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RESEARCH ARTICLE

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On the Lie algebra structure of integrable derivations

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Abstract

Building on work of Gerstenhaber, we show that the space of integrable derivations on an Artin algebra A forms a Lie algebra, and a restricted Lie algebra if A contains a field of characteristic p. We deduce that the space of integrable classes in $\mathrm{HH}^1(A)$ forms a (restricted) Lie algebra that is invariant under derived equivalences, and under stable equivalences of Morita type between self-injective algebras. We also provide negative answers to questions about integrable derivations posed by Linckelmann and by Farkas, Geiss and Marcos. Along the way, we compute the first Hochschild cohomology of the group algebra of any symmetric group.

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1 | INTRODUCTION

For any algebra A over a commutative ring k, we consider the Lie algebra of k-linear derivations on A. The subspace of integrable derivations was introduced by Hasse and Schmidt in [12], and has since been important in geometry and commutative algebra, especially in regards to deformations, jet spaces, and automorphism groups [21, 22, 29].

More recently, integrable derivations have been used as source of invariants in representation theory [8, 18]. The first Hochschild cohomology Lie algebra $\mathrm{HH}^1(A)$, consisting of all derivations modulo inner derivations, is a critically important invariant, and the subspace $\mathrm{HH}^1_{\mathrm{int}}(A)$ spanned by integrable derivations is the main object of interest in this work. This class of derivations is known to have good invariance properties: Farkas, Geiss and Marcos prove that $\mathrm{HH}^1_{\mathrm{int}}(A)$ is an invariant under Morita equivalences [8], and Linckelmann proves, for self-injective algebras, that $\mathrm{HH}^1_{\mathrm{int}}(A)$ is an invariant under stable equivalences of Morita type [18].

In the first part of this work, we survey some results of Gerstenhaber on integrable derivations [10], which seem not to be well-known. Building on this, we prove that for Artin algebras, the integrable derivations form a (restricted) Lie algebra.

Theorem A (See Corollary 1). *If A is an Artin algebra over a commutative Artinian ring k, then $\mathrm{HH}^1_{\mathrm{int}}(A)$ is a Lie subalgebra of $\mathrm{HH}^1(A)$. If moreover A contains a field of characteristic p, then $\mathrm{HH}^1_{\mathrm{int}}(A)$ is a restricted Lie subalgebra of $\mathrm{HH}^1(A)$.*

We also show that this (restricted) Lie algebra is invariant under derived equivalences and stable equivalences of Morita type, extending the work of Farkas, Geiss and Marcos [8] and Linckelmann [18].

Theorem B (See Theorem 2). *Let A and B be two finite-dimensional split algebras over a field k. Assume either that A and B are derived equivalent, or that A and B are self-injective and stably equivalent of Morita type. Then $\mathrm{HH}^1_{\mathrm{int}}(A) \cong \mathrm{HH}^1_{\mathrm{int}}(B)$ as Lie algebras, and this is an isomorphism of restricted Lie algebras if k is of positive characteristic.*

In the second part of this work, we answer several questions posed in [8] and [17] about integrable derivations on group algebras. We show by example that the Lie algebra of integrable derivations is not always solvable for every block of a finite group, proving a negative answer to [17, Question 8.2].

Theorem C (See Theorem 4). *Let k be a field of characteristic $p \geqslant 3$. Let P be an elementary abelian p-group of rank greater than 1. Then $\mathrm{HH}^1_{\mathrm{int}}(kP)$ is not solvable.*

By [8, Theorem 2.2], the group algebra of a finite p-group, over a field of characteristic p, always admits a non-integrable derivation. The authors of this work ask whether this is true of all finite groups with order divisible by p, and our next example shows that this is not the case.

Theorem D (See Theorem 8). *Let k a field of characteristic $p \geqslant 3$ and let kS_p be the group algebra of the symmetric group on p letters. Then $\dim_k(\mathrm{HH}^1(kS_p)) = 1$ and $\mathrm{HH}^1(kS_p) = \mathrm{HH}^1_{\mathrm{int}}(kS_p)$.*

Along the way, in Theorem 5, we give a formula for the dimension of $\mathrm{HH}^1(kS_n)$ for any n. The same formula has also been obtained independently in the recent work [4]. The first part of Theorem 8, that $\dim_k(\mathrm{HH}^1(kS_p)) = 1$, is an immediate consequence.

