



A vertical Sato-Tate law for $GL(4)$

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Abstract. We establish a doubly-weighted vertical Sato-Tate law for $GL(4)$ with explicit error terms. The main ingredient is an extension of the orthogonality relation for Maass cusp forms on $GL(4)$ of Goldfeld, Stade, and Woodbury from spherical to general forms, and without their assumption of the Ramanujan conjecture for the error term.

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1. Introduction. The Kuznetsov trace formula is a fundamental tool in the analytic theory of automorphic forms. The case of $GL(n)$ for $n = 2, 3$ is by now well-established, and we refer to [2] for an excellent survey. Recently Goldfeld, Stade, and Woodbury established a Kuznetsov formula for $n = 4$, and using it deduced an orthogonality relation for Maass cusp forms [5]. The orthogonality relation was proved for $n = 3$ by Zhou, and was used to prove a doubly-weighted vertical Sato-Tate with rate of convergence. The goal of this note is to establish the same result for $n = 4$.

As was already remarked in [5], the techniques to obtain such an application are already well-established. This note confirms that expectation. In order to do so, one has to generalize the orthogonality relation established in [5, Theorem 1.1.1] that was proved only for spherical Maass cusp forms. We shall provide crude estimates here for general Maass cusp forms. Let $\{\phi_j\}$ be an orthogonal basis of even Maass cusp forms for $SL_4(\mathbb{Z})$ tempered at infinity with associated Langlands parameters

$$\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \alpha_3^{(j)}, \alpha_4^{(j)}) \in (i\mathbb{R})^4,$$

and L -th Fourier coefficient $A_j(L)$ where $L = (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$. Let $\mathcal{L}_j = L(1, \text{Ad}, \phi_j)$. Our first result is then the following extension of [5, Theorem 1.1.1] to general Fourier coefficients.

Theorem 1.1. *Let $T \geq 1$ and $R \geq 14$. Let $h_{T,R}$ be the test function defined in (4), and define $\omega_j(T) = h_{T,R}(\alpha^{(j)})/\mathcal{L}_j$. Then as $T \rightarrow \infty$, we have*

$$\begin{aligned} \sum_{j=1}^{\infty} A_j(L) \overline{A_j(M)} \omega_j(T) &= \delta_{L,M} \left(c_1 T^{9+8R} + c_2 T^{8+8R} + c_3 T^{7+8R} \right) \\ &\quad + O_{\epsilon,R} \left((LM)^{\frac{1}{2}+\epsilon} T^{6+8R+\epsilon} + (LM)^{\frac{1}{2}+\epsilon} T^{5+8R+\epsilon} \right. \\ &\quad \left. + (\ell_1 m_1)^{15/2} (\ell_2 m_2)^7 (\ell_3 m_3)^{15/2} T^{4+8R+\epsilon} \right), \end{aligned}$$

where $LM = \ell_1 \ell_2 \ell_3 m_1 m_2 m_3$. Also, $c_1, c_2, c_3 > 0$ are constants that depend at most on R , $\epsilon > 0$, and $\delta_{L,M} = 1$ if $L = M$ and zero otherwise.

Note that [5, Theorem 1.1.1] is the case of $\ell_2 = \ell_3 = m_2 = m_3 = 1$. It relies on the assumption of the Ramanujan conjecture for $n = 2, 3$ due to [5, Theorem 7.0.7], but this assumption should be able to be removed at the cost of a slightly worse error term, depending on bounds towards Ramanujan, such as shown in [1, Appendix] for $\text{GL}(3)$. In our theorem, we do not assume the Ramanujan conjecture, instead we use weaker estimates based on work of Chandee and Li [3], namely a Ramanujan on average bound.

As an application, the orthogonality relation implies the weighted vertical Sato-Tate law by a result of Zhou [9]. Denote by dx the Sato-Tate measure by pushing forward the normalized Haar measure on $\text{SU}(4)$ to $\hat{T}_0/W$ where $\hat{T}_0$ is the diagonal torus and W the Weyl group. Also, let $\hat{T}$ be the standard maximal torus of $\text{SL}_4(\mathbb{C})$.

