



Constitutive relations for anisotropic porous solids undergoing small strains whose material moduli depend on the density and the pressure

K.R. Rajagopal^a, R. Bustamante^{b,*}

^a Department of Mechanical Engineering, University of Texas A&M, College Station, TX, USA

^b Departamento de Ingeniería Mecánica, Universidad de Chile, Beauchef 851, Santiago Centro, Santiago, Chile

ABSTRACT

Recently, Arumugam et al. (2023) developed a constitutive relation for the response of isotropic inhomogeneous compressible elastic solids in order to describe the response of the trabecular bone. Since porous solids such as bones, cement concrete, rocks, metallic alloys, etc., are anisotropic, in this short note we develop a constitutive relation for such bodies that exhibit transverse isotropy and also having two preferred directions of symmetry. Another characteristic of bones is that they exhibit different response characteristics in tension and compression, and hence any constitutive relation that is developed has to be capable of describing this. Also, the material moduli depend on both the density and the mean value of the stress (mechanical pressure), as is to be expected in a porous solid. In the constitutive relation that is developed in this paper, though the stress and the linearized strain appear linearly in the constitutive relation, the relationship is nonlinear. We also derive the response of such solids when undergoing uniaxial extension and compression, simple shear and torsion.

1. Introduction

Materials such as bones are porous and the porosity affects their mechanical response (see Bell et al. (1999), Helgason et al. (2008), McCalden, McGeough, Barker, and Court-Brown (1993), Schaffler and Burr (1988) and Vanleene, Rey, and Tho (2008)) and such a porous structure is essential for the flow of interstitial fluids (see Cardoso, Fritton, Gailani, Benalla, and Cowin (2013)) within the bones. Moreover, the pore structure varies over the body, and thus such a body needs to be described by a constitutive relation for an inhomogeneous compressible elastic body. The material moduli of porous elastic bodies are assumed to depend on the porosity, even when such bodies are undergoing small strains. Since porosity is related to the density, that is, more the porosity the lower the local density and vice-versa, the material moduli depending on the porosity is tantamount to the material moduli depending on the density. While such a situation is possible in nonlinear Cauchy elastic bodies (see Cauchy (1823, 1828)), and in Green elastic bodies (see Green (1842)) as they are a sub-class of Cauchy elastic bodies, it is not possible in the linearization of Cauchy elasticity under the assumption that the displacement gradient is small, as dependence on the density would make the material moduli depend on the trace of the linearized strain in virtue of the balance of mass, and the resulting approximation would involve nonlinear terms in the linearized strain that are to be ignored. However, as shown by Rajagopal (2021), the dependence of the material moduli in the porosity and hence density and thereby the trace of the linearized strain is possible in the linearization of the class of implicit elastic bodies introduced by Rajagopal (see Rajagopal (2003, 2007, 2010, 2011)).

The response of many bones is anisotropic (see Dong and Guo (2004), Hosseini, Maquer, and Zysset (2017), Kersh et al. (2013), Larsson et al. (2014) and Yassine and Hamade (2018)) thus necessitating the constitutive relation to reflect this characteristic. Furthermore, bones exhibit nonlinear response characteristics even within the context of small displacement gradients (see Morgan,

* Corresponding author.

E-mail address: rogbusta@ing.uchile.cl (R. Bustamante).

Yeh, Chang, and Keaveny (2001))¹ and complicating matters further, bones exhibit different response in tension and compression even when subject to small strains (see Rohl, Larsen, Linde, Odgaard, and Jorgensen (1991)), like rocks.²

Arumugam, Alagappan, Bird, Moreno, and Rajagopal (2023) developed a class of constitutive relations to describe the response of porous isotropic inhomogeneous compressible elastic bodies whose material moduli depend on the mean value of the stress (mechanical pressure³) and the density. Bustamante and Rajagopal (2022a) developed implicit constitutive relations to describe the response of homogeneous isotropic compressible elastic bodies and then obtained approximations based on the displacement gradients being small. Also, recently, Bustamante and Rajagopal (2022b) developed constitutive relations for response of anisotropic porous elastic bodies that are transversely isotropic and that have two preferred directions of symmetry.⁴ They first developed general constitutive relations that are valid for large deformation gradients, and then they obtained linearization based on the assumption that the displacement gradients are small, the constitutive relation being nonlinear in the stress. However, as the linearized constitutive relations in both these studies contain many constants, the aim of this study is to try and reduce the number of material moduli that appear in the constitutive relations but yet have sufficient robustness to capture the response exhibited by the material under consideration.

The starting point for the studies in Arumugam et al. (2023) and Bustamante and Rajagopal (2022a, 2022b) are the implicit constitutive relations to describe the response of elastic bodies introduced by Rajagopal (2003, 2007, 2010).⁵ Here, we introduce a sub-class of the constitutive relations developed in Bustamante and Rajagopal (2022a) for porous solids that are transversely isotropic or that have two preferred directions of symmetry, wherein the Cauchy stress and the linearized strain appear linearly. Also, the material moduli that appear in the constitutive relation depend on the density and the mean value of the stress. It is important to recognize that care has to be exercised with regard to the linearization of implicit constitutive relations (see Rajagopal (2018)) as the operations of linearization and inversion do not commute.

