



Residual stress and material symmetry

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ABSTRACT

There seems to be a basic misconception in several recent papers concerning the material symmetry of bodies in configurations that are pre-stressed. In this short paper we point to the source of the error and show that the material symmetry that is possible depends on the nature of the pre-stress. We also extend the results for material symmetry which have been well-known within the context of simple elastic solids to the general class of simple materials. This generalization has relevance to the material symmetry of biological solids that are viscoelastic.

1. Introduction

The assumption that the reference configuration of a body, whether it be a laboratory specimen or the object of a study that is relevant to a practical problem, that is subject to deformation, is in a state free of stress and strain, is invariably not true and more a matter of mathematical and computational convenience, as most bodies that one deals with have been subject to processing either by nature or humans. Thus, the body usually has pre-stresses or pre-strain (the terms initial stress/residual stress and permanent set are more often used for these two notions. In this short note we shall refer to initial stress/residual stress as pre-stress). How a body responds from the reference configuration, when subject to external stimuli depends on the state of the reference configuration. Since as we have just observed, the reference configuration might be pre-stressed, the response depends on the nature of the pre-stress. Moreover, the pre-stress in the reference configuration determines a set of transformations, namely the set of the gradient of the mappings between the reference configuration and other configurations of the body from which the response remains the same, and this set of transformation forms a group, usually referred to as the material symmetry group or peer group (see Truesdell (1992)). The material symmetry associated with a particular reference configuration that is residually pre-stressed cannot be chosen at will, only the material symmetry group that is consistent with the reference configuration can be assumed. For instance, a consequence of the works of Coleman and Noll (1964) and Hoger (1985) is the fact that an isotropic simple elastic material cannot support prestresses.

Recently, there have been numerous papers that assume that the body under consideration is a pre-stressed “Simple”¹ elastic solid and that its configuration in this pre-stressed reference configuration has associated with it, a particular symmetry group. Unfortunately, there seems to be a basic misconception concerning the material symmetry of bodies that are in a pre-stressed state. One can only discuss the material symmetry of the body with regard to a specific configuration, and as mentioned above the material symmetry group is the group of transformations of the gradients of the mapping from the configuration under consideration to another configuration, which leaves the response unchanged. Thus, one could say that the body with respect to the configuration $\kappa(B)$ is isotropic, transversely isotropic, orthotropic, etc., depending on the material symmetry group with the clear understanding

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¹ Cauchy elastic solids (Cauchy, 1823, 1828; Truesdell & Noll, 1965) are Simple materials in the sense of Noll (1957, 1958). Implicit elastic solids introduced by Rajagopal (2003, 2007) are not Simple materials in the sense of Noll (1957, 1958).

that this group depends on $\kappa(B)$. For instance, let us consider the statement (see Shams et al. [Shams, Destrade, and Ogden \(2011\)](#)) “On the basis of the nonlinear theory of elasticity, the general constitutive equation for an isotropic hyperelastic solid in the presence of initial stress is derived”. The only way in which the sentence can be interpreted is that in the configuration in which the body is initially stressed, the body is isotropic. This statement is incorrect as a body with initial stresses/pre-stresses/residual stresses cannot be isotropic in such a configuration in which it is pre-stressed as a consequence of the results of [Coleman and Noll \(1964\)](#) and [Hoger \(1985\)](#). If the body were to be isotropic in some other reference configuration, then its material symmetry group in the initially stressed configuration would have to be determined using Noll’s rule (see [Noll \(1958\)](#), [Truesdell \(1992\)](#)) that relates the material symmetry groups of the configuration in which the body is isotropic and the one in which it has initial stress. In any event, the body cannot be isotropic in an initially stressed state. This same misconception occurs in many papers. For instance, if the material symmetry group of the body is the isotropy group in a particular configuration, then if such a body is uniaxially extended, its new material symmetry group contains the transverse isotropy group as well as unimodular transformations that are not orthogonal (see [Wineman, Rajagopal, and Negahban \(1988\)](#)), that is one cannot assume whatever symmetry one pleases.