Theorem 1.2. *Suppose ϕ_j corresponds to an irreducible unramified automorphic representation π_j of $\text{PGL}_4(\mathbb{A}_{\mathbb{Q}})$ with Satake parameter $a_j(\pi_p)$ at p . For any continuous test function f in $\hat{T}/W$, we have the equality*

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} f(a_j(\pi_p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \int_{\hat{T}_0/W} f(x) dx. \quad (1)$$

If further f is a monomial function as in (8), we have

$$\begin{aligned} \frac{\sum_{j=1}^{\infty} f(a_j(\pi_p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} &= \int_{\hat{T}_0/W} f(x) dx \\ &\quad + O(P^{\frac{1}{2}+\epsilon} (T^{6+8R+\epsilon} + T^{5+8R+\epsilon}) + P^8 T^{4+8R+\epsilon}), \end{aligned} \quad (2)$$

with $P = p^{i_1+i'_1+i_2+i'_2+i_3+i'_3}$.

Similar results were established by Matz and Templier [8] and Jana [6] for general n , but without explicit error terms, while more recently [7] proved related results as an application of Ramanujan on average.

2. The general orthogonality relation.

2.1. Notation. We shall first recall some basic definitions and properties that we shall need. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 5/2$, the L-function associated to ϕ_j is given by

$$\begin{aligned} L(s, \phi_j) &= \sum_{m=1}^{\infty} A_j(m, 1, 1) m^{-s} \\ &= \prod_p \left(1 - \frac{A_j(p, 1, 1)}{p^s} + \frac{A_j(1, p, 1)}{p^{2s}} - \frac{A_j(1, 1, p)}{p^{3s}} + \frac{1}{p^{4s}} \right)^{-1} \end{aligned}$$

and

$$L(s, \phi_j) = \prod_p L_p(s, \phi_j)$$

where

$$L_p(s, \phi_j) = \prod_{j=1}^4 \left(1 - \frac{\alpha_j(p)}{p^s} \right)^{-1} = \sum_{m \geq 0} \frac{A_j(p^m, 1, 1)}{p^{ms}}.$$

In the special case of $\mathrm{SL}_4(\mathbb{Z})$, the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ associated to $s = \frac{1}{4} + (v_1, v_2, v_3)$ are given by

$$\begin{aligned} \alpha_1 &= 3v_1 + 2v_2 + v_3, & \alpha_2 &= -v_1 + 2v_2 + v_3 \\ \alpha_3 &= -v_1 - 2v_2 + v_3, & \alpha_4 &= -v_1 - 2v_2 - 3v_3; \end{aligned}$$

where

$$v_1 = \frac{\alpha_1 - \alpha_2}{4}, \quad v_2 = \frac{\alpha_2 - \alpha_3}{4}, \quad v_3 = \frac{\alpha_3 - \alpha_4}{4}.$$

Recall from Theorems 9.3.11 and 10.8.1 of [4] for $n = 4$, the Hecke relations

$$A(m_1 m'_1, m_2 m'_2, m_3 m'_3) = A(m_1, m_2, m_3) A(m'_1, m'_2, m'_3)$$

for $(m_1 m_2 m_3, m'_1 m'_2 m'_3) = 1$, and

$$A(m, 1, 1) A(m_1, m_2, m_3) = \sum_{\substack{c_1 c_2 c_3 = m \\ c_i | m_i}} A\left(\frac{m_1 c_3}{c_1}, \frac{m_2 c_1}{c_2}, \frac{m_3 c_2}{c_3}\right). \quad (3)$$

Also $A(m_1, m_2, m_3) = \overline{A(m_3, m_2, m_1)}$.

2.2. Test functions. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{C}^4$ with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$. Let $T > 1$ with $T \rightarrow \infty$ and $R \geq 14$ with R fixed. We consider the test function

$$p_{T,R}^\sharp(\alpha) := e^{\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2}{2T^2}} \mathcal{F}_R(\alpha) \prod_{1 \leq j \neq k \leq 4} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right),$$

where

$$\mathcal{F}_R(\alpha) = \left(\prod_{\sigma \in S_4} \left(1 + \alpha_{\sigma(1)} - \alpha_{\sigma(2)} - \alpha_{\sigma(3)} + \alpha_{\sigma(4)} \right) \right)^{\frac{R}{24}}.$$

