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ORIGINAL ARTICLE

Stability of the Volume Preserving Mean Curvature Flow in Hyperbolic Space

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Abstract
We consider the dynamic property of the volume preserving mean curvature flow. This flow was introduced by Huisken (J Reine Angew Math 382:35–48, 1987) who also proved it converges to a round sphere of the same enclosed volume if the initial hypersurface is strictly convex in Euclidean space. We study the stability of this flow in hyperbolic space. In particular, we prove that if the initial hypersurface is hyperbolically mean convex and close to an umbilical sphere in the L^2-sense, then the flow exists for all time and converges exponentially to an umbilical sphere.

Keywords Volume preserving mean curvature flow · h-mean convex in hyperbolic space · Dynamical stability

Mathematics Subject Classification Primary 53E10; Secondary 58J35

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1 Introduction

1.1 Background and Main Theorem

Let M^n be a smooth, embedded, closed (compact, no boundary) n-dimensional manifold in hyperbolic space $\mathbb{H}^{n+1}$ ($n \geq 2$), and we evolve it by the volume preserving mean curvature flow (VPMCF),

$$\frac{\partial F}{\partial t} = (h - H)\,\nu, \quad F(\cdot, 0) = F_0(\cdot), \tag{1.1}$$

where $F_0 : M^n \to \mathbb{H}^{n+1}$ is the initial embedding, $H = H(x, t)$ is the mean curvature and $\nu = \nu(x, t)$ is the outward unit normal vector of the evolving surface $M_t = F(\cdot, t)$ at point (x, t) (for simplicity, we write $(x, t) \in M_t$). The function h is the average of the mean curvature on M_t, given by

$$h = h(t) = ⨍_{M_t} H\, d\mu = \frac{\int_{M_t} H\, d\mu}{\int_{M_t} d\mu}, \tag{1.2}$$

where $d\mu = d\mu_t$ denotes the surface area element of the evolving surface M_t with respect to the induced metric $g(t)$.

In this paper, we use the convention that the mean curvature is the sum of all principal curvatures. Clearly, we have $H \not\equiv 0$ on M_0 since there is no closed minimal hypersurface in hyperbolic space. The presence of the global term h in the VPMCF equation (1.1) forces the flow to behave quite differently from the usual mean curvature flow (MCF).

Hypersurfaces of constant mean curvature are critical points of the area functional under the constraint of fixed enclosed volume. These hypersurfaces are also static state for the VPMCF equation (1.1). A remarkable theorem of Huisken–Yau [19] on the existence of a foliation of spheres outside of some large compact set in asymptotic flat manifolds was achieved by studying a parameter family of VPMCFs. This flow, and the surface area preserving mean curvature flow studied in [14, 21], are special cases of so-called mixed volume preserving mean curvature flow. They are closely related to convex geometry and classical inequalities, see for instance [1, 9, 22, 26], also see [5, 8] for other geometric settings.

We denote $A = \{a_{ij}\}$ the second fundamental form of M_t and $\mathring{A} = A - \frac{H}{n} g$ its traceless part. Then, we have $|\mathring{A}|^2 = |A|^2 - \frac{1}{n} H^2$. This quantity measures the roundness of a (closed, immersed) hypersurface Σ in $\mathbb{H}^{n+1}$: if $\mathring{A} \equiv 0$, i.e., Σ is umbilic at every point, then by a classical Codazzi's theorem in differential geometry, it is a geodesic sphere, see, e.g., [24, Theorem 29]. We also remark that, in $\mathbb{R}^3$, De Lellis and Müller [7] generalized Codazzi's theorem by showing a version of the following quantitative rigidity:

$$\inf_{\lambda \in \mathbb{R}} \|A - \lambda\, \mathrm{Id}\|_{L^2(\Sigma)} \leq C \|\mathring{A}\|_{L^2(\Sigma)},$$

for some universal constant C. Such quantitative rigidity is not available for hyperbolic space to our knowledge.

Strict convexity (i.e., all principal curvatures are positive) plays a fundamental role in classical works of several types of MCFs, especially in Euclidean space. Huisken [15] proved that an initial smooth closed and strictly convex hypersurface will stay convex and flow into a round point along the MCF in Euclidean space. He [17] also showed, in the case of the VPMCF, the flow of an initial smooth closed and strictly convex hypersurface will exist for all time and flow into a round sphere in Euclidean space. The parallel result for the surface area preserving mean curvature flow is also true, showed by McCoy [21]. Though natural in Euclidean geometry, this notion of convexity is not the most natural in hyperbolic space. The presence of horospheres in hyperbolic space poses strong restrictions on the geometry of hypersurfaces (via Hopf's maximum principle): for instance any closed constant mean curvature hypersurface has mean curvature greater than n in $\mathbb{H}^{n+1}$.

Definition 1.1 We call a hypersurface of an $(n+1)$-dimensional hyperbolic manifold (strictly) *h-convex* if every principal curvature of the hypersurface at every point is greater than 1, and call it (strictly) *h-mean convex* or hyperbolically mean convex if its mean curvature at every point is greater than n.

The "h-convexity" was introduced in [4], where the authors proved that h-convexity is preserved along the VPMCF in hyperbolic space. Moreover, under the assumption of closed initial hypersurface being h-convex, they showed that the volume preserving mean curvature flow exists for all time and converges to an umbilical sphere. The "h-mean convexity", or the notion of being hyperbolically mean convex, is much weaker than h-convexity, and it is not known to be preserved along the VPMCF. But it turns out this condition plays a very important role in proving the dynamic stability of the VPMCF.

Unlike the regular MCF, the VPMCF (1.1) has a global forcing term in the equation which greatly complicates the analysis of the flow. How the singularities of the flow may form remains elusive at the current stage of study, even in Euclidean space. Moreover, in our hyperbolic space setting, the negative curvature of the ambient space presents significant challenges in analyzing the evolution equations involved in the study. As a first step to understand the long-term behavior of the flow, in this paper, we study the dynamical property of the VPMCF (1.1) in hyperbolic space in the situation that the initial hypersurface is not necessarily h-convex, yet close to an umbilical sphere in the L^2-sense. More precisely, we show the stability of the flow with initial *h-mean convex* hypersurface (namely the initial mean curvature at every point is greater than n) and small L^2-norm of the traceless part of the second fundamental form. Our main theorem is the following.

Theorem 1.2 *Let $M_t^n \subset \mathbb{H}^{n+1}$ of $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) for $t \in [0, T)$ with $T \leq \infty$. Assume that M_0 is h-mean convex with $H - n \geq \frac{1}{\Lambda_0^2}$ and*

$$\max\left\{|M_0|^2,\ \max_{M_0}|A|^2,\ \int_{M_0}|\nabla^m A|^2\,d\mu\right\} \leq \Lambda_0^2, \tag{1.3}$$

for some $\Lambda_0 \gg 1$ *and all* $m \in [1, n+3]$, *where* $|M_t|$ *is the n-dimensional surface area of* M_t *with the induced metric. Then, there exists some* $\epsilon_0 = \epsilon_0(n, \Lambda_0) > 0$ *such that if*

$$\int_{M_0} |\mathring{A}|^2 \, d\mu < \epsilon_0, \tag{1.4}$$

then $T = \infty$ *and the flow converges exponentially to an umbilical sphere which encloses the same volume as* M_0.

Remark 1.3 The VPMCF has been an interesting research topic since the introduction in [17], natural questions as studied in this paper have attracted much attention for a while. Let us provide some background here.

(1) Our strategy is similar to that of [28], both inspired by the work in [20, 29]. While our method is self-contained and more direct, there are some subtle differences. For instance, as in Lemma 2.12, we work with integral bounds instead of the much stronger L^∞-bound for the higher covariant derivatives of the second fundamental form, together with the reduction in Sect. 3.2 and further estimates in Sect. 3, we are able to follow through with the strategy.
(2) Furthermore, in previous treatments of similar stability problems for nonlocal flows in Euclidean or hyperbolic spaces, a key estimate often needed is to what degree one has control on the higher covariant derivatives of the second fundamental form along the flow. Such approaches often rely on L^∞-control of these covariant derivatives from only C^0-bound on the initial hypersurface. Such estimates are not available in the current literature for a nonlocal flow such as the VPMCF, even in the Euclidean case to our knowledge. In the appendix, we provide the Shi-type estimates for both the Euclidean and hyperbolic spaces, using relevant calculations in this paper. This extends Huisken's arguments [17] to obtain pointwise control on higher covariant derivatives of the second fundamental form with only continuous data on the initial hypersurface.
(3) There are other very interesting recent works on the stability of other nonlocal geometric flows similar to the VPMCF in hyperbolic space, see for instance [3, 10, 13] where different initial conditions were imposed for the specific flows to ensure the convergence.

1.2 Outline of the Proof

We would like to stress that there are several serious complications in order to investigate the dynamic stability for VPMCF: with a forcing term h, the flow is global in nature, therefore, it is difficult to localize the analysis and it is essential to keep track of h along the flow; we are working in the hyperbolic space where h-mean convexity is likely not preserved along the flow in general and the negative curvature of hyperbolic space makes the analysis of the flow substantially more involved. To overcome these difficulties, we use iteration techniques in combination of several new tools to prove the main theorem.

