The Ramanujan Journal
https://doi.org/10.1007/s11139-024-00880-z

The density of the graph of elliptic Dedekind sums

Stephen Bartell[1] · Abby Halverson[2] · Brenden Schlader[3] · Siena Truex[4] · Tian An Wong[5]

Received: 30 August 2023 / Accepted: 26 March 2024

Abstract

We show that the graph of normalized elliptic Dedekind sums is dense in its image for arbitrary imaginary quadratic fields, generalizing a result of Ito in the Euclidean case. We also derive some basic properties of Martin's continued fraction algorithm for arbitrary imaginary quadratic fields.

Keywords Elliptic Dedekind sums · Complex continued fractions

Mathematics Subject Classification 11F67 · 11F41

This work was partially supported the NSF grant DMS-1950102 and NSA grant H98230-19. The last author was also supported by NSF grant DMS-2212924.

✉ Tian An Wong
tiananw@umich.edu

Stephen Bartell
sbartell@princeton.edu

Abby Halverson
halver2@stolaf.edu

Brenden Schlader
schlbre2@bvu.edu

Siena Truex
struex@middlebury.edu

1 Princeton University, 4002 Frist Campus Center, Princeton, NJ 08544, USA

2 St. Olaf College, 1500 St. Olaf Ave, Northfield, MN 55057, USA

3 Buena Vista University, 610 W. 4th St., Storm Lake, IA 50588, USA

4 3682 Middlebury College, 14 Old Chapel Rd., Middlebury, VT 05753, USA

5 2002 CASL Building, University of Michigan-Dearborn, 4901 Evergreen Rd, Dearborn, MI 48128, USA

Published online: 22 June 2024

1 Introduction

1.1 The density of elliptic Dedekind sums

The classical Dedekind sum $s(m, n)$ is defined for $m, n \in \mathbb{Z}$, $(m, n) = 1$, $n \neq 0$, by

$$s(m, n) := \frac{1}{4n} \sum_{k=1}^{n-1} \cot\left(\pi \frac{mk}{n}\right) \cot\left(\pi \frac{k}{n}\right).$$

They are of arithmetic interest, arising originally from the transformation law of the Dedekind eta function. It has found applications to the special values of L-functions, and also areas outside of number theory such as knot theory, geometric topology, and combinatorics. Grosswald and Rademacher conjectured that the values of $s(m, n)$ are dense in $\mathbb{R}$, and moreover the graph $\{(m/n, s(m/n)) : m/n \in \mathbb{Q}\}$ is dense in $\mathbb{R}^2$ [12], where $s(m/n) := s(m, n)$ whenever $(m, n) = 1$. The latter statement (which implies the former) was first proved by Hickerson [2].

Many generalizations of Dedekind sums have been made, and in this paper we study elliptic Dedekind sums. These are generalizations of classical Dedekind sums to complex lattices, or imaginary quadratic number fields. Let L be a non-degenerate lattice in $\mathbb{C}$. We define

$$E_k(z) = \sum_{\substack{x \in L, \\ x+z \neq 0}} (x + z)^{-k} |x + z|^{-s} \Big|_{s=0},$$

where the value of the sum at $s = 0$ is evaluated by means of analytic continuation. Define the ring of multipliers for L as $\mathcal{O}_L = \{m \in \mathbb{C} : mL \subset L\}$. It is either equal to the ring of integers or to an order in an imaginary quadratic field. Then, following Sczech [13], the elliptic Dedekind sums for L are defined as

$$D(a, c) = \frac{1}{c} \sum_{\mu \in L/cL} E_1\Big(\frac{a\mu}{c}\Big) E_1\Big(\frac{\mu}{c}\Big)$$

for $a, c \in \mathcal{O}_L$, $c \neq 0$.

Assume for the rest of this paper that $\mathcal{O}_L$ is the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$. If K has class number 1, one can again define $D(a/c) = D(a, c)$ for $a/c \in K$ and a, c coprime in $\mathcal{O}_K$. (Note that if the class number is greater than 1, the fraction is no longer well-defined.) Ito [3] showed that the graph

$$\{(a/c, \tilde{D}(a/c)) : a/c \in K\}$$

of the normalized elliptic Dedekind sum

$$\tilde{D}(a/c) = \tilde{D}(a, c) = (i\sqrt{|d_K|} E_2(0))^{-1} D(a, c)$$

is dense in $\mathbb{C} \times \mathbb{R}$ when $D = 2, 5, 7$, where d_K is the discriminant of K. (That is, when $\mathcal{O}_K$ is Euclidean and $D \neq 1, 3$; note $\tilde{D}$ is trivial if and only if $D = 1, 3$.) It was also shown recently by the last author and others [1] that the image of $\tilde{D}(a, c)$ is dense in $\mathbb{R}$ for general lattices L in $\mathbb{C}$ (and in particular for arbitrary class number) using a recent method of Kohnen [5].