Given $2 \leq n \leq 4$, we define

$$p_{T,R}^{\sharp,(n)}(\alpha) := \begin{cases} e^{\frac{\alpha_1^2 + \dots + \alpha_n^2}{2T^2}} \prod_{1 \leq j \neq k \leq n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right) & \text{if } n = 2, 3, \\ e^{\frac{\alpha_1^2 + \dots + \alpha_n^2}{2T^2}} \mathcal{F}_R(\alpha) \prod_{1 \leq j \neq k \leq n} \Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right) & \text{if } n = 4. \end{cases}$$

Suppose that ϕ is a Maass cusp form for $\mathrm{GL}(n)$ with Langlands parameter $\alpha(\phi) := \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. Then we define the test function

$$h_{T,R}^{(n)}(\phi) := \frac{|p_{T,R}^{\sharp,(n)}(\alpha)|^2}{\prod_{1 \leq j \neq k \leq n} \Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)}. \quad (4)$$

We shall often omit the superscript (n) above when $n = 4$.

2.3. The orthogonality relation. The orthogonality conjecture [5, Conjecture 1.0.2] states that, for some constant $0 < \theta < 1$, the relation

$$\sum_{j=1}^{\infty} A_j(M) \overline{A_j(M')} \frac{h_{T,R}(\alpha^{(j)})}{\mathcal{L}_j} = \delta_{M,M'} \sum_{j=1}^{\infty} \frac{h_{T,R}(\alpha^{(j)})}{\mathcal{L}_j} + O_{M,M'} \left(\frac{h_{T,R}(\alpha^{(j)})}{\mathcal{L}_j} \right)^{\theta} \quad (5)$$

holds. In the case $n = 4$, [5] prove this for the test function $h_{T,R}(\alpha)$ above, which can be written as

$$e^{\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2}{T^2}} \prod_{\substack{1 \leq j, k \leq 4 \\ j \neq k}} \frac{\Gamma\left(\frac{2+R+\alpha_j-\alpha_k}{4}\right)^2}{\Gamma\left(\frac{1+\alpha_j-\alpha_k}{2}\right)} \prod_{\sigma \in S_4} \left(1 + \alpha_{\sigma(1)} - \alpha_{\sigma(2)} - \alpha_{\sigma(3)} + \alpha_{\sigma(4)} \right)^{\frac{R}{12}}.$$

We assume each Maass cusp form ϕ_j is normalized so that its first Fourier coefficient $A_j(1, 1, 1) = 1$. Let $\ell, m \in \mathbb{Z}$ with $\ell m \neq 0$.

Theorem 2.1 ([5, Theorem 1.1.1]). *With notation as above, we have for $T \rightarrow \infty$,*

$$\sum_{j=1}^{\infty} A_j(\ell, 1, 1) \overline{A_j(m, 1, 1)} \frac{h_{T,R}(\alpha^{(j)})}{\mathcal{L}_j} = \delta_{\ell,m} \left(c_1 T^{9+8R} + c_2 T^{8+8R} + c_3 T^{7+8R} \right) + O_{\epsilon,R} \left(|\ell m|^{\frac{2}{5}+\epsilon} T^{6+8R+\epsilon} + |\ell m|^{\frac{7}{32}+\epsilon} T^{5+8R+\epsilon} + |\ell m|^{\frac{15}{2}} T^{4+8R+\epsilon} \right),$$

where $\delta_{\ell,m}$ is the Kronecker symbol and $c_1, c_2, c_3 > 0$ are absolute constants which depend at most on R . Note that $h_{T,R}$ is of size T^{8R} on the relevant support. The error term assumes the Ramanujan conjecture for $\mathrm{GL}(n)$ for $n \leq 3$.

This orthogonality relation is obtained by evaluating the Kuznetsov trace formula for $\mathrm{GL}(4)$. The cuspidal contribution to the Kuznetsov formula is given by

$$\mathcal{C} = \sum_{j=1}^{\infty} \frac{A_j(L) \overline{A_j(M)} \left| p_{T,R}^{\#} \left(\overline{\alpha^{(j)}} \right) \right|^2}{\mathcal{L}_j \prod_{1 \leq j \neq k \leq 4} \Gamma \left(\frac{1+\alpha_j-\alpha_k}{2} \right)}.$$

The main theorem of [5] shows that for $N = 4$, $M = (m, 1, 1)$, and $L = (\ell, 1, 1)$, $\mathcal{C}$ is equal to the sum of

$$\mathcal{M} = \delta_{M,L} (c_1 T^{9+8R} + c_2 T^{8+8R} + c_3 T^{7+8R} + O(T^{6+8R})) \quad (6)$$

and

$$\mathcal{K} - \mathcal{E} = O_{\epsilon,R} \left(|\ell m|^{\frac{2}{5}+\epsilon} T^{6+8R+\epsilon} + |\ell m|^{\frac{7}{32}+\epsilon} T^{5+8R+\epsilon} + |\ell m|^{\frac{15}{2}} T^{4+8R+\epsilon} \right) \quad (7)$$

where $\mathcal{M}, \mathcal{K}, \mathcal{E}$ are referred to as the main term, the Kloosterman contribution, and the Eisenstein contribution, respectively.

