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Article

Moduli Spaces of Quadratic Maps: Arithmetic and Geometry

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We establish an implication between two long-standing open problems in complex dynamics. The roots of the nth Gleason polynomial $G_n \in \mathbb{Q}[c]$ comprise the 0-dimensional moduli space of quadratic polynomials with an n-periodic critical point. $\mathrm{Per}_n(0)$ is the 1-dimensional moduli space of quadratic rational maps on $\mathbb{P}^1$ with an n-periodic critical point. We show that if G_n is irreducible over $\mathbb{Q}$, then $\mathrm{Per}_n(0)$ is irreducible over $\mathbb{C}$. To do this, we exhibit a $\mathbb{Q}$-rational smooth point on a projective completion of $\mathrm{Per}_n(0)$, using the admissible covers completion of a Hurwitz space. In contrast, the Uniform Boundedness Conjecture in arithmetic dynamics would imply that for sufficiently large n, $\mathrm{Per}_n(0)$ itself has no $\mathbb{Q}$-rational points.

1 Introduction

The *nth Gleason polynomial* $G_n \in \mathbb{Q}[c]$ is the monic polynomial whose roots are the set of $c \in \mathbb{C}$ such that, under $f_c(z) = z^2 + c$, the critical point 0 has a periodic orbit of exact period n. A long-standing open question in complex dynamics asks whether G_n is irreducible over $\mathbb{Q}$ for all n; see [5, 7, 15, 21].

The roots of G_n comprise the 0-dimensional moduli space of quadratic polynomials having an n-periodic critical point. Milnor [20] initiated the study of a natural generalization: the curve $\mathbf{Per}_n(0)$ parametrizing quadratic rational maps with an n-periodic critical point. Another long-standing open question asks whether $\mathbf{Per}_n(0)$ is irreducible over $\mathbb{C}$ for all n.

Theorem 1.1. Fix $n \geq 4$. If G_n is irreducible over $\mathbb{Q}$, then $\mathbf{Per}_n(0)$ is irreducible over $\mathbb{C}$.

Irreducibility of G_n has been shown for $n \leq 19$ [9] using *Magma*. We conclude:

Corollary 1.2. For $n \leq 19$, $\mathbf{Per}_n(0)$ is irreducible over $\mathbb{C}$.

The strategy is as follows. Milnor [20] shows that every irreducible component of $\mathbf{Per}_n(0)$ contains a polynomial, that is, a root of G_n. This implies (see Corollary 2.2) that if G_n is irreducible over $\mathbb{Q}$, then so is $\mathbf{Per}_n(0)$. In order to upgrade irreducibility over $\mathbb{Q}$ to irreducibility over $\mathbb{C}$, we show:

Theorem 1.3. For every $n \geq 4$, there exists a projective completion of $\mathbf{Per}_n(0)$ that has a smooth $\mathbb{Q}$-rational point $f_\star$.

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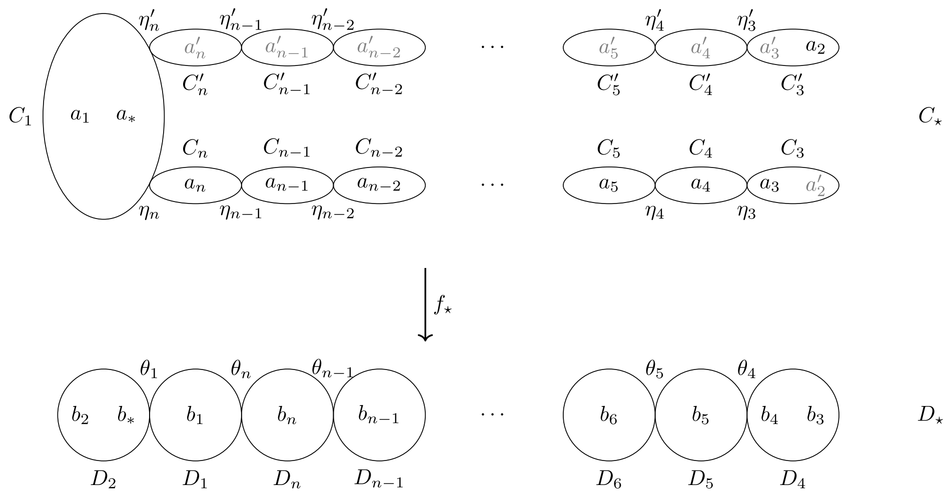

Fig. 1. The point $f_\star \in \overline{\mathcal{H}}_n$; see Section 3. On $C_1 \cong \mathbb{P}^1_{\mathbb{Q}}$, coordinates are chosen so that η_n is at "1", and η'_n at "−1", a_* at "∞" and a_1 at "0".