1.2 Main result

In this paper, we extend these results to the density of the graph for imaginary quadratic fields of arbitrary class number. Note that the reduced fraction a/c associated to any $\alpha \in K$ is only uniquely defined when K has class number 1. Also, the set of factorizations of an element in $\mathcal{O}_K$ is bounded [7]. For general class number, it will suffice for us to fix any nonunique representative a/c of α and consider $\tilde{D}(\alpha) = \tilde{D}(a, c)$. Note that $\tilde{D}(a, c) = \tilde{D}(\lambda a, \lambda c)$ for any nonzero $\lambda \in \mathcal{O}_K$ by [13, (15)]. With this convention, we state our main result.

Theorem 1.1 *Let $K = \mathbb{Q}(\sqrt{-D})$ with $D \neq 1, 3$. Then the graph of the normalized elliptic Dedekind sum*

$$\{(\alpha, \tilde{D}(\alpha)) : \alpha \in K\} \tag{1.1}$$

is dense in $\mathbb{C} \times \mathbb{R}$.

Following the method of Ito and Hickerson, our proof first relies on a generalization of the continued fraction algorithm recently established by Martin [6] for imaginary quadratic fields of general class number. The second ingredient in our proof is Sczech's homomorphism $\Phi : SL_2(\mathcal{O}_K) \to \mathbb{C}^+$ [13] given by

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} E_2(0) I\left(\dfrac{a+d}{c}\right) - D(a, c), & c \neq 0 \\ E_2(0) I\left(\dfrac{b}{d}\right), & c = 0 \end{cases}$$

where $I(z) := z - \bar{z} = 2\,\mathrm{Im}(z)$. It was extended to $GL_2(\mathcal{O}_K)$ by Obaisi [10] as follows: for a more general matrix $A \in GL_2(\mathcal{O}_K)$, we have

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} E_2(0) I\left(\dfrac{a+\det(A)d}{c}\right) - D(a, c), & c \neq 0 \\ E_2(0) I\left(\dfrac{b}{d}\right), & c = 0 \end{cases}$$

by evaluating [10, (4.2)] at the point $u = (0, 0)$. [1]

[1] Actually, Obaisi's generalization of the Sczech cocycle in [10] should be multiplied by -1, but this does not affect the computations.

Let $A \in GL_2(\mathcal{O})$ act on the extended complex plane by Möbius transformations. If $d \neq 0$, then we see $A(0) = \frac{b}{d}$. If $c \neq 0$, we also have $A(\infty) = \frac{a}{c}$ and

$$A^{-1}(\infty) = \begin{pmatrix} \frac{d}{\det(A)} & \frac{-b}{\det(A)} \\ \frac{-c}{\det(A)} & \frac{a}{\det(A)} \end{pmatrix} (\infty) = -\frac{d}{c}. \tag{1.2}$$

Note that c and d cannot both be zero because $A \in GL_2(\mathcal{O})$. In addition, $c = 0$ if and only if $A(\infty) = \infty$, and $d = 0$ if and only if $A(\infty) \neq \infty$. Therefore, we can rewrite the homomorphism extension as

$$\Phi(A) = \begin{cases} E_2(0)I(A(\infty) - \det(A)(A^{-1}(\infty))) - D(A(\infty)), & A(\infty) \neq \infty \\ E_2(0)I(A(0)), & A(\infty) = \infty \end{cases}. \tag{1.3}$$

This is the form of the homomorphism that we shall use.

A natural question to ask is whether the graph of normalized elliptic Dedekind sums is equidistributed. It would be natural to expect such a result based on the earlier works [4] and [9]. We also note that the conjecture of Ito in the same paper [3, §3] on the bias of Dedekind sums was recently proved in the case of classical Dedekind sums [8]; it would be interesting to explore the conjecture for elliptic Dedekind sums.

We conclude with a brief summary of the contents of this paper. In Sect. 2, we recall Martin's continued fraction algorithm. We also derive some properties of Martin's continued fraction algorithm generalizing the properties of the Hurwitz continued fraction algorithm in [3] and classical continued fractions [2]. In Sect. 3 we prove an approximation property for the generalized continued fractions analogous to [3, Lemma 1]. In Sect. 4 we prove the main theorem.