The paper is arranged in the following manner: In Section 2 the basic equations for the kinematics of a continuum media implicit constitutive relations are given. In Section 3 implicit constitutive relations are presented for porous transversely isotropic bodies, in the case of small gradient of displacement field. In Section 4 implicit relations for bodies with two preferred directions of symmetry are obtained. In Section 5 some simple boundary value problems are studied for the case of transversely isotropic bodies. The note ends with some final remarks in the final Section 6.

2. Preliminaries

Let $\kappa_R(\mathcal{B})$ and $\kappa_t(\mathcal{B})$ denote the reference and current configurations at time t of an abstract body $\mathcal{B}$. Let $\mathbf{X}$ and $\mathbf{x}$ denote the position of a typical particle $p \in \mathcal{B}$, in the reference configuration and current configuration, respectively. Let

$$\mathbf{x} = \chi(\mathbf{X}, t) \quad (1)$$

denote the motion of the body that is one to one, and sufficiently smooth to make all the derivatives taken, meaningful. Let

$$\mathbf{u} := \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \frac{\partial \chi}{\partial \mathbf{X}}, \quad \nabla \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}}, \quad (2)$$

where $\mathbf{u}$, $\mathbf{F}$, $\nabla \mathbf{u}$ denote the displacement, deformation gradient and displacement gradient, respectively. The linearized strain $\boldsymbol{\varepsilon}$ is defined through

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \quad (3)$$

The balance of mass

$$\rho_R = \rho \det \mathbf{F}, \quad (4)$$

where ρ_R , ρ are the densities in the reference and current configuration, reduces to

$$\rho_R \approx \rho(1 + \text{tr } \boldsymbol{\varepsilon}), \quad (5)$$

under the assumption of small displacement gradient, in virtue of the Cayley-Hamilton theorem.

¹ This display of nonlinear response even within the context of small strains is observed in ceramics, rocks, metallic alloys, see Lydon and Balendran (1986), Munro (2004), Nguyen, Beaucour, Ortola, and Noumowé (2014) and Pauw (1960) for cement concrete, Hao et al. (2005), Li, Morris, Nagasako, Kuramoto, and Chrzan (2007), Saito et al. (2003), Sakaguchi, Niinomi, and Akahori (2004), Sakaguchi, Niinomi, Akahori, Takeda, and Toda (2005), Talling, Dashwood, Jackson, Kuramoto, and Dye (2008), Withey et al. (2008) and Zhang, Li, Jia, Hao, and Yang (2009) for metallic alloys, Luo and Stevens (1999) and Zhang, Gao, Elias, Dong, and Chen (2014), for ceramics, Hirose, Tanaka, Tanaki, and Asami (2004), Manoylov, Borodich, and Evans (2013) and Kováčik (1999), for sintered materials, and Cristescu (2012) for rocks.

² That rocks exhibit different response characteristics in tension and compression has been well documented (see Fuenkajorn and Klanphumeesri (2010), Haimson and Tharp (1974), Johnson and Rasolofosaon (1996) and Patel and Martin (2018)). Bustamante and Ortiz (2021) and Sundaram and Corrales (1980)).

³ The concept of pressure has been often times misused in mechanics (see Rajagopal (2003)).

⁴ Classification of anisotropy for implicit bodies, which includes implicit elastic bodies as a special subclass, can be found in Rajagopal (2015).

⁵ The constitutive relations developed in Rajagopal (2003, 2007, 2010) appeal to standard representation theorems (see Spencer (1971)). Rajagopal and Saccomandi (2022) developed a constitutive relation that is linear in the stress and the linearized strain, starting from a generalization of the Blatz-Ko constitutive relation, on purely physical grounds.

The left and right Cauchy–Green tensors $\mathbf{B}$ and $\mathbf{C}$ are defined respectively through

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \quad (6)$$

where $\mathbf{U}$ and $\mathbf{V}$ are the stretches that appear in the polar-decomposition theorem

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad (7)$$

where $\mathbf{R}$ is an orthogonal transformation.

The above brief kinematical definitions suffice for our purpose. A detailed discussion of kinematics can be found in [Truesdell \(1977\)](#) and [Truesdell and Noll \(2004\)](#).

Let us consider the class of implicit elastic bodies introduced by [Rajagopal \(2003\)](#) through

$$\mathbf{f}(\rho, \mathbf{T}, \mathbf{F}, \mathbf{X}) = \mathbf{0}. \quad (8)$$

Frame-indifference requires that

$$\mathbf{f}(\rho, \mathbf{R}^T\mathbf{T}\mathbf{R}, \mathbf{C}, \mathbf{X}) = \mathbf{0}. \quad (9)$$

3. Transversely anisotropic porous solids

Let $\mathbf{a}$ denote a preferred direction for the body with regards to its response in the current configuration. It follows that [Spencer \(1971\)](#)

$$\begin{aligned} & \alpha_0 \mathbf{I} + \alpha_1 \mathbf{T} + \alpha_2 \mathbf{T}^2 + \alpha_3 \mathbf{B} + \alpha_4 \mathbf{B}^2 + \alpha_5 (\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) + \alpha_6 (\mathbf{T}^2\mathbf{B} + \mathbf{B}\mathbf{T}^2) + \alpha_7 (\mathbf{T}\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}) + \alpha_8 (\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2) + \alpha_9 \mathbf{a} \otimes \mathbf{a} \\ & + \alpha_{10} [\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}] + \alpha_{11} [\mathbf{a} \otimes (\mathbf{T}^2\mathbf{a}) + (\mathbf{T}^2\mathbf{a}) \otimes \mathbf{a}] + \alpha_{12} [\mathbf{a} \otimes (\mathbf{B}\mathbf{a}) + (\mathbf{B}\mathbf{a}) \otimes \mathbf{a}] + \alpha_{13} [\mathbf{a} \otimes (\mathbf{B}^2\mathbf{a}) + (\mathbf{B}^2\mathbf{a}) \otimes \mathbf{a}] = \mathbf{0}, \end{aligned} \quad (10)$$