A $\mathbb{Q}$-rational point of a $\mathbb{Q}$-irreducible variety must lie on every $\mathbb{C}$-irreducible component (because the Galois group acts transitively on the set of $\mathbb{C}$-irreducible components, but fixes $\mathbb{Q}$-rational points). On the other hand, a smooth point can lie on at most one $\mathbb{C}$-irreducible component. We conclude that if $\mathrm{Per}_n(0)$ is irreducible over $\mathbb{Q}$, then it is irreducible over $\mathbb{C}$.

1.1 Finding the smooth $\mathbb{Q}$-rational point at infinity in a Hurwitz space

The strategy of upgrading $\mathbb{Q}$-irreducibility to $\mathbb{C}$-irreducibility by finding a smooth $\mathbb{Q}$-rational point has been fruitful in dynamics [4, 6]. However, in our setting, there is a major obstacle to finding a smooth $\mathbb{Q}$-rational point, as follows. The Uniform Boundedness Conjecture for arithmetic dynamics would imply that for large n, $\mathrm{Per}_n(0)$ has no $\mathbb{Q}$-rational point [22]—indeed, $\mathrm{Per}_5(0)$ has no $\mathbb{Q}$-rational point [8]. Milnor [20] originally defined $\mathrm{Per}_n(0)$ as a subset of the moduli space $\mathcal{M}_2 \cong \mathbb{C}^2$ parametrizing quadratic rational maps up to conjugacy, and studied the closure of $\mathrm{Per}_n(0)$ in $\mathbb{CP}^2 \supset \mathbb{C}^2$. The intersection of this closure with the line at infinity in $\mathbb{CP}^2$ contains $\mathbb{Q}$-rational points, but they are very singular in general [20, 27].

We circumvent this obstacle by instead considering a birational model $\mathcal{E}_n$ of $\mathrm{Per}_n(0)$ that embeds naturally in a Hurwitz space $\mathcal{H}_n$ ([12, 17]; see Section 2). The Hurwitz space $\mathcal{H}_n$ has a completion $\overline{\mathcal{H}}_n$ whose points correspond to *admissible covers*, which are branched coverings of nodal curves [1, 18]. We apply techniques developed in [24] to study the completion $\overline{\mathcal{E}_n}$ of $\mathcal{E}_n$ in $\overline{\mathcal{H}}_n$, and find a smooth $\mathbb{Q}$-rational point $f_\star \in \overline{\mathcal{E}_n} \setminus \mathcal{E}_n$. As $f_\star$ is an admissible cover, it admits an interpretation as a "degenerate quadratic map" (see Figure 1 and Section 3). The primary novel aspect of the present paper is the use of this modular interpretation to establish both $\mathbb{Q}$-rationality and smoothness of $f_\star$. This is a proof-of-concept for the usefulness of moduli spaces of admissible covers in studying dynamics on $\mathbb{P}^1$.

Remark 1.4. This technique appears to generalize well. We have found two other smooth $\mathbb{Q}$-rational points on $\overline{\mathcal{E}_n}$, as well as a smooth $\mathbb{Q}$-rational point on an analogous completion of the space of quadratic rational maps on $\mathbb{P}^1$ with a pre-periodic critical point of pre-period 2 (and arbitrary period).

1.2 Notes and references

The moduli space $\mathrm{Per}_n(0)$ is an example of a *critical-orbit-relation* space of rational maps, that is, it parametrizes conjugacy classes of rational maps of fixed degree, with a prescribed condition on a subset of critical orbits. Critical-orbit-relation spaces have been conjectured by Baker-DeMarco [3] to be the only families of rational maps containing a Zariski-dense subset of post-critically finite rational maps; this conjecture has been proved in special cases [3, 11, 13].

