sigma}^n)\mathbf{I} \quad \text{for } n = 1, 2. \tag{2.18}$$

Subtracting (2.18) for $n = 1$ and $n = 2$, with the help of (2.16) and using the notation $C_{\boldsymbol{\sigma}}$ from (2.12), we estimate

$$(\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^2) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) = E_1 \|(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2)^*\|^2 + E_3 \big(B(\mathrm{tr}\boldsymbol{\sigma}^1) - B(\mathrm{tr}\boldsymbol{\sigma}^2)\big)\mathrm{tr}(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \geq 2C_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2\|^2.$$

This justifies the monotone property (2.13) for $\mathfrak{G}$. Moreover, this estimate establishes strong monotonicity of $\mathcal{F}$, that is, uniqueness of the solution.

Now, for fixed $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in (\mathbb{R}^{d\times d}_{\mathrm{sym}})^2$, we assume that the inequality in (2.14) holds for all $(\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}$ satisfying (2.18). For arbitrary $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and small $\delta > 0$, let

$$\boldsymbol{\sigma}^{\delta} := \boldsymbol{\sigma}^1 \pm \delta \boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}^{\delta} := E_1 (\boldsymbol{\sigma}^{\delta})^* + E_3 B(\mathrm{tr}\boldsymbol{\sigma}^{\delta})\mathbf{I}, \tag{2.19}$$

which belongs to the graph: $(\boldsymbol{\sigma}^\delta, \boldsymbol{\varepsilon}^\delta) \in \mathfrak{G}$. Testing (2.14) with $(\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) = (\boldsymbol{\sigma}^\delta, \boldsymbol{\varepsilon}^\delta)$ and substituting (2.19) yields

$$0 \le (\boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^\delta) \cdot (\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^\delta) = \mp\delta\big[\boldsymbol{\varepsilon}^1 - E_1(\boldsymbol{\sigma}^1 \pm \delta\boldsymbol{\sigma})^* - E_3 B\big(\mathrm{tr}(\boldsymbol{\sigma}^1 \pm \delta\boldsymbol{\sigma})\big)\mathbf{I}\big] \cdot \boldsymbol{\sigma},$$

which after division by δ yields

$$\mp\big[\boldsymbol{\varepsilon}^1 - E_1(\boldsymbol{\sigma}^1)^* \mp \delta E_1\boldsymbol{\sigma}^* - E_3 B\big(\mathrm{tr}(\boldsymbol{\sigma}^1 \pm \delta\boldsymbol{\sigma})\big)\mathbf{I}\big] \cdot \boldsymbol{\sigma} \ge 0.$$

On taking the limit as $\delta \to 0$, the continuity of B in (2.15) leads to the variational equality

$$\big[\boldsymbol{\varepsilon}^1 - E_1(\boldsymbol{\sigma}^1)^* - E_3 B(\mathrm{tr}\boldsymbol{\sigma}^1)\mathbf{I}\big] \cdot \boldsymbol{\sigma} = 0 \tag{2.20}$$

for all $\boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. This justifies the equation (2.18) for $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1)$, thus $(\boldsymbol{\sigma}^1, \boldsymbol{\varepsilon}^1) \in \mathfrak{G}$, and the maximal monotone property (2.14) of the graph $\mathfrak{G}$ is valid.

The inclusion (2.10) is evident. The proof is completed. □

In the next section, we provide a variational formulation for the quasi-linear viscoelastic problem (2.1)–(2.4), (2.6) and (2.8).

3. Variational formulation within maximal monotone and coercive graphs

We start by recalling the Korn–Poincaré inequality:

$$\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega)} \le \|\mathbf{u}\|_{H^1(\Omega)} \le C_{\mathrm{KP}}\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^2(\Omega)} \quad \text{if } \mathbf{u} = \mathbf{0} \text{ on } \Gamma_{\mathrm{D}} \tag{3.1}$$

with constant $C_{\mathrm{KP}} \ge 1$, and the boundary trace theorem

$$\|\mathbf{u}\|_{L^2(\partial\Omega)} \le C_{\mathrm{tr}}\|\mathbf{u}\|_{H^1(\Omega)}, \quad C_{\mathrm{tr}} > 0. \tag{3.2}$$