We note that the estimates on $\mathcal{M}$ and $\mathcal{K}$ are obtained for general M, L by Proposition 3.5.1 and Proposition 4.0.4 of [5]. Therefore, only the $\mathcal{E}$ estimate needs to be generalized to arbitrary M, L . In [5, Remark 1.1.3], the authors note that it is possible, using the Hecke relations, to obtain a more general version of Theorem 2.1 but the formulas get quite complex and messy. We shall use the Hecke relations to get the more general estimate parallel to [5, Theorem 7.0.7]. The main term $\mathcal{M}$ being as above, we shall now describe $\mathcal{K}$ and $\mathcal{E}$ more explicitly.

2.4. The Kloosterman contribution. Let $M = (m_1, m_2, m_3), L = (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3$. Denote by $W_4 \simeq S_4$ the Weyl group of $\mathrm{GL}_4(\mathbb{R})$. The Kloosterman contribution to the Kuznetsov trace formula is given by

$$\mathcal{K} = C_{L,M}^{-1} \sum_{w \in W_4} \mathcal{I}_w$$

where

$$C_{L,M} = c_4 (\ell_1 m_1)^3 (\ell_2 m_2)^4 (\ell_3 m_3)^3,$$

and $\mathcal{I}_w$ is a sum of Kloosterman integrals defined in [5, (4.0.1)] and c_4 is a positive absolute constant. Let $r \geq 1$ be an integer. Then for R sufficiently large and any $\epsilon > 0$, we have from [5, Proposition 4.0.4] that the Kloosterman contribution is bounded by

$$C_{L,M}^{-1} |\mathcal{I}_{w_j}| \ll_{\epsilon,R} (\ell_1 m_1)^{2r-1/2} (\ell_2 m_2)^{2r-1} (\ell_3 m_3)^{2r-1/2} B_j(T),$$

where

$$B_j(T) = \begin{cases} T^{\epsilon+8R+20-4r} & \text{if } j = 2, 3, 4, \\ T^{\epsilon+8R+19-5r} & \text{if } j = 6, 7, \\ T^{\epsilon+8R+18-6r} & \text{if } j = 5, 8. \end{cases}$$

The index $j = 1, \dots, 8$ runs over elements in W_4 . Optimising at $r = 4$, we have that

$$\mathcal{K} = O_{\epsilon,R}((\ell_1 m_1)^{15/2} (\ell_2 m_2)^7 (\ell_3 m_3)^{15/2} T^{4+8R+\epsilon}).$$

2.5. The Eisenstein contribution. Suppose that $4 = n_1 + \dots + n_r$ is a partition of 4 and $\Phi = (\phi_1, \dots, \phi_r)$ where, for $1 \leq j \leq r$, ϕ_j is a Maass cusp form for $\mathrm{SL}(n_j, \mathbb{Z})$ if $n_j > 1$, while ϕ_j is the constant function (properly normalized) if $n_j = 1$. Let $\mathcal{P} = \mathcal{P}_{n_1, \dots, n_r}$. Then we define

$$\mathcal{E}_{\mathcal{P}, \Phi} = \int_{\mathrm{Re}(s_1)=0} \dots \int_{\mathrm{Re}(s_{r-1})=0} A_{E_{\mathcal{P}, \Phi}}(L, s) \overline{A_{E_{\mathcal{P}, \Phi}}(M, s)} \left| p_{T,R}^{\#}(\alpha_{\mathcal{P}, \Phi}(s)) \right|^2 ds_1 \dots ds_{r-1},$$

and denote

$$\mathcal{E}_{\mathcal{P}_{\min}} = \mathcal{E}_{\mathcal{P}_{1,1,1,1}, \Phi},$$