2 Martin's algorithm

Martin [6] provides a continued fraction algorithm that can be executed in an arbitrary imaginary quadratic field K with ring of integers $\mathcal{O} = \mathcal{O}_K$. For any $z \in \mathbb{C} \setminus K$, terminating the algorithm after n iterations produces an approximation $\frac{p_n}{q_n} \in K$ of z. The algorithm is implemented as follows: For $a, b \in \mathbb{C}$, let

$$S(a, b) = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}.$$

Fix $\varepsilon \in (0, 1)$ Let $B \subset \mathcal{O}\setminus\{0\}$ be a finite admissible subset for $varepsilon$ in the sense of [6, Definition 2.4]. Intuitively, an admissible set is taken to be a set B of denominators that are enough so that the collection of discs $D(a/b, \varepsilon/|b|)$ with $a \in K$ cover the complex plane. By [6, Theorem 4.3], we may take $B = \{1, 2, \ldots, \lfloor\sqrt{|d_K|}\rfloor\}$.

We define recursively as in Algorithm 1 in [6, §2.2] the sequence of matrices

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_n = M_{n-1} S\left(\frac{a_n}{b_{n-1}}, \frac{b_n}{b_{n-1}}\right), \ n \geq 1,$$

where the coefficients $a_i \in \mathcal{O}$, $b_i \in B$ can be determined by Algorithm 2 in [6, §3.2]. Note that we are following Martin's convention in Algorithm 1 in [6, §2.2] that $b_0 = 1$.

We then define the n-th convergents p_n and q_n as the left column entries of M_n. By the definition of S, the right column entries of M_n are then p_{n-1} and q_{n-1}. Hence, we have

$$M_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

and viewing M_n as a Möbius transformation on the extended complex plane, we can write $M_n(\infty) = \frac{p_n}{q_n}$. It follows inductively from the definition of S that

$$\det M_n = p_n q_{n-1} - p_{n-1} q_n = (-1)^n b_n. \tag{2.1}$$

Martin verifies that the n-th approximation $\frac{p_n}{q_n}$ can be written in the continued fraction form

$$\frac{p_n}{q_n} = \frac{a_1}{b_1} + \cfrac{b_0/b_1}{\frac{a_2}{b_2} + \cfrac{b_1/b_2}{\ddots + \frac{a_{n-1}}{b_{n-1}} + \frac{b_{n-2}/b_{n-1}}{a_n/b_n}}}.$$

2.1 Properties

We next recall the properties of Martin's continued fraction algorithm proved in [6] that we shall require. Define $\mu = \max_B |b|$ for a fixed finite admissible set B with $\varepsilon \in (0, 1)$. Then for any $n \geq 1$, the following properties hold.

(1) (Lemma 3.1)

$$|z_n| \geq 1/\varepsilon. \tag{2.2}$$

(2) (Proposition 3.2)

$$|q_n z - p_n| \leq \varepsilon |q_{n-1} z - p_{n-1}|.$$

(3) (Theorem 3.11) If $0 \leq n' < n$, then

$$|q_n| > \frac{(1-\varepsilon^2)^2 |q_{n'} z_{n'}|}{4\varepsilon^{n-n'} \mu^2}. \tag{2.3}$$

where $z_n = \dfrac{q_{n-1} z - p_{n-1}}{p_n - q_n z}$. In particular, $|q_n| > (1-\varepsilon^2)^2 / 4\varepsilon^n \mu^2$. This implies $\lim_{n\to\infty} |q_n| = \infty$.

(4) (Corollary 3.12) For all $n \geq 1$,

$$\left| z - \frac{p_n}{q_n} \right| < \frac{4\varepsilon^{2n} \mu^2}{(1-\epsilon^2)^2}, \tag{2.4}$$

which implies that $\lim_{n\to\infty} |z - \frac{p_n}{q_n}| = 0$.

We next derive several additional identities for Martin's generalized continued fractions. They are not used for the main theorem, but they were developed in the course of its proof. We include them here as they establish the general analogues of the Euclidean case. Let $a_n \in \mathcal{O}$ and $b_n \in B$ as before. Now define recursively the following notation

$$[b_0] = b_0, \qquad [b_0, a_1, b_1] = a_1$$

$$[b_0, \ldots, a_n, b_n] = \frac{a_n}{b_{n-1}}[b_0, \ldots, a_{n-1}, b_{n-1}] + \frac{b_n}{b_{n-1}}[b_0, \ldots, a_{n-2}, b_{n-2}].$$

The first property gives a continued fraction expansion identity for the convergents.