We apply standard variational arguments, namely the equilibrium equation (2.2) is multiplied by $\mathbf{v} = (v_1, \ldots, v_d)(x)$ and integrated by parts over Ω using Green's formula

$$-\int_\Omega \sum_{i,j=1}^{d} \frac{\partial\sigma_{ij}}{\partial x_j} v_i \,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \,\mathrm{d}\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{v} \,\mathrm{d}S_{\mathbf{x}},$$

where $\boldsymbol{\varepsilon}(\mathbf{v})$ is defined according to (2.1). Applying the Neumann boundary condition (2.4), this yields the variational equation

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} \mathbf{g} \cdot \mathbf{v} \,\mathrm{d}S_{\mathbf{x}} \tag{3.3}$$

for all test functions $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$ such that $\mathbf{v} = \mathbf{0}$ on Γ_{D}. Conversely, for H^1-smooth stress $\boldsymbol{\sigma}$, pointwise relations (2.2) and (2.4) follow from (3.3).

For the stress $\boldsymbol{\sigma} \in C([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ and displacements $\mathbf{u}, \mathbf{w} \in C([0,T]; H^1(\Omega; \mathbb{R}^d))$ such that $\mathbf{u} = \mathbf{w} = \mathbf{0}$ on Γ^T_{D}, for every $t \in [0,T]$, we express the variational equation (2.6) in the form:

$$\int_\Omega \big(\mathbf{u} - \mathcal{I}[\mathbf{w}]\big) \cdot \boldsymbol{\xi} \,\mathrm{d}\mathbf{x} = 0 \tag{3.4}$$

for all test functions $\boldsymbol{\xi} \in L^2(\Omega; \mathbb{R}^d)$, and set a selection on the graph $\mathfrak{G}$ which fulfill the inclusion in (2.7) as

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{w})) \in \mathfrak{G} \Leftrightarrow \int_\Omega \big(\boldsymbol{\varepsilon}(\mathbf{w}) - \mathcal{F}[\boldsymbol{\sigma}]\big) \cdot \boldsymbol{\eta} \,\mathrm{d}\mathbf{x} = 0 \tag{3.5}$$

for all test functions $\boldsymbol{\eta} \in L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$.

We prove existence of the weak solution $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{w})$ satisfying the variational relations (3.3)–(3.5) based on Lemma 2.2. It is worth noting that the variable $\mathbf{w}$ is redundant and later can be reduced from the solution due to Lemma 2.1.

Theorem 3.1. (Well-posedness) *The unique solution $\boldsymbol{\sigma} \in C([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym}))$ and $\mathbf{u}, \mathbf{w} \in C([0,T]; H^1(\Omega; \mathbb{R}^d))$ such that $\mathbf{u} = \mathbf{w} = \mathbf{0}$ on $\Gamma_{\rm D}^T$ satisfying for all $t \in [0,T]$ the Dirichlet boundary condition (2.3) and variational relations (3.3)–(3.5) exists and fulfills the following a-priori estimate:*

$$C_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}\|^2_{C([0,T];L^2(\Omega))} + \frac{C_\varepsilon}{2C^2_{\rm KP}} \|\mathbf{w}\|^2_{C([0,T];H^1(\Omega))} \leq \frac{C^2_{\rm KP}}{2C_\varepsilon} \|C^2(\mathbf{f}, \mathbf{g})\|_{C([0,T])}, \tag{3.6}$$

where the constant $C_{\boldsymbol{\sigma}}$ and C_ε are defined in (2.12), and the forces determine

$$C(\mathbf{f}, \mathbf{g}) := \|\mathbf{f}\|_{L^2(\Omega)} + C_{\rm tr}\|\mathbf{g}\|_{L^2(\Gamma_{\rm N})}. \tag{3.7}$$

Proof. The theorem is proved in three steps: from the Galerkin approximation, we derive uniform estimate, and then pass to the limit.

k-dimensional Galerkin approximation. Let subspaces S^k and V^k of finite dimensions $k \in \mathbb{N}$ build the conforming approximation of the admissible stress $\boldsymbol{\sigma}$ and displacement $\mathbf{w}$ preserving $\mathbf{w} = \mathbf{0}$ on $\Gamma_{\rm D}$. We assume that $\cup_{k=1}^\infty S^k$ is dense in $L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym})$, the union $\cup_{k=1}^\infty V^k$ is dense in $H^1(\Omega; \mathbb{R}^d)$, and inclusion $\mathbf{v}^k \in V^k$ implies that $\boldsymbol{\varepsilon}(\mathbf{v}^k) \in S^k$.