where the α_i , $i = 0, 1, 2, \dots, 13$ depend on the density ρ and the invariants

$$I_1 = \text{tr} \mathbf{T}, \quad I_2 = \text{tr} (\mathbf{T}^2), \quad I_3 = \text{tr} (\mathbf{T}^3), \quad I_4 = \text{tr} \mathbf{B}, \quad I_5 = \text{tr} (\mathbf{B}^2), \quad I_6 = \text{tr} (\mathbf{B}^3), \quad I_7 = \text{tr} (\mathbf{T}\mathbf{B}), \quad I_8 = \text{tr} (\mathbf{T}^2\mathbf{B}), \quad (11)$$

$$I_9 = \text{tr} (\mathbf{T}\mathbf{B}^2), \quad I_{10} = \text{tr} (\mathbf{T}^2\mathbf{B}^2), \quad I_{11} = \mathbf{a} \cdot (\mathbf{T}\mathbf{a}), \quad I_{12} = \mathbf{a} \cdot (\mathbf{T}^2\mathbf{a}), \quad I_{13} = \mathbf{a} \cdot (\mathbf{B}\mathbf{a}), \quad I_{14} = \mathbf{a} \cdot (\mathbf{B}^2\mathbf{a}), \quad (12)$$

$$I_{15} = \mathbf{a} \cdot [(\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T})\mathbf{a}], \quad I_{16} = \mathbf{a} \cdot [(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2)\mathbf{a}]. \quad (13)$$

The above constitutive relation is totally inapplicable as we have as many as 14 material functions α_i , $i = 0, 1, 2, \dots, 13$, depending on 16 variables plus the density! We have to cull the constitutive relation down so that it can capture the physical phenomenon of interest, using the minimum number of material constants. In view of the constitutive relation used in [Arumugam et al. \(2023\)](#) having had some success in describing experimental data, we could make the following choice for a transversely isotropic material to describe transversely isotropic bones undergoing small deformations⁶ in which $\boldsymbol{\epsilon}$ and $\mathbf{T}$ appear linearly:

$$\begin{aligned} & (1 + \lambda_1 \text{tr} \mathbf{T})\boldsymbol{\epsilon} + \beta_1 (1 + \lambda_2 \text{tr} \boldsymbol{\epsilon})\mathbf{T} + \beta_2 (1 + \lambda_3 \text{tr} \boldsymbol{\epsilon})(\text{tr} \mathbf{T})\mathbf{I} + \beta_3 [\lambda_4 \text{tr} \boldsymbol{\epsilon} + \lambda_5 \text{tr} \mathbf{T} + \lambda_6 (\text{tr} \boldsymbol{\epsilon})(\text{tr} \mathbf{T}) + \lambda_7 \mathbf{a} \cdot (\mathbf{T}\mathbf{a}) + \lambda_8 \mathbf{a} \cdot (\boldsymbol{\epsilon}\mathbf{a})]\mathbf{a} \otimes \mathbf{a} \\ & + \beta_4 (1 + \lambda_9 \text{tr} \boldsymbol{\epsilon})[\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}] + \beta_5 (1 + \lambda_{10} \text{tr} \mathbf{T})[\mathbf{a} \otimes (\boldsymbol{\epsilon}\mathbf{a}) + (\boldsymbol{\epsilon}\mathbf{a}) \otimes \mathbf{a}] = \mathbf{0}. \end{aligned} \quad (14)$$

Even the above constitutive relation involves too many material constants that can be determined within the context of a reasonable experimental program. Hence, a more reasonable choice might be

$$\begin{aligned} & (1 + \lambda \text{tr} \mathbf{T})\boldsymbol{\epsilon} + \beta_1 (1 + \gamma \text{tr} \boldsymbol{\epsilon})\mathbf{T} + \beta_2 (1 + \gamma \text{tr} \boldsymbol{\epsilon})(\text{tr} \mathbf{T})\mathbf{I} + \beta_3 [\lambda \mathbf{a} \cdot (\mathbf{T}\mathbf{a}) + \gamma \mathbf{a} \cdot (\boldsymbol{\epsilon}\mathbf{a})]\mathbf{a} \otimes \mathbf{a} + \beta_4 [\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}] \\ & + \beta_5 [\mathbf{a} \otimes (\boldsymbol{\epsilon}\mathbf{a}) + (\boldsymbol{\epsilon}\mathbf{a}) \otimes \mathbf{a}] = \mathbf{0}, \end{aligned} \quad (15)$$

with 7 constants λ , γ , β_i , $i = 1, 2, 3, 4, 5$. When $\beta_3 = \beta_4 = \beta_5 = 0$ the constitutive relation (15) reduces to that studied in [Arumugam et al. \(2023\)](#).⁷ When in addition $\lambda = 0$, the constitutive relation reduces to the one studied by [Erbas, Kaplunov, and Rajagopal \(2013\)](#), [Itou, Kovtunen, and Rajagopal \(2021, 2022\)](#), [Murru and Rajagopal \(2021a, 2021b\)](#), [Průša, Rajagopal, and Wineman \(2022\)](#) and [Vajipeyajula, Murru, and Rajagopal \(2023a, 2023b\)](#). Simplifying the constitutive relation any further might render it useless with regard to describing transversely isotropic solids undergoing small strains as even in the case of classical linearized elasticity, a classical transversely isotropic Green elastic solid is described by 5 material constants.