In general, very little is known about the geometry of critical-orbit-relation spaces, but irreducibility has been proved in some important infinite families. Arfeux-Kiwi [2] proved that the analog of $\mathrm{Per}_n(0)$

in the space of cubic polynomials is irreducible over $\mathbb{C}$, and Buff-Epstein-Koch [4] proved irreducibility over $\mathbb{C}$ of spaces of quadratic rational maps (as well as cubic polynomials) with a pre-fixed critical point.

Finally, we note that there are natural transcendental covering spaces of $\mathbf{Per}_n(0)$ (more precisely, of $\mathcal{E}_n$) called *deformation spaces* [12, 23], whose connectedness and contractibility have been studied in [14, 16, 17].

2 Background: The Moduli Spaces

2.1 Quadratic rational maps

We denote by $\mathcal{M}_2$ the space of quadratic rational self-maps of $\mathbb{P}^1$, up to conjugacy. In fact, $\mathcal{M}_2$ is isomorphic to $\mathbb{A}^2$ over $\mathbb{C}$ [20] and over $\mathbb{Q}$ [25]. For $n \geq 1$, let $\mathbf{Per}_n(0) \subsetneq \mathcal{M}_2$ be the one-dimensional $\mathbb{Q}$-subvariety parametrizing maps with an n-periodic critical point. $\mathbf{Per}_1(0)$ is the locus of quadratic polynomials, and is a (vertical) line in $\mathcal{M}_2 \cong \mathbb{A}^2$. For $n \geq 2$, under the natural identification of $\mathbf{Per}_1(0)$ with the family $\{f_c(z) = z^2 + c\}$, $\mathbf{Per}_n(0) \cap \mathbf{Per}_1(0)$ is the set of roots of G_n. Lemma 4.1 of [20] establishes that for $n \geq 2$, $\mathbf{Per}_n(0)$ only intersects the line at infinity in $\mathbb{P}^2$ at points whose coordinates are roots of unity. In particular, for $n \geq 2$, $\mathbf{Per}_n(0)$ does not meet $\mathbf{Per}_1(0)$ at the line at infinity, which implies:

Theorem 2.1. [Milnor [20], Lemma 4.1 and Theorem 4.2] Every irreducible component of $\mathbf{Per}_n(0)$ has non-empty intersection with $\mathbf{Per}_1(0)$.

As an immediate consequence, we obtain:

Corollary 2.2. If G_n is irreducible over $\mathbb{Q}$, then so is $\mathbf{Per}_n(0)$.

Proof. We apply Theorem 2.1 to note that a non-trivial factorization of the equation cutting out $\mathbf{Per}_n(0)$ in $\mathcal{M}_2$ would restrict to a non-trivial factorization of G_n on $\mathbf{Per}_1(0)$. ■

2.2 The Hurwitz space

We assume here that $n \geq 3$. Let $M_{0,n}$ be the moduli space of n-pointed genus-0 curves, i.e., $M_{0,n}$ parametrizes tuples $(C, p_1, \ldots, p_n)$, where C is a smooth genus-0 curve, and $p_1, \ldots, p_n \in C$ are distinct. Let $\mathcal{H}_n$ be the space parametrizing tuples

$$(C, D, f, a_*, a_1, \ldots, a_n, a'_2, \ldots, a'_n, b_*, b_1, \ldots, b_n),$$

where C and D are smooth genus-0 curves, $a_*, a_1, \ldots, a_n, a'_2, \ldots, a'_n \in C$ are distinct, $b_*, b_1, \ldots, b_n \in D$ are distinct, and $f : C \to D$ is a degree-2 map for which a_* and a_1 are critical and such that $f(a_*) = b_*, f(a_1) = b_2, f(a_n) = f(a'_n) = b_1$, and $f(a_i) = f(a'_i) = b_{i+1}$ for $i = 2, \ldots, n-1$. There are two maps $\pi_a, \pi_b : \mathcal{H}_n \to \mathcal{M}_{0,n}$, where

$$\pi_a((C, D, f, a_*, a_1, \ldots, a_n, a'_2, \ldots, a'_n, b_*, b_1, \ldots, b_n)) = (C, a_1, \ldots, a_n)$$
$$\pi_b((C, D, f, a_*, a_1, \ldots, a_n, a'_2, \ldots, a'_n, b_*, b_1, \ldots, b_n)) = (D, b_1, \ldots, b_n).$$

Let $\Delta \subseteq M_{0,n} \times M_{0,n}$ be the diagonal, and $\mathcal{E}_n := (\pi_a \times \pi_b)^{-1}(\Delta)$.