First, we look for a discrete solution $\boldsymbol{\sigma}^k \in C([0,T]; S^k)$ and $\mathbf{w}^k \in C([0,T]; V^k)$ fulfilling for all $t \in [0,T]$ the semi-discrete in space problem (3.3):

$$\int_\Omega \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\mathbf{v}^k)\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}^k\, d\mathbf{x} + \int_{\Gamma_{\rm N}} \mathbf{g} \cdot \mathbf{v}^k\, dS_{\mathbf{x}}, \tag{3.8}$$

which is endowed with the constitutive equation according to (3.5):

$$\int_\Omega \big(\boldsymbol{\varepsilon}(\mathbf{w}^k) - \mathcal{F}[\boldsymbol{\sigma}^k]\big) \cdot \boldsymbol{\eta}^k\, d\mathbf{x} = 0 \tag{3.9}$$

for all test functions $\mathbf{v}^k \in V^k$ and $\boldsymbol{\eta}^k \in S^k$. Its unique solution exists in virtue of the Browder–Minty theorem, because the Lipschitz continuity (2.15) and strong monotony (2.16) of the nonlinear function B provide the coercive, strictly monotone, bounded, and hemi-continuous properties of the operator $\mathcal{F}$ in the mixed variational problem (3.8) and (3.9).

Uniform in k estimate. Since the stress $\boldsymbol{\sigma}^k$ and strain $\boldsymbol{\varepsilon}(\mathbf{w}^k)$ are connected by the relation (3.9), the selection $(\boldsymbol{\sigma}^k, \boldsymbol{\varepsilon}(\mathbf{w}^k)) \in \mathfrak{G}$ holds according to definition (3.5). Therefore, applying the coercivity (2.11) and using the Korn–Poincaré inequality (3.1) yields the lower bound

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^k) \cdot \boldsymbol{\sigma}^k\, d\mathbf{x} \geq C_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}^k\|^2_{L^2(\Omega)} + \frac{C_\varepsilon}{C^2_{\rm KP}} \|\mathbf{w}^k\|^2_{H^1(\Omega)}. \tag{3.10}$$

On the other side, we test the variational equation (3.8) with $\mathbf{v}^k = \mathbf{w}^k$, apply the Cauchy–Schwarz inequality and trace theorem (3.2), then the weighted Young inequality provides the upper bound, where the constant $C(\mathbf{f}, \mathbf{g})$ is from (3.7):

$$\int_\Omega \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\mathbf{w}^k)\, d\mathbf{x} \leq C(\mathbf{f}, \mathbf{g}) \|\mathbf{w}^k\|_{H^1(\Omega)} \leq \frac{C_\varepsilon}{2C^2_{\rm KP}} \|\mathbf{w}^k\|^2_{H^1(\Omega)} + \frac{C^2_{\rm KP}}{2C_\varepsilon} C^2(\mathbf{f}, \mathbf{g}). \tag{3.11}$$

Combining together (3.10) and (3.11) and taking maximum over $t \in [0,T]$ leads to the uniform in k estimate:

$$C_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}^k\|^2_{C([0,T];L^2(\Omega))} + \frac{C_\varepsilon}{2C^2_{\rm KP}} \|\mathbf{w}^k\|^2_{C([0,T];H^1(\Omega))} \leq \frac{C^2_{\rm KP}}{2C_\varepsilon} \|C^2(\mathbf{f}, \mathbf{g})\|_{C([0,T])}. \tag{3.12}$$

Passage to the limit $k \to \infty$. From the uniform estimate (3.12), we obtain a weakly convergent subsequence still denoted by k such that as $k \to \infty$:

$$\boldsymbol{\sigma}^k \rightharpoonup \boldsymbol{\sigma} \text{ in } C([0,T]; L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})), \quad \mathbf{w}^k \rightharpoonup \mathbf{w} \text{ in } C([0,T]; H^1(\Omega; \mathbb{R}^d)). \tag{3.13}$$

Taking the limit of the linear equation (3.8), we get the equilibrium equation (3.3) for $\boldsymbol{\sigma}$ and $\mathbf{w}$ from (3.13). Next, we derive the nonlinear equation in (3.5).