where $\alpha_{\mathcal{P}, \Phi}(s)$, $\alpha_{\mathcal{P}_{\min}}(s)$ denote the Langlands parameters of the Eisenstein series $E_{\mathcal{P}, \Phi}(g, s)$, $E_{\mathcal{P}_{\min}}(g, s)$, respectively. Also, $A_{E_{\mathcal{P}, \Phi}}(L, s)$, $A_{E_{\mathcal{P}, \Phi}}(M, s)$ denote the L -th and M -th Fourier coefficients of $E_{\mathcal{P}, \Phi}(g, s)$, and similarly for $E_{\mathcal{P}_{\min}}(g, s)$. The following estimate follows from a Ramanujan on average bound of Chandee and Li.

Lemma 2.2. *For positive integers k, l, n and $\epsilon > 0$, we have*

$$|A(k, l, n)| \ll (kln)^{\frac{1}{2}+\epsilon}.$$

Proof. This follows from [3, Lemma 3.4], namely, the formula

$$A(k, l, n) = \sum_{d|(k,l)} \sum_{e|(d,k/d)} \sum_{f|(k,n)} \mu(d)\mu(e) A\left(\frac{k}{def}, 1, 1\right) A\left(1, \frac{l}{d}, \frac{dn}{ef}\right)$$

and an application of [3, Lemma 3.5]. □

Now define

$$\mathcal{E}_{\mathcal{P}} = \sum_{\Phi} c_{L,M,\mathcal{P}} \mathcal{E}_{\mathcal{P}, \Phi}.$$

Then the contribution to the Kuznetsov trace formula coming from the Eisenstein series is given by

$$\mathcal{E} = c_1 \mathcal{E}_{\mathcal{P}_{\min}} + c_2 \mathcal{E}_{\mathcal{P}_{2,1,1}} + c_3 \mathcal{E}_{\mathcal{P}_{2,2}} + c_4 \mathcal{E}_{\mathcal{P}_{3,1}}$$

for constants $c_1, c_2, c_3, c_4 > 0$. We shall be satisfied with the ineffective estimate below.

Proposition 2.3. *Let $L = (\ell_1, \ell_2, \ell_3)$, $M = (m_1, m_2, m_3)$, and set $LM = \ell_1 \ell_2 \ell_3 m_1 m_2 m_3$. Then*

$$\begin{aligned} |\mathcal{E}_{\mathcal{P}_{\min}}| &\ll_{\epsilon} (LM)^{\frac{1}{2}+\epsilon} T^{3+8R+\epsilon}, & |\mathcal{E}_{\mathcal{P}_{2,1,1}}| &\ll_{\epsilon} (LM)^{\frac{1}{2}+\epsilon} T^{2+8R+\epsilon}, \\ |\mathcal{E}_{\mathcal{P}_{2,2}}| &\ll_{\epsilon} (LM)^{\frac{1}{2}+\epsilon} T^{5+8R+\epsilon}, & |\mathcal{E}_{\mathcal{P}_{3,1}}| &\ll_{\epsilon} (LM)^{\frac{1}{2}+\epsilon} T^{6+8R+\epsilon}, \end{aligned}$$

as $T \rightarrow \infty$ for any fixed $\epsilon > 0$.

Proof. The crucial step is to bound the product of Fourier coefficients, and the remainder of the analysis will follow in the same manner as in the proof of [5, Theorem 7.0.7]. That is, we bound

$$A_{E_{\mathcal{P}, \Phi}}(L, s) \overline{A_{E_{\mathcal{P}, \Phi}}(M, s)}$$

and

$$A_{E_{\mathcal{P}_{\min}}}(L, s) \overline{A_{E_{\mathcal{P}_{\min}}}(M, s)}$$

using Lemma 2.2. □

Remark 2.4. We note that our estimates here are much weaker, even in the spherical case, than that of [5, Theorem 7.0.7]. The reason for this is that the general Fourier coefficients require a calculation of the Hecke eigenvalues for general Hecke operators, which is not known, whereas the spherical case has been long known due to work of Goldfeld.

Then putting the estimates for $\mathcal{M}, \mathcal{K}$, and $\mathcal{E}$ together in (6) and (7), we obtain Theorem 1.1.