Lemma 2.3 (See also [17, 24]). There is a birational map $\nu : \mathcal{E}_n \to \mathbf{Per}_n(0)$, defined over $\mathbb{Q}$.

Proof. There are universal curves $\mathcal{C}$ and $\mathcal{D}$ over $\mathcal{H}_n$, together with sections $\mathbf{a}_*, \mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{a}'_2, \ldots, \mathbf{a}'_n$ of $\mathcal{C}$, and sections $\mathbf{b}_*, \mathbf{b}_1, \ldots, \mathbf{b}_n$ of $\mathcal{D}$, and a universal degree-2 map $\mathbf{f} : \mathcal{C} \to \mathcal{D}$, all defined over $\mathbb{Q}$ (Theorem 10 of [18]; see also [1]). Because $\mathcal{C}$ and $\mathcal{D}$ have more than 3 disjoint sections, they are each isomorphic to $\mathbb{P}^1 \times \mathcal{H}_n$. There is an isomorphism $\mathcal{C}|_{\mathcal{E}_n} \to \mathcal{D}|_{\mathcal{E}_n}$, defined over $\mathbb{Q}$, identifying $\mathbf{a}_i$ with $\mathbf{b}_i$. Under this identification, $\mathbf{f}$ restricts to a family of quadratic rational self-maps of $\mathbb{P}^1$ parametrized by $\mathcal{E}_n$. This induces a morphism $\nu : \mathcal{E}_n \to \mathcal{M}_2$, defined over $\mathbb{Q}$ [25]. Since every rational map in the family has an n-periodic critical point, ν factors through the inclusion of $\mathbf{Per}_n(0)$ in $\mathcal{M}_2$. To see that ν is birational, we construct a rational inverse. Let $\mathbf{Per}_n(0)^\circ \subsetneq \mathbf{Per}_n(0)$ be the non-empty Zariski-open subset (defined over $\mathbb{Q}$) where exactly one critical point is n-periodic. Since exactly one critical point of $f \in \mathbf{Per}_n(0)^\circ$, is periodic, any automorphism of f

must fix the periodic critical point, and therefore fix every point in the forward orbit of that critical point. Since $n \geq 0$, $f \in \mathrm{Per}_n(0)^\circ$ has no nontrivial automorphisms. This implies that by [25], $\mathrm{Per}_n(0)^\circ$ carries a universal family $\mathcal{C}'$ whose geometric fibres are isomorphic to $\mathbb{P}^1$, together with a universal degree-2 morphism $\mathbf{f}' : \mathcal{C}' \to \mathcal{C}'$. One can mark on $\mathcal{C}'$ the n-periodic critical point as p_1, and also mark its forward orbit $p_i := (\mathbf{f}')^{i-1}(p_i)$ for $i = 2, \ldots, n$. For $i = 2, \ldots, n$, one can also mark p_i' as the inverse image of p_{i+1} that is not equal to p_i. Finally, one can mark the non-n-periodic critical point x' and its image y'. This induces a map $\mu : \mathrm{Per}_n(0)^\circ \to \mathcal{H}_n$, which factors through $\mathcal{E}_n$. Since μ and ν induce isomorphisms between the universal families coming from $\mathcal{M}_2$ and $\mathcal{H}$, we see that μ and ν are inverses. ■

Corollary 2.4. $\mathrm{Per}_n(0)$ is irreducible over $\mathbb{Q}$ (resp. $\mathbb{C}$) if and only if $\mathcal{E}_n$ is irreducible over $\mathbb{Q}$ (resp. $\mathbb{C}$).

