The Quarterly Journal of Mathematics, 2024, **00**, 1–20
doi:https://doi.org/10.1093/qmath/haae025
Article

OXFORD

PLECTIC JACOBIANS

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ABSTRACT

Looking for a geometric framework to study plectic Heegner points, we define a collection of abelian varieties – called *plectic Jacobians*—using the middle-degree cohomology of quaternionic Shimura varieties (QSVs). The construction is inspired by the definition of Griffiths' intermediate Jacobians and rests on Nekovář–Scholl's notion of plectic Hodge structures. Moreover, we construct exotic Abel–Jacobi maps sending certain zero cycles on QSVs to plectic Jacobians.

1. INTRODUCTION

For a long time number theorists have been looking for suitable generalizations of Heegner points to tackle the BSD conjecture for elliptic curves of rank greater than 1. Motivated by that problem, a conjectural construction of determinants of global points was recently proposed [10, 9] combining Darmon's pioneering work [3] with the powerful insights of Nekovář–Scholl's plectic conjectures [22]. These *plectic Stark–Heegner (PSH) points* are constructed using p-adic integration, and their peculiar appearance is motivated by the uniformization of quaternionic Shimura varieties (QSVs) by certain p-adic symmetric domains. To explain how PSH points should arise from global points, precise conjectures were formulated in [9, Conjectures 1.3 and 1.5]. In a nutshell, they claim that given an elliptic curve of algebraic rank r, a PSH point constructed using r different p-adic places is in the image of the top exterior power of the Mordell–Weil group via a p-adic determinant map. Notably, for elliptic curves of rank 1 those conjectures recover the expectation that classical Stark–Heegner points are images of global points under p-adic localization.

Those aforementioned conjectures were substantiated by numerical and theoretical evidence. On the computational side, in [10] they were verified (up to precision) for several elliptic curves of rank 2 defined over $\mathbb{Q}(\sqrt{13})$ and $\mathbb{Q}(\sqrt{37})$. On the theoretical side, instances of the conjectures were proved in the setting of polyquadratic CM extensions [8], leveraging higher p-adic Gross–Zagier formulas for anticyclotomic p-adic L-functions ([9], Theorem A). Moreover, it is reasonable to expect that the work of Molina–Hernandez [16] will help in clarifying the connection between PSH points and generalized Kato classes [5].

As is the case for Darmon's Stark–Heegner points, one cannot usually guarantee that PSH points arise from global points because of their inherently analytic construction. More than 20 years after the introduction of Stark–Heegner points, our understanding of their conjectural global properties remains quite unsatisfactory in general. There is a notable exception: for CM extensions Darmon's

Received 11 December 2023; **Revised** 13 March 2024

points recover classical Heegner points, whose global features have long been understood using Jacobian varieties and the theory of complex multiplication. One of the appealing traits of PSH points is that they are already interesting and new for CM extensions. Thus, from now on, we will refer to PSH points for CM extensions as *plectic Heegner points*, and we will try to shed some light on their attributes using Nekovář–Scholl's *plectic Hodge theory*.

1.1. Main results

Nekovář and Scholl observed [**22**, **23**] that Hodge structures of Hilbert modular varieties carry more information than those of general Kähler manifolds. In particular, they showed the existence of a Künneth-like structure that reflects the canonical decomposition of the tangent bundle of Hilbert modular varieties.

Definition 1.1 Let $n \geq 1$ be an integer. An n-plectic Hodge structure on a finite free $\mathbb{Z}$-module H consists of a decomposition

$$H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{\alpha,\beta\in\mathbb{Z}^n} H^{\alpha,\beta} \quad \text{such that} \quad H^{\alpha,\beta} = \overline{H^{\beta,\alpha}}.$$

Remark 1.2 There is a natural procedure that produces a Hodge structure from the data of an n-plectic Hodge structure. Given $\alpha = (\alpha_j)_{j=1}^n \in \mathbb{Z}^n$ set $|\alpha| = \sum_{j=1}^n \alpha_j$, and let H be an n-plectic Hodge structure. The classical Hodge structure arising from H is defined by setting

$$H^{p,q} := \bigoplus_{|\alpha|=p,|\beta|=q} H^{\alpha,\beta} \qquad \forall\, p,q \in \mathbb{Z}.$$

In this case we say that the n-plectic Hodge structure refines the associated Hodge structure.

Our first main theorem shows that plectic Hodge structures arise in the cohomology of compact rigidified Kähler manifolds, that is, compact complex manifolds endowed with certain foliations and compatible Kähler metrics (see Definitions 2.2 and 3.3 and Corollary 3.8).

Theorem A *Let X be an n-dimensional compact rigidified Kähler manifold. The cohomology of X is endowed with a canonical n-plectic Hodge structure refining its classical Hodge structure.*

We note here that the compatibility conditions between the foliation and the Kähler metric are singled out to ensure that the Laplacian operator associated with the Kähler metric respects the decomposition of harmonic differential forms induced by the foliation.

1.1.1. Plectic Jacobians and exotic Abel–Jacobi maps

Our work on PSH points was inspired by Nekovář and Scholl's belief that CM points on higher dimensional QSVs could be used to study the arithmetic of elliptic curves of higher rank. While previous articles leveraged p-adic techniques, this paper begins to develop an Archimedean framework to study plectic Heegner points following Oda's trailblazing work on periods of Hilbert modular varieties ([**24**, **25**]). The aim is to understand a form of geometric modularity where Jacobians of Shimura curves are replaced by *plectic Jacobians* of higher-dimensional QSVs. As the Jacobian of a curve C can be constructed from the weight 1 Hodge structure $\mathrm{H}^1(C,\mathbb{Z})$, so plectic Jacobians of a QSV X are defined using the plectic Hodge structure appearing in the middle-degree cohomology group $\mathrm{H}^{\dim X}(X,\mathbb{Z})$.

Definition 1.3 An n-plectic Hodge structure H is *effective* and has *weight* $\underline{1} = (1,..,1) \in \mathbb{Z}^n$ if

$$H^{\alpha,\beta} \neq 0 \quad\Longrightarrow\quad \alpha,\beta \in \mathbb{N}^n \quad \& \quad \alpha+\beta = \underline{1}.$$

An effective n-plectic Hodge structure of weight $\underline{1} \in \mathbb{Z}^n$ can be thought of as a collection of n effective Hodge structures of weight 1 on the same underlying module by setting

$$F^{1_j} = F^{1_j}(H) := \bigoplus_{\alpha_j \geq 1} H^{\alpha,\beta} \qquad \text{for any } j = 1,\dots,n.$$

It is then natural to make the following definition.

Definition 1.4 Let H be an effective n-plectic Hodge structure of weight $\underline{1} \in \mathbb{Z}^n$. For any $j = 1,\dots,n$ the plectic Jacobian $\mathrm{J}_{\infty}(H,j)$ associated with H is the complex torus defined by

$$\mathrm{J}_{\infty}(H,j) := H \backslash (H \otimes_{\mathbb{Z}} \mathbb{C})/F^{1_j}.$$

Systems of Hecke eigenvalues of modular elliptic curves appear in the cohomology of QSVs only in middle degree, and the cuspidal part of those middle-degree cohomology groups can be shown to carry a canonical effective plectic Hodge structure of weight $\underline{1}$ (Lemma 5.1). Therefore, an r-dimensional compact QSV X determines r plectic Jacobians $\{\mathrm{J}_{\infty}(X,j)\}_{j=1}^{r}$ which are abelian varieties (Proposition 5.4) and conjecturally are closely related to modular elliptic curves (Conjectures 5.5 & 5.6).

Remark 1.5 When the QSV X has odd dimension r, all middle-degree cohomology classes are cuspidal. Thus, the real torus

$$\mathrm{H}^r(X,\mathbb{R})/\mathrm{H}^r(X,\mathbb{Z})$$

can be endowed with several complex structures: those arising from our definitions and those considered by Weil ([**29**] and [**17**], Section 3) and Griffiths ([**15**], Section 3). However, while Weil's and Griffiths' definitions of intermediate Jacobians exclude even cohomological degrees, our definition applies unchanged to the middle-degree cohomology of even dimensional QSVs.

To add details to our discussion, let us consider a totally real number field F of narrow class number one, and a non-split quaternion algebra B/F with $\Sigma := \{\nu_1,\dots,\nu_r\}$ as its set of split Archimedean places. Recall that a QSV X_B, associated with B/F and a choice of Eichler order, has a canonical model over the reflex field $\mathbb{Q}(\sum_{j=1}^{r}\nu_j(x)|x \in F) \subset \mathbb{C}$. The following conjecture aims at elucidating the relations between the plectic Jacobians of X_B.

Conjecture 1.6. *There is an abelian variety $\mathrm{J}_{\infty}(X_B)$ defined over F and canonical isomorphisms*

$$(\mathrm{J}_{\infty}(X_B) \otimes_{F,\nu_j} \mathbb{C})^{\mathrm{an}} \cong \mathrm{J}_{\infty}(X_B,j) \qquad \forall\, j = 1,\dots,r.$$

Remark 1.7 Conjecture 1.6 is well-known when $r = 1$, that is, whenever X_B is a Shimura curve, while it is wide open for all $r \geq 2$.