Outline

The sections of this paper can be read independently. In Section 2, we survey parts of the work of Gerstenhaber on integrable derivations and note some consequences. Here we prove Theorems A and B. The main result of Section 3 is Theorem C that provides the first counter-example. In Section 4, we give a formula for the dimension of $\mathrm{HH}^1(kS_n)$. The main result in Section 5 is Theorem D that provides the second counter-example. The Appendix contains a dictionary explaining the more general terminology used by Gerstenhaber in [10], and how it relates to the setting considered here.

2 | INTEGRABLE DERIVATIONS

Let A be an algebra over a commutative ring k, and let $\mathrm{Der}(A)$ denote the space of k-linear derivations on A. Gerstenhaber investigated integral derivations in [10], but as his work was written in substantial generality, in the language of 'composition complexes', many of its results are not well-known. In this section, we present some of the results of [10] in more familiar terms (but in particular Lemma 1 and its consequences are new to this work). The main result of this section is that if A is an Artin algebra, the class of integrable derivations forms a Lie subalgebra of $\mathrm{Der}(A)$.

To define integrable derivations we consider the k-algebras $A[t]/(t^n)$ and their limit $A[\![t]\!]$. We denote by $\mathrm{Aut}_1(A[t]/(t^n))$ the group of $k[t]/(t^n)$-algebra automorphisms α that yield the identity modulo t. Any such automorphism can be expanded

$$\alpha = \mathrm{id} + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1}$$

for some k-linear maps $\alpha_i : A \to A$. The first of these, α_1, is a derivation on A, and in general the maps satisfy $\alpha_i(xy) = x\alpha_i(y) + \alpha_1(x)\alpha_{i-1}(y) + \cdots + \alpha_i(x)y$ for all i. A sequence $\alpha_1, \ldots, \alpha_{n-1}$ of linear endomorphisms of A satisfying these identities is called a Hasse–Schmidt derivation of order n. One similarly interprets elements $\alpha \in \mathrm{Aut}_1(A[\![t]\!])$ as infinite sequences $\alpha_1, \alpha_2, \ldots$ of linear endomorphisms of A, and these are called Hasse–Schmidt derivations of infinite order. These were studied in [12] under the name higher derivation.

Extending this slightly, we set $\mathrm{Aut}_m(A[t]/(t^n))$ to be the group of $k[t]/(t^n)$-algebra automorphisms α that yield the identity modulo t^m. Any such automorphism can be expanded

$$\alpha = \mathrm{id} + \alpha_m t^m + \alpha_{m+1} t^{m+1} + \cdots + \alpha_{n-1} t^{n-1}$$

for some $\alpha_i : A \to A$. The first non-vanishing coefficient α_m is always a derivation on A. The same goes for the power series algebra $A[\![t]\!]$ and the corresponding automorphisms in $\mathrm{Aut}_m(A[\![t]\!])$.

Definition 1 [10, 12, 26]. A k-linear derivation D on A is called $[m,n)$-integrable if there is an automorphism $\alpha \in \mathrm{Aut}_m(A[t]/(t^n))$ such that $D = \alpha_m$. We say that D is $[m,\infty)$-integrable if it is $[m,n)$-integrable for all n. And we say that D is $[m,\infty]$-integrable if there is an automorphism $\alpha \in \mathrm{Aut}_m(A[\![t]\!])$ such that $D = \alpha_m$. We will write

$$\mathrm{Der}_{[m,n)}(A) = \left\{ k\text{-linear } [m,n)\text{-integrable derivations on } A \right\},$$

and we will use similar notation for $[m,\infty)$- and $[m,\infty]$-integrable derivations.

The derivations that are $[1,\infty]$-integrable are simply known as integrable, and we will also use the notation $\mathrm{Der}_{\mathrm{int}}(A) = \mathrm{Der}_{[1,\infty]}(A)$.

Remark 1. The $[m,n)$-integrable derivations are closed under addition and subtraction, that is, $\mathrm{Der}_{[m,n)}(A)$ is an additive subgroup of $\mathrm{Der}(A)$. If $\alpha, \alpha' \in \mathrm{Aut}_m(A[t]/(t^n))$ then the automorphisms

$$\alpha\alpha' = \mathrm{id} + (\alpha_m + \alpha'_m)t^m + \cdots \quad \text{and} \quad \alpha^{-1} = \mathrm{id} - \alpha_m t^m + \cdots$$

are witness to fact that $\alpha_m + \alpha'_m$ and $-\alpha_m$ are $[m,n)$-integrable. The same argument holds for $[m,\infty)$- and $[m,\infty]$-integrable derivations.