⁶ The representation provided in [Bustamante and Rajagopal \(2022a\)](#) is not the complete representation for general implicit elastic bodies that are transversely isotropic or have two preferred directions of symmetry. Also, when one linearizes (10) under the assumption of small displacement gradient we have to replace $\mathbf{B}$ by $\mathbf{I} + 2\boldsymbol{\epsilon}$, and then ignore terms that are not linear in $\boldsymbol{\epsilon}$.

⁷ This constitutive relation (14), when β_3 , β_4 and β_5 equal to zero, was also studied within the context of inhomogeneous bodies by [Alagappan, Arumugam, and Rajagopal \(2023\)](#).

4. Implicit elastic body with two preferred directions of symmetry

We shall not provide a general representation and then derive a simplified constitutive relation. Instead, we directly provide a constitutive relation for a porous elastic body in which both the linearized strain and the Cauchy stress appear linearly, namely:

$$(1 + \lambda \text{tr } \mathbf{T})\boldsymbol{\varepsilon} + \beta_1(1 + \gamma \text{tr } \boldsymbol{\varepsilon})\mathbf{T} + \beta_2(1 + \gamma \text{tr } \boldsymbol{\varepsilon})(\text{tr } \mathbf{T})\mathbf{I} + \beta_3\mathbf{a} \otimes \mathbf{a} + \beta_4[\mathbf{a} \otimes (\mathbf{T}\mathbf{a}) + (\mathbf{T}\mathbf{a}) \otimes \mathbf{a}] + \beta_5[\mathbf{a} \otimes (\boldsymbol{\varepsilon}\mathbf{a}) + (\boldsymbol{\varepsilon}\mathbf{a}) \otimes \mathbf{a}] \\ + \beta_6\mathbf{b} \otimes \mathbf{b} + \beta_7[\mathbf{b} \otimes (\mathbf{T}\mathbf{b}) + (\mathbf{T}\mathbf{b}) \otimes \mathbf{b}] + \beta_8[\mathbf{b} \otimes (\boldsymbol{\varepsilon}\mathbf{b}) + (\boldsymbol{\varepsilon}\mathbf{b}) \otimes \mathbf{b}] = \mathbf{0}, \quad (16)$$

where $\mathbf{a}$, $\mathbf{b}$ are the two preferred directions in the current configuration, and β_3 , β_6 can depend linearly in $\mathbf{T}$ and $\boldsymbol{\varepsilon}$, whose expressions are not shown here for the sake of brevity. It is also important to know that above constitutive relation is not the complete constitutive relation, wherein the linearized strain and the Cauchy stress appear linearly. Even this simple constitutive relation (16) has at least as many as 10 constants. It might not be possible to reduce the number of material constants further as the orthotropic classical linearized elastic is characterized by 12 constants in the case of Cauchy elastic bodies and 9 constants in the case of Green elastic bodies.

5. Boundary value problems

In this section we study some simple boundary value problems. For the sake of brevity we only consider the implicit constitutive relation for porous transversely isotropic solids (15). For real bone, $\mathbf{a}$ and other material properties can depend on the position $\mathbf{x}$ (inhomogeneous solids), but for simplicity for the examples to be shown in this section, we assume that $\mathbf{a}$ and other material properties do not depend on $\mathbf{x}$. We shall not solve any of the boundary value problems numerically as we cannot corroborate the same as there is no experimental data available to do so. However, our calculations can be used in the future were such experiments are to be available.

5.1. Uniform extension/compression of a cylinder

We study the behavior of the cylinder

$$0 \leq r \leq r_0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq L, \quad (17)$$

where (r, θ, z) are cylindrical coordinates. We assume:

$$\mathbf{T} = \sigma_z \mathbf{e}_z \otimes \mathbf{e}_z, \quad \mathbf{a} = \mathbf{e}_z. \quad (18)$$

We assume that the above stress causes the strain distribution $\boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{e}_r \otimes \mathbf{e}_r + \varepsilon_2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \varepsilon_3 \mathbf{e}_z \otimes \mathbf{e}_z$. It follows from (15) and (18) that $\varepsilon_2 = \varepsilon_1$, then using the notation $\varepsilon_1 = \varepsilon_r$ and $\varepsilon_3 = \varepsilon_z$, thus

$$\boldsymbol{\varepsilon} = \varepsilon_r(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta) + \varepsilon_z \mathbf{e}_z \otimes \mathbf{e}_z, \quad (19)$$

where ε_r and ε_z are constants. Using (19) and (18) in (15) after some manipulations we obtain:

$$\varepsilon_z(1 + \lambda \sigma_z) + (\beta_1 + \beta_2)[1 + \gamma(2\varepsilon_r + \varepsilon_z)]\sigma_z + \beta_3(\gamma\varepsilon_z + \lambda \sigma_z) + 2\beta_4\sigma_z + 2\beta_5\varepsilon_z = 0, \quad \varepsilon_r(1 + \lambda \sigma_z) + \beta_2[1 + \gamma(2\varepsilon_r + \varepsilon_z)]\sigma_z = 0. \quad (20)$$

These two relations can be used to obtain ε_r and ε_z in terms of σ_z .