2.3 Stable curves and admissible covers

Given a finite set S, an *S-marked stable genus-0 curve* is a connected nodal genus-0 curve C, together with an injection from S into the smooth locus of C such that every irreducible component has at least three points that are either marked or nodes. $\overline{M}_{0,n}$ is the Deligne–Mumford completion of $M_{0,n}$; its geometric points correspond to n-marked stable genus-0 curves. Let $\overline{\mathcal{H}_n}$ be the projective completion of $\mathcal{H}_n$ whose geometric points correspond to *admissible covers* [1, 18], that is, tuples

$$(C, D, f, a_*, a_1, \ldots, a_n, a_2', \ldots, a_n', b_*, b_1, \ldots, b_n),$$

where

- C is a stable $\{a_*, a_1, \ldots, a_n, a_2', \ldots, a_n'\}$-marked genus-0 curve;
- D is a stable $\{b_*, b_1, \ldots, b_n\}$-marked genus-0 curve; and
- $f : C \to D$ is a finite degree-2 map for which a_* and a_1 are critical, satisfying:
  - $f(a_*) = b_*, f(a_1) = b_2$, and $f(a_n) = f(a_n') = b_1$, for $i = 2, \ldots, n-1$, $f(a_i) = f(a_i') = b_{i+1}$,
  - nodes of C map to nodes of D, and smooth points of C map to smooth points of D,
  - away from a_* and a_1, the only ramification of f is at nodes of C, and
  - (Balancing condition) at each node $\eta \in C$, the two different branches at η map to the two different branches at $f(\eta) \in D$, and map with equal local degree.

The maps π_a and π_b extend to maps from $\overline{\mathcal{H}_n}$ to $\overline{M}_{0,n}$. Let $\overline{\Delta} \subseteq \overline{M}_{0,n} \times \overline{M}_{0,n}$ be the diagonal, and let $\overline{\mathcal{E}_n}^+ := (\pi_a \times \pi_b)^{-1}(\overline{\Delta})$. Note that $\overline{\mathcal{E}_n}^+$ contains the Zariski closure $\overline{\mathcal{E}_n}$ of $\mathcal{E}_n$, but in general also contains irreducible components supported on $\overline{\mathcal{H}_n} \setminus \mathcal{H}_n$.

3 The Smooth $\mathbb{Q}$-rational Point

In this section, we fix $n \geq 4$. We describe a $\mathbb{Q}$-rational point $(C_\star, D_\star, f_\star) \in \overline{\mathcal{H}_n}$, as depicted in Figure 1. Precisely:

- the C_is, C_i's, and D_is label irreducible components of $C_\star$ and $D_\star$ (depicted as ellipses),
- the a_is, a_i's, and b_is inside an ellipse are the marked points on that irreducible component,
- $f_\star$ maps $C_\star$ to $D_\star$ "vertically", and the restriction of $f_\star$ to any C_i or C_i' *except* for C_1 is the unique isomorphism sending marked points to marked points and nodes to nodes as depicted.
- On $C_1 \cong \mathbb{P}^1_{\mathbb{Q}}$, coordinates are chosen so that η_n is at "1", and η_n' at "−1", a_* at "∞" and a_1 at "0". The map $f_\star|_{C_1} : C_1 \to D_2$ is the map $z \mapsto z^2$, with coordinates on D_2 so that b_* is at ∞, b_2 is at 0, and θ_1 is at 1.

Lemma 3.1. The point $(C_\star, D_\star, f_\star) \in \overline{\mathcal{H}_n}$ is $\mathbb{Q}$-rational.

Proof. The above construction was over $\mathbb{Q}$. In particular, C_1 is the only component of $C_\star$ with more than three special points, and the cross-ratio of the four special points on C_1 is −1. Every component of $D_\star$ has exactly three special points. This implies that $C_\star$ and $D_\star$ are algebraic curves defined over $\mathbb{Q}$, and their marked points $\mathbb{Q}$-rational. The restriction of the map $f_\star$ to each irreducible component of $C_\star$ is also clearly defined over $\mathbb{Q}$ (is either an isomorphism to an irreducible component of $D_\star$ or is the squaring map). This implies that the tuple $(C_\star, D_\star, f_\star)$ is an admissible cover over $\mathrm{Spec}(\mathbb{Q})$. By [1], $\overline{\mathcal{H}_n}$ represents the functor from schemes to sets that sends a scheme to the set of admissible covers over that scheme.

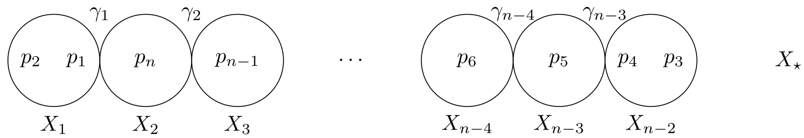

Fig. 2. The stable curve $(X_\star, p_1, \dots, p_n) \in \overline{M}_{0,n}$ that is the common stabilization of $(C_\star, a_1, \dots, a_n)$ and $(D_\star, b_1, \dots, b_n)$.