We organize the iteration argument in the following steps:

- Step 1, based on the initial bounds, we derive bounds on some short time interval for several geometric quantities (Lemmas 3.1 and 3.2) such as $H, \nabla H, |\mathring{A}|^2$, etc. As a consequence, we show that the h-mean convexity is preserved on some definite time interval provided the initial hypersurface is close enough to an umbilical sphere in the L^2 sense;
- Step 2, we reduce to the case of the mean curvature H of the evolving hypersurface is close to n;
- Step 3, we prove exponential decay for these quantities on the time interval obtained in Step 1 (Theorem 3.5), which allows us to obtain uniform bounds for these quantities on the interval;
- Step 4, we see that the above arguments allows to extend the time interval (Theorem 3.6) repeatedly with the amount extended each time only depends on the initial conditions, and conclude the main theorem.

2 Technical Preparations

In this section, we collect basic evolution equations for key quantities, and derive some preliminary estimates that will be used in the proof.

2.1 Evolution Equations

Let us first fix some notations of the following geometric quantities that will be used in this study:

(1) the induced metric of the evolving hypersurface M_t: $\{g_{ij}(t)\}$;
(2) the second fundamental form of M_t: $A(\cdot, t) = \{a_{ij}(\cdot, t)\}$, and its square norm is given by

$$|A(\cdot, t)|^2 = g^{ij} g^{kl} a_{ik} a_{jl};$$

(3) the mean curvature of M_t with respect to the outward unit normal vector: $H(\cdot, t) = g^{ij} a_{ij}$;
(4) the traceless part of the second fundamental form: $\mathring{A} = A - \frac{H}{n} g$;
(5) the area element of the evolving hypersurface M_t: $d\mu_t = \sqrt{\det(g_{ij})} dx$.

The evolution equations for these quantities are as follows:

Lemma 2.1 [17, 19] *The metric of M_t satisfies the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = 2(h - H) a_{ij}. \tag{2.1}$$

Therefore,

$$\frac{\partial}{\partial t} g^{ij} = -2(h - H) a^{ij} \tag{2.2}$$

and the area element satisfies:

$$\frac{\partial}{\partial t}(d\mu_t) = H(h - H)d\mu_t. \tag{2.3}$$

Moreover, the outward unit normal vector ν *to* M_t *satisfies*

$$\frac{\partial \nu}{\partial t} = \nabla H, \tag{2.4}$$

where $\frac{\partial \nu}{\partial t}$ *is a conventional way of writing down* $\overline{\nabla}_{\frac{\partial}{\partial t}} \nu$ *for the connection* $\overline{\nabla}$ *of the ambient space.*

By (2.3), we have the following geometrical properties of the VPMCF:

Corollary 2.2 [17]

(1) *The* $(n+1)$*-dimensional volume* V_t *of the region enclosed by* M_t *remains unchanged along the flow, i.e.,*

$$\frac{d}{dt}V_t = \int_{M_t} (h - H)\, d\mu = 0.$$

(2) *The* n*-dimensional surface area* $|M_t|$ *of* M_t *is non-increasing along the flow:*

$$\frac{d}{dt}|M_t| = \frac{d}{dt}\int_{M_t} d\mu = \int_{M_t} H(h - H)\, d\mu = -\int_{M_t} (h - H)^2\, d\mu \leq 0.$$

Following Huisken's calculations for the MCF in general Riemannian manifolds [16], we have the following evolution equations for key quantities in our setting. See also [19] for the case $n = 2$ in the setting of asymptotic flat manifolds and [4] for equivalent formulas in hyperbolic space setting.

Theorem 2.3 *We have the evolution equations for* H *and* $|A|^2$*:*

(i)

$$\frac{\partial}{\partial t}H = \Delta H + (H - h)(|A|^2 - n); \tag{2.5}$$

(ii)

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2(|A|^2 + n) - 2h\mathrm{tr}(A^3) + 2H(h - 2H). \tag{2.6}$$

where $\mathrm{tr}(A^3) = g^{ij}g^{kl}g^{mn}a_{ik}a_{lm}a_{nj}$.

We include a short proof for readers' convenience.

Proof Let $\bar{g} = \{\bar{g}_{\alpha\beta}\}$ be the metric on $\mathbb{H}^{n+1}$, $\bar{\nabla}$ and $\bar{R}_{\alpha\beta\gamma\delta}$ be covariant derivative and Riemannian curvature tensor with respect to $\bar{g}$. Equation (2.5) is clear since $\overline{\mathrm{Ric}}(\nu, \nu) = -n$ in $\mathbb{H}^{n+1}$. For (2.6), we first follow [4, 16] to find that the second fundamental form $\{a_{ij}\}$ of M_t satisfies the following evolution equation:

$$\frac{\partial}{\partial t}a_{ij} = \Delta a_{ij} + (h - 2H)a_{i\ell}a_{j\ell} + |A|^2 a_{ij} + a_{ij}\bar{R}_{0\ell 0\ell} - h\bar{R}_{0i0j} \\ -a_{j\ell}\bar{R}_{\ell mim} - a_{i\ell}\bar{R}_{\ell mjm} + 2a_{\ell m}\bar{R}_{\ell imj} - \bar{\nabla}_j\bar{R}_{0\ell i\ell} - \bar{\nabla}_\ell\bar{R}_{0ij\ell}. \quad (2.7)$$

The last two terms which involve the covariant derivatives of the curvature tensor drop out as we are in a constant curved space. Furthermore, since $\mathbb{H}^{n+1}$ has constant sectional curvature -1, the Riemannian curvature tensor is given by

$$\bar{R}_{\alpha\beta\gamma\delta} = (-1)\cdot(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\gamma\beta}). \quad (2.8)$$

Now, (2.6) follows from contraction and (2.1). □

The covariant derivatives for A satisfy the following.

Corollary 2.4 *We have the evolution equation for $|\nabla^m A|^2$ with $m \geq 1$:*

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 + \nabla^m A * \nabla^m A \\ + \sum_{i+j+k=m}\nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A + \sum_{r+s=m}\nabla^r A * \nabla^s A * \nabla^m A, \quad (2.9)$$

where $S * \Omega$ denotes any linear combination of tensors formed by contraction on S and Ω by the metric g. Here, in addition to constants, $h = h(t)$ (having only time variable) may be involved in the contraction coefficients, which is essentially bounded by $|A|$.

Proof We have the following evolution of the second fundamental form from the proof of Theorem 2.3:

$$\frac{\partial}{\partial t}A = \Delta A + A * A * A + A * A + *A.$$

Meanwhile, the time derivative of the Christoffel symbols Γ^i_{jk} is equal to

$$\frac{\partial}{\partial t}\Gamma^i_{jk} = \frac{1}{2}g^{il}\left\{\nabla_j\left(\frac{\partial}{\partial t}g_{kl}\right) + \nabla_k\left(\frac{\partial}{\partial t}g_{jl}\right) - \nabla_l\left(\frac{\partial}{\partial t}g_{jk}\right)\right\} \\ = g^{il}\left\{\nabla_j((h-H)a_{kl}) + \nabla_k\big((h-H)a_{jl}\big) - \nabla_l\big((h-H)a_{jk}\big)\right\} \\ = *\nabla A + A * \nabla A\,, \quad (2.10)$$

where $*T$ denotes contraction of T by the metric g. Note that we have used the evolution equation (2.1) for the metric.

Now, we proceed as in [11, Sect. 13] (see also [15, Sect. 7]) to obtain Eq. (2.9). In particular, using (2.10), if S and Ω are tensors satisfying the evolution equation $\frac{\partial}{\partial t}S = \Delta S + \Omega$, then the covariant derivative ∇S, which involves the Christoffel symbols, satisfies an equation of the following form:

$$\frac{\partial}{\partial t}\nabla S = \Delta(\nabla S) + S * A * \nabla A + S * \nabla A + A * A * \nabla S + \nabla\Omega. \tag{2.11}$$

Therefore, by (2.7), we find

$$\frac{\partial}{\partial t}\nabla A = \Delta\nabla A + \sum_{i+j+k=1} \nabla^i A * \nabla^j A * \nabla^k A + \sum_{r+s=1} \nabla^r A * \nabla^s A + *\nabla A.$$

Then, by induction we have for $m \geq 1$,

$$\frac{\partial}{\partial t}\nabla^m A = \Delta\nabla^m A + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A + \sum_{r+s=m} \nabla^r A * \nabla^s A + *\nabla^m A. \tag{2.12}$$

Then, Eq. (2.9) follows from the following identity essentially from (2.1)

$$\frac{\partial}{\partial t}|\nabla^m A|^2 = 2\left\langle \nabla^m A, \frac{\partial}{\partial t}\nabla^m A\right\rangle + A * \nabla^m A * \nabla^m A + A * A * \nabla^m A * \nabla^m A$$

and the standard identity

$$\Delta|\nabla^m A|^2 = 2\langle \nabla^m A, \Delta\nabla^m A\rangle + 2|\nabla^{m+1} A|^2.$$

□

We also have the time derivative for the average of mean curvature $h(t)$.