In the case $m = 1$, $\mathrm{Der}_{[1,n)}(A)$ is moreover a submodule of $\mathrm{Der}(A)$ over $\mathrm{Z}(A)$, the centre of A. This can be seen from the automorphism

$$\mathrm{id} + z\alpha_1 t + z^2\alpha_2 t^2 + \cdots$$

that exists for any $\alpha \in \mathrm{Aut}_1(A[t]/(t^n))$ and $z \in \mathrm{Z}(A)$.

Remark 2. All inner derivations are integrable. Indeed if $a \in A$, then $1 + at$ is a unit in $A[\![t]\!]$ and the automorphism $\mathrm{ad}(1+at) = \mathrm{id} + [a,-]t + \cdots$ shows that $[a,-]$ is integrable.

Remark 3. Integrable derivations preserve the Jacobson radical (see [8, Corollary 2.1]). However, the converse is not true. Let k be a field of characteristic 2 and $A = k[x,y]/(x^2,y^3)$. Consider the derivation $y\partial_x$ where ∂_x is the unique k-linear derivation such that $\partial_x(x) = 1$ and $\partial_x(y) = 0$. Then $y\partial_x$ is not an integrable derivation but it preserves the Jacobson radical. Indeed, assume that $y\partial_x$ is integrable. Then there exists an automorphism α such that $0 = \alpha(x^2) = \alpha(x)^2 = y^2t^2 + \cdots$ which is a contradiction.

Definition 2. The first Hochschild cohomology of A over k is the quotient $\mathrm{HH}^1(A)$ of $\mathrm{Der}(A)$ by the space of inner derivations. We denote by $\mathrm{HH}^1_{\mathrm{int}}(A)$ the image of $\mathrm{Der}_{\mathrm{int}}(A)$ in $\mathrm{HH}^1(A)$. By the previous two remarks, a class in Hochschild cohomology is integrable if and only if all or any one of its representatives is integrable.

Remark 4. If D and D' are derivations that are $[m,n)$-integrable and $[m',n)$-integrable, respectively, then the commutator $[D,D']$ is an $[m+m',n)$-integrable derivation. Indeed, a computation shows that

$$\alpha\alpha'\alpha^{-1}\alpha'^{-1} = \mathrm{id} + (\alpha_m\alpha'_{m'} - \alpha'_{m'}\alpha_m)t^{m+m'} + \cdots$$

for $\alpha \in \mathrm{Aut}_m(A[t]/(t^n))$ and $\alpha' \in \mathrm{Aut}_{m'}(A[t]/(t^n))$.

If A contains a field of characteristic p, then the pth power of any derivation is again a derivation, and together with the commutator bracket this gives $\mathrm{Der}(A)$ the structure of a restricted Lie algebra. If D is an $[m,n)$-integrable derivation then D^p is a $[pm,n)$-integrable derivation. Indeed, this time one computes

$$\alpha^p = \mathrm{id} + \alpha_m^p t^{pm} + \cdots$$

for $\alpha \in \mathrm{Aut}_m(A[t]/(t^n))$. This structure is studied in [26].

This remark shows that $[m,n)$-integrable derivations arise even if one is interested only in $[1,n)$-integrable derivations.

Gerstenhaber works locally in [10], assuming that k is an algebra over $\mathbb{Z}_p$ (the integers localised at p) for some prime p. Using the next lemma, in which we write $k_p = k \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $A_p = A \otimes_{\mathbb{Z}} \mathbb{Z}_p$, we can readily reduce to this case. Readers interested in algebras over local rings or fields can disregard the lemma.

Lemma 1. *Let A be a Noether algebra over a commutative Noetherian ring k. A k-linear derivation on A is $[m,n)$-integrable if and only if for all primes p, the induced k_p-linear derivation on A_p is $[m,n)$-integrable.*

Proof. The forward implication is clear. Conversely, assume that for each prime p there is an automorphism $\alpha = \mathrm{id} + \alpha_m t^m + \cdots + \alpha_{n-1}t^{n-1}$ in $\mathrm{Aut}_m(A_p[t]/(t^n))$ such that $\alpha_m = D$.