Lemma 2.1 *The convergents p_n and q_n from Martin's algorithm can be written as*

$$p_n = [b_0, a_1, b_1, a_2, \ldots, a_{n-1}, b_{n-1}, a_n, b_n]$$

$$q_n = [b_1, a_2, b_2, a_3, \ldots, a_{n-1}, b_{n-1}, a_n, b_n].$$

Proof To prove these statements we will use induction. Beginning with p_n, we use two base cases where $n = 0$ and $n = 1$. When $n = 0$, we have that $p_0 = 1$, and since we assume that $b_0 = 1$, $p_0 = 1 = b_0 = [b_0]$. Additionally when $n = 1$, we have that $p_1 = a_1 = [b_0, a_1, b_1]$. We then assume that the $n-1$ and $n-2$ cases hold, and examine the definition of $[b_0, a_1, b_1, a_2, \ldots, a_{n-1}, b_{n-1}, a_n, b_n]$. We have that

$$\begin{aligned}
&[b_0, a_1, b_1, a_2, \ldots, a_{n-1}, b_{n-1}, a_n, b_n] \\
&= \frac{a_n}{b_{n-1}}[b_0, \ldots, a_{n-1}, b_{n-1}] + \frac{b_n}{b_{n-1}}[b_0, \ldots, a_{n-2}, b_{n-2}] \\
&= \frac{a_n}{b_n} p_{n-1} + \frac{b_n}{b_{n-1}} p_{n-2} = p_n,
\end{aligned}$$

which proves the equality for p_n.

Next we do the same thing for q_n, instead using base cases $n = 1$ and $n = 2$. When $n = 1$, we have that $q_1 = b_1 = [b_1]$. When $n = 2$, we have that $q_2 = a_2 = [b_1, a_2, b_2]$. Now we assume that the $n-1$ and $n-2$ cases hold, and have by definition and the assumption of the $n-1$ and $n-2$ cases by a similar computation that $[b_1, a_2, b_2, a_3, \ldots, a_{n-1}, b_{n-1}, a_n, b_n] = q_n$. □

The following properties then generalize those of Hurwitz continued fractions, as stated in [3, §2]. These properties are analogous to those of Hurwitz continued fractions, with the differences that our coefficients are no longer necessarily in the ring of integers, that b_n is now part of the third property, and that the index is increased by one in (2.7) because we start at b_0 without an a_0.[2]

Lemma 2.2 *The following identities hold:*

$$[b_0, \ldots, a_n, b_n] = \frac{a_1}{b_1}[b_1, \ldots, a_n, b_n] + \frac{b_0}{b_1}[b_2, \ldots, a_n, b_n], \tag{2.5}$$

[2] We thank the reviewer for also pointing out to us that these hold as formal identities in the field of fractions of $\mathbb{Z}$ adjoined with sufficiently many variables.

$$[b_0, a_1, b_1, \ldots, a_n, b_n] = [b_n, a_n, \ldots, b_1, a_1, b_0], \tag{2.6}$$

$$[b_m, \ldots, b_n][b_{m+1}, \ldots, b_{n-1}] - [b_m, \ldots, b_{n-1}][b_{m+1}, \ldots, b_n] = b_n b_m (-1)^{n-m}. \tag{2.7}$$

Proof Beginning with the bracket notation for $\frac{p_n}{q_n}$, one checks that

$$\frac{[b_0, a_1, \ldots, a_n, b_n]}{[b_1, a_2, \ldots, a_n, b_n]} = \frac{a_1}{b_1} + \frac{b_0}{b_1} \cdot \frac{[b_2, a_3, \ldots, a_n, b_n]}{[b_1, a_2, \ldots, a_n, b_n]}.$$

Multiply the left and rightmost sides of this equality to obtain

$$[b_0, \ldots, a_n, b_n] = \frac{a_1}{b_1}[b_1, \ldots, a_n, b_n] + \frac{b_0}{b_1}[b_2, \ldots, a_n, b_n].$$

This proves the first identity (2.5).

We prove the second property by induction. To start, we establish the base cases of $n = 0$ and $n = 1$. For $n = 0$, we have that $[b_0] = b_0 = [b_0]$. For $n = 1$ we have that $[b_0, a_1, b_1] = a_1 = [b_1, a_1, b_0]$. Having established the base cases, we assume that the $n - 1$ and $n - 2$ cases hold. We have that

$$[b_0, a_1, \ldots, a_n, b_n] = \frac{a_1}{b_1}[b_1, a_2, \ldots, a_n, b_n] + \frac{b_0}{b_1}[b_2, a_3, \ldots, a_n, b_n]$$

by (2), and because we are assuming the $n - 1$ and $n - 2$ cases, this is equal to

$$\frac{a_1}{b_1}[b_n, a_n, \ldots, a_2, b_1] + \frac{b_0}{b_1}[b_n, a_n, \ldots, a_3, b_2] = [b_n, a_n, \ldots, a_1, b_0].$$