Griffiths' style definition of plectic Jacobians allows us to define an exotic Abel–Jacobi map with a subgroup of zero cycles which we now describe. We begin by recalling the complex uniformization of a QSV. Let us fix an isomorphism $\iota_j\colon B \otimes_{F,\nu_j} \mathbb{R} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{R})$ for every $\nu_j \in \Sigma$, then, if the level of the Eichler order is large enough, there is a torsion-free arithmetic subgroup $\Gamma \leq B^\times/F^\times$ such that

$$X_B = \Gamma \backslash \mathcal{H}_\Sigma$$

is a complex manifold where Γ acts on the product $\mathcal{H}_\Sigma = \prod_{\nu_j \in \Sigma} \mathcal{H}_{\nu_j}$ of Poincaré's upper-half planes via Möbius transformations. For technical reasons (see Equation (14)) our exotic Abel-Jacobi map is only defined for zero cycles supported at 'generic' points: denoting by $\mathcal{H}^\circ_{\nu_j} \subseteq \mathcal{H}_{\nu_j}$ the subset of those points with trivial stabilizer in $\iota_j(\Gamma) \le \mathrm{PGL}_2(\mathbb{R})$, we can define

$$\mathcal{H}^\circ_\Sigma := \prod_{\nu_j \in \Sigma} \mathcal{H}^\circ_{\nu_j} \qquad \text{and} \qquad X^\circ_B := \Gamma \backslash \mathcal{H}^\circ_\Sigma.$$

Note that this is not a serious restriction for arithmetic applications since the set X°_B contains all CM points. The free group $\mathbb{Z}[\mathcal{H}^\circ_\Sigma]$ of the product $\mathcal{H}^\circ_\Sigma$ is canonically isomorphic to $\otimes_{j=1}^r \mathrm{Div}(\mathcal{H}^\circ_{\nu_j})$ by mapping generators $[(\tau_1,\dots,\tau_r)]$ to elementary tensors $\otimes_{j=1}^r [\tau_j]$. If we denote by $\mathrm{Div}^0(\mathcal{H}^\circ_{\nu_j})$ the subgroup of degree-zero elements of $\mathrm{Div}(\mathcal{H}^\circ_{\nu_j})$, we can define *plectic zero cycles* supported on X°_B by setting

$$\mathbb{Z}_\infty[X^\circ_B] := \mathrm{Im}\Big(\otimes_{j=1}^r \mathrm{Div}^0(\mathcal{H}^\circ_{\nu_j}) \to \mathbb{Z}[X^\circ_B] \Big).$$

Following Darmon–Logan [4] we consider the homomorphism

$$\int^r : \otimes_{j=1}^r \mathrm{Div}^0(\mathcal{H}^\circ_{\nu_j}) \longrightarrow \mathrm{H}^r_{\mathrm{dR}}(X_B)^\vee, \qquad \otimes_{j=1}^r([x_j]-[y_j]) \mapsto \int_{y_1}^{x_1} \cdots \int_{y_r}^{x_r} (-)$$

mapping an elementary tensor to the linear functional computing a series of line integrals. We are now ready to state our second main theorem, a first step towards an Archimedean construction of plectic Heegner points.

Theorem B *The homomorphism $\int^r$ induces a well-defined Abel–Jacobi map*

$$\mathrm{AJ}^j_\infty \colon \mathbb{Z}_\infty[X^\circ_B] \longrightarrow \mathrm{J}_\infty(X_B, j) \qquad \forall\, j = 1,\dots,r.$$

In future work, we plan to perform numerical experiments to understand the feasibility of enlarging the domain of our exotic Abel–Jacobi maps to contain canonically defined zero cycles supported on CM points.

2. RIGIDIFIED COMPLEX MANIFOLDS

Definition 2.1 Let $U, V \subseteq \mathbb{C}^n$ be open subsets. We say that a holomorphic function $\phi\colon U \to V$ is *rigid* if there exist holomorphic functions $\{\phi^j\}_{j=1}^n$ in one variable such that

$$\phi(u_1,\dots,u_n) = (\phi^1(u_1),\dots,\phi^n(u_n)).$$

Let X be a Hausdorff topological space. An n-dimensional chart (U,ϕ) in X consists of an open subset $U \subseteq X$ and an homeomorphism $\phi\colon U \to D$ onto an open subset $D \subseteq \mathbb{C}^n$. We say that two charts $(U,\phi),(V,\psi)$ are *compatible* if the transition function

$$\phi \circ \psi^{-1} \colon \psi(U\cap V) \to \phi(U \cap V)$$

and its inverse are both rigid. A covering of X consisting of pairwise compatible n-dimensional charts is called an n-dimensional *rigidified atlas* of X. Moreover, two such atlases $\mathscr{A}_1, \mathscr{A}_2$ are called *equivalent* if any two charts $(U,\phi) \in \mathscr{A}_1$ and $(V,\psi) \in \mathscr{A}_2$ are compatible. Finally, an equivalence class of n-dimensional rigidified atlases on X is called an n-dimensional *rigidified holomorphic structure* on X. It contains a maximal atlas which is the union of the atlases in the equivalence class.

Definition 2.2 An n-dimensional *rigidified complex manifold* consists of a Hausdorff space X with a countable basis, equipped with an n-dimensional rigidified holomorphic structure.

2.1 Examples

Any open subset $\Omega \subseteq \mathbb{C}^n$ has a natural structure of rigidified complex manifold given by the atlas

$$\mathscr{A} = \{(U, \mathrm{id}_U) \mid U \text{ open subset of } \Omega\}.$$

The class of examples most relevant for our arithmetic applications consists of quotients $\Gamma\backslash\Omega$ of an open subset $\Omega \subseteq \mathbb{C}^n$ by a discrete group Γ.

LEMMA 2.3 *Let Γ be a discrete group acting on a connected open subset $\Omega \subseteq \mathbb{C}^n$. Suppose*

- *Γ acts smoothly, freely and properly on Ω,*
- *there exists a homomorphism $\Gamma \to \mathrm{GL}_2(\mathbb{C})^n$, $\gamma \mapsto (\gamma_1, \dots, \gamma_n)$, such that*

$$\gamma \cdot (x_1, \dots, x_n) = (\gamma_1(x_1), \dots, \gamma_n(x_n)) \qquad \forall \gamma \in \Gamma,$$

then $\Gamma\backslash\Omega$ has a structure of rigidified complex manifold.

Proof. Let $\pi : \Omega \to \Gamma\backslash\Omega$ be the quotient map and $\mathscr{A}$ an atlas in the canonical rigidified holomorphic structure of Ω. We define a rigidified atlas $\mathscr{A}_\Gamma$ for $\Gamma\backslash\Omega$ as the collection of all pairs $(\pi(U), \pi_{|U}^{-1})$ such that (U, id_U) belongs to $\mathscr{A}$ and $\pi_{|U} : U \to \pi(U)$ is injective. First, as Γ acts smoothly, freely and properly on Ω the quotient $\Gamma\backslash\Omega$ is a complex manifold. Then, $\mathscr{A}_\Gamma$ is a rigidified atlas because its transition functions are given by the action of elements of the group, since Γ is discrete. □

REMARK 2.4 Lemma 2.3 shows that complex tori and QSVs are natural examples of rigidified complex manifolds. Moreover, we note here that the notion of rigidified complex manifolds could be generalized to include symplectic and unitary Shimura varieties over totally real number fields.

2.2. Foliations

In this section we explain why the tangent bundle of a rigidified complex manifold admits a natural decomposition. For readers interested in the relation between split tangent bundles and product structures of the universal covering space, we refer to the articles [1, 6].

Let X be an n-dimensional rigidified complex manifold. For any index $j = 1, \dots, n$ we define the j-th sub-vector bundle T_X^j of the tangent bundle T_X of X as follows. Let $\mathscr{U} = \{U_k\}_k$ be an open covering of X, and set $U_{k,\ell} := U_k \cap U_\ell$. Then the rank 2 real vector bundle T_X^j is covered by open sets $\{U_k \times \mathbb{R}^2\}_k$ and the transition morphism between

$$U_{k,\ell} \times \mathbb{R}^2 \subseteq U_k \times \mathbb{R}^2 \qquad \text{and} \qquad U_{k,\ell} \times \mathbb{R}^2 \subseteq U_\ell \times \mathbb{R}^2$$

is given by $(u, v) \mapsto (u, d\phi_{ik}^j(u)(v))$, where $\phi_{k,\ell} = (\phi_{k,\ell}^1, \dots, \phi_{k,\ell}^n)$ is the rigid transition map between $\phi_k(U_{k,\ell}) \subseteq \mathbb{C}^n$ and $\phi_\ell(U_{k,\ell}) \subseteq \mathbb{C}^n$ and $d\phi_{k,\ell}^j(u) \colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the Jacobian matrix of $\phi_{k,\ell}^j$ at point u. We note that there is a direct sum decomposition of the tangent bundle

$$T_X = \bigoplus_{j=1}^n T_X^j.$$

Since X is a complex manifold, each vector bundle T_X^j is equipped with an almost complex structure I_j. Therefore, there is a decomposition of the extension of scalars

$$T_X^j \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1_j,0_j} \oplus T_X^{0_j,1_j},$$

where $T_X^{1_j,0_j}$ (resp. $T_X^{0_j,1_j}$) is the sub-bundle of $T_X^j \otimes_{\mathbb{R}} \mathbb{C}$ on which the involution I_j acts with eigenvalue i (resp. $-i$).

Remark 2.5 Let $(z_1, \ldots, z_n)$ be local complex coordinates on an open subset $U \subseteq X$ trivializing T_X^j, and denote by $\mathscr{C}^\infty(U)$ the ring of smooth $\mathbb{C}$-valued functions on U. If we write $z_j = x_j + iy_j$, then we can explicitly describe smooth sections as

$$T_X^j \otimes_{\mathbb{R}} \mathbb{C}(U) = \mathscr{C}^\infty(U) \cdot \frac{\partial}{\partial x_j} \oplus \mathscr{C}^\infty(U) \cdot \frac{\partial}{\partial y_j}.$$

Moreover, the modules of smooth sections of $T_X^{1_j,0_j}$ and $T_X^{0_j,1_j}$ over U are free of rank one over $\mathscr{C}^\infty(U)$ with respective basis elements

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

2.2.1. *Refined types of differential forms*

The classical Hodge decomposition of differential form on complex manifolds uses the factorization of the exterior differential d_X into a holomorphic and an anti-holomorphic component $\mathrm{d}_X = \partial_X + \overline{\partial}_X$. The decomposition of the tangent bundle of rigidified complex manifolds further refines the types of differential forms and a fortiori the factorization of the exterior differential.