5.2. Simple shear stress in a slab

In this section we study the behavior of the slab

$$-\frac{L_i}{2} \leq x_i \leq \frac{L_i}{2}, \quad i = 1, 2, 3, \quad (21)$$

subject to simple shear, where we assume the presence of the stress tensor

$$\mathbf{T} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1), \quad (22)$$

where τ is constant. As well as this We assume

$$\mathbf{a} = \cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_2. \quad (23)$$

Considering the above expressions for the stress and $\mathbf{a}$ we assume that

$$\boldsymbol{\varepsilon} = \sum_{i=1}^3 \varepsilon_i \mathbf{e}_i \otimes \mathbf{e}_i + \varepsilon_{12}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1). \quad (24)$$

Replacing (22)–(24) in (15) it is possible to show that $\varepsilon_3 = 0$ and:

$$\varepsilon_1 + \beta_3 \cos^2 \vartheta [2\lambda \tau \sin \vartheta \cos \vartheta + \gamma(\varepsilon_1 \cos^2 \vartheta + \varepsilon_2 \sin^2 \vartheta + 2\varepsilon_{12} \sin \vartheta \cos \vartheta)] + 2\beta_4 \tau \cos \vartheta \sin \vartheta + 2\beta_5(\varepsilon_1 \cos^2 \vartheta + \varepsilon_{12} \cos \vartheta \sin \vartheta) = 0, \quad (25)$$

$$\varepsilon_2 + \beta_3 \sin^2 \vartheta [2\lambda \tau \sin \vartheta \cos \vartheta + \gamma(\varepsilon_1 \cos^2 \vartheta + \varepsilon_2 \sin^2 \vartheta + 2\varepsilon_{12} \sin \vartheta \cos \vartheta)] + 2\beta_4 \tau \cos \vartheta \sin \vartheta + 2\beta_5(\varepsilon_2 \sin^2 \vartheta + \varepsilon_{12} \cos \vartheta \sin \vartheta) = 0, \quad (26)$$

$$\begin{aligned} \varepsilon_{12} + \beta_1 [1 + \gamma(\varepsilon_1 + \varepsilon_2)]\tau + \beta_3 \cos \vartheta \sin \vartheta [2\lambda\tau \sin \vartheta \cos \vartheta + \gamma(\varepsilon_1 \cos^2 \vartheta + \varepsilon_2 \sin^2 \vartheta + 2\varepsilon_{12} \sin \vartheta \cos \vartheta)] + \beta_4 \tau \\ + \beta_5 [\varepsilon_{12} + (\varepsilon_1 + \varepsilon_2) \cos \vartheta \sin \vartheta] = 0. \end{aligned} \quad (27)$$

The above 3 relations can be used to find ε_1 , ε_2 and ε_{12} in terms of τ for different angles ϑ .

5.3. Torsion of a cylinder

In the above two problems we assumed homogeneous distributions for the stresses and strains. The above two problems can be used to fit the material constants from experiments. In the problem studied in this section we have inhomogeneous distributions for the stresses and strains.

Let us consider the cylinder described in (17) with the stress tensor

$$\mathbf{T} = \sigma_r(r)\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_\theta(r)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_z(r)\mathbf{e}_z \otimes \mathbf{e}_z + \tau_{\theta z}(r)(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_\theta \otimes \mathbf{e}_z). \quad (28)$$

For the above stress tensor to satisfy the equation of equilibrium without body forces we need

$$\sigma_\theta = \frac{d}{dr}(r\sigma_r). \quad (29)$$

We assume that the above stress tensor causes the displacement field

$$u_r = u_r(r), \quad u_\theta = krz, \quad u_z = (\lambda_z - 1)z, \quad (30)$$

where k , λ_z are positive constants. It follows that the linearized strain tensor takes the form

$$\varepsilon = \frac{du_r}{dr} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{u_r}{r} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + (\lambda_z - 1)\mathbf{e}_z \otimes \mathbf{e}_z + \frac{kr}{2}(\mathbf{e}_\theta \otimes \mathbf{e}_z + \mathbf{e}_\theta \otimes \mathbf{e}_z). \quad (31)$$

For the vector $\mathbf{a}$ we consider 3 cases, namely:

$$\mathbf{a} = \mathbf{e}_r, \quad \mathbf{a} = \mathbf{e}_\theta, \quad \mathbf{a} = \mathbf{e}_z. \quad (32)$$

Using (28), (31) and (32) in (15) we obtain the following relations:

Case $\mathbf{a} = \mathbf{e}_r$: In this case we get (using the notation u'_r for $\frac{du_r}{dr}$):

$$\begin{aligned} u'_r [1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_r + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) \\ + \beta_3 (u'_r \gamma + \lambda \sigma_r) + 2\beta_4 \sigma_r + 2\beta_5 u'_r = 0, \end{aligned} \quad (33)$$

$$\frac{u_r}{r} [1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_\theta + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) = 0, \quad (34)$$

$$(\lambda_z - 1)[1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_z + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) = 0, \quad (35)$$

and

$$\frac{kr}{2} [1 + \beta_5 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_4 \tau_{\theta z} + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \tau_{\theta z} = 0. \quad (36)$$

Case $\mathbf{a} = \mathbf{e}_\theta$: In this problem we obtain:

$$u'_r [1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_r + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) = 0, \quad (37)$$

$$\begin{aligned} \frac{u_r}{r} [1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_\theta + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) \\ + \beta_3 \left(\frac{u_r}{r} \gamma + \lambda \sigma_\theta \right) + 2\beta_4 \sigma_\theta + 2\beta_5 \frac{u_r}{r} = 0, \end{aligned} \quad (38)$$

$$\frac{kr}{2} [1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \tau_{\theta z} + \beta_4 \tau_{\theta z} + \beta_5 \frac{kr}{2} = 0, \quad (39)$$

and we also get the same relation (35).