Consider the element $(\alpha_i) \in \bigoplus_{i=m}^{n-1} \mathrm{Hom}_{k_p}(A_p, A_p)$. As k is Noetherian and A is finitely generated as a k-module, $\bigoplus_{i=m}^{n-1} \mathrm{Hom}_{k_p}(A_p, A_p) \cong (\bigoplus_{i=m}^{n-1} \mathrm{Hom}_k(A, A))_p$, and therefore there is a sequence $(\beta_i) \in \bigoplus_{i=m}^{n-1} \mathrm{Hom}_k(A, A)$ and an integer u coprime to p such that $(\alpha_i) = (\frac{\beta_i}{u})$ in $\bigoplus_{i=m}^{n-1} \mathrm{Hom}_{k_p}(A_p, A_p)$, and we can assume that $\beta_m = uD$. There is then an equality

$$\mathrm{id} + u^m\alpha_m t^m + \cdots + u^{n-1}\alpha_{n-1}t^{n-1} = \mathrm{id} + u^{m-1}\beta_m t^m + \cdots + u^{n-2}\beta_{n-1}t^{n-1}$$

in $\mathrm{Aut}_m(A_p[t]/(t^n))$. In particular, for each $m \leqslant i < n$ the Hasse–Schmidt identity

$$u^{i-1}\beta_i(xy) - u^{i-1}x\beta_i(y) - u^{i-2}\beta_m(x)\beta_{i-m}(y) - \cdots - u^{i-1}\beta_i(x)y = 0$$

holds, when interpreted in $\mathrm{Hom}_{k_p}(A_p \otimes_{k_p} A_p, A_p)$; here $\alpha_i = \beta_i = 0$ if $0 < i < m$, by convention. As $\mathrm{Hom}_{k_p}(A_p \otimes_{k_p} A_p, A_p) \cong \mathrm{Hom}_k(A \otimes_k A, A)_p$ we may find an integer v_i coprime to p such that

$$v_i\left[u^{i-1}\beta_i(xy) - u^{i-1}x\beta_i(y) - u^{i-2}\beta_m(x)\beta_{i-m}(y) - \cdots - u^{i-1}\beta_i(x)y\right] = 0$$

holds in $\mathrm{Hom}_k(A \otimes_k A, A)$. Now set $v = v_m \cdots v_{n-1}$. It follows that the sequence

$$(v^m u^{m-1}\beta_m, \dots, v^{n-1}u^{n-2}\beta_{n-1})$$

satisfies the Hasse–Schmidt identities in $\mathrm{Hom}_k(A \otimes_k A, A)$ for all $m \leqslant i < n$, and therefore defines an element of $\mathrm{Aut}_m(A[t]/(t^n))$. Now set

$$\gamma_p = \mathrm{id} + v^m u^{m-1}\beta_m t^m + \cdots + v^{n-1}u^{n-2}\beta_{n-1}t^{n-1} \quad \text{and} \quad w_p = v^m u^m$$

(up until now our notation has not indicated the dependence on p). Note that $w_p D = v^m u^{m-1}\beta_m$ by construction.

The ideal $(w_p : p \text{ is a prime}) \subseteq \mathbb{Z}$ contains for each prime p an element coprime to p, and it is consequently the unit ideal. This means there are primes $p_1, \dots, p_i$ and integers $a_1, \dots, a_i$ such that $a_1 w_{p_1} + \cdots + a_i w_{p_i} = 1$. The automorphism

$$\gamma = \gamma_{p_1}^{a_1} \cdots \gamma_{p_i}^{a_i} \in \mathrm{Aut}_m(A[t]/(t^n))$$

has as its t^m coefficient $a_1 w_{p_1} D + \cdots + a_i w_{p_i} D = D$. Therefore, γ shows that D is $[m, n)$-integrable. □

Certainly all $[m, \infty]$-integrable derivations are $[m, \infty)$-integrable. It seems to be open in general whether the inclusion $\mathrm{Der}_{[m,\infty)}(A) \subseteq \mathrm{Der}_{[m,\infty]}(A)$ can be strict. However, the next result, essentially due to Gerstenhaber, shows that equality holds for Artin algebras.

Lemma 2. *Let A be an Artin algebra over a commutative Artinian ring k. A k-linear derivation on A is $[1, \infty)$-integrable if and only if it is $[1, \infty]$-integrable.*

Proof. By Lemma 1, we may assume that k is an algebra over $\mathbb{Z}_p$, so that the results of [10] apply. As $\mathrm{Der}(A)$ is an Artinian k-module, the sequence of submodules $\mathrm{Der}_{[1,n)}(A)$ must eventually stabilise, and so there is an n such that $\mathrm{Der}_{[1,n)}(A) = \mathrm{Der}_{[1,\infty)}(A)$. The proof of [10, Theorem 5] shows that this implies $\mathrm{Der}_{[1,\infty)}(A) = \mathrm{Der}_{[1,\infty]}(A)$ (the statement of [10, Theorem 5] is the seemingly weaker equality $\mathrm{Der}_{[1,\infty)}(A) = \bigcup_m \mathrm{Der}_{[m,\infty]}(A)$). □

It is easy to show that a $[1, \infty]$-integrable derivation is $[m, \infty]$-integrable for every m (see [27, Theorem 3.6.6]). The next result of Gerstenhaber's shows that the converse is also true.