Definition 2.6 Let X be an n-dimensional rigidified complex manifold. For $j \in \{1, \ldots, n\}$ set

$$\mathcal{A}_X^{1_j,0_j} := \mathrm{Hom}_{\mathbb{C}}\left(T_X^{1_j,0_j}, \mathbb{C}\right) \qquad \text{and} \qquad \mathcal{A}_X^{0_j,1_j} := \mathrm{Hom}_{\mathbb{C}}\left(T_X^{0_j,1_j}, \mathbb{C}\right).$$

Then, for an ordered pair (α, β) of elements in $\{0,1\}^n$, we define the vector bundle of $\mathbb{C}$-valued smooth differential forms of type (α, β) by

$$\mathcal{A}_X^{\alpha,\beta} := \bigotimes_{\alpha_j=1} \mathcal{A}_X^{1_j,0_j} \otimes \bigotimes_{\beta_j=1} \mathcal{A}_X^{0_j,1_j}. \tag{1}$$

Remark 2.7 Exterior powers of smooth differential forms $\mathcal{A}_X := \mathrm{Hom}_{\mathbb{C}}(T_X, \mathbb{C})$ admit direct sum decomposition of the form

$$\wedge^k \mathcal{A}_X = \bigoplus_{|\alpha+\beta|=k} \mathcal{A}_X^{\alpha,\beta}.$$

Let $j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{0,1\}^n$. If $\alpha_j = 0$ there is a differential operator $\xi_j : \mathcal{A}_X^{\alpha,\beta} \to \mathcal{A}_X^{\alpha+1_j,\beta}$ defined by the diagram

$$\begin{array}{ccc} \mathcal{A}_X^{\alpha,\beta} & \overset{\xi_j}{\dashrightarrow} & \mathcal{A}_X^{\alpha+1_j,\beta} \\ \big\downarrow & & \big\uparrow \\ \wedge^{|\alpha+\beta|}\mathcal{A}_X & \xrightarrow{\mathrm{d}_X} & \wedge^{|\alpha+\beta|+1}\mathcal{A}_X . \end{array} \tag{2}$$

If we simply set $\xi_j\colon \mathcal{A}_X^{\alpha,\beta} \to \{0\}$ when $\alpha_j = 1$, we can write $\partial_X = \sum_{j=1}^n \xi_j$. On a local chart with coordinates $(z_1, \dots, z_n)$, any differential form $\omega \in \mathcal{A}_X^{\alpha,\beta}$ can be written as

$$\omega = f \cdot dz_\alpha \wedge d\bar{z}_\beta, \qquad \text{where} \qquad dz_\alpha = \wedge_{\{j:\, \alpha_j=1\}} dz_j, \quad d\bar{z}_\beta = \wedge_{\{j:\, \beta_j=1\}} d\bar{z}_j,$$

and the differential operator ξ_j is given by the formula

$$\xi_j(\omega) = \frac{\partial}{\partial z_j} f \cdot dz_j \wedge dz_\alpha \wedge d\bar{z}_\beta. \tag{3}$$

3. REFINED HODGE DECOMPOSITION

To promote the refined decomposition of differential forms into a refinement of the Hodge decomposition of de Rham cohomology, it is necessary to understand when the Laplacian operator associated with a Kähler metric respects the refined types of differential forms. The next definition singles out a sufficient condition.

Definition 3.1 We say that a hermitian metric ds^2 on an n-dimensional rigidified complex manifold X is *distinctive Kähler* if in a neighborhood of every point $x \in X$ there are a holomorphic coordinate system $(z_1, \dots, z_n)$ and a unitary coframe $\varphi_1, \dots, \varphi_n$ for the metric, such that

$$\varphi_j = f_j(z_j) \cdot dz_j \qquad \& \qquad \frac{\partial}{\partial \bar{z}_j} f_j(x) = 0 \qquad \forall\, j = 1, \dots, n.$$

Therefore, φ_j is a differential form of type $(1_j, 0)$ which satisfies $\frac{\partial}{\partial z_k} f_j \equiv 0 \equiv \frac{\partial}{\partial \bar{z}_k} f_j \quad \forall\, k \neq j$.

Remark 3.2 A distinctive Kähler metric is also Kähler. Indeed, one of the equivalent conditions for a metric on an n-dimensional complex manifold to be Kähler is the existence, for any $x \in X$, of a unitary coframe $\varphi_1, \dots, \varphi_n$ in a neighborhood of x such that $d_X \varphi_j(x) = 0$ for every $j = 1, \dots, n$.

Definition 3.3 A *rigidified Kähler manifold* is a rigidified complex manifold admitting a distinctive Kähler metric.

We now show that rigidified Kähler manifolds exist in nature. For every $j = 1, \dots, n$ consider a connected open subset $\Omega_j \subseteq \mathbb{C}$ admitting a Kähler metric ds_j^2. Write $\Omega = \prod_{j=1}^n \Omega_j$ and suppose a discrete group Γ acts on it satisfying the assumptions of Lemma 2.3.

Lemma 3.4 *If the* Γ*-action preserves the metric* $ds^2 = \sum_j ds_j^2$*, then* $\Gamma \backslash \Omega$ *has a natural structure of rigidified Kähler manifold.*

Proof. Lemma 2.3 shows that $\Gamma \backslash \Omega$ has a structure of rigidified complex manifold. We just have to prove that the metric $ds^2 = \sum_j ds_j^2$ induces a distinctive Kähler metric on the quotient. For any point $x \in \Gamma \backslash \Omega$ we choose a lift $\tilde{x} = (x_1, \dots, x_n) \in \Omega$ and open neighborhoods $U_j \subseteq \Omega_j$ of x_j such that the projection map $\pi\colon \Omega \to \Gamma \backslash \Omega$ is injective when restricted to $U = \prod_j U_j$. As each (Ω_j, ds_j^2) is a Kähler manifold, we can suppose that there is a holomorphic coordinate z_j on U_j and a unitary coframe $\varphi_j = f_j(z_j) \cdot dz_j$ for the metric ds_j^2 satisfying $d\varphi_j(x_j) = 0$. Then, the collection of all the pull-backs of the φ_j's to the product U gives the sought-after unitary coframe. □

Remark 3.5 Lemma 3.4 implies that complex tori and QSVs admit natural structures of rigidified Kähler manifolds.

3.1. Refined Hodge identities

Following [11, Ch. 0, Section 6], we recall the definitions of formal adjoint differential operators and apply it to the special case of rigidified Kähler manifolds. For a connected, compact, n-dimensional rigidified Kähler manifold X, we fix a distinctive Kähler metric $\mathrm{d}s^2$ with associated $(1,1)$-form ω locally given in a unitary coframe by

$$\omega = \frac{i}{2}\sum_{j=1}^{n}\varphi_j \wedge \overline{\varphi}_j.$$

The distinctive Kähler metric induces a Hermitian metric on the space of differential forms which can be used in combination with the volume form ω^n to define the inner product

$$\langle \psi, \eta\rangle := \int_X \big(\psi(x), \eta(x)\big)\frac{\omega^n}{n!} \qquad \forall\ \psi, \eta \in A^{p,q}(X).$$

Using Hodge's star operator ([11], pp. 82 and 101–102), we can then define the formal adjoint differential operator $\xi_j^\star : A^{p,q}(X) \to A^{p-1,q}(X)$ by setting

$$\xi_j^\star = -\star\,\xi_j\star,$$

and check that it satisfies

$$\langle \xi_j^\star\psi,\ \eta\rangle = \langle \psi,\ \xi_j\eta\rangle \qquad \forall\ \eta \in A^{p-1,q}(X)$$

by adapting slightly the computation beginning at the end of [11, p. 82].