The purpose of this paper is

1. To emphasize that care has to be exercised in the assumption of the nature of the anisotropy of the body, when the reference configuration of the body is in a state of pre-stress.
2. To show that the results of [Coleman and Noll \(1964\)](#) and [Hoger \(1985\)](#) that a “Simple” elastic material that is isotropic cannot support pre-stress can be extended to all “Simple” materials. This result has important consequences to polymeric bodies as well as biological matter that is often times modeled as “Simple” viscoelastic materials.
3. To show that if a “Simple” material that is supporting pre-stresses is anisotropic, it has to have a special symmetry group associated with the pre-stress. A particular anisotropy group that is chosen to exist might not be consistent with the assumed form of pre-stress distribution.
4. Conversely, in a “Simple” material, a specific anisotropy can only support a specific pre-stress distribution over the body.

The above results only apply to “Simple” materials and thus caution has to be exercised in applying them to theories of inelasticity that are invariably non-Simple, and to implicit constitutive relations. In the case of “Simple” elastic materials, problems involving reference configurations with pre-stress have been studied in two different ways. In the first approach, one assumes that the stress depends on the deformation gradient from a reference configuration which is assumed to be in a state of pre-stress. The tacit assumption is that there is a possible configuration in which the body is free of stress. In the second approach, the response from the pre-stressed configuration is supposed to depend on the deformation gradient from the pre-stressed configuration as well as the pre-stress. That is, in the response function for the body, the pre-stress acts as an internal variable. We show that irrespective of the manner in which you study the problem, the results outlined above in (1)-(4) hold.

The organization of this short note is as follows. In Section 2, we have a brief discussion of the kinematics, and in Section 3, we assume that the stress depends on the deformation gradient from a reference configuration which is assumed to be in a state of pre-stress and show that if the body is isotropic then it cannot support pre-stress. In the following Section, we assume that the stress depends on the deformation gradient and the pre-stress (used as an internal variable), and once again show that if the body in a configuration is isotropic in that configuration, then it cannot support pre-stress. The final section is devoted to some concluding remarks.

2. Stress depends on the deformation gradient from a reference configuration which is assumed to be in a state of pre-stress.

Let t denote the current time and s denote an earlier time, $0 \leq s \leq t$. In the reference configuration of a body, $\kappa_R(B)$, the position of a material particle is $\mathbf{X}$. Its position in the configuration at current time t , $\kappa_t(B)$, is $\mathbf{x}$. The deformation gradient at time s is $\mathbf{F}_{\kappa_R}(s) = \mathbf{F}(s) = \frac{\partial \mathbf{x}(s)}{\partial \mathbf{X}}$ where we dropped the suffix κ_R for convenience. The residual stress distribution in $\kappa_R(B)$ is $\mathbf{T}^{res}$.

If a body is composed of a simple material the constitutive assumption is

$$\mathbf{T}(t) = \mathcal{R}_{\kappa_R}[\mathbf{F}(s)]_{s=0}^{s=t}. \quad (1)$$

$\mathcal{R}_{\kappa_R}$ is a response functional that depends on the choice of the reference configuration and the deformation gradient history $\mathbf{F}(s)$, $0 \leq s \leq t$, (also denoted by $\mathbf{F}(s)|_{s=0}^{s=t}$). There are no *a priori* assumptions about $\mathcal{R}_{\kappa_R}$. It can be any rule of correspondence that determines the Cauchy stress $\mathbf{T}(t)$ in the configuration $\kappa_t(B)$ once the deformation gradient history is given.