3. Sato-Tate with rate of convergence.

3.1. Preliminaries. We follow the setup of [9]. We first recall some facts about the representation theory of $\mathrm{GL}(n)$. We specialize to $n = 4$ for concreteness. Let $\pi = \otimes_{p \leq \infty} \pi_p$ be a cuspidal automorphic representation of $\mathrm{PGL}_4(\mathbb{A})$. For each finite prime $p < \infty$, we take π_p to be an unramified admissible representation with Satake parameter

$$a(\pi_p) = \begin{pmatrix} \alpha_{p,1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \alpha_{p,4} \end{pmatrix} \in \hat{T}/W,$$

where $\hat{T}$ is the standard maximal torus of $\mathrm{SL}_4(\mathbb{C})$, and $W \simeq S_4$ is the usual Weyl group. The generalized Ramanujan conjecture implies that $|\alpha_{p,i}| = 1$ for all $1 \leq i \leq 4$ and p . It is equivalent to the statement that $a(\pi_p)$ lies in the subtorus $\hat{T}_0/W$ of $\mathrm{SU}(4)$.

Let ϵ_i be the i -th standard basis vector in $\mathbb{R}^4$. Denote the root system of $\mathrm{GL}_4(\mathbb{C})$ by $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$, and the set of positive roots $\Phi^+ = \{\epsilon_i - \epsilon_j : i < j\}$. The set of integral weights Λ of $\mathrm{GL}_4(\mathbb{C})$ is generated by the set of

$$\epsilon_i - \frac{1}{4} \sum_{j=1}^n \epsilon_j, \quad 1 \leq i \leq 3.$$

The Weyl chamber associated to Φ^+ is then given by

$$C = \left\{ \sum_{i=1}^4 a_i \epsilon_i : a_1 \geq \cdots \geq a_4, a_i \in \mathbb{R}, \sum_{i=1}^4 a_i = 0 \right\}.$$

Then given a weight μ in $\Lambda \cap C$, let V_μ be the highest weight representation of μ and χ_μ its character. Formally, we may write χ_μ as a finite sum of e^λ for $\lambda \in \Lambda$ with nonnegative integer coefficients, invariant under the action of W . Let V be the representation given by the standard embedding of $\mathrm{SL}_4(\mathbb{C})$ in $\mathrm{GL}_4(\mathbb{C})$, and let $V_k = \wedge^k V$ be the k -th exterior product for $k = 1, \dots, 3$, so that V_k corresponds to the highest weight representation of

$$\sum_{i=1}^k \left(\epsilon_i - \frac{1}{4} \sum_{j=1}^n \epsilon_j \right).$$

Its character is given by

$$\chi_k \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_4 \end{pmatrix} \right) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k},$$

which is an elementary symmetric polynomial on $\hat{T}/W$. We define a map from $\hat{T}/W \rightarrow \mathbb{C}^3$ by

$$\rho : \mathrm{diag}(\alpha_1, \dots, \alpha_4) \rightarrow (\chi_1(\mathrm{diag}(\alpha_1, \dots, \alpha_4)), \dots, \chi_n(\mathrm{diag}(\alpha_1, \dots, \alpha_4))).$$

And define a monomial function $f(\mathrm{diag}(\alpha_1, \dots, \alpha_4))$ of the form

$$\begin{aligned} & \left(\sum_i \alpha_j \right)^{i_1} \left(\sum_i \bar{\alpha}_i \right)^{i'_1} \left(\sum_{i < j} \alpha_i \alpha_j \right)^{i_2} \left(\sum_{i < j} \bar{\alpha}_i \bar{\alpha}_j \right)^{i'_2} \cdots \\ & \left(\sum_{i < j < k} \alpha_i \alpha_j \alpha_k \right)^{i_3} \left(\sum_{i < j < k} \bar{\alpha}_i \bar{\alpha}_j \bar{\alpha}_k \right)^{i'_3} \end{aligned} \quad (8)$$

for nonnegative integers $i_1, i'_1, i_2, i'_2, i_3, i'_3$. Define a bijection between $\Lambda \cap C$ and the set $\mathbb{Z}_{\geq 0}^3$ of integer nonnegative triples

$$\omega : (l_1, l_2, l_3) \mapsto \sum_{i=1}^3 \left(\sum_{k=1}^{3-i} l_k \right) \left(\epsilon_i - \frac{1}{4} \sum_{j=1}^3 \epsilon_j \right).$$

Then by the Casselman-Shalika formula, we have that the Fourier coefficients of π are given by $A(p^{l_1}, \dots, p^{l_k}) = \chi_{\omega(l_1, \dots, l_{n-1})}(a(\pi_p))$.