Proposition 3.6 *If X is a rigidified Kähler manifold, then*

$$\xi_j \cdot \xi_k^\star + \xi_k^\star \cdot \xi_j = 0 \qquad \forall\ j \neq k.$$

Proof. We adapt the proof of the classical Hodge identities ([11], p. 111) and begin by verifying the claim for $\mathbb{C}^n$ with the Euclidean metric. The idea is to write the differential operators ξ_js as a composition of simpler operators. For each index $j = 1, \dots, n$ we consider the operator $e_j : A_c^{a,b}(\mathbb{C}^n) \to A_c^{a+1_j,b}(\mathbb{C}^n)$ on compactly supported forms defined by

$$e_j(\psi) = \mathrm{d}z_j \wedge \psi.$$

Let $e_j^\star$ denote the formal adjoint of e_j, and note that the operators $e_j, e_j^\star$ are $\mathscr{C}^\infty(\mathbb{C}^n)$-linear. By a direct computation one verifies that

$$e_j^\star \cdot e_k + e_k \cdot e_j^\star = 0 \qquad \forall\ j \neq k.$$

For $j = 1, \dots, n$ we also consider the operator ∂_j on $A_c^{a,b}(\mathbb{C}^n)$ defined by

$$\partial_j(f \cdot \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_b) = \frac{\partial}{\partial z_j} f \cdot \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_b.$$

The operators ∂_j's commute with each other, with all $e_k, e_k^\star$s, and satisfy $\partial_j^\star = -\overline{\partial}_j$, that is,

$$\partial_j^\star(f \cdot \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_b) = -\frac{\partial}{\partial \overline{z}_j} f \cdot \mathrm{d}z_a \wedge \mathrm{d}\overline{z}_b.$$

We can then write

$$\xi_j = \partial_j \cdot e_j, \qquad \xi_j^\star = -\overline{\partial}_j \cdot e_j^\star.$$

For $j \neq k$, the straightforward computation

$$\begin{aligned}\xi_j \cdot \xi_k^\star &= \partial_j \cdot e_j \cdot (-\overline{\partial}_k \cdot e_k^\star)\\ &= \overline{\partial}_k \cdot \partial_j \cdot e_k^\star \cdot e_j\\ &= -\xi_k^\star \cdot \xi_j\end{aligned}$$

proves the proposition for $\mathbb{C}^n$. We claim that the computations with the Euclidean metric suffice to deduce the result for any rigidified Kähler manifold X. Indeed, recall that by assumption, in a neighborhood of any point $x \in X$, we can find a holomorphic coordinate system $(z_1, \ldots, z_n)$ and a unitary coframe $\{\varphi_j = f_j(z_j) \cdot \mathrm{d}z_j\}_{j=1}^n$ for the metric, such that

$$\frac{\partial}{\partial z_k} f_j \equiv 0 \equiv \frac{\partial}{\partial \overline{z}_k} f_j \qquad \& \qquad \frac{\partial}{\partial \overline{z}_j} f_j(x) = 0 \qquad \forall\, j = 1, \ldots, n, \ \forall\, k \neq j.$$

In general, the operator ξ_j equals $\partial_j \cdot e_j$ up to terms that only involve the first-order derivative of the function f_j. Then, as the operators $e_j, e_j^\star$s are linear with respect to the algebra of $\mathscr{C}^\infty$-functions, we deduce that $\xi_j \cdot \xi_k^\star + \xi_k^\star \cdot \xi_j$ for $k \neq j$ coincides at $x \in X$ with the zero operator up to terms that involve first derivatives, products of first derivatives and mixed second derivatives of the functions f_j and f_k. The terms containing the first-order derivatives or their products vanish at $x \in X$ because of the usual Kähler condition, while the terms containing the mixed partial derivatives vanish identically because each function f_j depends on a single holomorphic coordinate. □

Recall that given an operator P on differential forms, the associated Laplacian is the degree zero operator given by the formula $\Delta_P = P \cdot P^\star + P^\star \cdot P$. For a Kähler manifold X we set $\Delta_{\mathrm{d}} = \Delta_{\mathrm{d}_X}$.

Corollary 3.7 If X is an n-dimensional rigidified Kähler manifold, then

$$\Delta_{\mathrm{d}} = \frac{1}{2} \sum_{j=1}^{n} \Delta_{\xi_j}.$$

In particular, the Laplacian operator Δ_{d} respects differential forms of refined types.

Proof. Since X is a Kähler manifold we know that

$$\Delta_{\mathrm{d}} = 2\Delta_{\partial}.$$

The claim then follows from a direct computation using Proposition 3.6:

$$\begin{aligned}\Delta_\partial &= \partial \cdot \partial^\star + \partial^\star \cdot \partial\\ &= \Big(\sum_j \xi_j\Big) \cdot \Big(\sum_k \xi_k^\star\Big) + \Big(\sum_k \xi_k^\star\Big) \cdot \Big(\sum_j \xi_j\Big)\\ &= \sum_{j,k} \xi_j \cdot \xi_k^\star + \sum_{j,k} \xi_k^\star \cdot \xi_j\\ &= \sum_j \big(\xi_j \cdot \xi_j^\star + \xi_j^\star \cdot \xi_j\big) + \sum_{j \neq k} \big(\xi_j \cdot \xi_k^\star + \xi_k^\star \cdot \xi_j\big)\\ &= \sum_j \Delta_{\xi_j}.\end{aligned}$$

□

We denote by $\mathcal{H}^{\alpha,\beta}(X)$ the space of harmonic differential forms of refined type (α,β), that is,

$$\mathcal{H}^{\alpha,\beta}(X) := \{\psi \in A^{\alpha,\beta}(X) \mid \Delta_{\mathrm{d}}\psi = 0\}.$$

Note that $\mathcal{H}^{\alpha,\beta}(X) = \overline{\mathcal{H}^{\beta,\alpha}(X)}$ since the Laplacian operator Δ_{d} is real.

Corollary 3.8 The de Rham cohomology of any compact rigidified Kähler manifold X admits a canonical direct sum decomposition

$$\mathrm{H}^k_{\mathrm{dR}}(X/\mathbb{C}) = \bigoplus_{|\alpha+\beta|=k} \mathcal{H}^{\alpha,\beta}(X), \qquad \alpha,\beta \in \{0,1\}^{\dim X}$$

in terms of harmonic differential forms of refined types.

Proof. The usual Hodge decomposition describes the de Rham cohomology of Kähler manifolds in terms of harmonic differential forms. As the Laplacian operator Δ_{d} respects differential forms of refined types when X is rigidified Kähler (Corollary 3.7), the claim follows. □

Remark 3.9 When X is a compact QSV, Corollary 3.8 can also be deduced from Matsushima–Shimura [**21**] or Nekovář–Scholl [**22**, Section 2]. The case of non-compact Hilbert modular varieties was studied by Davidescu–Scholl in [7].

Lemma 3.10 *The refined Hodge decomposition does not depend on the choice of a distinctive Kähler metric, that is, it only depends on the rigidified complex structure.*

Proof. Voisin's proof ([**28**], Proposition 6.11) of the independence of the Hodge decomposition from the choice of Kähler metric can be easily adapted to our setting. We aim to show that the subspace of de Rham cohomology classes which are representable by a closed form of refined type (α,β) coincides with $\mathcal{H}^{\alpha,\beta}(X)$. Let ψ be a closed form of refined type (α,β). We can write in a unique way $\psi = \eta + \Delta_{\mathrm{d}}\zeta$ with η harmonic. Since Δ_{d} respects refined types (Proposition 3.7), we can further suppose that both η and ζ are of refined type (α,β). As $\ker(\Delta_{\mathrm{d}}) \subseteq \ker(\mathrm{d})$, we see that $\mathrm{d}\eta = 0$ and compute that $\Delta_{\mathrm{d}}\zeta$ is closed:

$$\mathrm{d} \circ \Delta_{\mathrm{d}}\zeta = \mathrm{d}\eta + \mathrm{d} \circ \Delta_{\mathrm{d}}\zeta = \mathrm{d}\psi = 0.$$

Since $\mathrm{d}^2 = 0$, we further deduce that $\mathrm{d}^\star\mathrm{d}\zeta \in \ker(\mathrm{d})$. As the intersection $\ker(\mathrm{d}) \cap \mathrm{Im}(\mathrm{d}^\star)$ is trivial, we obtain $\mathrm{d}^\star\mathrm{d}\zeta = 0$. Hence, $\psi = \eta + \mathrm{d}\mathrm{d}^\star\zeta$, that is, ψ and η are in the same de Rham class. □

In other words, we have shown that the cohomology of a compact rigidified Kähler manifold is endowed with a canonical plectic Hodge structure refining its classical Hodge structure.

3.2. Functoriality of plectic Hodge structures

Definition 3.11 Let X, Y be n-dimensional rigidified complex manifolds. A morphism from X to Y is a function $\varphi\colon X \to Y$ such that for every point $x \in X$ there is a chart (U,ϕ) on X with $x \in U$ and a chart (V,ψ) on Y with $\varphi(x) \in V$ such that $\varphi(U) \subseteq V$ and the composition $\psi \circ \varphi \circ \phi^{-1}$ is a rigid holomorphic function.

Remark 3.12 In the rest of this section we will commit a slight abuse of notation and denote the maximal torsion-free quotients of the cohomology groups $\mathrm{H}^k(X,\mathbb{Z})$ by the same symbols.

What follows is a simple adaptation of [28, Section 7.3.2]. Let X, Y be n-dimensional compact rigidified Kähler manifolds and $\varphi\colon X \to Y$ a morphism of rigidified complex manifolds. Using the de Rham cohomology description of pullbacks, it is clear that $\varphi^*\colon \mathrm{H}^k(Y,\mathbb{Z}) \to \mathrm{H}^k(X,\mathbb{Z})$ is a morphism of n-plectic Hodge structures of degree 0.

Lemma 3.13 *For every $k \geq 0$ the pushforward $\varphi_*\colon \mathrm{H}^k(X,\mathbb{Z}) \to \mathrm{H}^k(Y,\mathbb{Z})$ is a morphism of n-plectic Hodge structures of degree 0.*

Proof. Recall that Poincaré duality $\langle\ ,\ \rangle_X\colon \mathrm{H}^k(X,\mathbb{Z}) \times \mathrm{H}^{2n-k}(X,\mathbb{Z}) \to \mathbb{Z}$ gives an isomorphism $\mathrm{H}^k(X,\mathbb{Z}) \cong \mathrm{H}^{2n-k}(X,\mathbb{Z})^\vee$ between the maximal torsion-free quotients of the cohomology groups which is used to characterize the pushforward φ_* map:

$$\langle \varphi_*\kappa, \eta \rangle_Y = \langle \kappa, \varphi^*\eta \rangle_X \qquad \forall\, \kappa \in \mathrm{H}^k(X,\mathbb{Z}),\ \eta \in \mathrm{H}^{2n-k}(Y,\mathbb{Z}).$$

Suppose now that $\kappa \in \mathcal{H}^{\alpha,\beta}(X)$, then to prove the lemma we need to show that $\varphi_*\kappa \in \mathcal{H}^{\alpha,\beta}(Y)$. This is a direct consequence of equality (4), and the fact that the pullback map φ^* is a morphism of plectic Hodge structures of weight 0.