We prove (2.7) both when m is fixed and when n is fixed. When we fix m, we have by direct computation that

$$\begin{aligned}
&[b_{m+1}, a_m, \ldots, a_{n-1}, b_{n-1}]\left(\frac{a_n}{b_{n-1}}[b_m, a_{m+1}, \ldots, a_{n-1}, b_{n-1}]\right.\\
&\quad\left.+\frac{b_n}{b_{n-1}}[b_m, a_{m+1}, \ldots, a_{n-2}, b_{m-2}]\right)\\
&-[b_m, a_{m+1}, \ldots, a_{n-1}, b_{n-1}]\left(\frac{a_n}{b_{n-1}}[b_{m+1}, a_{m+2}, \ldots, a_{n-1}, b_{n-1}]\right.\\
&\quad\left.+\frac{b_n}{b_{n-1}}[b_{m+1}, a_{m+2}, \ldots, a_{n-2}, b_{n-2}]\right)
\end{aligned}$$

equals

$$\frac{b_n}{b_{n-1}}\left((-1)^{(n-1)-m} b_{n-1} b_m\right) = (-1)^{(n-m)} b_n b_m.$$

When we fix n, we have similarly that

$$
\begin{aligned}
&[b_{m+1}, a_m, \ldots, a_{n-1}, b_{n-1}] \left(\frac{a_{m+1}}{b_{m+1}} [b_{m+1}, a_{m+2}, \ldots, a_n, b_n \right.\\
&\quad \left. + \frac{b_m}{b_{m+1}} [b_{m+2}, a_{m+3}, \ldots, a_n, b_n] \right)\\
&-[b_{m+1}, a_{m+2}, \ldots, a_n, b_n] \left(\frac{a_{m+1}}{b_{m+1}} [b_{m+1}, a_{m+2}, \ldots, a_{n-1}, b_{n-1}] \right.\\
&\quad \left. + \frac{b_m}{b_{m+1}} [b_{m+2}, a_{m+3}, \ldots, a_{n-1}, b_{n-1}] \right)
\end{aligned}
$$

equals

$$
\frac{b_m}{b_{m+1}} \left((-1)^{n-(m+1)} b_{m+1} b_n \right) = (-1)^{(n-m)} b_n b_m.
$$

□

3 Approximation by generalized continued fractions

We now prove an approximation result generalizing [3, Lemma 1] to the convergents produced by Martin's algorithm. Denote

$$
\zeta = \frac{(1-\varepsilon^2)^2}{4\varepsilon^2 \mu^2},
$$

where ε and μ are defined as above.

Lemma 3.1 *Consider the continued fraction expansion of $z \in \mathbb{C} \backslash K$. Let $\delta \in (0, \zeta)$ and $W \in GL_2(\mathcal{O})$ be such that $W(\infty) \neq \infty$ and $|W(\infty)| \geq \frac{1}{\zeta - \delta}$. Then, $M_n W(\infty) \neq \infty$, and for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ independent of W such that*

$$
|z - M_n W(\infty)| < \epsilon
$$

for all $n \geq N$.

Proof Since $0 < \frac{1}{\zeta-\delta} \leq |W(\infty)| < \infty$, we can define $w := \frac{1}{W(\infty)}$. Note that $|w| \leq \zeta - \delta$ by the assumption of the lemma. In addition, we have

$$
\begin{aligned}
M_n W(\infty) &= \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} (W(\infty)) \\
&= \frac{p_n W(\infty) + p_{n-1}}{q_n W(\infty) + q_{n-1}} \\
&= \frac{p_n + \frac{1}{W(\infty)} p_{n-1}}{q_n + \frac{1}{W(\infty)} q_{n-1}}
\end{aligned}
$$

$$= \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}}.$$

In order to show that $M_n W(\infty) \neq \infty$, we must show that $q_n + wq_{n-1} \neq 0$. Setting $n' = n - 1$ in (2.3) and combining with (2.2), we see that for $n \geq 2$,

$$|q_n| > \frac{(1-\varepsilon^2)^2}{4\varepsilon\mu^2}|q_{n-1}z_{n-1}| \geq \frac{(1-\varepsilon^2)^2}{4\varepsilon^2\mu^2}|q_{n-1}| = \zeta|q_{n-1}|.$$

Using the triangle inequality, we have

$$\begin{aligned} |q_n + wq_{n-1}| &\geq |q_n| - |wq_{n-1}| \\ &> \zeta|q_{n-1}| - |w||q_{n-1}| \\ &\geq \delta|q_{n-1}|. \end{aligned}$$

Since $|q_n + wq_{n-1}| > \delta|q_{n-1}| > 0$, we can conclude that $q_n + wq_{n-1} \neq 0$, as desired.

Moving on to the second part of the proof, we must show that there exists $N \in \mathbb{N}$ such that

$$|z - M_n W(\infty)| = \left| z - \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}} \right| < \epsilon$$

for all $n \geq N$.