Let $\bar{\mathbf{I}}(s)|_{s=0}^{s=t}$ denote the deformation gradient history in which the body has always been in its reference configuration,

$$\mathbf{F}(s) = \bar{\mathbf{I}}(s) = \mathbf{I}, \quad 0 \leq s \leq t, \quad (2)$$

where $\mathbf{I}$ is the identity tensor. Then

$$\mathbf{T}^{res} = \mathcal{R}_{\kappa_R}[\bar{\mathbf{I}}(s)]_{s=0}^{s=t}. \quad (3)$$

Note that, since $\mathbf{I}(s)$ is the identity tensor for all s , in all the configurations at time s , the current configuration is the same as the reference configuration modulo rotation, as the body is not deformed. Let $\mathbf{Q}(s)$, $0 \leq s \leq t$, where $\mathbf{Q}(s)$ is an orthogonal tensor, be a rotation history. Then observer invariance (frame indifference) imposes the restriction

$$\mathbf{Q}(t)\mathcal{R}_{\kappa_R}[\mathbf{F}(s)]_{s=0}^{s=t}\mathbf{Q}(t)^T = \mathcal{R}_{\kappa_R}[\mathbf{Q}(s)\mathbf{F}(s)]_{s=0}^{s=t}. \quad (4)$$

If $\mathbf{H}$ is an orthogonal transformation² that belongs to the symmetry group $\mathcal{G}$ of the material, then

$$\mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}] = \mathcal{R}_{\kappa_R}[\mathbf{F}(s)\mathbf{H}|_{s=0}^{s=t}], \quad (5)$$

for all transformations $\mathbf{H}$ in $\mathcal{G}$. Combining (4) and (5) gives

$$\mathbf{Q}(t)\mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}]\mathbf{Q}(t)^T = \mathcal{R}_{\kappa_R}[\mathbf{Q}(s)\mathbf{F}(s)\mathbf{H}|_{s=0}^{s=t}]. \quad (6)$$

Let $\mathbf{Q}(s) = \mathbf{H}^T$, $0 \leq s \leq t$ in (6) and then let $\mathbf{F}(s) = \bar{\mathbf{I}}(s)$. With (3) and (5), the result can be expressed as

$$\mathbf{H}^T \mathbf{T}^{res} \mathbf{H} = \mathbf{T}^{res} \quad (7)$$

or

$$\mathbf{H} \mathbf{T}^{res} = \mathbf{T}^{res} \mathbf{H} \quad (8)$$

for each orthogonal transformation in the material symmetry group in $\mathcal{G}$.

The Eq. (8) implies that the pre-stress has to be isotropic,³ that is $\mathbf{T}^{res} = \alpha \mathbf{I}$. Hoger (1985) has shown that if an elastic body is to be traction free in a particular configuration and also meet the equilibrium equations, then $\alpha = 0$. That is, the material symmetry of the body in the configuration under consideration cannot be isotropic.

3. Stress depends on the deformation gradient and the pre-stress (used as an internal variable)

In Hoger (1993) and several subsequent works, it has been assumed that the stress depends on the deformation gradient and the residual stress. In this case, (1) is revised to

$$\mathbf{T}(t) = \mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}]. \quad (9)$$

and (3) becomes

$$\mathbf{T}^{res} = \mathcal{R}_{\kappa_R}[\bar{\mathbf{I}}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}] \quad (10)$$

The restriction (4) due to observer invariance is now

$$\mathbf{Q}(t)\mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}]\mathbf{Q}(t)^T = \mathcal{R}_{\kappa_R}[\mathbf{Q}(s)\mathbf{F}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}]. \quad (11)$$

and the material symmetry restriction (5) becomes

$$\mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}] = \mathcal{R}_{\kappa_R}[\mathbf{F}(s)\mathbf{H}|_{s=0}^{s=t}; \mathbf{H}\mathbf{T}^{res}\mathbf{H}^T]. \quad (12)$$

Combining (11) and (12) and recognizing that $\mathbf{H}$ is a member of the symmetry group leads to

$$\mathbf{Q}(t)\mathcal{R}_{\kappa_R}[\mathbf{F}(s)|_{s=0}^{s=t}; \mathbf{T}^{res}]\mathbf{Q}(t)^T = \mathcal{R}_{\kappa_R}[\mathbf{Q}(s)\mathbf{F}(s)\mathbf{H}|_{s=0}^{s=t}; \mathbf{H}\mathbf{T}^{res}\mathbf{H}^T], \quad (13)$$

and since $\mathbf{H}$ is a proper orthogonal transformation and since $\mathbf{T}^{res}$ is in the reference configuration, it gets appropriately rotated as shown in (13). As before, let $\mathbf{Q}(s) = \mathbf{H}^T$, $0 \leq s \leq t$ and then $\mathbf{F}(s) = \bar{\mathbf{I}}(s)$. With (10), it is found that once again the pre-stress must satisfy (7) or (8).