Testing the equilibrium equation (3.3) with the limit function $\mathbf{v} = \mathbf{w}$ implies

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w}\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g} \cdot \mathbf{w}\,\mathrm{d}S_\mathbf{x}. \tag{3.14}$$

For finite k, inserting $\mathbf{v}^k = \mathbf{w}^k$ into (3.8), we obtain

$$\int_\Omega \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\mathbf{w}^k)\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w}^k\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{N}} \mathbf{g} \cdot \mathbf{w}^k\,\mathrm{d}S_\mathbf{x}. \tag{3.15}$$

By the virtue of weak convergences in (3.13), from (3.14) and (3.15), we conclude that

$$\lim_{k\to\infty} \int_\Omega \boldsymbol{\sigma}^k \cdot \boldsymbol{\varepsilon}(\mathbf{w}^k)\,\mathrm{d}\mathbf{x} = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x}. \tag{3.16}$$

Arbitrary $(\boldsymbol{\sigma}^2, \boldsymbol{\varepsilon}^2) \in \mathfrak{G}$ and $(\boldsymbol{\sigma}^k, \boldsymbol{\varepsilon}(\mathbf{w}^k)) \in \mathfrak{G}$ on the graph fulfill (2.13), that is

$$\left[\boldsymbol{\varepsilon}(\mathbf{w}^k) - \boldsymbol{\varepsilon}^2\right] \cdot (\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^2) \geq 0.$$

This allows us to estimate from below the scalar product

$$\int_\Omega \left[\boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}^2\right] \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^2)\,\mathrm{d}\mathbf{x} \geq \int_\Omega \Big(\boldsymbol{\varepsilon}(\mathbf{w} - \mathbf{w}^k) \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^2) + \left[\boldsymbol{\varepsilon}(\mathbf{w}^k) - \boldsymbol{\varepsilon}^2\right] \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^k)\Big)\,\mathrm{d}\mathbf{x}.$$

On taking the limit based on the convergences in (3.13) and (3.16) leads to

$$\begin{aligned}&\int_\Omega \left[\boldsymbol{\varepsilon}(\mathbf{w}) - \boldsymbol{\varepsilon}^2\right] \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^2)\,\mathrm{d}\mathbf{x} \\ &\quad \geq \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}) \cdot \boldsymbol{\sigma}\,\mathrm{d}\mathbf{x} - \limsup_{k\to\infty} \int_\Omega \boldsymbol{\varepsilon}(\mathbf{w}^k) \cdot \boldsymbol{\sigma}^k\,\mathrm{d}\mathbf{x} = 0.\end{aligned}$$

Then, the maximal monotone property (2.14) of the graph guarantees the inclusion $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{w})) \in \mathfrak{G}$, i.e., relation (3.5) holds for the limit functions from (3.13).

The uniqueness of $\boldsymbol{\sigma}$ and $\mathbf{w}$ can be derived from the strong monotone property (2.16) of B entering the term $\mathcal{F}$ as shown in the estimate between (2.18) and (2.19). The displacement $\mathbf{u}$ is determined uniquely from the equation (3.4). This completes the proof. □

4. Semi-analytical solution for isotropic extension or compression

In dimension $d = 3$, our consideration is simplified under the assumption of isotropic extension or compression independent of $\mathbf{x}$ such that

$$\boldsymbol{\sigma} = -p(t)\mathbf{I}, \quad \boldsymbol{\varepsilon} = \frac{1}{3}e(t)\mathbf{I}. \tag{4.1}$$

Formula (4.1) implies the deviatoric parts $\boldsymbol{\sigma}^* = \boldsymbol{\varepsilon}^* = \mathbf{0}$. Therefore, the unknowns are the scalar time-dependent functions for pressure p according to (1.5) and dilatation $e = \mathrm{tr}\boldsymbol{\varepsilon}$ according to the deviatoric-spherical decomposition (1.3). The sign $p < 0$ implies extension, and $p > 0$ compression. Since the stress tensor is space-independent, the equilibrium equation (2.2) is satisfied identically with the body force $\mathbf{f} \equiv \mathbf{0}$.