Applying the triangle inequality, we have

$$\left| z - \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}} \right| \leq \left| z - \frac{p_n}{q_n} \right| + \left| \frac{p_n}{q_n} - \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}} \right|.$$

By (2.4), the first term above tends to zero, so we can define $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have

$$\left| z - \frac{p_n}{q_n} \right| < \frac{\epsilon}{2}.$$

Now we are left to bound the second term above. By combining fractions, we see that

$$\begin{aligned} \left| \frac{p_n}{q_n} - \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}} \right| &= \left| \frac{p_n q_n + wp_n q_{n-1} - p_n q_n - wp_{n-1}q_n}{q_n(q_n + wq_{n-1})} \right| \\ &= \left| \frac{w(p_n q_{n-1} - p_{n-1}q_n)}{q_n(q_n + wq_{n-1})} \right|. \end{aligned}$$

By (2.1), we see that

$$\left| \frac{w(p_n q_{n-1} - p_{n-1}q_n)}{q_n(q_n + wq_{n-1})} \right| = \left| \frac{wb_n(-1)^n}{q_n(q_n + wq_{n-1})} \right| = \left| \frac{wb_n}{q_n(q_n + wq_{n-1})} \right|.$$

Finally, using that $|q_n + wq_{n-1}| > \delta|q_{n-1}|$, we have

$$\left| \frac{wb_n}{q_n(q_n + wq_{n-1})} \right| < \left| \frac{wb_n}{\delta q_n q_{n-1}} \right| \leq \frac{\mu|w|}{|\delta|} \frac{1}{|q_n q_{n-1}|}.$$

Since the rightmost side of this inequality tends to 0 as n approaches infinity by (2.3), there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have

$$\left|\frac{p_n}{q_n} - \frac{p_n + wp_{n-1}}{q_n + wq_{n-1}}\right| < \frac{\mu|w|}{|\delta|} \frac{1}{|q_n q_{n-1}|} < \frac{\epsilon}{2}.$$

Letting $N = \max\{N_1, N_2\}$ completes the proof. □

4 Proof of density

We now prove the main theorem. We utilize the homomorphism (1.3) to prove density in the general case. Following along with Ito's proof of [3, Theorem 2], we will show that the set

$$\{(x, (\sqrt{|d|}i)^{-1} I(x - z)) : x, z \in \mathbb{C} - K\}$$

is contained in the closure of the set

$$\{(\alpha, \tilde{D}(\alpha)) : \alpha \in K\}.$$

Recall from the discussion in the introduction that any element of K can be written as a/c with $a, c \in \mathcal{O}_K$. This representation is non-unique but there are only finitely many choices, and here we are fixing a choice of representative. Our proof is independent of this choice.

Let $\epsilon > 0$ and $x, z \in \mathbb{C} - K$. Suppose first that for $|\alpha - x| < \epsilon$, we have

$$|\tilde{D}(\alpha) - (\sqrt{|d|}i)^{-1} I(x - z)| \leq 4(\sqrt{|d|})^{-1}\epsilon. \tag{4.1}$$

It then follows that

$$(x, (\sqrt{|d|}i)^{-1} I(x - z)) \in \overline{\{(\alpha, \tilde{D}(\alpha)) : \alpha \in K\}},$$

since that K is dense in $\mathbb{C}$ (see for example [11]). It thus remains to prove (4.1). We shall do this by constructing an $A \in GL_2(\mathcal{O})$ such that $\alpha = A(\infty)$, which approximates x, and $\beta = A^{-1}(\infty)$ approximates z. Then (4.1) will follow from evaluating $\Phi(A)$ in two ways.

By Lemma 3.1, we can choose sufficiently large $m, n \in \mathbb{N}$ such that for any $W \in GL_2(\mathcal{O})$ with $|W(\infty)| \geq \frac{2}{\zeta}$, we have

$$|x - M_{m,x} W(\infty)| < \epsilon$$

and

$$|z - M_{n,z} W(\infty)| < \epsilon,$$

where $M_{m,x}$ denotes the m-th convergent matrix in the continued fraction representation of x and similarly for $M_{n,z}$.

Let $S = M_{m,x}^{-1} M_{n,z}$, and let S^* be given by

$$S^* = \begin{cases} S, & |S(\infty)| \neq \infty \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S, & |S(\infty)| = \infty. \end{cases}$$

We claim that $|S^*(\infty)| \neq \infty$. First suppose that $|S(\infty)| \neq \infty$. In this case, we simply have $|S^*(\infty)| = |S(\infty)| \neq \infty$. Next suppose $|S(\infty)| = \infty$. In this case, we have

$$|S^*(\infty)| = \left| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S(\infty) \right| = \left| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\infty) \right| = 0 \neq \infty.$$

In both cases, this also tells us that $|{S^*}^{-1}(\infty)| \neq \infty$ because $|{S^*}^{-1}(\infty)| = \infty$ would imply $|S^*(\infty)| = \infty$, a contradiction.