Theorem 1. *Let A be an Artin algebra over a commutative Artinian ring k and let $m \in \mathbb{N}$. A k-linear derivation on A is $[1, \infty]$-integrable if and only it is $[m, \infty]$-integrable.*

Proof. By Lemma 1, we may assume that k is an algebra over $\mathbb{Z}_p$. Let D be an $[m, \infty]$-integrable derivation on A. By Lemma 2, there is an n such that $\mathrm{Der}_{[1,n)}(A) = \mathrm{Der}_{[1,\infty]}(A)$, so it suffices to show that D is $[1, n)$-integrable. There is an automorphism

$$\alpha = \mathrm{id} + Dt^m + \alpha_{m+1} t^{m+1} + \cdots + \alpha_{mn} t^{mn}$$

in $\mathrm{Aut}_m(A[t]/(t^{mn+1}))$, and applying [10, Theorem 3] to this yields an automorphism

$$\alpha' = \mathrm{id} + Dt^m + \alpha'_{2m} t^{2m} + \cdots + \alpha'_{mn} t^{mn}$$

involving only powers of t^m. Replacing t^m with t finishes the proof. □

At this point we can deduce that $\mathrm{Der}_{\mathrm{int}}(A) = \mathrm{Der}_{[1,\infty]}(A)$ forms a Lie algebra; this was originally proven by Gerstenhaber [10, Corollary 1], assuming that $k = k_p$ for some prime p.

Corollary 1. *If A is an Artin algebra over a commutative Artinian ring k, then $\mathrm{Der}_{\mathrm{int}}(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$, and if moreover A contains a field of characteristic p, then $\mathrm{Der}_{\mathrm{int}}(A)$ is a restricted Lie subalgebra of $\mathrm{Der}(A)$. By the same token, $\mathrm{HH}^1_{\mathrm{int}}(A)$ is a (restricted) Lie subalgebra of $\mathrm{HH}^1(A)$.*

Proof. This follows from Remark 4 and Theorem 1. □

The class of integrable derivations is known to have good invariance properties, and we may use Corollary 1 to upgrade these invariance results to statements about Lie algebras. The next result builds on the work of Rouquier, Huisgen-Zimmermann, Saorín, Keller, and Linckelmann. When k has characteristic zero all derivations are integrable, so that $\mathrm{HH}^1_{\mathrm{int}}(A) = \mathrm{HH}^1(A)$, and the theorem is well-known in this case.

Theorem 2. *Let A and B two finite-dimensional split algebras over a field k. Assume either that A and B are derived equivalent, or that A and B are self-injective and stably equivalent of Morita type. Then $\mathrm{HH}^1_{\mathrm{int}}(A) \cong \mathrm{HH}^1_{\mathrm{int}}(B)$ as Lie algebras, and this is an isomorphism of restricted Lie algebras if k is of positive characteristic.*

Proof. Assume first that A and B are derived equivalent. The identity component $\mathrm{Out}(A)^\circ$ of the group scheme of outer automorphisms of A is a derived invariant by [13, 25]. Therefore, the set

$$\mathrm{Out}_1(A[\![t]\!])^\circ = \{f \in \mathrm{Hom}_{k\text{-scheme}}(\mathrm{Spec}(k[\![t]\!]), \mathrm{Out}(A)^\circ) \,:\, f(0) = 1\}$$

is a derived invariant as well, and we have $\mathrm{Out}_1(A[\![t]\!])^\circ \cong \mathrm{Out}_1(B[\![t]\!])^\circ$.

As $\mathrm{HH}^1(A) \cong \mathrm{Ext}^1_{A^{\mathrm{op}}\otimes A}(A, A)$ the first Hochschild cohomology is also a derived invariant, and we have $\mathrm{HH}^1(A) \cong \mathrm{HH}^1(B)$.

Both isomorphisms above are realised by tensoring with a complex of A-B bimodules [23], and it follows that the *map* below is a derived invariant:

$$\pi :\ \mathrm{Out}_1(A[\![t]\!])^\circ \longrightarrow \mathrm{HH}^1(A) \quad (\alpha = \mathrm{id} + \alpha_1 t + \alpha_2 t^2 + \cdots) \mapsto \alpha_1.$$

Therefore, the image $\mathrm{HH}^1_{\mathrm{int}}(A) = \mathrm{im}(\pi)$ is a derived invariant as well (cf. the proof of [18, Theorem 5.1] for a similar argument). The fact that the isomorphism $\mathrm{HH}^1_{\mathrm{int}}(A) \cong \mathrm{HH}^1_{\mathrm{int}}(B)$ is one of Lie algebras now follows from Corollary 1 and [16, section 4]. In positive characteristic this respects the p-power structure by [15, (3.2)] combined with [5, Theorem 2].