Lemma 2.5

$$h'(t) = \frac{\int_{M_t}(H-h)(|A|^2 - H^2 + hH)d\mu}{\int_{M_t} d\mu}. \tag{2.13}$$

Proof An easy calculation using Eqs. (2.3) and (2.5). Note that the expression does not contain terms involving ∇H. □

The following inequalities for gradients are useful and we record them here:

Lemma 2.6 (cf. [16]) *The following inequalities hold:*

(i)

$$|\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2;$$

(ii)

$$|\nabla \mathring{A}|^2 \geq \frac{n-1}{2n+1}|\nabla A|^2 \geq \frac{3(n-1)}{(n+2)(2n+1)}|\nabla H|^2.$$

2.2 Intuitive Decay of $|\mathring{A}|^2$

One of the key estimates for us is an exponential decay for $|\mathring{A}|^2$ on some time interval. We now give a heuristic argument to show why this is the case when $|\mathring{A}|^2$ is small and h-mean convexity is preserved.

Since $|\mathring{A}|^2 = |A|^2 - \frac{1}{n}H^2$ and $|\nabla \mathring{A}|^2 = |\nabla A|^2 - \frac{1}{n}|\nabla H|^2$, we obtain the evolution equation for $|\mathring{A}|^2$ as follows.

Lemma 2.7

$$\begin{aligned}\frac{\partial}{\partial t}|\mathring{A}|^2 &= \Delta|\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + 2|\mathring{A}|^2(|A|^2+n) + \frac{2h}{n}\big(H|A|^2 - n\,\mathrm{tr}(A^3)\big)\\ &= \Delta|\mathring{A}|^2 - 2|\nabla \mathring{A}|^2 + 2|\mathring{A}|^2(|A|^2+n) - 2h\left\{\mathrm{tr}(\mathring{A}^3) + \frac{2}{n}|\mathring{A}|^2 H\right\}. \quad (2.14)\end{aligned}$$

Proof The evolution equation for H^2 can be easily derived from (2.5):

$$\frac{\partial}{\partial t}H^2 = \Delta H^2 - 2|\nabla H|^2 + 2H(H-h)(|A|^2-n). \quad (2.15)$$

Then, (2.14) follows easily from the identity (see, e.g., page 335 of [20]):

$$\mathrm{tr}(A^3) - \frac{1}{n}|A|^2 H = \mathrm{tr}(\mathring{A}^3) + \frac{2}{n}|\mathring{A}|^2 H.$$

□

To see a heuristic argument on exponential decay of $|\mathring{A}|^2$, we examine Eq. (2.14) more closely, provided $|\mathring{A}|^2$ is small and $|h-H|$ is also very small. Obviously, one can apply the maximum principle to (2.14) to obtain the exponential decay of $|\mathring{A}|^2$, if for some small $\epsilon > 0$, we have

$$2|\mathring{A}|^2(|A|^2+n) - 2h\left\{\mathrm{tr}(\mathring{A}^3) + \frac{2}{n}|\mathring{A}|^2 H\right\} \leq -\epsilon|\mathring{A}|^2.$$

Since $|\mathrm{tr}(\mathring{A}^3)| \leq |\mathring{A}|^3$, it suffices to show

$$|\mathring{A}|^2 + \frac{H^2}{n} + n + |h|\cdot|\mathring{A}| - \frac{2hH}{n} < -\frac{\epsilon}{2}, \quad (2.16)$$

which can be rewritten as

$$\frac{H^2}{n} = H + \frac{H(H-n)}{n} > |\mathring{A}|^2 + n + |h|\cdot|\mathring{A}| + \frac{2H(H-h)}{n} + \frac{\epsilon}{2}.$$

This inequality holds once we establish $H > n + \sigma$ for some $\sigma > 0$ (i.e., h-mean convexity) provided that $|\mathring{A}|^2$ and $|h - H|$ are both sufficiently small. We will make the argument precise in Sect. 3.3.

2.3 Technical Tools

For the sake of self-containedness of the paper, we now collect tools that will be used in the proof: a version of maximum principle, Hamilton's interpolation inequalities for tensors, a generalization of Topping's theorem in hyperbolic space, and an L^2-bound for covariant derivatives of A along the VPMCF. First, the following version of maximum principle is useful in our iteration scheme.

Theorem 2.8 (Maximum Principle, see, e.g., [6, Lemma 2.11]) *Suppose* $u : M \times [0, T] \to \mathbb{R}$ *satisfies*

$$\frac{\partial}{\partial t} u \leq a^{ij}(t)\nabla_i \nabla_j u + \langle B(t), \nabla u \rangle + F(u),$$

where the coefficient matrix $\left(a^{ij}(t)\right) > 0$ *for all* $t \in [0, T]$, $B(t)$ *is a time-dependent vector field and* F *is a Lipschitz function. If* $u \leq c$ *at* $t = 0$ *for some* $c \in \mathbb{R}$, *then* $u(x, t) \leq U(t)$ *for all* $(x, t) \in M \times \{t\}$, $t \in [0, T]$, *where* $U(t)$ *is the solution to*

$$\frac{d}{dt} U(t) = F(U) \quad \text{with} \quad U(0) = c.$$

We also need the following Hamilton's interpolation inequalities for tensors. These inequalities will be used inductively for us to obtain integral bounds of covariant derives of $\mathring{A}$.

Theorem 2.9 [11] *Let* M *be an* n*-dimensional compact Riemannian manifold and* Ω *be any tensor on* M. *Suppose*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \quad \text{with} \quad r \geq 1.$$

Then, we have the estimate:

$$\left(\int_M |\nabla \Omega|^{2r} \, d\mu\right)^{\frac{1}{r}} \leq (2r - 2 + n) \left(\int_M |\nabla^2 \Omega|^p \, d\mu\right)^{\frac{1}{p}} \left(\int_M |\Omega|^q \, d\mu\right)^{\frac{1}{q}}.$$

Theorem 2.10 [11] *Let* M *and* Ω *be the same as in Theorem 2.9. If* $1 \leq i \leq m - 1$ *and* $m \geq 1$, *then there exists a constant* $C = C(n, m)$ *independent of the metric and connection on* M, *such that*

$$\int_M |\nabla^i \Omega|^{\frac{2m}{i}} \, d\mu \leq C \max_M |\Omega|^{2(\frac{m}{i} - 1)} \int_M |\nabla^m \Omega|^2 \, d\mu.$$

As an application of these inequalities and Corollary 2.4, we have the following.

Lemma 2.11 *For any $m \geq 0$, we have the estimate*

$$\left(\frac{d}{dt}\int_{M_t}|\nabla^m A|^2 d\mu\right) + 2\int_{M_t}|\nabla^{m+1}A|^2 d\mu$$
$$\leq C\max_{M_t}\left(1+|A|+|A|^2\right)\int_{M_t}|\nabla^m A|^2 d\mu\,,$$

where $C = C(n, m, |h|)$.

Proof When $m = 0$, the inequality is obvious in light of (2.6). Now we consider $m \geq 1$. By integrating (2.9) of Corollary 2.4 and using the generalized Hölder inequality, we have

$$\left(\frac{d}{dt}\int_{M_t}|\nabla^m A|^2\, d\mu\right) - \int_{M_t}(h-H)H|\nabla^m A|^2\, d\mu + 2\int_{M_t}|\nabla^{m+1}A|^2\, d\mu$$
$$\leq C\Bigg\{\sum_{i+j+k=m}\left(\int_{M_t}|\nabla^i A|^{\frac{2m}{i}}\, d\mu\right)^{\frac{i}{2m}}\left(\int_{M_t}|\nabla^j A|^{\frac{2m}{j}}\, d\mu\right)^{\frac{j}{2m}}\left(\int_{M_t}|\nabla^k A|^{\frac{2m}{k}}\, d\mu\right)^{\frac{k}{2m}}$$
$$+\sum_{r+s=m}\left(\int_{M_t}|\nabla^r A|^{\frac{2m}{r}}\, d\mu\right)^{\frac{r}{2m}}\left(\int_{M_t}|\nabla^s A|^{\frac{2m}{s}}\, d\mu\right)^{\frac{s}{2m}}\Bigg\}\left(\int_{M_t}|\nabla^m A|^2\, d\mu\right)^{\frac{1}{2}}$$
$$+C\int_{M_t}|\nabla^m A|^2\, d\mu\,,$$

where all the indices now take values from 1 and up and the terms in the original sums with 0 indices being absorbed by other sums and C's.

Applying Theorem 2.10 for A, we have

$$\left(\int_{M_t}|\nabla^q A|^{\frac{2m}{q}}\, d\mu\right)^{\frac{q}{2m}} \leq C\max_{M_t}|A|^{1-\frac{q}{m}}\left(\int_{M_t}|\nabla^m A|^2\, d\mu\right)^{\frac{1}{2m}},$$

where q can be i, j, k, r or s. We also notice

$$\int_{M_t}|(h-H)H|\nabla^m A|^2 d\mu \leq \max_{M_t}\left\{|h||H|+H^2\right\}\int_{M_t}|\nabla^m A|^2 d\mu$$
$$\leq C(n,|h|)\max_{M_t}\left(|A|^2+|A|\right)\int_{M_t}|\nabla^m A|^2 d\mu.$$

Combining these inequalities, we complete the proof. □

It is known from [17] that L^∞ bound for $|A|$ along the VPMCF in Euclidean space and the initial L^∞ bounds of its covariant derivatives will give L^∞ bounds for the covariant derivatives. Adapting the argument there, we have the following lemma for explicit L^2 bounds for our situation.