Case $\mathbf{a} = \mathbf{e}_z$: In this final case we have:

$$\begin{aligned} (\lambda_z - 1)[1 + \lambda(\sigma_r + \sigma_\theta + \sigma_z)] + \beta_1 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] \sigma_z + \beta_2 \left[1 + \gamma \left(u'_r + \frac{u_r}{r} + \lambda_z - 1 \right) \right] (\sigma_r + \sigma_\theta + \sigma_z) \\ + \beta_3 [\lambda \sigma_z + \gamma(\lambda_z - 1)] + 2\beta_4 \sigma_z + 2\beta_5 (\lambda_z - 1) = 0, \end{aligned} \quad (40)$$

and we obtain the same relations (34), (37) and (39).

In each case we have 4 relations, from which we can find $\sigma_r(r)$, $\sigma_z(r)$, $\tau_{\theta z}(r)$ and $u_r(r)$. For example, in the case $\mathbf{a} = \mathbf{e}_z$ from (39) and (40) we can find $\tau_{\theta z}$ and σ_z in terms of σ_r , $\sigma_\theta = \frac{d}{dr}(r\sigma_r)$, u_r and u'_r (for the sake of brevity we do not show the relevant expressions explicitly here, as they are very long). Replacing them in (34) and (37) we obtain two first order coupled nonlinear ordinary differential equations for $u_r(r)$ and $\sigma_r(r)$. About boundary conditions, we can assume $u_r(0) = 0$ and $\sigma_r(0) = 0$. We do not try to solve such nonlinear equations here, since we do not have numerical values for the different material constants in (15).

6. Concluding remarks

In this short note we have provided constitutive relations for the response of porous transversely isotropic and porous orthotropic elastic bodies. We have developed the system of equations for three boundary value problems, uniaxial extension/compression, simple shear and torsion, but we have not solved any of the boundary value problems using these constitutive relations as the material moduli, which appear in these constitutive relations are not available, and such a study would be at best a parametric study as no systematic experimental program has been carried out using which data the utility of the constitutive relation can be assessed. While we have discussed idealized simple boundary value problems such as uniform extension/compression, simple shear and torsion of transversely isotropic porous elastic bodies in this short paper, we plan to carry out a study of such a body in a realistic geometry that is pertinent to a human bone in a subsequent study.

CRediT authorship contribution statement

K.R. Rajagopal: Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization. **R. Bustamante:** Writing – review & editing, Writing – original draft, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

K.R. Rajagopal thanks the Office of Naval Research and the National Science Foundation for its support. R. Bustamante thanks the support provided by Anid-Fondecyt Chile 1210002.

References

- Alagappan, J., Arumugam, J., & Rajagopal, K. R. (2023). A short note on the response of elastic bodies whose material moduli depend on the density and the mechanical pressure. *Applications in Engineering Science*, 16, Article 100162.
- Arumugam, J., Alagappan, P., Bird, J., Moreno, M., & Rajagopal, K. R. (2023). A new constitutive relation to describe the response of bone. *International Journal of Non-Linear Mechanics*, submitted for publication.
- Bell, K., Loveridge, N., Power, J., Garrahan, N., Meggitt, B., & Reeve, J. (1999). Regional differences in cortical porosity in the fractured femoral neck. *Bone*, 24, 57–64.
- Bustamante, R., & Ortiz, C. (2021). A bimodular nonlinear constitutive equation for rock. *Applications in Engineering Science*, 8, Article 100067.
- Bustamante, R., & Rajagopal, K. R. (2022a). On the response of anisotropic elastic bodies described by implicit constitutive relations. *Zeitschrift für Angewandte Mathematik und Mechanik*, 102, Article 202200029.
- Bustamante, R., & Rajagopal, K. R. (2022b). Application of implicit constitutive theory for describing the elastic response of rocks and concrete. *Archives of Mechanics*, 74, 513–547.
- Cardoso, L., Fritton, S. P., Gailani, G., Benalla, M., & Cowin, S. C. (2013). Advances in assessment of bone porosity, permeability and interstitial fluid flow. *Journal of Biomechanics*, 46, 253–265.
- Cauchy, A. (1823). Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. *Bulletin de la Société Philomathique*, 913, 2, 300–304, Oeuvres (2).
- Cauchy, A. (1828). Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique. *Exercices de Mathématiques*, 3, 160–187.
- Cristescu, N. (2012). *Rock rheology*, Vol. 7. Springer Science and Business Media.
- Dong, X. N., & Guo, X. E. (2004). The dependence of transversely isotropic elasticity of human femoral cortical bone on porosity. *Journal of Biomechanics*, 37, 1281–1287.
- Erbas, B., Kaplunov, J., & Rajagopal, K. R. (2013). 2D asymptotic analysis of a thin elastic beam with density-dependent generalized Young's modulus. In H. Altenbach, G. Bruno, V. A. Eremeyev, M. Y. Gutkin, & W. H. Müller (Eds.), *Mechanics of heterogeneous materials* (pp. 501–513). Springer.
- Fuenkajorn, K., & Klanphumeesri, S. (2010). Direct tension tests of intact rocks using compression to-tension load converter. *Engineering Journal of Research and Development*, 21, 51–57.
- Green, G. (1842). On the laws of the reflection and refraction of light at the common surface of two non-crystallized media. *Transactions of the Cambridge Philosophical Society*, 7(Part I), 1–24.
- Haimson, B., & Sharp, T. (1974). Stresses around boreholes in bilinear elastic rock. *Society of Petroleum Engineers Journal*, 14, 145–151.
- Hao, Y. L., Li, S. J., Sun, S. Y., Zheng, C. Y., Hu, Q. M., & R., Yang. (2005). Super-elastic titanium alloy with unstable plastic deformation. *Applied Physics Letters*, 87, Article 091906.