For $\alpha \in \{0,1\}^n$ denote by $\alpha^c \in \{0,1\}^n$ the unique element such that $\alpha + \alpha^c = \underline{1}$. We claim that for any (α,β) satisfying $|\alpha+\beta| = k$ we have

$$\mathcal{H}^{\alpha,\beta} = \left(\bigoplus_{(\gamma,\delta)\neq(\alpha^c,\beta^c)} \mathcal{H}^{\gamma,\delta} \right)^{\perp}, \tag{4}$$

where the pairs (γ,δ) also satisfy $|\gamma+\delta| = 2n-k$ and the orthogonality is taken with respect to Poincaré pairing. To prove the claim we note that $\mathcal{H}^{\alpha,\beta}$ is contained on the right-hand side (RHS), and the dimensions of the two spaces coincide

$$\dim \mathrm{RHS} = \dim \mathcal{H}^{\alpha^c,\beta^c} = \dim \mathcal{H}^{\alpha,\beta}. \tag{5}$$

The first equality of (5) follows from the non-degeneracy of the Poincaré pairing, while the second arises from the isomorphism $\star\colon \mathcal{H}^{\alpha,\beta} \cong \mathcal{H}^{\alpha^c,\beta^c}$ given by the Hodge star operator induced by the *distinguished Kähler metric.* □

4. ALGEBRAICITY OF COMPLEX TORI WITH REAL MULTIPLICATION

While the content of this section is well-known to experts, we decided to include it for the convenience of the reader. Recall that for any complex torus $T = V/\Lambda$ there is an injective ring homomorphism

$$\mathrm{End}(T) \hookrightarrow \mathrm{End}_{\mathbb{Z}}(\Lambda).$$

In particular, the ring $\mathrm{End}(T)$ is torsion-free and finitely generated as a $\mathbb{Z}$-module.

Definition 4.1 A complex torus T has real multiplication if there exists a totally real field L with $[L:\mathbb{Q}] = \dim T$ and a unital ring homomorphism $\theta\colon L \to \mathrm{End}(T)_{\mathbb{Q}}$. We say that T has real multiplication by an order $\mathcal{O}$ if T has real multiplication and $\mathcal{O} := \theta^{-1}(\mathrm{End}(T))$.

Lemma 4.2 *Any complex torus T with real multiplication is isogenous to a complex torus T' with real multiplication by the ring of integers of a totally real number field.*

Proof. Let $T = V/\Lambda$ be a complex torus with real multiplication given by $\theta\colon L \to \mathrm{End}(T)_{\mathbb{Q}}$. As $\mathcal{O} := \theta^{-1}(\mathrm{End}(T))$ is an order in L, it has finite index in the ring of integers $\mathcal{O}_L$. Write

$n=[\mathcal{O}_L:\mathcal{O}]$ and consider

$$\Lambda' := \bigcup_{x\in\mathcal{O}_L/\mathcal{O}} x\Lambda.$$

We have $\Lambda\subseteq\Lambda'\subseteq\frac{1}{n}\Lambda$, hence Λ' is a lattice in V commensurable with Λ. Moreover, $T' := V/\Lambda'$ is a complex torus with real multiplication by $\mathcal{O}_L$, isogenous to T. □

Proposition 4.3 *([19], Theorem 7.2) Let R be a Dedekind ring. For a finitely generated R-module M the following are equivalent:*

- *M is R-projective,*
- *M is R-flat,*
- *M is R-torsion-free.*

Moreover, any finitely generated and torsion-free R-module M is isomorphic to $R^{n-1}\oplus\mathfrak{A}$ *for some fractional ideal* $\mathfrak{A}$ *of R and* $n=\mathrm{rk}_R M$.

Corollary 4.4 Let $T=V/\Lambda$ be a complex torus with real multiplication by $\mathcal{O}_L$, the ring of integers of a totally real number field. Then, Λ is $\mathcal{O}_L$-projective of rank two.

Proof. By considering the inclusion $\mathrm{End}(T)\hookrightarrow\mathrm{End}_{\mathbb{Z}}(\Lambda)\cong\mathrm{M}_{\mathrm{rk}\Lambda}(\mathbb{Z})$, we see that the non-zero elements of $\mathcal{O}_L$ act on Λ via elements of $\mathrm{M}_{\mathrm{rk}\Lambda}(\mathbb{Z})\cap\mathrm{GL}_{\mathrm{rk}\Lambda}(\mathbb{Q})$. Hence, Λ is $\mathcal{O}_L$-torsion-free. Now, [12, Corollary 2.6] tells us that $V=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$ is a free $\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{C}$-module of rank one. Therefore, we compute that

$$\mathrm{rk}_{\mathcal{O}_L}\Lambda=\mathrm{rk}_{\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{R}}V=2\cdot\mathrm{rk}_{\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{C}}V=2.$$

Proposition 4.3 then finishes the proof. □

The following result can also be found in [27, Ch. IX, Lemma 1.4].

Theorem 4.5 *Every complex torus T with real multiplication is an abelian variety.*

Proof. Up to isogeny ([2], Ch. 2, Proposition 1.1), we can assume that $T=V/\Lambda$ has real multiplication by a maximal order $\mathcal{O}_L$ (Lemma 4.2), then $\Lambda\cong\mathcal{O}_L\oplus\mathfrak{A}$ for some fractional ideal $\mathfrak{A}$ (Corollary 4.4 & Proposition 4.3). Set $\Sigma=\mathrm{Hom}_{\mathbb{Q}}(L,\mathbb{C})$. As V is a free $\mathcal{O}_L\otimes_{\mathbb{Z}}\mathbb{C}$-module of rank one, it admits a decomposition into one-dimensional $\mathbb{C}$-subspaces

$$V=\bigoplus_{\sigma\in\Sigma}V^\sigma,$$

such that an element $x\in\mathcal{O}_L$ acts on V^σ as multiplication by $\sigma(x)$. From a choice of isomorphism $\phi\colon\Lambda\xrightarrow{\sim}\mathcal{O}_L\oplus\mathfrak{A}$ of $\mathcal{O}_L$-modules, we obtain an $\mathbb{R}$-linear isomorphism $(\phi\otimes1)^\sigma\colon V^\sigma\xrightarrow{\sim}\mathbb{R}^2$ for every $\sigma\in\Sigma$ by extending scalars to $\mathbb{R}$. Moreover, transporting the complex structure from V^σ to $\mathbb{R}^2$, we can promote them to $\mathbb{C}$-linear identifications $(\phi\otimes1)^\sigma\colon V^\sigma\xrightarrow{\sim}\mathbb{C}$. Thus, there are $\lambda,\mu\in(\mathbb{C}^\times)_\Sigma$ such that $\lambda_\sigma,\mu_\sigma\in\mathbb{C}^\times$ are $\mathbb{R}$-linearly independent $\forall\sigma\in\Sigma$ and

$$\mu^{-1}(\phi\otimes1)\Lambda=\mathcal{O}_L\cdot(\lambda\mu^{-1})+\mathfrak{A}\cdot\underline{1}.$$

As L is dense in $L\otimes_{\mathbb{Q}}\mathbb{R}$, the composition $L^\times\to(L\otimes_{\mathbb{Q}}\mathbb{R})^\times\to\{\pm1\}_\Sigma$ is surjective, and we can choose $x\in L$ such that every component of $x(\lambda\mu^{-1})\in\mathbb{C}_\Sigma$ has positive imaginary part. Hence, we have shown the existence of $z \in \mathcal{H}_\Sigma$, a fractional ideal $\mathfrak{B}$ of $\mathcal{O}_L$ and an isomorphism of complex tori

$$T \cong \mathbb{C}_\Sigma/\Lambda_z \quad \text{where} \quad \Lambda_z = \mathcal{O}_L \cdot z + \mathfrak{B}.$$

We deduce that T is an abelian variety using [12, Corollary 2.10]. □

Remark 4.6 Every compact complex manifold has at most one algebraic structure because the analytification functor is fully faithful ([13], Corollaire 4.5).

Definition 4.7 A rational Hodge structure $H_\mathbb{Q}$ of weight 1 has real multiplication if there is a totally real field L with $2[L : \mathbb{Q}] = \dim_\mathbb{Q} H_\mathbb{Q}$ and a unital ring homomorphism $\theta \colon L \to \mathrm{End}(H_\mathbb{Q})$.

Corollary 4.8 The isogeny class of complex tori associated with an effective rational Hodge structure of weight 1 with real multiplication consists of abelian varieties.