Lemma 2.12 *Along the VPMCF, for $k \geq 0$, if*

$$\max\left\{|M_0|, \max_{M_t, t\in[0,T]} |A|^2, \max_{m\leq k} \int_{M_0} |\nabla^m A|^2 \, d\mu\right\} \leq \Lambda_0^2,$$

then uniformly for $t \in [0, T]$ and $m \leq k$, we have

$$\int_{M_t} |\nabla^m A|^2 \, d\mu \leq C(\Lambda_0, k),$$

where $C(\Lambda_0, k)$ is independent of T.

Proof Along the VPMCF, we have $|M_t| \leq |M_0|$ by Corollary 2.2. Therefore, the conclusion is clear for $m = 0$ for any fixed $k \geq 0$. We can then prove the lemma by induction on m. Suppose the conclusion is true for $m \geq 0$, to see this holds for $m + 1 \leq k$, note that by Lemma 2.11, we know for $m \geq 0$,

$$\frac{d}{dt}\int_{M_t} |\nabla^m A|^2 \, d\mu \leq C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 \, d\mu - 2\int_{M_t} |\nabla^{m+1} A|^2 \, d\mu\,,$$
$$\frac{d}{dt}\int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \leq C(\Lambda_0) \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu\,.$$

Let $G(t) = C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 \, d\mu + \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu$. Then, we have

$$G'(t) \leq C(\Lambda_0) \left(C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 \, d\mu - \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \right). \tag{2.17}$$

Consider the maximum of $G(t)$ achieved at $t = \bar{t} \in [0, T]$. If $\bar{t} = 0$ then for all $t \in [0, T]$,

$$G(t) \leq G(0) \leq (C(\Lambda_0) + 1)\Lambda_0^2. \tag{2.18}$$

Otherwise by (2.17),

$$C(\Lambda_0) \int_{M_{\bar{t}}} |\nabla^m A|^2 \, d\mu - \int_{M_{\bar{t}}} |\nabla^{m+1} A|^2 \, d\mu \geq 0,$$

and thus, we have for all $t \in [0, T]$

$$G(t) \leq G(\bar{t}) \leq C(\Lambda_0, k). \tag{2.19}$$

Therefore, by (2.18) and (2.19),

$$\int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \leq C(\Lambda_0, k),$$

which is independent of T. □

In the Euclidean space, Topping [25] discovered a relation between the intrinsic diameter and the mean curvature H of any closed, connected and smoothly immersed submanifold. This result has been extended to a more general Riemannian setting by Wu–Zheng [27], using Hoffman–Spruck's generalization [12] of the Michael–Simon's inequality [23]. We formulate their result in our setting below.

Theorem 2.13 [27] *Let M be an n-dimensional closed, connected manifold smoothly isometrically immersed in $\mathbb{H}^N$, where $N \geq n+1$. There exists a constant $C = C(n)$ such that the intrinsic diameter and the mean curvature H of M are related by the following inequality:*

$$\operatorname{diam}(M) \leq C(n) \int_M |H|^{n-1}\, d\mu.$$

2.4 Hyperbolic Mean Convexity

The h-mean convexity is a very important geometric ingredient in our main result. Note that mean convexity and h-mean convexity are not known to be preserved along the VPMCF. The strict h-convexity is however preserved along the VPMCF in $\mathbb{H}^{n+1}$ [4]. We give an alternative proof for this result by following very closely Huisken's tensor calculations in [15, 17] and highlighting the role of the curvature for the ambient space. Unlike the preserved mean convexity along the MCF in Euclidean space, this shows the subtlety of the h-mean convexity in hyperbolic space and the negative-curvature effects of the ambient space.

Proposition 2.14 [4] *Let M^n be a smooth, embedded, closed hypersurface moving by the VPMCF (1.1) in a smooth, complete, hyperbolic manifold N^{n+1}. If the initial hypersurface M^n is strictly h-convex, then each evolving hypersurface M^n_t is also strictly h-convex along the flow (1.1).*

Proof Let $M_{ij} = a_{ij} - g_{ij}$. Recall the evolution equations for a_{ij} and g_{ij} along the mean curvature flow (1.1) as (2.7) and (2.1):

$$\begin{aligned}\frac{\partial}{\partial t}a_{ij} - \Delta a_{ij} = &(h-2H)a_{i\ell}a_{j\ell} + |A|^2 a_{ij} - n a_{ij} - h\bar{R}_{0i0j}\\ &-a_{j\ell}\bar{R}_{\ell mim} - a_{i\ell}\bar{R}_{\ell mjm} + 2a_{\ell m}\bar{R}_{\ell imj},\end{aligned}$$

where the covariant derivatives for the curvature tensor disappear since the sectional curvature is -1, and

$$\frac{\partial}{\partial t}g_{ij} = 2(h-H)a_{ij}.$$

Therefore, we obtain the evolution equation for the symmetric tensor M_{ij}:

$$\frac{\partial}{\partial t}M_{ij} = \Delta M_{ij} + N_{ij},$$

where we have used $\Delta g = 0$ and

$$N_{ij} = (h-2H)a_{i\ell}a_{j\ell} + \big(|A|^2 - n\big)a_{ij} - h\bar{R}_{0i0j} - a_{j\ell}\bar{R}_{\ell mim} \\ -a_{i\ell}\bar{R}_{\ell mjm} + 2a_{\ell m}\bar{R}_{\ell imj} + 2(H-h)a_{ij}. \tag{2.20}$$

Now, recall from (2.8),

$$\bar{R}_{\alpha\beta\gamma\delta} = (-1)\cdot(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\gamma\beta}).$$

Let X be a null-eigenvector of M_{ij} at some (x_0, t_0). We arrange the coordinates such that at (x_0, t_0), $X = e_1$, $g_{ij} = \delta_{ij}$ and $a_{ij} = \lambda_i\delta_{ij}$. This is justified as $\{g_{ij}\}$ is a symmetric positive-definite matrix, $\{a_{ij}\}$ is a symmetric matrix, and so they can be simultaneously diagonalized.

We examine term by term from (2.20) to arrive at

$$N_{11} = (h-2H)\lambda_1^2 + \big(|A|^2 - n\big)\lambda_1 + h + 2(n-1)\lambda_1 + 2(\lambda_1 - H) + 2(H-h)\lambda_1.$$

Meanwhile, with $X = e_1$ being a null-eigenvector of M_{ij}, we have $\lambda_1 = 1$ since $M_{11} = a_{11} - g_{11} = 0$. Thus, we have

$$N_{11} = |A|^2 + n - 2H \geq \frac{1}{n}H^2 - 2H + n = \frac{1}{n}(H-n)^2 \geq 0.$$

The conclusion follows from Hamilton's maximum principle for tensors [11]. □

3 Proof of Main Theorem

We are now ready to use iteration method to prove our main theorem. It is divided into four steps discussed in four subsections accordingly.

3.1 Step 1: Short Time Bounds

We start by bounding important geometric quantities for short time, with the bounds depending on the initial conditions. This is certainly expected for a smooth flow. However, one expects such bounds to hold only for a short time, and as the flow evolves such bounds would deteriorate by extending the time interval.

The first technical lemma is as follows:

Lemma 3.1 *Let $M_t^n \subset \mathbb{H}^{n+1}$, $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) for $t \in [0, T)$ with $T \leq \infty$. Assume*

$$\max\left\{|M_0|^2,\ \max_{M_0}|A|^2,\ \int_{M_0}|\nabla^m A|^2\,d\mu\right\} \leq \Lambda_0^2 \tag{3.1}$$

for some $\Lambda_0 \gg 1$ and all $m \in [1, n+3]$, where $|M_t|$ is the n-dimensional surface area of M_t with the induced metric. There exist constants $\epsilon_0 = \epsilon_0(n, \Lambda_0) > 0$ and

$t_1 = t_1(n, \Lambda_0) \in (0, 1)$ *such that if*

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon < \epsilon_0, \tag{3.2}$$

then for any $t \in [0, t_1]$ *and any* $m \in [0, n+3]$, *we have*

$$\max \left\{ \max_{M_t} |A|^2, \int_{M_t} |\nabla^m A|^2 \, d\mu \right\} \leq 2\Lambda_0^2. \tag{3.3}$$

Moreover, there exist $C_1 = C_1(n, \Lambda_0)$ *and some universal constant* $\alpha \in (0, 1)$ *such that for any* $t \in [0, t_1]$

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H| \right) \leq C_1 \epsilon^\alpha. \tag{3.4}$$

Proof Recall from (2.6) the evolution equation for $|A|^2$ is given by

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2\left(|A|^2 + n\right) - 2h\text{tr}(A^3) + 2H(h - 2H).$$

Using the facts that $|\text{tr}(A^3)| \leq |A|^3$ (see [18, Lemma 2.2]), and $H^2 \leq n|A|^2$, we obtain the following inequality on M_t for all $t \in [0, T)$:

$$\frac{\partial}{\partial t}|A|^2 \leq \Delta|A|^2 + 2|A|^4 + 2n|A|^2 + 2|h|\left(|A|^3 + \sqrt{n}|A|\right). \tag{3.5}$$

Set $f(t) = \max_{M_t} |A|^2$, then $f(t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} f &\leq 2f^2 + 2nf + 2|h|\left(|A|^3 + \sqrt{n}|A|\right) \\ &\leq 2f^2 + 2nf + 2\sqrt{n} f^2 + 2nf \\ &\leq 4nf^2 + 4nf. \end{aligned} \tag{3.6}$$

One solves the comparison ODE explicitly to get $U(t) > 0$ satisfying

$$\log\left(1 + \frac{1}{U(t)}\right) = \log\left(1 + \frac{1}{U(0)}\right) - 4nt,$$

with $U(0) = f(0) = \max_{M_0} |A|^2 \leq \Lambda_0^2$ by (3.1). Therefore, by Theorem 2.8, $f(t) \leq U(t)$ for all $t \in [0, T)$.