This means that the admissible cover $(C_\star, D_\star, f_\star)$ over $\mathbf{Spec}(\mathbb{Q})$ corresponds to a morphism from $\mathbf{Spec}(\mathbb{Q})$ to $\overline{\mathcal{H}}_n$, whose image is the point $(C_\star, D_\star, f_\star) \in \overline{\mathcal{H}}_n$. Since $(C_\star, D_\star, f_\star)$ is the image of $\mathbf{Spec}(\mathbb{Q})$, it is a $\mathbb{Q}$-rational point of $\overline{\mathcal{H}}_n$. ■

Lemma 3.2. The point $(C_\star, D_\star, f_\star)$ lies in $\overline{\mathcal{E}_n}^+$.

Proof. The common stabilization of $(C_\star, a_1, \dots, a_n)$ and $(D_\star, b_1, \dots, b_n)$ is the stable curve $(X_\star, p_1, \dots, p_n)$ depicted in Figure 2, so

$$\pi_a((C_\star, D_\star, f_\star)) = \pi_b((C_\star, D_\star, f_\star)) = (X_\star, p_1, \dots, p_n) \in \overline{M}_{0,n}.$$

■

Lemma 3.3. The point $(C_\star, D_\star, f_\star)$ lies in $\overline{\mathcal{E}_n}$, and is a smooth point of $\overline{\mathcal{E}_n}$.

Proof. The proof is a computation in local coordinates on $\overline{\mathcal{H}}_n$, as developed in Sections 3.4 and 3.5 in [24], based on deformation theory arguments from [10, 18, 19]. We base change to $\mathbb{C}$, and use notation from Figures 1 and 2.

In a (formal) neighbourhood of $(C_\star, D_\star, f_\star)$, $\overline{\mathcal{H}}_n$ admits local coordinates $(s_1, \dots s_{n-2})$, where s_i is a *node-smoothing parameter* for η_{n+1-i}—as well as for η'_{n+1-i} and for $f_\star(\eta_{n+1-i})$. (This last fact is because all of the nodes η_i are unramified.) In a (formal) neighbourhood of $(X_\star, p_1, \dots, p_n)$, $\overline{M}_{0,n}$ admits local coordinates $t_1, \dots, t_{n-3}$, where t_i is a node-smoothing parameter for γ_i. Using Figures 1 and 2, we can write $\pi_a^*(t_j)$ and $\pi_b^*(t_j)$ in terms of the coordinates s_i, as follows. Observe:

- On $X_\star$, the only node separating p_1 and p_2 from $p_3, \dots, p_n$ is γ_1, for which t_1 is a node-smoothing parameter.
- On $C_\star$, the only node that separates a_1 and a_2 from $a_3, \dots, a_n$ is η_n, for which s_1 is a node-smoothing parameter.
- On $D_\star$, the only node that separates b_1 and b_2 from $b_3, \dots, b_n$ is θ_n, for which s_2 is a node-smoothing parameter.

We infer that $\pi_a^*(t_1) = \alpha_1 s_1$ and $\pi_b^*(t_1) = \beta_1 s_2$, where α_1 and β_1 are non-vanishing regular functions on a formal neighbourhood of $f_\star \in \overline{\mathcal{H}}_n$. Similar computations tell us that for all $i = 1, 2, \dots n-3$, we have

$$\pi_a^*(t_i) = \alpha_i s_i \qquad \text{and} \qquad \pi_b^*(t_i) = \beta_i s_{i+1},$$

where α_i and β_i are non-vanishing regular functions on a formal neighbourhood of $f_\star$.