For the case of self-injective algebras that are stably equivalent of Morita type, there is by [18, Theorem 5.1] an isomorphism $\mathrm{HH}^1_{\mathrm{int}}(A) \cong \mathrm{HH}^1_{\mathrm{int}}(B)$ induced by a transfer map, and this is an isomorphism of restricted Lie algebras by Corollary 1 together with [26, Theorem 1.1] and [5, Theorem 1]. □

2.1 | Obstructions to integrability

The results of [10] are proven using an obstruction theory for integrability, which we explain now.

For any automorphism $\alpha = \mathrm{id} + \alpha_1 t^1 + \cdots + \alpha_{n-1} t^{n-1}$ in $\mathrm{Aut}_1(A[t]/(t^n))$ we define a k-linear map $\mathrm{obs}(\alpha) :\ A \otimes A \to A$ by the rule

$$\widetilde{\mathrm{obs}}(\alpha)(x \otimes y) = \alpha_1(x)\alpha_{n-1}(y) + \cdots + \alpha_{n-1}(x)\alpha_1(y).$$

Viewed as a degree 2 element in the Hochschild cochain complex $C^*(A)$, one checks that $\widetilde{\mathrm{obs}}(\alpha)$ is a cycle, and therefore defines a cohomology class

$$\mathrm{obs}(\alpha) = [\widetilde{\mathrm{obs}}(\alpha)] \quad \text{in} \quad \mathrm{HH}^2(A).$$

To extend α to an element of $\mathrm{Aut}_1(A[t]/(t^{n+1}))$, we must find a k-linear endomorphism $\alpha_n : A \to A$ satisfying the Hasse–Schmidt identity

$$\alpha_n(xy) = x\alpha_n(y) + \alpha_1(x)\alpha_{n-1}(y) + \cdots + \alpha_n(x)y.$$

This may be rearranged and formulated using the Hochschild cochain complex:

$$\partial(\alpha_n) = \widetilde{\mathrm{obs}}(\alpha) \quad \text{in} \quad C^2(A).$$

We obtain the statement of [10, Proposition 5]: an automorphism $\alpha \in \mathrm{Aut}_1(A[t]/(t^n))$ can be extended to $\mathrm{Aut}_1(A[t]/(t^{n+1}))$ if and only if $\mathrm{obs}(\alpha) = 0$.

For any $[m,n)$-integrable derivation D, there is by definition an automorphism $\alpha = \mathrm{id} + \alpha_m t^m + \cdots \alpha_{n-1}t^{n-1}$ in $\mathrm{Aut}_m(A[t]/(t^n))$ with $\alpha_m = D$. The above result shows that if $\mathrm{obs}(\alpha) = 0$, then D is $[m, n+1)$-integrable. We caution that D being $[m, n+1)$-integrable does not necessarily imply that $\mathrm{obs}(\alpha) = 0$, only that *some* automorphism in $\mathrm{Aut}_m(A[t]/(t^n))$ extending $\mathrm{id} + Dt^m$ is unobstructed.

The *obstruction order* of an automorphism α in $\mathrm{Aut}_1(A[t]/(t^i))$ is n if it admits an extension to an automorphism in $\mathrm{Aut}_1(A[t]/(t^n))$, but admits no extension to an automorphism in $\mathrm{Aut}_1(A[t]/(t^{n+1}))$.

Gerstenhaber proves the following two facts about the above obstruction theory.

Theorem 3 [10, Theorem 1, Theorem 2, and Corollary 1]. *Let A be an algebra over a commutative ring k.*

(1) *If* $\alpha, \alpha' \in \mathrm{Aut}_1(A[t]/(t^n))$ *then* $\mathrm{obs}(\alpha\alpha') = \mathrm{obs}(\alpha) + \mathrm{obs}(\alpha')$ *in* $\mathrm{HH}^2(A)$.
(2) *Assume $k = k_p$ for some prime p. If D is a derivation then the obstruction order of* $\mathrm{id} + Dt$ *is (if finite) of the form p^e. Moreover, in this case the obstruction order of* $\mathrm{id} + Dt^m$ *is mp^e.*

Because of (2), a derivation D is said to have *obstruction exponent e* if $\mathrm{id} + Dt$ has obstruction order p^e.