3.2. Proof of Theorem 1.2. The identity (1) follows immediately from [9, Theorem 8.4]. The proof of (2) is essentially the same as that of [9, Theorem 9.1], as an application of the orthogonality relation and the Casselman-Shalika formula. We sketch the proof here.

Let $A_j(M)$ be the M -th Fourier coefficient of ϕ_j . Denote $A_j(1, \dots, p, \dots, 1)$ with p in the $(4-k)$ -th position as $A_j[k]$. It follows from the proof of [9, Lemma 8.3] that

$$\begin{aligned} \prod_{k=1}^3 A_j[k]^{i_k} \overline{A_j[k]}^{i'_k} &= \prod_{k=1}^3 A_j[k]^{i_k} A_j[4-k]^{i'_k} \\ &= \prod_{k=1}^3 \chi_k(a_k(\pi_p))^{i_k} \chi_{4-k}(a_k(\pi_p))^{i'_k} \\ &= \sum_{\mu \in \Lambda \cap C} a_\mu \chi_\mu(a_j(\pi_p)). \end{aligned}$$

The last line uses the property that

$$\prod_{k=1}^3 \chi_k^{i_k} \chi_{3-k}^{i'_k} = \sum_{\mu \in \Lambda \cap C} a_\mu \chi_\mu$$

where a_μ is the multiplicity of V_μ in the decomposition of the finite-dimensional representation

$$\bigotimes_{k=1}^3 (V_k^{\otimes i_k} \otimes V_k^{\otimes i'_k}) = \bigoplus_{\mu \in \Lambda \cap C} V_\mu^{\oplus a_\mu}.$$

Then the assumption that f is monomial translates to $f \circ \rho$ being given by

$$(z_1, z_2, z_3) \mapsto z_1^{i_1} \bar{z}_1^{i'_1} z_2^{i_2} \bar{z}_2^{i'_2} z_3^{i_3} \bar{z}_3^{i'_3}, \quad i_1, \dots, i'_3 \in \mathbb{Z}_{\geq 0},$$

which implies by the analogous computation as in [9, p.423] that

$$\begin{aligned} &\frac{\sum_{j=1}^{\infty} f(X_j(p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} - \int_{\hat{T}_0/W} f(x) dx \\ &= \sum_{l_1, l_2, l_3 \geq 0} a_{\omega((l_1, l_2, l_3))} \left(\frac{A_j(p^{l_1}, p^{l_2}, p^{l_3}) \omega_j(T)}{\sum_j \omega_j(T)} - \delta_{l_1, 0} \delta_{l_3, 0} \delta_{l_3, 0} \right), \quad (9) \end{aligned}$$

and applying the orthogonality relation of Theorem 1.1, we may bound the inner expression by

$$O_{\epsilon, R} \left((p^{l_1 + l_2 + l_3})^{\frac{1}{2} + \epsilon} T^{5 + 8R + \epsilon} (T + 1) + p^{\frac{15l_1}{2} + 7l_2 + \frac{15l_3}{2}} T^{4 + 8R + \epsilon} \right).$$

Also, by [9, p. 424], we have that, for any $\alpha > 0$,

$$\sum_{l_1, l_2, l_3 \geq 0} a_{\omega((l_1, l_2, l_3))} p^{\alpha(l_1 + l_2, l_3)} = O(p^{\alpha(i_1 + i'_1 + \dots + i_3 + i'_3)}),$$

and it follows that (9) equals

$$\begin{aligned} &O \left(\sum_{l_1, l_2, l_3 \geq 0} a_{\omega((l_1, l_2, l_3))} ((p^{l_1 + l_2 + l_3})^{\frac{1}{2} + \epsilon} (T^{6 + 8R + \epsilon} + T^{5 + 8R + \epsilon}) + p^{8(l_1 + l_2 + l_3)} T^{4 + 8R + \epsilon}) \right) \\ &= O(P^{\frac{1}{2} + \epsilon} (T^{6 + 8R + \epsilon} + T^{5 + 8R + \epsilon}) + P^8 T^{4 + 8R + \epsilon}), \end{aligned}$$

with $P = p^{i_1 + i'_1 + \dots + i_3 + i'_3}$ as desired.

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