In a formal neighbourhood of $f_\star$, $\overline{\mathcal{E}_n}^+$ is cut out by the equations $\pi_a^*(t_i) = \pi_b^*(t_i)$, which can be rewritten as $\alpha_i s_i = \beta_i s_{i+1}$. (In these coordinates, $f_\star$ is identified with the origin.) These equations describe a (germ of a) curve, smooth at the origin, that does not lie in any of the coordinate hypersurfaces $\{s_i = 0\}$. On the other hand, in these coordinates, the boundary $\overline{\mathcal{H}}_n \setminus \mathcal{H}_n$ is identified with the union of the $(n-2)$ coordinate hypersurfaces $\{s_i = 0\}$. We conclude two things: First, near $f_\star$, $\overline{\mathcal{E}_n}^+$ does not generically lie in $\overline{\mathcal{H}}_n \setminus \mathcal{H}_n$. In other words, $f_\star$ is in $\overline{\mathcal{E}_n}$. Second, $\overline{\mathcal{E}_n}$ is smooth at $f_\star$. ■

We tie together the various strands to complete the proof of Theorem 1.1.

Proof. of Theorem 1.1 Suppose $n \geq 4$ is such that G_n is irreducible over $\mathbb{Q}$. Then by Theorem 2.2, $\mathbf{Per}_n(0)$ is irreducible over $\mathbb{Q}$. By Corollary 2.4, $\mathcal{E}_n$ is irreducible over $\mathbb{Q}$, and therefore so is $\overline{\mathcal{E}_n}$. By [26, Tag 04KY], since $\overline{\mathcal{E}_n}$ is $\mathbb{Q}$-irreducible, the action of the absolute Galois group on $\overline{\mathcal{E}_n}$ induces a transitive action on its set of $\mathbb{C}$-irreducible components. The point $f_\star \in \overline{\mathcal{E}_n}$, being $\mathbb{Q}$-rational (Lemma 3.1), is fixed by the Galois action and must therefore lie on every $\mathbb{C}$-irreducible component. On the other hand, $f_\star$ is a smooth point (Lemma 3.3) and so can lie on at most one $\mathbb{C}$-irreducible component. We conclude that $\overline{\mathcal{E}_n}$ is irreducible over $\mathbb{C}$. By Corollary 2.4, $\mathbf{Per}_n(0)$ is irreducible over $\mathbb{C}$. ■

Remark 3.4 (Proof of Theorem 1.3). We have not quite proved Theorem 1.3—doing so was not strictly necessary for Theorem 1.1. $\overline{\mathcal{E}_n}$ has a smooth $\mathbb{Q}$-rational point, but it is not a completion of $\mathbf{Per}_n(0)$, merely a birational model. Set $\overline{\mathbf{Per}_n(0)}$ to be the Zariski closure of $\mathbf{Per}_n(0)$ inside the $\mathbb{P}^2$ compactification of $\mathcal{M}_2$. The birational morphism $\nu : \mathcal{E} \to \mathbf{Per}_n(0)$ extends without indeterminacy to $\mathcal{E} \cup \{f_\star\} \to \overline{\mathbf{Per}_n(0)}$, since $f_\star$ is a smooth point on a curve. The point $\nu(f_\star) \in \overline{\mathbf{Per}_n(0)}$ is singular—we take a partial desingularization, $\widetilde{\mathbf{Per}_n(0)} \to \overline{\mathbf{Per}_n(0)}$, defined over $\mathbb{Q}$, by normalizing over $\nu(f_\star)$ but not modifying $\mathbf{Per}_n(0)$. The induced birational map $\mathcal{E} \to \mathbf{Per}_n(0)$ extends without indeterminacy to a morphism $\mathcal{E} \cup \{f_\star\} \to \widetilde{\mathbf{Per}_n(0)}$; the image of $f_\star$ is now a smooth point, and is $\mathbb{Q}$-rational because $f_\star$ is $\mathbb{Q}$-rational and the morphism is defined over $\mathbb{Q}$.