Therefore, there exists some $t_1 = t_1(n, \Lambda_0) \in (0, 1)$ such that

$$\max_{M_t} |A|^2 \leq 2\Lambda_0^2 \quad \text{for all } t \in [0, t_1]. \tag{3.7}$$

Moreover, by choosing t_1 sufficiently small and integrating the inequality in Lemma 2.11 over $[0, t_1]$, we have

$$\int_{M_t} |\nabla^m A|^2 \, d\mu \le e^{C(n,\Lambda_0)t_1} \int_{M_0} |\nabla^m A|^2 \, d\mu \le 2\Lambda_0^2 \tag{3.8}$$

for all $t \in [0, t_1]$ and $m \in [1, n+3]$. Using the Sobolev embedding on compact manifolds [2], this yields

$$|A|_{C^2(M_t)} \le C(n, \Lambda_0) \quad \text{for all } t \in [0, t_1]. \tag{3.9}$$

In light of

$$|h| \le \max_{M_t} |H| \le \sqrt{n} \max_{M_t} |A| \le \sqrt{2n}\Lambda_0,$$
$$|\mathrm{tr}(\mathring{A}^3)| \le |\mathring{A}|^3 \le \sqrt{2}\Lambda_0 |\mathring{A}|^2,$$

we integrate the evolution equation (2.14) for $|\mathring{A}|^2$ over M_t for $t \in [0, t_1]$ to get

$$\frac{\partial}{\partial t} \int_{M_t} |\mathring{A}|^2 \, d\mu \le C(n, \Lambda_0) \int_{M_t} |\mathring{A}|^2 \, d\mu, \tag{3.10}$$

and so using (3.2), we have

$$\int_{M_t} |\mathring{A}|^2 \, d\mu \le \epsilon e^{C(n,\Lambda_0)t} \le C(n, \Lambda_0)\epsilon \quad \text{for all } t \in [0, t_1], \tag{3.11}$$

where the constant $C(n, \Lambda_0)$ can be different at places. We then apply Hamilton's interpolation inequalities (Theorem 2.9 with $r = 1,\ p = q = 2$):

$$\int_{M_t} |\nabla \mathring{A}|^2 \, d\mu \le n \left(\int_{M_t} |\mathring{A}|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_{M_t} |\nabla^2 \mathring{A}|^2 \, d\mu \right)^{\frac{1}{2}} \le C(n, \Lambda_0)\epsilon^{\frac{1}{2}}, \tag{3.12}$$

where we use $|\nabla^2 \mathring{A}| \le C(n)|\nabla^2 A|$ and the L^2-bound for $|\nabla^2 A|$ in (3.8). In fact, applying Theorem 2.9 inductively, we have for all $m \in [0, n+2]$,

$$\int_{M_t} |\nabla^m \mathring{A}|^2 \, d\mu \le C(n, \Lambda_0)\epsilon^{1/2^m} \quad \text{for all } t \in [0, t_1]. \tag{3.13}$$

Now, again by the Sobolev embedding [2], we have

$$|\mathring{A}|_{C^2(M_t)} \le C(n, \Lambda_0)\epsilon^{\alpha}, \tag{3.14}$$

for all $t \in [0, t_1]$ and some universal constant $\alpha \in (0, 1)$. Now, by (ii) of Lemma 2.6, for all $t \in [0, t_1]$, we have

$$\max_{M_t} |\nabla H| \le C(n) \max_{M_t} |\nabla \mathring{A}| \le C(n, \Lambda_0)\epsilon^{\alpha}. \tag{3.15}$$

Furthermore, by Corollary 2.2, the surface area $|M_t|$ is non-increasing along the flow, i.e.,

$$|M_t| \leq |M_0| \leq \Lambda_0^2. \tag{3.16}$$

Using Theorem 2.13, (3.7), (3.15) and (3.16), we arrive at

$$\begin{aligned} |h(t) - H(x,t)| &= \left(\int_{M_t} d\mu\right)^{-1} \left|\int_{M_t} \big(H(y,t) - H(x,t)\big)\, d\mu(y)\right| \\ &\leq \operatorname{diam}(M_t) \max_{M_t} |\nabla H| \\ &\leq C(n, \Lambda_0)\epsilon^{\alpha} \end{aligned} \tag{3.17}$$

for all $(x,t) \in M_t$ and $t \in [0, t_1]$. This together with (3.14) and (3.15) give (3.4), and we conclude the proof. □

With the above control of geometric quantities, we next show that the h-mean convexity is preserved for short time if the initial hypersurface is close to an umbilical sphere in the L^2-sense.

Lemma 3.2 *Let $M_t^n \subset \mathbb{H}^{n+1}$ for $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) as in Lemma 3.1 with the initial condition (3.1). Suppose*

$$\min_{M_0}(H - n) \geq c_0 > 0. \tag{3.18}$$

Then, there exist $\epsilon_1 = \epsilon_1(n, \Lambda_0) \in (0, \epsilon_0)$ and $T_1 = T_1(n, \Lambda_0) \in (0, t_1]$, where ϵ_0 and t_1 are as in Lemma 3.1, such that if

$$\int_{M_0} |\mathring{A}|^2\, d\mu \leq \epsilon < \epsilon_1,$$

then for $t \in [0, T_1]$, we have

$$\min_{M_t}(H - n) \geq \frac{c_0}{2} > 0. \tag{3.19}$$

Proof We start with the evolution equation for H (2.5):

$$H_t = \Delta H + (H - h)\big(|A|^2 - n\big).$$

By (3.7) and (3.9), for any $(x,t) \in M_t$, $t \in [0, t_1]$, we have

$$\left|\frac{\partial}{\partial t} H\right|(x,t) \leq C(n, \Lambda_0), \tag{3.20}$$

where we have also used $|\nabla^2 H| \leq C(n)|\nabla^2 A|$. Using (3.15) and (3.20) and choosing $T_1 = T_1(n, \Lambda_0) \in (0, t_1]$ and $\epsilon_1 = \epsilon_1(n, \Lambda_0) \in (0, \epsilon_0)$ sufficiently small, we have

$$\min_{M_t}(H - n) \geq \frac{1}{2} \min_{M_0}(H - n) \geq \frac{c_0}{2} > 0.$$

□

3.2 Step 2: Reduction

In the previous subsection, we have obtained estimates (3.3) and (3.4) on some time interval $[0, t_1]$, provided that the initial hypersurface is close to an umbilical sphere in the L^2-sense (see (1.4)). In this step, we make a key reduction. Namely, we show it suffices to prove the main theorem when the mean curvature H of the evolving hypersurface is close to n. In particular, we have the following.

Proposition 3.3 *Let $M_t^n \subset \mathbb{H}^{n+1}$ for $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) on $t \in [0, t_1]$ with $t_1 = t_1(n, \Lambda_0) \in (0, 1)$, where t_1 and Λ_0 are as in Lemma 3.1. If (3.1) and (3.2) hold, then*

(1) *either the evolving hypersurface M_t becomes strictly h-convex, and the flow (1.1) exists for all time and converges exponentially to an umbilical sphere,*
(2) *or there is a constant $C_2 = C_2(n, \Lambda_0) > 0$ such that for all $(x, t) \in M_t$, $t \in [0, t_1]$, we have*

$$|H(x, t) - n| \leq C_2 \epsilon^{\frac{\alpha}{2}}, \tag{3.21}$$

where ϵ is from (3.2) and $\alpha \in (0, 1)$ is from (3.4).