- Helgason, B., Perilli, E., Schile, E., Taddei, F., Brynjólfsson, S., & Viceconti, M. (2008). Mathematical relationships between bone density and mechanical properties: a literature review. *Clinical Biomechanics*, 23, 135–146.
- Hirose, N., Tanaka, S., Tanaki, T., & Asami, J. (2004). The relationship between elastic modulus and porosity of sintered Fe-Cu system alloys. *Journal of the Japan Society of Powder and Powder Metallurgy*, 51, 315–322.
- Hosseini, H. S., Maquer, G., & Zysset, P. K. (2017). μ CT-based trabecular anisotropy can be reproducibly computed from HR-pQCT scans using the triangulated bone surface. *Bone*, 97, 114–120.
- Itou, H., Kovtunen, V., & Rajagopal, K. R. (2021). On an implicit model linear in both stress and strain to describe the response of porous solids. *Journal of Elasticity*, 144, 107–118.
- Itou, H., Kovtunen, V. A., & Rajagopal, K. R. (2022). Investigation of implicit constitutive relations in which both the stress and strain appear linearly, adjacent to non-penetrating cracks. *Mathematical Models & Methods in Applied Sciences*, 32, 1475–1492.
- Johnson, P. D., & Rasolofosaon, P. (1996). Manifestation of nonlinear elasticity in rock: convincing evidence over large frequency and strain intervals from laboratory studies. *Nonlinear Processes in Geophysics*, 3, 77–88.
- Kersh, M. E., Zysset, P. K., Pahr, D. H., Wolfram, U., Larsson, D., & Pandy, M. G. (2013). Measurement of structural anisotropy in femoral trabecular bone using clinical-resolution CT images. *Journal of Biomechanics*, 4, 2659–2666.
- Kováčik, J. (1999). Correlation between Young's modulus and porosity in porous materials. *Journal of Material Science Letters*, 18, 1007–1010.
- Larsson, D., Luisier, B., Kersh, M. E., Dallara, E., Zysset, P. K., Pandy, M. G., et al. (2014). Assessment of transverse isotropy in clinical-level CT images of trabecular bone using gradient structure tensor. *Annals of Biomedical Engineering*, 42, 950–959.
- Li, T., Morris, J. W., Nagasako, N., Kuramoto, S., & Chrzan, D. C. (2007). 'Ideal' engineering alloys. *Physical Review Letters*, 98, Article 105503.
- Luo, J., & Stevens, R. (1999). Porosity-dependence of elastic moduli and hardness of 3Y-TZP ceramics. *Ceramics International*, 25, 281–286.
- Lydon, F. D., & Balendran, R. V. (1986). Some observations on elastic properties of plain concrete. *Cement and Concrete Research*, 16, 314–324.
- Manoylov, A. V., Borodich, F. M., & Evans, H. P. (2013). Modelling of elastic properties of sintered porous materials. *Proceedings of the Royal Society of London, Series A (Mathematical and Physical Sciences)*, 469, Article 20120689.
- McCalden, R. W., McGeough, J. A., Barker, M. B., & Court-Brown, C. M. (1993). Age-related changes in the tensile properties of cortical bone. The relative importance of changes in porosity, mineralization, and microstructure. *Journal of Bone & Joint Surgery*, 75, 1193–1205.
- Morgan, E. F., Yeh, O. C., Chang, W. C., & Keaveny, T. M. (2001). Nonlinear behavior of trabecular bone at small strains. *Journal of Biomechanical Engineering*, 123, 1–9.
- Munro, R. G. (2004). Analytical representations of elastic moduli data with simultaneous dependence on temperature and porosity. *Journal of Research of the National Institute of Standards and Technology*, 109, 497–503.
- Murru, P. T., & Rajagopal, K. R. (2021a). Stress concentration due to the presence of a hole within the context of elastic bodies. *Material Design & Processing Communications*, 3, Article e219.
- Murru, P. T., & Rajagopal, K. R. (2021b). Stress concentration due to the bi-axial deformation of a porous elastic body with a hole. *Zeitschrift für Angewandte Mathematik und Mechanik*, 101, Article e202100103.
- Nguyen, L. H., Beaucour, A. L., Ortola, S., & Noumowé, A. (2014). Influence of the volume fraction and the nature of fine lightweight aggregates on the thermal and mechanical properties of structural concrete. *Construction and Building Materials*, 51, 121–132.
- Patel, S., & Martin, C. D. (2018). Evaluation of tensile Young's modulus and Poisson's ratio of a bi-modular rock from the displacement measurements in a Brazilian test. *Rock Mechanics and Rock Engineering*, 51, 361–373.
- Pauw, A. (1960). Static modulus of elasticity of concrete as affected by density. *ACI Journal*, 32, 679–687.
- Průša, V., Rajagopal, K. R., & Wineman, A. (2022). Pure bending of an elastic prismatic beam made of a material with density-dependent material parameters. *Mathematics and Mechanics of Solids*, 27, 1546–1558.