In the representation (4.1), we can distinguish between elastic and viscoelastic dilatation response through

$$e^{\mathrm{e}} = \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{w}), \quad e^{\mathrm{v}} = \mathrm{tr}\boldsymbol{\varepsilon}(\mathbf{u}). \tag{4.2}$$

Inserting (4.2) into the governing equation (2.6) with the Volterra convolution operator $\mathcal{I}$ from (1.13) implies that

$$e^{\mathrm{v}}(t) = \mathcal{I}[e^{\mathrm{e}}](t) = \int_0^t J'(t-s)e^{\mathrm{e}}(s)\,\mathrm{d}s, \tag{4.3}$$

and the governing relation (2.8) using the cut-off function B from (1.7) takes the following form:

$$e^{\mathrm{e}} = 3E_3 B(-3p) = 9E_3 \begin{cases} \dfrac{-p}{\underline{M}}, & \text{if } 1-3\lambda_3 p < \underline{M} \\ \dfrac{-p}{1-3\lambda_3 p}, & \text{if } \underline{M} \le 1-3\lambda_3 p \le \overline{M} \\ \dfrac{-p}{\overline{M}}, & \text{if } 1-3\lambda_3 p > \overline{M} \end{cases}. \tag{4.4}$$

We will compare (4.4) with the unthresholded constitutive equation which exhibits unlimited strain, introduced according to (1.4) as

$$e^{\mathrm{e}} = 9E_3 \frac{-p}{1-3\lambda_3 p}. \tag{4.5}$$

The stress control formulation is studied. Namely, from the prescribed pressure $p(t)$ as $t \in [0,T]$, we find the evolution of dilatation $e^{\mathrm{e}}(t)$ and $e^{\mathrm{v}}(t)$ satisfying equations (4.3) and (4.4) (respectively, (4.5) in the case when there is no thresholding).

To solve (4.3), we discretize the problem on the time-grid of $M+1$ points

$$0 = t_0 < t_1 < \cdots < t_M = T,$$

and pick the piecewise-affine approximation

$$e^{\mathrm{e}}_M(t) = e^{\mathrm{e}}(t_{k-1}) + (t-t_{k-1})\delta e^{\mathrm{e}}_k \quad \text{as } t \in [t_{k-1}, t_k] \tag{4.6}$$

for $k = 1, \ldots, M$, where the differences

$$\delta e^{\mathrm{e}}_k := \frac{e^{\mathrm{e}}(t_k) - e^{\mathrm{e}}(t_{k-1})}{\delta t_k}, \quad \delta t_k = t_k - t_{k-1}.$$

Inserting (4.6) into (4.3) and using $(e^{\mathrm{e}}_M)'(t) = \delta e^{\mathrm{e}}_k$ as $t \in [t_{k-1}, t_k]$ provides the numerical quadrature for the Volterra convolution operator

$$e^{\mathrm{v}}_M(t) = \mathcal{I}[e^{\mathrm{e}}_M](t) = \sum_{k=1}^{M} I^M_k, \quad I^M_k := \int_{t_{k-1}}^{t_k} J'(t-s)e^{\mathrm{e}}_M(s)\,\mathrm{d}s, \tag{4.7}$$

which after integration by parts using $J'(t-s) = -\mathrm{d}J(t-s)/\mathrm{d}s$ yields

$$I^M_k = \delta e^{\mathrm{e}}_k \int_{t_{k-1}}^{t_k} J(t-s)\,\mathrm{d}s - J(t-t_k)e^{\mathrm{e}}(t_k) + J(t-t_{k-1})e^{\mathrm{e}}(t_{k-1}). \tag{4.8}$$