4. Comments

In Hoger (1985) and Hoger (1993), Hoger presented a succinct derivation of (8) and showed that it is a consequence of observer invariance (frame indifference) and material symmetry restrictions. The derivations were carried out assuming elastic response. Coleman and Noll obtained the forms for different pre-stress imposed by (8) for the symmetry groups. For isotropy, $\mathbf{T}^{res} = \alpha \mathbf{I}$. For transverse isotropy about unit vector $\mathbf{k}$, the residual stress has to have the specific structure,

$$\mathbf{T}^{res} = \alpha \mathbf{I} + \beta \mathbf{k} \otimes \mathbf{k}, \quad (14)$$

where α and β are scalars. Hoger (1985) showed that for an isotropic material, the condition that the residual stress satisfy the equilibrium condition and surface traction free boundary condition in the reference configuration leads to the result that $\alpha = 0$. Thus, there cannot be residual stresses in an isotropic elastic body. The paper also provides a detailed discussion for α and β for transversely isotropic materials. The calculations in this short note shows that these results carry over to the case of simple materials

² The notion of anisotropy was initially introduced only within the context of "turning" or rotations and thus there are those who will only consider the sub-group consisting in proper orthogonal transformations as the legitimate material symmetry group (see discussion in Rajagopal (2015) concerning relevant issues.) According to such an interpretation, the symmetry group of an isotropic material is the set of proper orthogonal tensors. On the other hand Noll, Truesdell and others assume that a material is isotropic if the material symmetry group contains the full orthogonal group (includes both proper and improper orthogonal transformations). The representation for the stress that arises from these two differing view points is different. The results established in this note are valid from either perspective.

³ Here we are referring to the state of stress being isotropic, that is spherical.

and hold in the case of viscoelastic bodies that are Simple materials (One can have viscoelastic materials that are not Simple wherein the history of the stress and the history of the relative deformation gradient are related implicitly (see Pruša and Rajagopal (2012)).

A procedure for constructing constitutive equations for hyperelastic materials with residual stress was developed in Hoger (1993). In Section 4, page 113, the assumption is made that the strain energy function W is an explicit function of the deformation gradient and the pre-stress:

$$W = W[\mathbf{F}, \mathbf{T}^{res}], \quad (15)$$

It is also stated in words that W is a joint invariant, This implies

$$W[\mathbf{F}, \mathbf{T}^{res}] = W[\mathbf{F}\mathbf{H}, \mathbf{H}\mathbf{T}^{res}\mathbf{H}]. \quad (16)$$

for each transformation $\mathbf{H}$ in the material symmetry group. This condition corresponds to (12). The strain energy is expressed as a function of the joint invariants of $\mathbf{C}$ and $\mathbf{T}^{res}$. Substitution into the relation between stress and strain energy for an elastic material then results in a constitutive relation that depends on $\mathbf{T}^{res}$.

The corresponding procedure for developing a constitutive equation for a simple material would then be: (1) Get the joint invariants of $\mathbf{C}(s)$ and $\mathbf{T}^{res}$ determined by (12) for the material symmetry group $\mathcal{G}$, (2) Using the method in Spencer (1971), a general representation for the constitutive equation is constructed that depends on $\mathbf{T}^{res}$. (3) Substitute the specific form for $\mathbf{T}^{res}$ for the material symmetry under consideration. For isotropic materials, this leads to nothing new since $\mathbf{T}^{res} = \alpha \mathbf{I}$ and $\alpha = 0$. In Hoger (1993), all three steps are carried out for a transversely isotropic elastic material

CRedit authorship contribution statement

K.R. Rajagopal: Conceptualization, Formal analysis, Investigation. **A. Wineman:** Conceptualization, Formal analysis, Investigation.

Declaration of competing interest

There are no competing interest for any of the authors.

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