To give an example of how the obstruction theory is applied we use it to give Gerstenhaber's beautiful proof of the analogue of Corollary 1 for finitely integrable derivations (which are especially connected with jet spaces).

Corollary 2 [10]. *If A is a Noether algebra over a commutative Noetherian ring k, then* $\mathrm{Der}_{[1,n)}(A)$ *is a Lie subalgebra of* $\mathrm{Der}(A)$*, and if moreover A contains a field of characteristic p, then* $\mathrm{Der}_{[1,n)}(A)$ *is a restricted Lie subalgebra of* $\mathrm{Der}(A)$.

Proof. By Lemma 1, we may assume that $k = k_p$, so that Theorem 3 part (2) applies and $\mathrm{Der}_{[1,n)}(A) = \mathrm{Der}_{[1,p^e)}(A)$ where p^e is the smallest power of p that is at least n. Therefore, we may assume that $n = p^e$.

Take $D, D' \in \mathrm{Der}_{[1,p^e)}(A)$ and suppose that $D = \alpha_1$ and $D' = \alpha'_1$ for two $\alpha, \alpha' \in \mathrm{Aut}_1(A[t]/(t^{p^e}))$. As in Remark 4 the automorphism $\alpha\alpha'\alpha^{-1}\alpha'^{-1}$ shows that $[D, D']$ is $[2, p^e)$-integrable. However, by Theorem 3 part (1) we have $\mathrm{obs}(\alpha'\alpha^{-1}\alpha'^{-1}) = 0$, therefore $[D, D']$ is in fact $[2, p^e + 1)$-integrable. By Theorem 3 part (2), it must therefore be $[2, 2p^e)$-integrable, as $2p^{e-1} < p^e + 1$. By [10, Theorem 3], this implies that $[D, D']$ is $[1, p^e)$-integrable.

A similar argument yields the second claim, as $\operatorname{obs}(\alpha^p) = p \operatorname{obs}(\alpha) = 0$ using Theorem 3 part (1). □

Remark 5. Linckelmann considers a more general notion of integrable derivation in [18], replacing $k[[t]]$ with any discrete valuation ring. It would be interesting to develop the obstruction theory using this definition, and to see whether the results above extend to this context as well.

3 | COUNTER-EXAMPLES TO SOLVABILITY

Notation

For the next three sections, we denote by S_n the symmetric group on n letters, by A_n the alternating group and by C_n the cyclic group of order n. When dealing with groups we use the superscript notation for the n-fold direct product and we denote by $H \rtimes N$ the semidirect product of N and H. We denote by k a field and by k^+ its additive group.

In this section, we give a counter-example concerning the solvability of integrable derivations.

Question 1 [17, Question 8.2]. When is the Lie algebra $\mathrm{HH}^1_{\mathrm{int}}(A)$ solvable?

In the same article, Linckelmann suggests that, based on examples, $\mathrm{HH}^1_{\mathrm{int}}(A)$ should be a solvable Lie algebra if A is a block of a finite group algebra kG over an algebraically closed field of prime characteristic. We provide a negative answer to this suggestion by considering the group algebra kP of an elementary abelian p-group P of rank greater than 1. In this case the group algebra kP coincides with its unique block.

Theorem 4. *Let k be a field of characteristic $p \geqslant 3$. Let P be an elementary abelian p-group of rank greater than 1. Then $\mathrm{HH}^1_{\mathrm{int}}(kP)$ is not solvable.*

Proof. Let P be an elementary abelian p-group of rank $n > 1$. Note that $A := kP \cong k[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$. For each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, set $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and set $J = \{(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n \mid 0 \leqslant \alpha_i < p\}$. A basis of A is given by x^α, where $\alpha \in J$.

Then $\mathrm{HH}^1(A)$ is a Jacobson–Witt algebra [14] and it admits a basis $\{x^\alpha \partial_{x_i}\}_{i=1}^n$ where ∂_{x_i} is the unique k-linear derivation such that $\partial_{x_i}(x_j) = \delta_{i,j}$ where $\delta_{i,j}$ denotes the Kronecker symbol.

Note that $\mathrm{HH}^1_{\mathrm{int}}(A)$ has the same k-basis of $\mathrm{HH}^1(A)$ excluding the set of derivations $\{\partial_{x_i}\}_{i=1}^n$. To prove that $x_i\partial_{x_i}$, for $1 \leqslant i \leqslant n$, is an integrable derivation we use the automorphism $\alpha = \mathrm{id} + x_i \partial_{x_i} t \in \mathrm{Aut}_1(A[[t]])$; it is easy to check that this is a well-defined automorphism. As the space of integrable derivations is a module over $A = \mathrm{Z}(A)$, the rest of the derivations in the basis, aside from $\{\partial_{x_i}\}_{i=1}^n$, are integrable as well. The derivations ∂_{x_i} for $1 \leqslant i \leqslant n$ (and any non-trivial linear combination of them) do not preserve the Jacobson radical, hence they are not integrable (see [8, Corollary 2.1]).