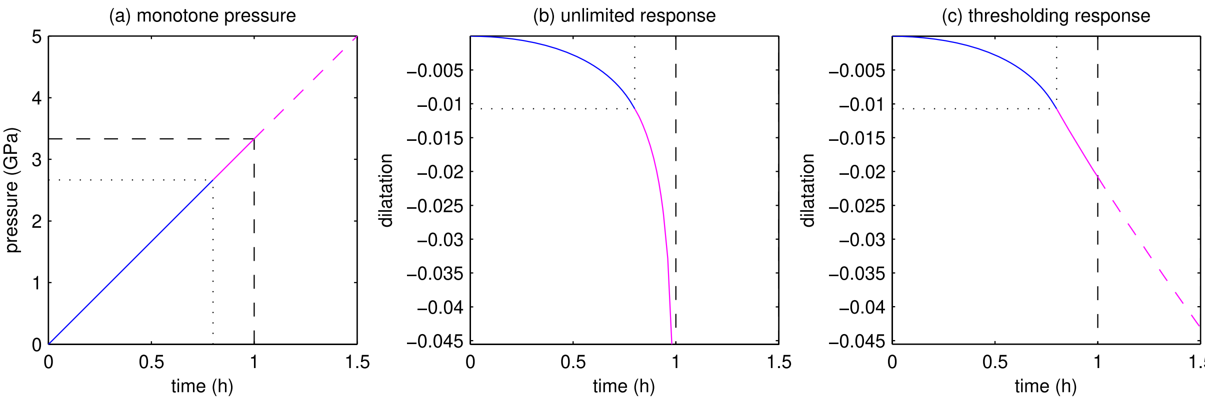

FIG. 1. Under linear pressure $p(t)$ in plot (**a**): quasi-linear viscoelastic dilatation $e^{\mathrm{v}}_M(t)$ for unlimited response (4.5) in plot (**b**), and (4.4) with thresholding in plot (**c**)

For the numerical example, we take the kernel $J(t) = J_1(1 - \exp(-t/\tau_1))$ as $N = 1$ in (1.15) such that

$$\int_{t_{k-1}}^{t_k} J(t-s)\,\mathrm{d}s = J_1 \left\{ \delta t_k - \tau_1 \left[\exp\left(\frac{t_k - t}{\tau_1}\right) - \exp\left(\frac{t_{k-1} - t}{\tau_1}\right) \right] \right\}$$

and (4.8) takes the explicit form of the piecewise-exponential function

$$\frac{I_k^M}{J_1} = \left[e^{\mathrm{e}}(t_k) - \tau_1 \delta e^{\mathrm{e}}_k\right] \exp\left(\frac{t_k - t}{\tau_1}\right) - \left[e^{\mathrm{e}}(t_{k-1}) - \tau_1 \delta e^{\mathrm{e}}_k\right] \exp\left(\frac{t_{k-1} - t}{\tau_1}\right). \tag{4.9}$$

The elastic moduli are set for concrete [27]: $E = 30$ (GPa) and $\nu = 0.2$ such that $E_1 = 0.04$, $E_3 = -E_2 = 0.00\overline{6}$ (1/GPa) in (1.12), $\lambda_3 = 0.1$ (1/GPa) in (4.4) and (4.5), the lower threshold is taken to be $\underline{M} = 0.2$. The creep parameters in (1.15) are $J_1 = 0.04$ and $\tau_1 = 0.4$ (h).

4.1. Monotone loading by pressure

First, we prescribe the pressure by the linearly increasing function

$$p(t) = gt \quad \text{for } t \in [0, T], \tag{4.10}$$

where the final time $T = 1.5$ (h), and the loading rate $g = 3.\overline{3}$ (GPa/h) as portrayed in the plot (a) of Fig. 1.

Inserting $p(t)$ defined as (4.10) into (4.4) (respectively, (4.5)) we obtain $e^{\mathrm{e}}(t)$. Then, the discrete solution $e^{\mathrm{v}}_M(t)$ to the corresponding equation is computed with the help of the quadrature formulas (4.7) and (4.9) on the uniform mesh with time-step $\delta t = 0.02$ as $M = 75$. The quasi-linear dilatation for the viscoelastic response is portrayed versus time in Fig. 1 in the plot (b) for the response (4.5), and in plot (c) for the thresholded equation (4.4). Blue lines indicate the part of $\underline{M} \le (1 - 3\lambda_3 p) \le \overline{M}$, red lines indicate the part of $(1 - 3\lambda_3 p) < \underline{M}$.