Proof. Let $H_\mathbb{Q}$ be an effective rational Hodge structure of weight 1 with real multiplication. Directly from the definitions, the Jacobian associated with any lattice $\Lambda \subset H_\mathbb{Q}$ is a complex torus with real multiplication. The claim then follows from Theorem 4.5. □

5. ARCHIMEDEAN PLECTIC JACOBIANS

Recall that an n-plectic Hodge structure H is called effective of weight $\underline{1} \in \mathbb{Z}^n$ if

$$H^{\alpha,\beta} \neq 0 \quad \Longrightarrow \quad \alpha,\beta \in \mathbb{N}^n \quad \& \quad \alpha + \beta = \underline{1}.$$

We can think of an n-plectic Hodge structure of weight $\underline{1} \in \mathbb{Z}^n$ as a collection of n classical Hodge structures of weight 1 on the same underlying module by setting for any $j = 1, \dots, n$

$$F^{1_j} = F^{1_j}(H) := \bigoplus_{\alpha_j \geq 1} H^{\alpha,\beta}. \tag{6}$$

Then, for any $j = 1, \dots, n$ the plectic Jacobian $\mathrm{J}_\infty(H,j)$ associated with an effective n-plectic Hodge structure H of weight $\underline{1} \in \mathbb{Z}^n$ is the complex torus defined by

$$\mathrm{J}_\infty(H,j) := H \backslash (H \otimes_\mathbb{Z} \mathbb{C})/F^{1_j}.$$

Note that if $H_j = (H, F^{1_j})$ denotes the j-th Hodge structure of weight 1, we deduce that points of plectic Jacobians parametrize extensions of mixed Hodge structures ([26], Example 3.34)

$$\mathrm{J}_\infty(H,j) = \mathrm{Ext}_{\mathrm{MHS}}(\mathbb{Z}(-1), H_j).$$

5.1. Example: products of complex tori

Recall that the Jacobian of the weight one Hodge structure appearing in the first cohomology group $\mathrm{H}^1(T,\mathbb{Z})$ of a complex torus T recovers the dual torus $T^\vee$ ([28], Section 7.2). If we consider a product $X = T_1 \times \cdots \times T_n$ of complex tori, then the tensor product

$$\mathrm{H}^n(X,\mathbb{Z})_{\underline{1}} := \bigotimes_{j=1}^n \mathrm{H}^1(T_j,\mathbb{Z})$$

is an effective n-plectic Hodge structure of weight $\underline{1} \in \mathbb{Z}^n$ satisfying

$$F^{1_j}\mathrm{H}^n(X,\mathbb{C})_{\underline{1}} = F^1\mathrm{H}^1(T_j,\mathbb{C}) \otimes_\mathbb{Z} \bigotimes_{k \neq j} \mathrm{H}^1(T_k,\mathbb{Z}),$$

and whose plectic Jacobians can be explicitly described. For any index $j = 1, \dots, n$ let us define the plectic Jacobian $\mathrm{J}_\infty(X,j)$ as the plectic Jacobian associated with the plectic Hodge structure $\mathrm{H}^n(X,\mathbb{Z})_{\underline{1}}$.

A direct calculation shows that

$$\mathrm{J}_{\varpi}(X,j)\cong T_j^{\vee}\otimes_{\mathbb{Z}}\bigotimes_{k\neq j}\mathrm{H}^1(T_k,\mathbb{Z}).$$

In particular, if T_j is an abelian variety defined over a number field, then $\mathrm{J}_{\varpi}(X,j)$ is also an abelian variety with a distinguished model over the same number field.

5.2. Compact QSVs

Let F be a totally real number field of degree $d=[F:\mathbb{Q}]$ and, to simplify the exposition, of narrow class number one. Let B/F be a non-split quaternion F-algebra, denote by $\Sigma=\{\nu_1,\dots,\nu_r\}$ the set of Archimedean places of F at which B/F splits, and fix isomorphisms

$$\iota_\nu\colon B\otimes_{F,\nu}\mathbb{R}\cong M_2(\mathbb{R})\quad \text{for}\quad \nu\in\Sigma.$$

Given an Eichler order R in B, we denote by $\widetilde{\Gamma}$ the subgroup of $R^\times$ consisting of those elements with totally positive reduced norm. The group $\widetilde{\Gamma}$ maps in $\prod_{\nu\in\Sigma}(B\otimes_{F,\nu}\mathbb{R})^\times\cong \mathrm{GL}_2(\mathbb{R})^\Sigma$, and hence it acts on r-copies of the Poincaré upper half plane $\mathcal{H}_\Sigma=\mathcal{H}_{\nu_1}\times\cdots\times\mathcal{H}_{\nu_r}$ via Möbius transformations. Let Z denote the center of $\mathrm{GL}_2(\mathbb{R})^\Sigma$. We suppose that $\Gamma:=\widetilde{\Gamma}/(\widetilde{\Gamma}\cap Z)$ is torsion-free so that the quotient $X_B:=\Gamma\backslash\mathcal{H}_\Sigma$ is a compact complex manifold. The holomorphic tangent bundle $\mathcal{T}$ of X_B has a canonical decomposition into line bundles

$$\mathcal{T}=\bigoplus_{\nu\in\Sigma}\mathcal{L}_\nu,$$

where $\mathcal{L}_\nu$ is the holomorphic line bundle associated with the ν-th automorphy factor, and the image of first Chern class $c_1(\mathcal{L}_\nu)\in\mathrm{H}^2(X_B,\mathbb{Z})$ in $\mathrm{H}^2_{\mathrm{dR}}(X_B/\mathbb{C})$ can be represented by

$$\frac{1}{4\pi i}\,\frac{\mathrm{d}z_\nu\wedge\mathrm{d}\bar{z}_\nu}{y_\nu^2}. \tag{7}$$

Furthermore, for every $k\geq 0$, the maximal torsion-free quotient of $\mathrm{H}^k(X_B,\mathbb{Z})$ carries a canonical r-plectic Hodge structure (Corollary 3.8) which is preserved by Hecke operators (Section 3.2). Denote by $L_\nu\colon \mathrm{H}^r(X_B,\mathbb{Z})\to\mathrm{H}^{r+2}(X_B,\mathbb{Z})$ the morphism of r-plectic Hodge structures of bidegree $(1_\nu,1_\nu)$ given by the cup product with the class $c_1(\mathcal{L}_\nu)$, and define the strongly primitive component

$$\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{Z}) \tag{8}$$

of $\mathrm{H}^r(X_B,\mathbb{Z})$ as the maximal torsion-free quotient of the kernel of $\oplus_{\nu\in\Sigma}L_\nu$.

Lemma 5.1 *The strongly primitive cohomology $\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{Z})$ carries a canonical effective r-plectic Hodge structure of weight $\underline{1}\in\mathbb{Z}^r$ and is preserved by the action of Hecke operators.*

Proof. Since the L_νs are morphisms of plectic Hodge structures of bidegree $(1_\nu,1_\nu)$ the strongly primitive cohomology is endowed with an r-plectic Hodge structure. We can determine its weight by noticing that the explicit formula given in (7) allows us to compute

$$\ker\left(L_\nu\colon \mathrm{H}^r(X_B,\mathbb{C})\to\mathrm{H}^{r+2}(X_B,\mathbb{C})\right)=\bigoplus_{\alpha_\nu=1\text{ or }\beta_\nu=1}H^{\alpha,\beta}.$$

To prove that Hecke operators preserve the strongly primitive cohomology, note that any prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ admits a totally positive generator $\varpi_{\mathfrak{p}}$ because F has narrow class number one. It is well-known that the maps $\pi,\pi_{\mathfrak{p}}$ describing the Hecke correspondence

$$T_{\mathfrak{p}} := \left[X_B \xleftarrow{\pi} X_B(\mathfrak{p}) \xrightarrow{\pi_{\mathfrak{p}}} X_B \right]$$

are, respectively, uniformized by the identity $\mathcal{H}_\Sigma \to \mathcal{H}_\Sigma$, $(z_\nu)_\nu \mapsto (z_\nu)_\nu$ and the morphism

$$\mathcal{H}_\Sigma \to \mathcal{H}_\Sigma, \quad (z_\nu)_\nu \mapsto (\nu(\varpi_{\mathfrak{p}}) \cdot z_\nu)_\nu.$$

Thus, from equation (7) we deduce that

$$(\pi)^* c_1(\mathscr{L}_\nu) = (\pi_{\mathfrak{p}})^* c_1(\mathscr{L}_\nu) \qquad \forall\, \nu \in \Sigma. \tag{9}$$

As the action of $T_{\mathfrak{p}}$ is given by the formula $T_{\mathfrak{p}} = (\pi_{\mathfrak{p}})_* \circ (\pi)^*$, equation (9) implies that

$$T_{\mathfrak{p}}(\omega) \cup c_1(\mathscr{L}_\nu) = T_{\mathfrak{p}}(\omega \cup c_1(\mathscr{L}_\nu)) \qquad \forall\, \omega \in \mathrm{H}^r(X_B, \mathbb{Z}),\ \forall\, \nu \in \Sigma,$$

that is, that the Hecke operator $T_{\mathfrak{p}}$ commutes with the morphisms L_νs. □

Remark 5.2 Matsushima and Shimura described the cohomology of X_B in terms of automorphic forms in [21]. An inspection of their result shows that the strongly primitive cohomology is the part of the cohomology of X_B which is controlled by cuspidal automorphic representations.