We also note that in either case we have

$$\Phi(S^*) = \Phi(S).$$

In the first case, this is clear because $S^* = S$. In the second case, we use the fact that Φ is a homomorphism to see that

$$\Phi(S^*) = \Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S \right) = \Phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \Phi(S) = \Phi(S),$$

since

$$4\Phi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 \right) = \Phi(I) = 0.$$

Finally, we note that $\det(S^*) = \det(S)$.

Now given $u \in \mathcal{O}$, we can define the matrix $A \in GL_2(\mathcal{O})$ by

$$A = M_{m,x} T^u S^* T^{-u} M_{n,z}^{-1}, \qquad T^u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \; T^{-u} = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}.$$

Using the property that $\det(S^*) = \det(S) = \det(M_{m,x}^{-1} M_{n,z})$, we can verify by direct computation that $\det(A) = 1$. Next, let

$$\alpha = A(\infty), \qquad \beta = A^{-1}(\infty) = M_{n,z} T^u {S^*}^{-1} T^{-u} M_{m,x}^{-1}(\infty)$$

and

$$W_1 = T^u S^* T^{-u} M_{n,z}^{-1}, \qquad W_2 = T^u {S^*}^{-1} T^{-u} M_{m,x}^{-1},$$

so that

$$\alpha = M_{m,x} W_1(\infty), \qquad \beta = M_{n,z} W_2(\infty).$$

We now pick $u \in \mathcal{O}$ sufficiently large so that

$$|W_1(\infty)| \geq \frac{2}{\zeta}, \qquad |W_2(\infty)| \geq \frac{2}{\zeta}.$$

Note that $\frac{2}{\zeta}$ can be written as $\frac{1}{\zeta - \delta}$ for $\delta = \frac{\zeta}{2} \in (0, \zeta)$.

We must make sure that we are always able to pick a coefficient u with this property. Let $S^* = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}$. Note that $M_{n,z}^{-1} = \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix}$, where these are convergents of z using Martin's algorithm. Before proceeding with any computation, we add the simple condition that $u \neq \frac{s_4 q_n - s_3 q_{n-1}}{s_3 q_n}$, noting that $q_n \neq 0$ and $s_3 \neq 0$ since $S^* \in GL_2(\mathcal{O})$ and $S^*(\infty) \neq \infty$. This condition implies that $s_3(q_{n-1} + q_n u) - s_4 q_n \neq 0$ and, as we will see below, ensures that $W_1(\infty) \neq \infty$. Now, we can see that

$$\begin{aligned} W_1 &= T^u S^* T^{-u} M_{n,z}^{-1} \\ &= \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} \\ &= \begin{pmatrix} s_1 + s_3 u & s_2 + s_4 u \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} q_{n-1} + q_n u & -p_{n-1} - p_n u \\ -q_n & p_n \end{pmatrix} \\ &= \begin{pmatrix} (s_1 + s_3 u)(q_{n-1} + q_n u) - (s_2 + s_4 u) q_n & (s_1 + s_3 u)(-p_{n-1} - p_n u) + (s_2 + s_4 u) p_n \\ s_3(q_{n-1} + q_n u) - s_4 q_n & s_3(-p_{n-1} - p_n u) + s_4 p_n \end{pmatrix}. \end{aligned}$$

We can use this, along with the triangle inequality, to see that

$$\begin{aligned} |W_1(\infty)| &= \left| \frac{(s_1 + s_3 u)(q_{n-1} + q_n u) - (s_2 + s_4 u) q_n}{s_3(q_{n-1} + q_n u) - s_4 q_n} \right| \\ &= \left| \frac{u(s_3(q_{n-1} + q_n u) - s_4 q_n) + s_1(q_{n-1} + q_n u) - s_2 q_n}{s_3(q_{n-1} + q_n u) - s_4 q_n} \right| \\ &= \left| u + \frac{s_1(q_{n-1} + q_n u) - s_2 q_n}{s_3(q_{n-1} + q_n u) - s_4 q_n} \right| \\ &\geq |u| - \left| \frac{s_1(q_{n-1} + q_n u) - s_2 q_n}{s_3(q_{n-1} + q_n u) - s_4 q_n} \right|. \end{aligned}$$

As $|u|$ tends to infinity, the second term in the inequality above tends to

$$-\left| \frac{s_1}{s_3} \right| = -|S^*(\infty)|,$$

which, by definition of S^*, is not infinite. Therefore, as $|u|$ tends to infinity, $|W_1(\infty)|$ also tends to infinity. We can use a very similar argument, which uses the m-th convergents of x and the fact that ${S^*}^{-1}(\infty) \neq \infty$, to show that $|W_2(\infty)|$ can also be made arbitrarily large as $|u|$ becomes arbitrarily large. Again, we can impose a simple condition on the value of u of the same form as above in order to ensure that $W_2(\infty) \neq \infty$.