Proof On the time interval $[0, t_1]$, we recall the estimate (3.4) from Lemma 3.1:

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) \leq C_1 \epsilon^{\alpha}$$

for some $C_1 = C_1(n, \Lambda_0) > 0$. Let $\{\lambda_i\}_{i=1,2,\ldots,n}$ be the principal curvatures of M_t at $(x, t) \in M_t$. Direct algebra gives

$$|\mathring{A}|^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2, \tag{3.22}$$

so there exists $C_3 = C_3(n, \Lambda_0) > 0$ such that for all $(x, t) \in M_t$, $t \in [0, t_1]$,

$$|\lambda_i(x, t) - \lambda_j(x, t)| \leq C_3 \epsilon^{\alpha}. \tag{3.23}$$

Therefore, for all $(x, t) \in M_t$, $t \in [0, t_1]$ and any fixed $i \in \{1, 2, \ldots, n\}$, we have

$$|H(x, t) - n\lambda_i(x)| \leq C_4 \epsilon^{\alpha}, \tag{3.24}$$

for some $C_4 = C_4(n, \Lambda_0) > 0$.

For some $C_5 = C_5(n, \Lambda_0) > 0$ which will be fixed shortly, suppose there is $\eta_0 = C_5\epsilon^{\frac{\alpha}{2}} > 0$ where $\epsilon \in (0, \epsilon_0)$ and some $(x_0, t_0) \in M_{t_0}$ where $t_0 \in [0, t_1]$ such that $H(x_0, t_0) < n - \eta_0$. Then, from (3.24), we have

$$n\lambda_i(x_0, t_0) - C_4\epsilon^\alpha \le H(x_0, t_0) < n - \eta_0 = n - C_5\epsilon^{\frac{\alpha}{2}}.$$

Since $\epsilon \in (0, \epsilon_0)$ is small, for properly chosen C_5 and $C_6 = C_6(n, \Lambda_0) > 0$, we have $\lambda_i(x_0, t_0) < 1 - C_6\epsilon^{\frac{\alpha}{2}}$ for all $i \in \{1, 2, \dots, n\}$. In light of $\max_{M_t} |\nabla H| \le C_1\epsilon^\alpha$, the smallness of ϵ and the diameter bound from Theorem 2.13, we have $H < n$ at every point of M_{t_0}. However, this contradicts the fact that any smooth closed hypersurface has at least one point whose mean curvature is greater than n in $\mathbb{H}^{n+1}$ by comparing with horospheres.

Similarly for some $C_5' = C_5'(n, \Lambda_0) > 0$ which will be fixed shortly, suppose there is some $\eta_0' = C_5'\epsilon^{\frac{\alpha}{2}} > 0$ where $\epsilon \in (0, \epsilon_0)$ and some $(x_0', t_0') \in M_{t_0'}$ such that $H(x_0', t_0') > n + \eta_0'$. We have

$$n\lambda_i(x_0', t_0') + C_4\epsilon^\alpha \ge H(x_0', t_0') > n + \eta_0' = n + C_5'\epsilon^{\frac{\alpha}{2}}.$$

Using again the smallness of ϵ, for properly chosen C_5' and C_6', we have $\lambda_i(x_0', t_0') > 1 + C_6'\epsilon^{\frac{\alpha}{2}}$ for any $i \in \{1, 2, \dots, n\}$. Using again the fact that $\max_{M_t} |\nabla H| \le C_1\epsilon^\alpha$, smallness of ϵ and the diameter bound from Theorem 2.13, we find $\lambda_i(x, t_0') > 1$ for all $i \in \{1, 2, \dots, n\}$ and all $(x, t_0') \in M_{t_0'}$. Namely, $M_{t_0'}$ is strictly *h-convex*. By the main theorem of [4], the VPMCF then exists for all time after $t = t_0'$, stays strictly *h-convex* and converges exponentially to an umbilical sphere in $\mathbb{H}^{n+1}$.

Finally, we are left with (3.21), which completes the proof. □

Remark 3.4 By Proposition 3.3, we can now assume H of M_t is very close to n on time interval $[0, t_1]$, namely the inequality (3.21), for the remaining proof for Theorem 1.2, and therefore, we now have $H > 0$ (hence, $h > 0$).

3.3 Step 3: Precise Decay

In the previous subsection, we have obtained estimates (3.3), (3.4) and (3.19) on some short time interval $[0, T_1]$, provided that the initial hypersurface is close to an umbilical sphere in the L^2 sense (see (1.4)) and h-mean convex (see (3.18)). These bounds will likely deteriorate along the flow if we iterate for later time intervals. For an iteration argument to work, we need to establish time-independent bound on these quantities for this short time interval.

In this subsection, we show that, if estimates similar to (3.3), (3.4) and (3.19) hold on some time interval $[0, T_1]$, then we can choose sufficiently small ϵ in the initial L^2-bound (1.4) on $\mathring{A}$, such that $|\mathring{A}|$, $|\nabla H|$ and $|h - H|$ exponentially decay on this time interval $[0, T_1]$. More precisely, we establish the following theorem.

Theorem 3.5 *Let $M_t^n \subset \mathbb{H}^{n+1}$ for $n \geq 2$ be a smooth closed solution to the VPMCF* (1.1) *with the initial condition*

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon.$$

Suppose for any $t \in [0, T_1]$ with $T_1 \leq \infty$ and all $m \in [1, n+3]$, we have

$$\max\left\{|M_0|^2, \max_{M_t} |A|^2, \int_{M_0} |\nabla^m A|^2 \, d\mu\right\} \leq \Lambda_1^2, \quad \min_{M_t}(H - n) \geq \sigma, \tag{3.25}$$

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) \leq C_1 \epsilon^\beta, \tag{3.26}$$

for constants $\Lambda_1 > 0, \sigma > 0, \beta \in (0, 1)$ and $C_1 > 0$. Then, there exists some $\epsilon_2 = \epsilon_2(n, \Lambda_0, \beta, C_1) > 0$ such that if $\epsilon < \epsilon_2$, then for all $t \in [0, T_1]$, we have

$$\max_{M_t} |\mathring{A}| \leq \max_{M_0} |\mathring{A}|, \tag{3.27}$$

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) \leq C_2(n, \Lambda_1, C_1) \left(\max_{M_0} |\mathring{A}|\right)^\alpha e^{-\alpha\sigma t}, \tag{3.28}$$

where $\alpha \in (0, 1)$ is the universal constant from Lemma 3.1.

Proof To start with, by Lemma 2.12 and (3.25), for $m \in [1, n+3]$ and $t \in [0, T_1]$, we have

$$\int_{M_t} |\nabla^m A|^2 \, d\mu \leq C(n, \Lambda_1),$$

which works as the replacement of (3.3) as in the proof of Lemma 3.1. Now, using (3.25), we compute

$$\begin{aligned} n - \frac{hH}{n} &= n - \frac{H \int_{M_t} H \, d\mu}{n \int_{M_t} d\mu} \\ &\leq n - \frac{(n+\sigma)^2}{n} \\ &< -2\sigma, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \left|\frac{1}{n} H^2 - \frac{hH}{n}\right|(x, t) &= \left|H(x, t) \cdot \frac{\int_{M_t} [H(x, t) - H(y, t)] \, d\mu(y)}{n \int_{M_t} d\mu}\right| \\ &\leq \frac{1}{n} \max_{M_t} H \cdot \operatorname{diam}(M_t) \cdot \max_{M_t} |\nabla H| \\ &\leq C(n, \Lambda_1, C_1) \epsilon^\beta, \end{aligned} \tag{3.30}$$

where we have used $|H| \leq \sqrt{n}|A| \leq \sqrt{n}\Lambda_1$ and Theorem 2.13.

Now, by (2.14), (3.29) and (3.30), we have

$$\begin{aligned}
\frac{\partial}{\partial t}|\mathring{A}|^2 &= \Delta|\mathring{A}|^2 - 2|\nabla\mathring{A}|^2 + 2|\mathring{A}|^2(|A|^2+n) - 2h\left\{\operatorname{tr}(\mathring{A}^3) + \frac{2}{n}|\mathring{A}|^2 H\right\} \\
&\leq \Delta|\mathring{A}|^2 + 2|\mathring{A}|^2\left(|\mathring{A}|^2 + \frac{1}{n}H^2 + n\right) + 2h\,|\mathring{A}|^3 - \frac{4hH}{n}|\mathring{A}|^2 \\
&= \Delta|\mathring{A}|^2 + 2\left(|\mathring{A}|^2 + h|\mathring{A}| + \frac{1}{n}H^2 + n - \frac{2hH}{n}\right)|\mathring{A}|^2 \\
&\leq \Delta|\mathring{A}|^2 - (4\sigma - \widehat{C}\epsilon^{\beta})|\mathring{A}|^2 \\
&\leq \Delta|\mathring{A}|^2 - \sigma|\mathring{A}|^2,
\end{aligned}$$

where $\widehat{C} = \widehat{C}(n, \Lambda_1, C_1) > 0$ and for the last step, we choose ϵ to be sufficiently small. Therefore, we conclude the exponential decay of $|\mathring{A}|$ from the maximum principle, i.e., Theorem 2.8,

$$\max_{M_t}|\mathring{A}|^2 \leq e^{-\sigma t}\max_{M_0}|\mathring{A}|^2,$$

and the estimate (3.27) also follows. This is where the h-mean convexity is essentially involved in our arguments, see (3.29). Afterwards, we can prove (3.28) by the exact arguments in the proof of Lemma 3.1, namely (3.12)–(3.17). □

3.4 Step 4: Time Extension

In this step, we use the exponential decay of $|\mathring{A}|$, $|\nabla H|$ and $|h - H|$ on some short time interval obtained in the previous step to extend the time interval of interest.