For every $\nu \in \Sigma$ let $\gamma_\nu \in R^\times$ be an element such that $\det(\iota_\nu(\gamma_\nu)) < 0$, and $\det(\iota_\mu(\gamma_\nu)) > 0$ if $\mu \neq \nu$. Such an element exists because F has narrow class number one. We define the non-holomorphic involution Fr_{ν_j} of X_B by setting

$$\mathrm{Fr}_{\nu_j}(\tau_{\nu_1}, \dots, \tau_{\nu_r}) = (\iota_{\nu_1}(\gamma_{\nu_j})\tau_{\nu_1}, \dots, \iota_{\nu_j}(\gamma_{\nu_j})\overline{\tau}_{\nu_j}, \dots, \iota_{\nu_r}(\gamma_{\nu_j})\tau_{\nu_r}). \tag{10}$$

The involutions $\{\mathrm{Fr}_\nu\}_{\nu\in\Sigma}$ acting on the cohomology of X_B all commute with each other and with the Hecke operators (see for example [14, Equation 5 and Section 3]), but they are not morphisms of plectic Hodge structures. Indeed, if we write $\mathrm{Fr}_\beta = \prod_{\nu,\,\beta_\nu=1} \mathrm{Fr}_\nu$ for any $\beta \in \{0,1\}^r$, we find that

$$\mathrm{H}^r_{\mathrm{sp}}(X_B, \mathbb{C}) = \bigoplus_{\alpha+\beta=\underline{1}} \mathrm{Fr}^*_\beta(H^{\underline{1},\underline{0}}), \tag{11}$$

that is, the base change to $\mathbb{C}$ of the strongly primitive cohomology is spanned by translates of holomorphic differential forms ([21] and [25, Theorem 1.3]). Nevertheless, for any fixed $\nu \in \Sigma$, the operators $\{\mathrm{Fr}_\mu\}_{\mu\neq\nu}$ are automorphisms of the ν-th Hodge structure of weight one

$$H_{\mathbb{Z}}(X_B, \nu) := (\mathrm{H}^r_{\mathrm{sp}}(X_B, \mathbb{Z}), F^{1_\nu}) \tag{12}$$

attached to the strongly primitive cohomology as in (6) since

$$F^{1_\nu}\mathrm{H}^r_{\mathrm{sp}}(X_B, \mathbb{C}) = \bigoplus_{\beta,\,\beta_\nu=0} \mathrm{Fr}^*_\beta(H^{\underline{1},\underline{0}}).$$

Definition 5.3 For every $\nu \in \Sigma$, we define the plectic Jacobian $\mathrm{J}_\varpi(X_B, \nu)$ as the Jacobian of the Hodge structure $H_{\mathbb{Z}}(X_B, \nu)$.

Proposition 5.4 *For every $\nu \in \Sigma$, the plectic Jacobian $\mathrm{J}_\varpi(X_B, \nu)$ is an abelian variety.*

Proof. There is a decomposition of rational Hodge structures

$$H_{\mathbb{Q}}(X_B,\nu)=\bigoplus_{\chi} H_{\mathbb{Q}}(X_B,\nu)^{\chi}$$

indexed by characters $\chi=\prod_{\mu\neq\nu}\chi_\mu\colon \prod_{\mu\neq\nu}\{\pm1\}\to\{\pm1\}$ such that Fr_μ acts on $H_{\mathbb{Q}}(X_B,\nu)^{\chi}$ as multiplication by $\chi_\mu(-1)$. Moreover,

$$\dim_{\mathbb{Q}} H_{\mathbb{Q}}(X_B,\nu)^{\chi}=2\cdot\dim_{\mathbb{C}} H^{\underline{1},\underline{0}}.$$

Let $\mathbb{T}^{\mathrm{good}}_{\mathbb{Q}}$ denote the $\mathbb{Q}$-algebra generated by the Hecke operators associated with primes not dividing $\mathrm{disc}(B)\cdot\mathrm{level}(R)$ acting faithfully on $\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{Q})$ via endomorphisms of plectic Hodge structures. We claim that

- $H_{\mathbb{Q}}(X_B,\nu)^{\chi}$ is a free $\mathbb{T}^{\mathrm{good}}_{\mathbb{Q}}$-module of rank 2 and
- $\mathbb{T}^{\mathrm{good}}_{\mathbb{Q}}$ is isomorphic to a product $\prod_\xi L_\xi$ of totally real number fields.

Indeed, equation (11) and the compatibility between the Hecke action and the action of the involutions $\{\mathrm{Fr}_\nu\}_\nu$ show that $\mathbb{T}^{\mathrm{good}}_{\mathbb{Q}}$ is determined by its faithful action on the space $H^{\underline{1},\underline{0}}$ of holomorphic differential forms. The Jacquet–Langlands correspondence produces a Hecke-equivariant (for the good Hecke operators) isomorphism between $H^{\underline{1},\underline{0}}$ and the space of Hilbert cuspforms of weight 2, level $\mathrm{disc}(B)\cdot\mathrm{level}(R)$, trivial character, which are new at the primes dividing $\mathrm{disc}(B)$. Then, the claim follows from Miyake's results ([**20**], Section 2). Note that $\mathbb{T}^{\mathrm{good}}_{\mathbb{Q}}$ is a product of *totally real* number fields because the relevant Hilbert modular forms have trivial character. The proposition now follows from Corollary 4.8 because $H_{\mathbb{Q}}(X_B,\nu)^{\chi}$ decomposes as a direct sum of effective rational Hodge structure of weight 1 with real multiplication. □

5.2.1. Plectic Oda conjecture

Let $E_{/F}$ be a modular elliptic curve corresponding to a quaternionic newform f of some level Γ for an indefinite quaternion algebra B/F. For every Archimedean place of F $\nu\in\Sigma$ where B/F is split, we consider the base change $E_\nu=E\times_{F,\nu}\mathbb{C}$, and we denote by c_ν the involution of $\mathrm{H}^1(E_\nu,\mathbb{Q})$ induced by action of complex conjugation on $E_\nu(\mathbb{C})$. Let $\mathbb{T}_{\mathbb{Q}}$ denote the Hecke $\mathbb{Q}$-algebra generated by Hecke operators for primes not dividing the discriminant of B/F acting faithfully on $\mathrm{H}^r(X_B,\mathbb{Q})$ and denote by $\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{Q})_f$ the f-isotypic component of the strongly primitive cohomology. The following is a refinement of a classical conjecture of Oda ([**25**], Conjecture A).

Conjecture 5.5. ([Plectic Oda]) *There is an isomorphism of rational r-plectic Hodge structures*

$$\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{Q})_f\cong\bigotimes_{\nu\in\Sigma}\mathrm{H}^1(E_\nu,\mathbb{Q})$$

intertwining the action of Fr_ν *with that of* c_ν *for every* $\nu\in\Sigma$.

We note that Conjecture 5.5 together with the computation in Section 5.1 implies the existence of a morphism of abelian varieties

$$\varphi_\nu\colon \mathrm{J}_\infty(X_B,\nu)\longrightarrow E_\nu(\mathbb{C})\otimes_{\mathbb{Z}}\bigotimes_{\mu\in\Sigma\setminus\{\nu\}}\mathrm{H}^1(E_\mu,\mathbb{Z}),\qquad(13)$$

which should be thought of as a generalization of the parameterization of elliptic curves by Jacobians of Shimura curves. Moreover, it suggests the following conjecture which aims at elucidating the relation between the various plectic Jacobians attached to X_B.

Conjecture 5.6. *There exists an abelian variety $\mathrm{J}_\infty(X_B)$ defined over F endowed with an F-rational Hecke action and a Hecke equivariant morphism $\varphi\colon \mathrm{J}_\infty(X_B) \to E^{2^{|\Sigma|-1}}$ such that the analytification of $\varphi \otimes_{F,\nu} \mathbb{C}$ is canonically isomorphic to φ_ν for every $\nu \in \Sigma$.*

6. PLECTIC ABEL–JACOBI MAPS

Recall that $X_B = \Gamma\backslash\mathcal{H}_\Sigma$ is a compact QSV. The group Γ is assumed to be torsion-free, and it acts on $\mathcal{H}_\nu$ through its image $\nu(\Gamma) \le \mathrm{PGL}_2(\mathbb{R})$.

Definition 6.1 For any $\nu \in \Sigma$ we denote by $\mathcal{H}_\nu^\circ \subseteq \mathcal{H}_\nu$ the subset of those points with trivial stabilizer in Γ. We set $\mathcal{H}_\Sigma^\circ := \prod_{\nu\in\Sigma} \mathcal{H}_\nu^\circ$ and $X_B^\circ := \Gamma\backslash\mathcal{H}_\Sigma^\circ$,

Remark 6.2 Let $X_B^{\mathrm{CM}} \subseteq X_B$ denote the subset of CM points, then $X_B^{\mathrm{CM}} \subseteq X_B^\circ$.

Let M be a Γ-module. The subset $\mathcal{H}_\nu^\circ$ has been singled out because the higher homology groups of the tensor product $\mathrm{Div}(\mathcal{H}_\nu^\circ) \otimes_{\mathbb{Z}} M$ with diagonal Γ-action vanish, that is,

$$\mathrm{H}_k(\Gamma,\ \mathrm{Div}(\mathcal{H}_\nu^\circ) \otimes_{\mathbb{Z}} M) = 0 \qquad \forall\, k \ge 1. \tag{14}$$

Indeed, by definition there is an isomorphism of Γ-modules

$$\mathrm{Div}(\mathcal{H}_\nu^\circ) \cong \bigoplus_{x\in\Gamma\backslash\mathcal{H}_\nu^\circ} \mathbb{Z}[\Gamma],$$

and the Γ-module $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} M$ with diagonal Γ-action is isomorphic to the induced Γ-module $\mathrm{Ind}_{\{1\}}^{\Gamma}(M)$.

Definition 6.3 Let $S \subseteq \Sigma$ be a subset with complement denoted by S^c. We define

$$\mathbb{Z}_S[\mathcal{H}_\Sigma^\circ] := \bigotimes_{\nu\in S} \mathrm{Div}(\mathcal{H}_\nu^\circ) \otimes \bigotimes_{\nu\in S^c} \mathrm{Div}^0(\mathcal{H}_\nu^\circ).$$

where $\mathrm{Div}^0(\mathcal{H}_\nu^\circ)$ denotes the group of divisors of degree zero.