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References

1. Abramovich, D., A. Corti, and A. Vistoli. "Twisted bundles and admissible covers." *Comm. Algebra* **31**, no. 8 (2003): 3547–618. https://doi.org/10.1081/AGB-120022434.
2. Arfeux, M., and J. Kiwi. "Irreducibility of periodic curves in cubic polynomial moduli space." *Proc. Lond. Math. Soc.* **127**, no. 3 (2023): 792–835. https://doi.org/10.1112/plms.12553.
3. Baker, M., and L. DeMarco. "Special curves and postcritically finite polynomials." *Forum Math. Pi* **1** (2013): e3. https://doi.org/10.1017/fmp.2013.2.
4. Buff, X., A. L. Epstein, and S. Koch. "Prefixed curves in moduli space." *Amer. J. Math.* **144**, no. 6 (2022): 1485–509. https://doi.org/10.1353/ajm.2022.0036.
5. Buff, X., W. Floyd, S. Koch, and W. Parry. "Factoring Gleason polynomials modulo 2." *J. Théor. Nombres Bordeaux*. **34**, no. (2022): 787–812.
6. Bousch, T. *Sur quelques problemes de dynamique holomorphe*. PhD thesis. Paris, 1992.
7. Buff, X. "On postcritically finite unicritical polynomials." *New York J. Math.* **24** (2018): 1111–22.
8. Canci, J. K., and S. Vishkautsan. "Quadratic maps with a periodic critical point of period 2." *Int. J. Number Theory* **13**, no. 06 (2017): 1393–417. https://doi.org/10.1142/S1793042117500786.
9. John Doyle, Paul Fili, and Bella Tobin. Personal communication.
10. Deligne, P., and D. Mumford. "The irreducibility of the space of curves of given genus." *Publ. Math. Inst. Hautes Études Sci.* **36**, no. 1 (1969): 75–109. https://doi.org/10.1007/BF02684599.
11. DeMarco, L., X. Wang, and H. Ye. "Bifurcation measures and quadratic rational maps." *Proc. Lond. Math. Soc.* **111** (2015): 149–80. https://doi.org/10.1112/plms/pdv024.
12. Epstein, A. Transversality in holomorphic dynamics. http://homepages.warwick.ac.uk/~mases/Transversality.pdf.
13. Favre, C., and T. Gauthier. "Distribution of postcritically finite polynomials." *Isr. J. Math.* **209** (2015): 235–92. https://doi.org/10.1007/s11856-015-1218-0.

14. Firsova, T., J. Kahn, and N. Selinger. "On deformation spaces of quadratic rational functions." *Int. Math. Res. Not.* **2023** (2022): 6703–38. https://doi.org/10.1093/imrn/rnab320.
15. Goksel, V. "On the orbit of a post-critically finite polynomial of the form $x^d + c$." *Funct. Approx. Comment. Math* **62**, no. 1 (2020): 95–104.
16. Hironaka, E. "The augmented deformation space of rational maps." *Contemp. Math. Amer. Math. Soc.* **742**, (2020): 85–107. https://doi.org/10.1090/conm/742/14940.
17. Hironaka, E., and S. Koch. "A disconnected deformation space of rational maps." *J. Mod. Dyn.* **11** (2017): 409–23. https://doi.org/10.3934/jmd.2017016.
18. Harris, J., and D. Mumford. "On the Kodaira dimension of the moduli space of curves." *Invent. Math.* **67** (1982): 23–86. https://doi.org/10.1007/BF01393371.
19. Knudsen, F. F. "The projectivity of the moduli space of stable curves, II: The stacks $M_{g,n}$." *Math. Scand.* **52**, no. 2 (1983): 161–99.
20. Milnor, J., Milnor, and T. Lei. "Geometry and dynamics of quadratic rational maps, with an appendix by the author and Lei Tan." *Exp. Math.* **2**, no. 1 (1993): 37–83. https://doi.org/10.1080/10586458.1993.10504267.
21. Milnor, J. "Arithmetic of unicritical polynomial maps." *Frontiers in Complex Dynamics: In Celebration of John Milnor's 80th Birthday (PMS-51)*, edited by Bonifant A., Lyubich M., and Sutherland S. Princeton University Press, 2014.
22. Morton, P., and J. H. Silverman. "Rational periodic points of rational functions." *Int. Math. Res. Not.* **1994**, no. 2 (1994): 97–110. https://doi.org/10.1155/S1073792894000127.
23. Rees, M. "A fundamental domain for v_3." *Mém. Soc. Math. Fr.* **1** (2009).
24. Ramadas, R., and R. Silversmith. "Equations at infinity for critical-orbit-relation families of rational maps." *Exp. Math.*, no. 0 (2022): 1–21.
25. Silverman, J. H. "The space of rational maps on $\mathbb{P}^1$." *Duke Math. J.* **94**, no. 1 (1998): 41–77.
26. The Stacks Project Authors. Stacks Project. https://stacks.math.columbia.edu, 2018.
27. Stimson, J. R. P. "Degree two rational maps with a periodic critical point." PhD thesis. University of Liverpool, 1993.