Theorem 3.6 *Let $M_t^n \subset \mathbb{H}^{n+1}$ for $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) with the initial hypersurface satisfying*

$$|M_0| \leq \Lambda_0, \quad \max_{M_0}|H| \leq \Lambda_0, \quad \int_{M_0}|\nabla^m A|^2\,d\mu \leq \Lambda_0^2, \quad \min_{M_0}(H-n) \geq \frac{1}{\Lambda_0^2} > 0$$

for all $m \in [1, n+3]$. Suppose for any $t \in [0, T]$ with $T < \infty$, we have

$$\max_{M_t}|A|^2 \leq \Lambda_0^2, \quad \min_{M_t}(H-n) \geq \frac{1}{2\Lambda_0^2} > 0 \tag{3.31}$$

and

$$\max_{M_t}\left(|\mathring{A}| + |\nabla H| + |h-H|\right) \leq C_*\epsilon^{\frac{\alpha^2}{2}}e^{-\alpha\sigma t} \leq C_*\epsilon^{\frac{\alpha^2}{2}}, \tag{3.32}$$

where $\alpha \in (0,1)$ is the universal constant from Lemma 3.1 and $\sigma = \frac{1}{2\Lambda_0^2}$ is as in Theorem 3.5. Then, there exist $\epsilon_3 = \epsilon_3(n, \Lambda_0, \alpha, C_*) > 0$ and $T_2 = T_2(n, \Lambda_0) > 0$

such that if

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon < \epsilon_3, \tag{3.33}$$

then (3.31) *and* (3.32) *hold for* $t \in [0, T + T_2]$.

Proof We begin by applying Lemmas 3.1 and 3.2 to obtain $\epsilon_4 = \epsilon_0(n, \Lambda_0^2)$ and $T_2 = T_1(n, \Lambda_0^2)$ such that if

$$\int_{M_0} |\mathring{A}|^2 \, d\mu \leq \epsilon < \epsilon_4,$$

then for all $t \in [T, T + T_2]$, we have

$$\max\left\{\max_{M_t} |A|^2, \int_{M_t} |\nabla^m A|^2 \, d\mu\right\} \leq 2\Lambda_0^2 \quad \text{and} \quad \min_{M_t}(H - n) \geq \frac{1}{4\Lambda_0^2}, \tag{3.34}$$

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) \leq C_1(n, \Lambda_0)\epsilon^{\alpha}, \tag{3.35}$$

where C_1 and α are from Lemma 3.1. Then, choose $\epsilon_5 = \epsilon_5(n, \Lambda_0, \alpha, C_*) > 0$ sufficiently small so that for any $\epsilon < \epsilon_5$, we have

$$C_1(n, \Lambda_0)\epsilon^{\alpha - \frac{\alpha^2}{2}} \leq C_*.$$

Therefore, for all $t \in [0, T + T_2]$, we have (3.34) and also

$$\max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) \leq C_* \epsilon^{\frac{\alpha^2}{2}}. \tag{3.36}$$

By Corollary 2.2, the surface area $|M_t|$ is non-increasing along the flow, therefore, $|M_t| \leq \Lambda_0 < \Lambda_0^2$ by the initial condition (1.3) as long as the flow exists, in particular, on $[0, T + T_2]$. Now, we apply the Theorem 3.5 on $[0, T + T_2]$ with $\Lambda_1^2 = 2\Lambda_0^2$, $C_1 = C_*$, $\beta = \frac{\alpha^2}{2}$ and $\sigma = \frac{1}{4\Lambda_0^2}$ to conclude that for some $\epsilon_6 := \epsilon_2(n, \Lambda_0, \alpha, C_*) > 0$ sufficiently small, if $\epsilon < \epsilon_6$, then for all $t \in [0, T + T_2]$, we have

$$\begin{aligned} \max_{M_t} \left(|\mathring{A}| + |\nabla H| + |h - H|\right) &\leq C_2(n, \Lambda_0, C_*)(\max_{M_0} |\mathring{A}|)^{\alpha} e^{-\alpha\sigma t} \\ &\leq C_2(n, \Lambda_0, C_*)[C_1(n, \Lambda_0)\epsilon^{\alpha}]^{\alpha} e^{-\alpha\sigma t}, \end{aligned} \tag{3.37}$$

where we have used (3.4) at $t = 0$. Now, choose $\epsilon_7 = \epsilon_7(n, \Lambda_0, \alpha, C_*) > 0$ small enough so that

$$C_2(n, 2\Lambda_0^2, C_*)[C_1(n, \Lambda_0)]^{\alpha} \epsilon^{\frac{\alpha^2}{2}} \leq C_*, \tag{3.38}$$

thus (3.32) holds for all $t \in [0, T + T_2]$.

We are left to show (3.31) for $t \in [0, T + T_2]$. Let us examine each term in (3.31). Consider $\max_{M_t} |A|$. Recall the time derivative formula for $h(t)$ (2.13) is given by

$$h'(t) = \frac{\int_{M_t} (H - h)(|A|^2 - H^2 + hH) d\mu}{\int_{M_t} d\mu}.$$

Then, using (3.34) and (3.35), we have

$$|h'(t)| \le C_3(n, C_*, \Lambda_0)\epsilon^{\frac{\alpha^2}{2}} e^{-\alpha\sigma t}$$

for all $t \in [0, T + T_2]$. Note that, from the initial condition (1.3), we also have

$$h(0) = \frac{\int_{M_0} H \, d\mu}{\int_{M_0} d\mu} \le \max_{M_0} |H| \le \Lambda_0.$$

By choosing $\epsilon < \epsilon_8 = \epsilon_8(n, \Lambda_0, \alpha, C_*)$ sufficiently small, we then have for any $t \in [0, T + T_2]$:

$$|h(t)| \le \frac{6}{5}\Lambda_0. \tag{3.39}$$

Then, by (3.35) and $n \ge 2$, for sufficiently large Λ_0, we have

$$\max_{M_t} |A| = \max_{M_t} \sqrt{|\mathring{A}|^2 + \frac{1}{n}H^2} \le \max_{M_t} \left(|\mathring{A}| + \frac{1}{\sqrt{n}}|H - h|\right) + \frac{1}{\sqrt{n}}|h(t)| \le \Lambda_0. \tag{3.40}$$

Finally, we consider the term $\min_{M_t}(H - n)$. Using the evolution equations for H (see (2.5)) and $d\mu$ (see (2.3)), we have

$$\begin{aligned}\int_{M_t} H \, d\mu - \int_{M_0} H \, d\mu &= \int_0^t \int_{M_s} H^2(h - H) + (H - h)(|A|^2 - n) \, d\mu \, ds \\ &\ge -C(n, \Lambda_0, C_*)\epsilon^{\frac{\alpha^2}{2}} \int_0^t e^{-\alpha\sigma s} \, ds \\ &\ge -C_4(n, \Lambda_0, \alpha, C_*)\epsilon^{\frac{\alpha^2}{2}},\end{aligned}$$

where we have used again the bound on $|h - H|$ in (3.32) for $t \in [0, T + T_2]$. Therefore,

$$\int_{M_t} H \, d\mu \ge \left(n + \frac{1}{\Lambda_0^2}\right)|M_0| - C_4(n, \Lambda_0, \alpha, C_*)\epsilon^{\frac{\alpha^2}{2}} \ge \left(n + \frac{2}{3\Lambda_0^2}\right)|M_0|, \tag{3.41}$$

where we have chosen $\epsilon < \epsilon_{10} = \epsilon_{10}(n, \Lambda_0, \alpha, C_*)$ sufficiently small and used the initial condition $\min_{M_0}(H - n) \ge \frac{1}{\Lambda_0^2}$.

Now, applying the bound on $|\nabla H|$ in (3.32) which holds for all $t \in [0, T + T_2]$, we conclude from (3.41) and $|M_t| \leq |M_0|$ that if $\epsilon < \epsilon_{11} = \epsilon_{11}(n, \Lambda_0, \alpha, C_*)$ is chosen sufficiently small, then for all $t \in [0, T + T_2]$, we have

$$\min_{M_t}(H - n) \geq \frac{1}{2\Lambda_0^2}.$$

Choosing $\epsilon_3 = \min\{\epsilon_4, \ldots, \epsilon_{11}\} > 0$, we conclude the proof of the theorem. □

Now, we conclude the proof of our main theorem.

Proof of Theorem 1.2 In light of Lemma 3.1, Lemma 3.2 and Theorem 3.5, by choosing Λ_0 sufficiently large, we are in position to apply Theorem 3.6. Thus, we can keep extending the VPMCF and estimates (3.31) and (3.32) for a fixed amount of time depending only on the initial condition. Hence, the flow (1.1) exists for all time and converges exponentially to a closed umbilic hypersurface in $\mathbb{H}^{n+1}$ by (3.32), i.e., an umbilical sphere [24]. □

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Appendix

In this appendix, we provide the global Shi's type of estimates for VPMCF, i.e., the higher order estimates for the second fundamental form away from the initial time depending only on C^0-bound on the initial hypersurface. This is done for the cases of the ambient space being Euclidean or hyperbolic, which are often needed for the earlier consideration of the stability problem in previous approaches. Similar arguments yield similar estimates for some other nonlocal flows such as the surface area preserving mean curvature flows.