Proposition 6.4 *There is a short exact sequence*

$$0 \longrightarrow \mathrm{H}_{|\Sigma|}(\Gamma,\mathbb{Z}) \longrightarrow \mathrm{H}_0(\Gamma,\mathbb{Z}_\emptyset[\mathcal{H}_\Sigma^\circ]) \longrightarrow \mathbb{Z}[X_B^\circ].$$

Proof. Note that the free group $\mathbb{Z}[\mathcal{H}_\Sigma^\circ]$ equals $\mathbb{Z}_\Sigma[\mathcal{H}_\Sigma^\circ]$ and that $\mathbb{Z}[X_B^\circ] = \mathrm{H}_0(\Gamma,\mathbb{Z}[\mathcal{H}_\Sigma^\circ])$. Then, to prove the proposition, it suffices to show that for any non-empty $S \subseteq \Sigma$ and any $\nu \in S$

$$\ker\Big(\mathrm{H}_0(\Gamma,\ \mathbb{Z}_{S\backslash\{\nu\}}[\mathcal{H}_\Sigma^\circ]) \longrightarrow \mathrm{H}_0(\Gamma,\ \mathbb{Z}_S[\mathcal{H}_\Sigma^\circ])\Big) = \begin{cases} \mathrm{H}_r(\Gamma,\mathbb{Z}) & \text{if } S = \{\nu\}, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

To see that the claim holds, consider the short exact sequence of Γ-modules

$$0 \longrightarrow \mathbb{Z}_{S\backslash\{\nu\}}[\mathcal{H}_\Sigma^\circ] \longrightarrow \mathbb{Z}_S[\mathcal{H}_\Sigma^\circ] \xrightarrow{\deg_\nu\otimes 1} \bigotimes_{\mu\in S\backslash\{\nu\}} \mathrm{Div}(\mathcal{H}_\mu^\circ) \otimes \bigotimes_{\mu\in S^c} \mathrm{Div}^0(\mathcal{H}_\mu^\circ) \longrightarrow 0.$$

If $|S|>1$, then we are done thanks to the observation (14). If $S=\{\nu\}$ we are left to prove that

$$\mathrm{H}_1\Big(\Gamma,\bigotimes_{\mu\neq\nu}\mathrm{Div}^0(\mathcal{H}_\mu^\circ)\Big)\cong\mathrm{H}_r(\Gamma,\mathbb{Z}). \tag{16}$$

For this, let $S\subseteq\Sigma$ be arbitrary, $\mu\notin S$, and consider the short exact sequence of Γ-modules

$$0\longrightarrow\bigotimes_{\nu\in S\cup\{\mu\}}\mathrm{Div}^0(\mathcal{H}_\nu^\circ)\longrightarrow\mathrm{Div}(\mathcal{H}_\mu^\circ)\otimes\bigotimes_{\nu\in S}\mathrm{Div}^0(\mathcal{H}_\nu^\circ)\xrightarrow{\deg_\mu\otimes 1}\bigotimes_{\nu\in S}\mathrm{Div}^0(\mathcal{H}_\nu^\circ)\longrightarrow 0.$$

Once more, observation (14) shows that taking homology we obtain the connecting isomorphisms

$$\mathrm{H}_{m+1}\Big(\Gamma,\bigotimes_{\nu\in S}\mathrm{Div}^0(\mathcal{H}_\nu^\circ)\Big)\cong\mathrm{H}_m\Big(\Gamma,\bigotimes_{\nu\in S\cup\{\mu\}}\mathrm{Div}^0(\mathcal{H}_\nu^\circ)\Big)\qquad\forall\, m\geq 1. \tag{17}$$

Thus, the isomorphism in (16) follows by repeatedly applying (17). □

Using the cup product in de Rham cohomology we make the following identification:

$$\mathrm{J}_\infty(X_B,\nu)\cong\big(F^{1_\nu}\mathrm{H}^r_{\mathrm{sp}}(X_B,\mathbb{C})\big)^\vee/\mathrm{H}_r(X_B,\mathbb{Z}),$$

and consider the homomorphism

$$\int^\Sigma:\mathrm{H}_0(\Gamma,\mathbb{Z}_\emptyset[\mathcal{H}_\Sigma^\circ])\longrightarrow\mathrm{J}_\infty(X_B,\nu),\qquad\otimes_{j=1}^r(x_{\nu_j}-y_{\nu_j})\mapsto\left[\int_{y_{\nu_1}}^{x_{\nu_1}}\cdots\int_{y_{\nu_r}}^{x_{\nu_r}}(-)\right]. \tag{18}$$

It will be convenient for the proof of the next lemma to compute the singular homology of X_B with the chain complex $(C_\bullet^\infty(X_B),\partial_\bullet)$ of smooth cubical chains on X_B [18]. An element $c\in C_n^\infty(X_B)$ is a non-degenerate smooth function $c:[0,1]^n\to X_B$ and the differential $\partial_n\colon C_n^\infty(X_B)\to C_{n-1}^\infty(X_B)$ is given by the formula

$$\partial_n(c)=\sum_{k=1}^n(-1)^k[A_k(c)-B_k(c)], \tag{19}$$

where $A_k(c)(x_1,\dots,x_{n-1})=c(x_1,\dots,x_{k-1},0,x_{k+1},\dots,x_{n-1})$ can be thought of as the k-th 'front' face of the cubical n-chain c and $B_k(c)(x_1,\dots,x_{n-1})=c(x_1,\dots,x_{k-1},1,x_{k+1},\dots,x_{n-1})$ as the k-th 'back' face.

THEOREM 6.5 *We have*

$$\mathrm{H}_r(\Gamma,\mathbb{Z})\subseteq\ker\int^\Sigma.$$

Proof. Integration over smooth cubical n-chains gives a morphism $\Upsilon\colon C_r^\infty(X_B)\to\mathrm{H}^r_{\mathrm{dR}}(X_B/\mathbb{C})^\vee$. Moreover, Stokes' theorem implies that $\partial_{r+1}C_{r+1}^\infty(X_B)\subseteq\ker\Upsilon$, and de Rham's theorem gives the following exact sequence:

$$0\longrightarrow\mathrm{H}_r(X_B,\mathbb{Z})/\mathrm{tor}\longrightarrow C_r^\infty(X_B)/\ker\Upsilon\xrightarrow{\partial_r}C_{r-1}^\infty(X_B)/\partial_r(\ker\Upsilon).$$

The homomorphism $\int^{\Sigma} \colon \mathrm{H}_0(\Gamma, \mathbb{Z}_{\emptyset}[\mathcal{H}_{\Sigma}^{\circ}]) \longrightarrow \mathrm{J}_{\infty}(X_B, \nu)$ naturally factors through

$$\xi \colon \mathrm{H}_0(\Gamma, \mathbb{Z}_{\emptyset}[\mathcal{H}_{\Sigma}^{\circ}]) \longrightarrow C_r^{\infty}(X_B)/\ker \Upsilon.$$

Motivated by the description of the boundary map ∂_r given in (19), we define

$$\zeta \colon \mathrm{H}_0(\Gamma, \mathbb{Z}_{\emptyset}[\mathcal{H}_{\Sigma}^{\circ}]) \longrightarrow \bigoplus_{j=1}^{r} \mathrm{H}_0(\Gamma, \mathbb{Z}_{\{\nu_j\}}[\mathcal{H}_{\Sigma}^{\circ}]), \qquad \Delta \mapsto ((-1)^j \Delta)_{j=1}^{r}.$$

Note that $\ker \zeta = \mathrm{H}_r(\Gamma, \mathbb{Z})$ by equation (15). Then, the claim follows because there exists a morphism $\varpi \colon \mathrm{Im}(\zeta) \to \partial_r(C_r^{\infty}(X_B)/\ker \Upsilon)$ making the following diagram (with exact rows) commute

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}_r(\Gamma, \mathbb{Z}) & \longrightarrow & \mathrm{H}_0(\Gamma, \mathbb{Z}_{\emptyset}[\mathcal{H}_{\Sigma}^{\circ}]) & \xrightarrow{\zeta} & \mathrm{Im}(\zeta) \\
& \downarrow & & \downarrow \xi & & \downarrow \varpi \\
0 \longrightarrow & \mathrm{H}_r(X_B, \mathbb{Z})/\mathrm{tor} & \longrightarrow & C_r^{\infty}(X_B)/\ker \Upsilon & \xrightarrow{\partial_r} & \partial_r(C_r^{\infty}(X_B)/\ker \Upsilon).
\end{array}$$

□

DEFINITION 6.6 Let $\mathbb{Z}_{\infty}[X_B^{\circ}] := \mathrm{Im}(\mathbb{Z}_{\emptyset}[\mathcal{H}_{\Sigma}^{\circ}] \to \mathbb{Z}[X_B^{\circ}])$ denote the group of plectic zero cycles supported on X_B°. For any $\nu \in \Sigma$ the ν-th plectic Abel–Jacobi map is the homomorphism

$$\mathrm{AJ}_{\infty}^{\nu} \colon \mathbb{Z}_{\infty}[X_B^{\circ}] \longrightarrow \mathrm{J}_{\infty}(X_B, \nu)$$

obtained from (18) using Proposition 6.4 and Theorem 6.5.

ACKNOWLEDGEMENTS

I am grateful to G. Baldi, A. Doan, H. Esnault, L. Gehrmann, X. Guitart, D. Lilienfeldt, F. Lin, M. Masdeu, G. Oh, O. Rivero, A. Shnidman, J. Stelzig, M. Tamiozzo and S. Zhang for answering my questions and participating in many enriching conversations. I would like to especially thank L. Gehrmann and M. Tamiozzo for generously sharing their time and for suggesting ways to improve this paper. The work on this article began while the author was a Simons Junior Fellow at Columbia University and was supported by the National Science Foundation under Grant No. DMS-1 928 930 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2023 semester. We thank the anonymous referee for their valuable feedback.

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The Quarterly Journal of Mathematics, 2024, 00, 1–20, doi:https://doi.org/10.1093/qmath/haae025, Article