- Rajagopal, K. R. (2003). On implicit constitutive theories. *Applications of Mathematics*, 48, 279–319.
- Rajagopal, K. R. (2007). The elasticity of elasticity. *Zeitschrift für Angewandte Mathematik und Physik*, 58, 309–317.
- Rajagopal, K. R. (2010). On a new class of models in elasticity. *Mathematical and Computational Applications*, 15, 506–528.
- Rajagopal, K. R. (2011). Conspectus of concepts of elasticity. *Mathematics and Mechanics of Solids*, 16, 536–562, (2011).
- Rajagopal, K. R. (2015). A note on the classification of the anisotropy of bodies defined by implicit constitutive equations. *Mechanics Research Communications*, 64, 38–41.
- Rajagopal, K. R. (2018). A note on the linearization of the constitutive relations of non-linear elastic bodies. *Mechanics Research Communications*, 93, 132–137.
- Rajagopal, K. R. (2021). An implicit constitutive relation for describing the small strain response of porous elastic solids whose material moduli are dependent on the density. *Mathematics and Mechanics of Solids*, 26, 1138–1146.
- Rajagopal, K. R., & Saccomandi, G. (2022). Implicit nonlinear elastic bodies with density dependent material moduli and its linearization. *International Journal of Solids and Structures*, (1112), 234–23555.
- Rohl, L., Larsen, E., Linde, F., Odgaard, A., & Jørgensen, J. (1991). Tensile and compressive properties of cancellous bone. *Journal of Biomechanics*, 24, 1143–1149.
- Saito, T., Furuta, T., Hwang, J. H., Kuramoto, S., Nishino, K., Suzuki, N., et al. (2003). Multifunctional alloys obtained via a dislocation-free plastic deformation mechanism. *Science*, 300, 464–467.
- Sakaguchi, N., Niinomi, M., & Akahori, T. (2004). Tensile deformation behavior of Ti-Nb-Ta-Zr biomedical alloys. *Materials Transactions*, 45, 1113–1119.
- Sakaguchi, N., Niinomi, M., Akahori, T., Takeda, J., & Toda, H. (2005). Effect of Ta content on mechanical properties of Ti-30Nb-XTa-5Zr. *Materials Science and Engineering: C*, 25, 370–376.
- Schaffler, M. B., & Burr, D. B. (1988). Stiffness of compact bone: effects of porosity and density. *Journal of Biomechanics*, 21, 13–16.
- Spencer, A. J. M. (1971). Theory of invariants. In A. C. Eringen (Ed.), *Continuum physics, Vol. 1* (pp. 239–353). New York: Academic Press.
- Sundaram, P., & Corrales, J. (1980). Brazilian tensile strength of rocks with different elastic properties in tension and compression. *International Journal of Rock Mechanics and Mining Sciences & Geomechanics Abstracts*, 17, 131–133.
- Talling, R. J., Dashwood, R. J., Jackson, M., Kuramoto, S., & Dye, D. (2008). Determination of (C11–C12) in Ti-36Nb-2Ta-3Zr-0.3 (wt.%) (gum metal). *Scripta Materialia*, 59, 669–672.
- Truesdell, C. A. (1977). *A first course in rational continuum mechanics, Vol. I*. Academic Press.
- Truesdell, C. A., & Noll, W. (2004). In S. S. Antman (Ed.), *The non-linear field theories of mechanics*. Springer.
- Vajipeyajula, B., Murru, P., & Rajagopal, K. (2023a). Stress concentration due to the presence of a rigid elliptical inclusion in porous elastic solids described by a new class of constitutive relations. *Journal of Elasticity*, 154, 255–273.
- Vajipeyajula, B., Murru, P., & Rajagopal, K. R. (2023b). Stress concentration due to an elliptical hole in a porous elastic plate. *Mathematics and Mechanics of Solids*, 28, 854–869.
- Vanleene, M., Rey, C., & Tho, M. C. H. B. (2008). Relationships between density and Young's modulus with microporosity and physicochemical properties of wistar rat cortical bone from growth to senescence. *Medical Engineering & Physics*, 30, 1049–1056.
- Withey, E., Jin, M., Minor, A., Kuramoto, S., Chrzan, D. C., & Morris, J. W. (2008). The deformation of 'gum metal' in nanoindentation. *Materials Science & Engineering A*, 493, 26–32.
- Yassine, R. A., & Hamade, R. F. (2018). Transversely isotropic and isotropic material considerations in determining the mechanical response of geometrically accurate bovine tibia bone. *Medical & Biological Engineering & Computing*, 57, 2159–2178.
- Zhang, L., Gao, K., Elias, A., Dong, Z., & Chen, W. (2014). Porosity dependence of elastic modulus of porous Cr3C2 ceramics. *Ceramics International*, 40, 191–198.
- Zhang, S. Q., Li, S. J., Jia, M. T., Hao, Y. L., & Yang, R. (2009). Fatigue properties of a multifunctional titanium alloy exhibiting nonlinear elastic deformation behavior. *Scripta Materialia*, 60, 733–736.