Let $\delta_1 = \alpha - x$ and $\delta_2 = \beta - z$. We can now apply Lemma 3.1 to see that

$$|\delta_1| = |x - \alpha| = |x - M_{m,x}W_1(\infty)| < \epsilon.$$

By the same logic, we have $|\delta_2| < \epsilon$. Since we know α and β are within ϵ of x and z, respectively, we can be sure that α and β are both well-defined complex numbers and not infinite.

To conclude the proof, we shall evaluate $\Phi(A)$ in two different ways. Note that since $A(\infty) = \alpha \neq \infty$, we are in the first case of (1.3). If we apply Φ to A directly, we get

$$\begin{aligned}\Phi(A) &= E_2(0)I(A(\infty) - A^{-1}(\infty)) - D(A(\infty))\\ &= E_2(0)I(\alpha - \beta) - D(\alpha)\\ &= E_2(0)I(x + \delta_1 - z - \delta_2) - D(\alpha).\end{aligned}$$

If we apply Φ to A by breaking it up into smaller matrices in $GL_2(\mathcal{O})$, we see that

$$\begin{aligned}\Phi(A) &= \Phi(M_{m,x}T^uS^*T^{-u}M_{n,z}^{-1})\\ &= \Phi(M_{m,x}) + \Phi(T^u) + \Phi(M_{m,x}^{-1}M_{n,z}) + \Phi(T^{-u}) + \Phi(M_{n,z}^{-1})\\ &= 0.\end{aligned}$$

Therefore, we have

$$E_2(0)I(x + \delta_1 - z - \delta_2) - D(\alpha) = 0,$$

which gives us $D(\alpha) = E_2(0)I(x + \delta_1 - z - \delta_2)$ and hence

$$\tilde{D}(\alpha) = (\sqrt{|d|}i)^{-1}I(x + \delta_1 - z - \delta_2),$$

as required, which we note is independent of the choice of representation of α.

Acknowledgements The authors thank Daniel Martin for communications regarding their paper. This research was conducted at the REU site: Mathematical Analysis and Applications at the University of Michigan-Dearborn.

References

1. Berkopec, Nicolas, Branch, Jacob, Heikkinen, Rachel, Nunn, Caroline, Wong, Tian An: The density of elliptic Dedekind sums. Acta Arith. **205**(1), 33–40 (2022)
2. Hickerson, Dean: Continued fractions and density results for Dedekind sums. J. Reine Angew. Math. **290**, 113–116 (1977)
3. Ito, Hiroshi: A density result for elliptic Dedekind sums. Acta Arith. **112**(2), 199–208 (2004)
4. Kim Klinger-Logan and Tian An Wong: The equidistribution of elliptic Dedekind sums and generalized Selberg-Kloosterman sums. Res. Number Theory **10**(1), 19 (2024)
5. Kohnen, Winfried: A short note on Dedekind sums. Ramanujan J. **45**(2), 491–495 (2018)
6. Martin, D.E.: Continued fractions over non-Euclidean imaginary quadratic rings. J. Number Theory **243**, 688–714 (2023)

7. Martin, Kimball: Nonunique factorization and principalization in number fields. Proc. Am. Math. Soc. **139**(9), 3025–3038 (2011)
8. Minelli, Paolo, Sourmelidis, Athanasios, Technau, Marc: Bias in the number of steps in the Euclidean algorithm and a conjecture of Ito on Dedekind sums. Math. Ann. **387**(1–2), 291–320 (2023)
9. Myerson, Gerald: Dedekind sums and uniform distribution. J. Number Theory **28**(3), 233–239 (1988)
10. Obaisi, R.: Eisenstein cocycles for GL(2) and special values of Hecke L-functions over imaginary quadratic fields. ProQuest LLC, Ann Arbor, MI, 2000. Thesis (Ph.D.)—Rutgers The State University of New Jersey - Newark
11. Oswald, N.M.R.: Diophantine approximation of complex numbers. Šiauliai Math. Semin. **10**(18), 91–102 (2015)
12. Rademacher, H., Grosswald, E.: Dedekind sums. The Carus Mathematical Monographs, No. 16. The Mathematical Association of America, Washington, D.C. (1972)
13. Sczech, R.: Dedekindsummen mit elliptischen Funktionen. Invent. Math. **76**(3), 523–551 (1984)