In the following, the positive constant C depends on the dimension n and the initial C^0-bound of A and might vary at places by abusing the notations. For the global term h, it is useful to notice $|h| \leq C \max |A|$.

For the ambient space being Euclidean, the computations in [17] give

$$\left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 \leq -2|\nabla A|^2 + 2|A|^4 + \max |A|$$

and so for $t \in [0, \delta_0]$ with some small $\delta_0 > 0$, we have $|A| \leq C$.

We stay in the time interval $[0, \delta_0]$, and the calculations in [17] also give

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla A|^2 \leq -2|\nabla^2 A|^2 + C|\nabla A|^2,$$

and so

$$\left(\frac{\partial}{\partial t} - \Delta\right) (t|\nabla A|^2) \leq -2t|\nabla^2 A|^2 + Ct|\nabla A|^2 + |\nabla A|^2.$$

Together with

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 \leq -2|\nabla A|^2 + C,$$

which makes use of the bound on $|A|$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\big(t|\nabla A|^2 + |A|^2\big) \leq C + C\big(t|\nabla A|^2 + |\nabla A|^2\big),$$

which gives

$$t|\nabla A|^2 \leq C.$$

Let us pick some small $\epsilon_1 \in (0, \delta_0)$ as the new initial time. Therefore, we have uniform both bounds for $|A|$ and $|\nabla A|$ for the new time $t \in [0, \delta_1]$ which corresponds to $[\epsilon_1, \delta_1 + \epsilon_1]$ for the original time.

Now, we proceed by induction, assuming for $t \in [0, \delta_m]$ (for the original time $[\epsilon_m, \delta_m + \epsilon_m]$),

$$|\nabla^k A| \leq C$$

for $k = 0, \dots, m$ for $m \in \mathbb{N}$. Now, we have from calculations in [17],

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^{m+1} A|^2 \leq -2|\nabla^{m+2} A|^2 + C + C|\nabla^{m+1} A|^2$$

and so

$$\left(\frac{\partial}{\partial t} - \Delta\right)\big(t|\nabla^{m+1} A|^2\big) \leq -2t|\nabla^{m+2} A|^2 + Ct + Ct|\nabla^{m+1} A|^2 + |\nabla^{m+1} A|^2.$$

By the induction assumption,

$$\left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + C.$$

Thus, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\big(t|\nabla^{m+1} A|^2 + |\nabla^m A|^2\big) \leq C + C\big(t|\nabla^{m+1} A|^2 + |\nabla^m A|^2\big),$$

which gives

$$t|\nabla^{m+1} A|^2 \leq C$$

and we conclude the induction step.

For any sufficiently small $\epsilon > 0$ and some small $\delta > 0$, we can choose ϵ_m and δ_m for $0 < \epsilon_m < \epsilon < \delta < \epsilon_m + \delta_m$ for all m's which gives bound for all higher orders in the origina time $[\epsilon, \delta]$.

For the ambient space be hyperbolic, the calculations in our Sect. 2 give

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 \leq -2|\nabla A|^2 + C \max |A| \cdot \left(|A| + |A|^3\right)$$

and so for small time $t \in [0, \delta_0]$,

$$|A| \leq C.$$

The rest of the argument is the same as above in light of Corollary 2.4, and we conclude

$$|\nabla^m A| \leq C$$

for $m = 0, 1, \ldots$ for $t \in (\epsilon, \delta)$ for any sufficiently small $\epsilon > 0$ and some small $\delta > 0$. We summarize the estimates in the following theorem.

Theorem A.1 *For the VPMCF of closed hypersurface in either the Euclidean or hyperbolic space, if the initial second fundamental form A is bounded, i.e., $|A| \leq \Lambda$ for some $\Lambda > 0$, then there exists $\delta(\Lambda) > 0$ such that for any sufficiently small $\epsilon > 0$, we have in $[\epsilon, \delta(\Lambda)]$ for $m = 0, 1, \ldots$*

$$|\nabla^m A| \leq C(\Lambda, m, \epsilon).$$

Remark A.2 One certainly expects such result to hold for more general ambient space with explicit bounds away from the initial time. We choose this form to make use of clean classical calculations, which is sufficient for earlier studies of the VPMCF as mentioned in Remark 1.3.

References

1. Andrews, B., Chen, X., Wei, Y.: Volume preserving flow and Alexandrov–Fenchel type inequalities in hyperbolic space. J. Eur. Math. Soc. **23**(7), 2467–2509 (2021)
2. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer Monographs in Mathematics. Springer, Berlin (1998)
3. Bertini, M., Pipoli, G.: Volume preserving non-homogeneous mean curvature flow in hyperbolic space. Differ. Geom. Appl. **54**(part B), 448–463 (2017)
4. Cabezas-Rivas, E., Miquel, V.: Volume preserving mean curvature flow in the hyperbolic space. Indiana Univ. Math. J. **56**(5), 2061–2086 (2007)
5. Cabezas-Rivas, E., Miquel, V.: Volume preserving mean curvature flow of revolution hypersurfaces between two equidistants. Calc. Var. Partial Differ. Equ. **43**(1–2), 185–210 (2012)
6. Chow, B., Peng, L., Ni, L.: Hamilton's Ricci Flow. Graduate Studies in Mathematics, vol. 77. American Mathematical Society, Providence (2006)
7. De Lellis, C., Müller, S.: Optimal rigidity estimates for nearly umbilical surfaces. J. Differ. Geom. **69**(1), 75–110 (2005)
8. Eichmair, M., Metzger, J.: On large volume preserving stable CMC surfaces in initial data sets. J. Differ. Geom. **91**(1), 81–102 (2012)

9. Escher, J., Simonett, G.: The volume preserving mean curvature flow near spheres. Proc. Am. Math. Soc. **126**(9), 2789–2796 (1998)
10. Guo, S., Li, G., Wu, C.: Volume-preserving flow by powers of the m-th mean curvature in the hyperbolic space. Commun. Anal. Geom. **25**(2), 321–372 (2017)
11. Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**(2), 255–306 (1982)
12. Hoffman, D., Spruck, J.: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Commun. Pure Appl. Math. **27**, 715–727 (1974)
13. Hu, Y., Li, H., Wei, Y., Zhou, T.: Contraction of surfaces in hyperbolic space and in sphere. Calc. Var. Partial Differ. Equ. **59**(5), 172 (2020)
14. Huang, Z., Lin, L.: Stability of the surface area preserving mean curvature flow in Euclidean space. J. Geol. **106**(3), 483–501 (2015)
15. Huisken, G.: Flow by mean curvature of convex surfaces into spheres. J. Differ. Geom. **20**(1), 237–266 (1984)
16. Huisken, G.: Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. Invent. Math. **84**(3), 463–480 (1986)
17. Huisken, G.: The volume preserving mean curvature flow. J. Reine Angew. Math. **382**, 35–48 (1987)
18. Huisken, G., Sinestrari, C.: Mean curvature flow singularities for mean convex surfaces. Calc. Var. Partial Differ. Equ. **8**(1), 1–14 (1999)
19. Huisken, G., Yau, S.-T.: Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. Invent. Math. **124**(1–3), 281–311 (1996)
20. Li, H.: The volume-preserving mean curvature flow in Euclidean space. Pac. J. Math. **243**(2), 331–355 (2009)
21. McCoy, J.: The surface area preserving mean curvature flow. Asian J. Math. **7**(1), 7–30 (2003)
22. McCoy, J.: The mixed volume preserving mean curvature flow. Math. Z. **246**(1–2), 155–166 (2004)
23. Michael, J.H., Simon, L.M.: Sobolev and mean-value inequalities on generalized submanifolds of R^n. Commun. Pure Appl. Math. **26**, 361–379 (1973)
24. Spivak, M.: A Comprehensive Introduction to Differential Geometry, vol. IV, 2nd edn. Publish or Perish, Wilmington (1979)
25. Topping, P.: Relating diameter and mean curvature for submanifolds of Euclidean space. Comment. Math. Helv. **83**(3), 539–546 (2008)
26. Wang, G., Xia, C.: Isoperimetric type problems and Alexandrov–Fenchel type inequalities in the hyperbolic space. Adv. Math. **259**, 532–556 (2014)
27. Wu, J.-Y., Zheng, Y.: Relating diameter and mean curvature for Riemannian submanifolds. Proc. Am. Math. Soc. **139**(11), 4097–4104 (2011)
28. Xu, H., Leng, Y., Zhao, E.: Volume-preserving mean curvature flow of hypersurfaces in space forms. Int. J. Math. **25**(3), 1450021 (2014)
29. Ye, R.: Ricci flow, Einstein metrics and space forms. Trans. Am. Math. Soc. **338**(2), 871–896 (1993)