We observe that approaching the critical pressure $p_{\mathrm{cr}} := 1/(3\lambda_3)$ the right-hand side of equation (4.5) becomes unbounded. The rate g in (4.10) is chosen such that the critical pressure $p_{\mathrm{cr}} = 3.\overline{3}$ (GPa) is attained in time $t = 1$ hour as seen in Fig. 1 in plot (a). Therefore, approaching from the left $t = 1$, which is marked by the vertical dashed line, the solution $e^{\mathrm{v}}_M(t)$ to (4.5) marked by the solid line in the plot (b) blows up to minus infinity. Whereas this singularity is avoided within the equation (4.4) with thresholding, the solution $e^{\mathrm{v}}_M(t)$ continues after $t = 1$ as indicated by the marked dashed line in the plot

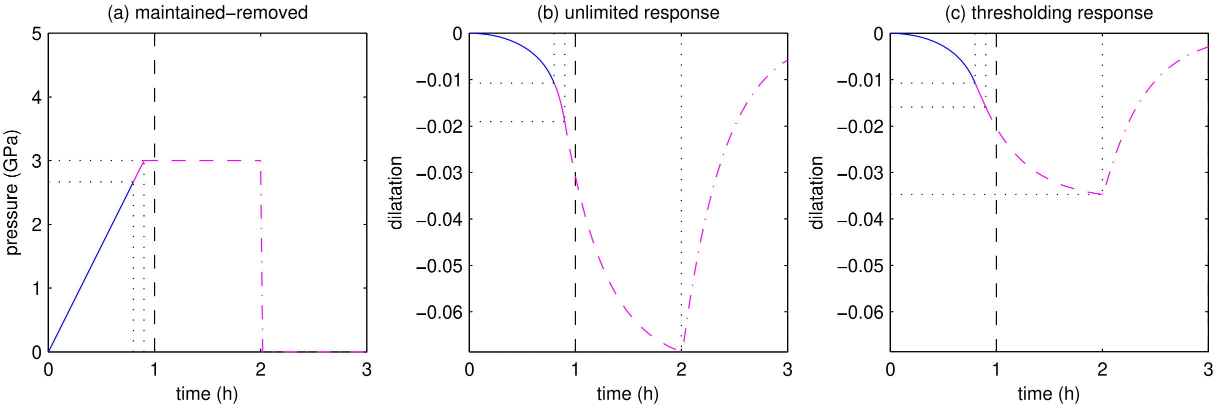

FIG. 2. Pressure $p(t)$ is increased linearly, then maintained constant and finally removed in plot (**a**): quasi-linear viscoelastic dilatation $e^{\mathrm{v}}_{M}(t)$ for unlimited response (4.5) in plot (**b**), and (4.4) with thresholding in plot (**c**)

(c). Both the solutions coincide for $t \in [0, 0.8]$ (h) and are distinguished only after reaching the critical pressure $p_{\mathrm{cr}}(1 - \underline{M}) = 2.\overline{6}$ (GPa), which is marked by the dotted lines in Fig. 1.

4.2. Creep test

In order to simulate the creep behavior, in the second test, the loading undergoes the three stages portrayed in the plot (a) of Fig. 2. Within $t \in [0, 0.9]$ (h), the pressure increases from zero linearly with the constant rate $g = 3.\overline{3}$ (GPa/h) as in (4.10). After reaching $0.9g = 3$ (GPa/h) at $t = 0.9$ before it reaches p_{cr}, the pressure is maintained within the time interval $t \in [0.9, 2]$, then immediately removed and kept zero for $t \in [2, 3]$, that is

$$p(t) = \begin{cases} gt & \text{for } t \in [0, 0.9] \\ 0.9g & \text{for } t \in [0.9, 2] \\ 0 & \text{for } t \in [2, 3] \end{cases}. \tag{4.11}$$

The quasi-linear solutions $e^{\mathrm{v}}_{M}(t)$ to the discrete Volterra convolution equation (4.7) and (4.9) with the uniform time-step $\delta t = 0.02$ are portrayed versus time in Fig. 2 in the plot (b) corresponding to the equation (4.5) without thresholding, and in the plot (c) to the equation (4.4) with thresholding. The responses are marked by the solid line for monotone loading, dashed line while the pressure is maintained constant, and dash-dotted line on the removal of the pressure. Here the variation of the solution to the equation that is thresholded is moderate compared to the one that is not thresholded.

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Declarations

Conflict of interest The authors declare no competing interests.

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