Let $\mathfrak{f}, \mathfrak{e}, \mathfrak{h}$ be a basis of $\mathfrak{sl}_2(k)$ satisfying $[\mathfrak{e}, \mathfrak{f}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{f}] = -2\mathfrak{f}$, and $[\mathfrak{h}, \mathfrak{e}] = 2\mathfrak{e}$. The derived subalgebra of $\mathrm{HH}^1_{\mathrm{int}}(A)$ contains the Lie algebra $\mathfrak{sl}_2(k)$ via the map sending $\mathfrak{f}$ to $x_2\partial_{x_1} = [x_2\partial_{x_1}, x_1\partial_{x_1}]$, $\mathfrak{h}$ to $x_1\partial_{x_1} - x_2\partial_{x_2} = [x_1\partial_{x_2}, x_2\partial_{x_1}]$, and $\mathfrak{e}$ to $x_1\partial_{x_2} = [x_1\partial_{x_2}, x_2\partial_{x_2}]$. Note that $\mathfrak{sl}_2(k)$ in characteristic different from 2 is not solvable. The statement follows. □

4 | THE FIRST HOCHSCHILD COHOMOLOGY OF THE SYMMETRIC GROUP

In this section, we give a formula for the dimension of $\mathrm{HH}^1(kS_n)$. We start by recalling some basic facts on the representation theory of the symmetric group.

Definition 3. A *partition* of a non-negative integer n is a decreasing sequence of positive integers $\lambda_1 > \lambda_2 > \dots > \lambda_s > 0$ and positive integers $e_1, \dots, e_s$ such that such that $e_1\lambda_1 + \cdots + e_s\lambda_s = n$. We use the notation $\lambda = (\lambda_1^{e_1}, \dots, \lambda_s^{e_s})$, and we say that λ_i is the *ith part* of λ, and e_i is the *multiplicity* of λ_i. We denote by $\mathcal{P}(n)$ the set of all partitions of n.

We recall that the conjugacy classes of S_n are in bijection with the partitions of n, with the conjugacy class of an element x corresponding to the partition λ determined by the cycle type of x. That is, if $x = c_{1,1} \dots c_{1,e_1} \dots c_{s,1} \dots c_{s,e_s}$ in disjoint cycle notation (including cycles of length one), where each $c_{i,j}$ is a cycle of length λ_i, then $\lambda = (\lambda_1^{e_1}, \dots, \lambda_s^{e_s})$.

We begin by computing $\mathrm{HH}^1(kS_n)$ using the centraliser decomposition of Hochschild cohomology. We note that as S_n/A_n is a trivial group when $n = 1$ and a cyclic group of order 2 otherwise, the dimensions of the vector spaces below depend on whether or not k has characteristic 2.

Theorem 5. *Let k a field of characteristic p and let S_n the symmetric group on n letters. Then we have the following decomposition:*

$$\mathrm{HH}^1(kS_n) \cong \bigoplus_{\lambda\in\mathcal{P}(n)} \mathrm{Hom}\left(\prod_{i=1}^{s}(C_{\lambda_i} \times (S_{e_i}/A_{e_i})), k^+\right).$$

Proof. Using the decomposition of $\mathrm{HH}^1(kS_n)$ into the direct sum of the first group cohomology of centraliser subgroups we have:

$$\mathrm{HH}^1(kS_n) = \bigoplus_{\lambda\in\mathcal{P}(n)} \mathrm{H}^1(C_{S_n}(x), k) = \bigoplus_{\lambda\in\mathcal{P}(n)} \mathrm{Hom}(C_{S_n}(x), k^+),$$

where x is a representative element in each conjugacy class of kS_n parameterised by λ. The first step is to study the centraliser $C_{S_n}(x)$. As consequence of the fact that conjugation permutes cycles of the same length we have that $C_{S_n}(x) = \prod_{i=1}^{s} C_{\lambda_i} \wr S_{e_i}$ where $\wr$ denotes the wreath product of C_{λ_i} by S_{e_i}. In fact, there are two groups that sit inside $C_{S_n}(x)$ and that generate $C_{S_n}(x)$. The first one is $E := S_{e_1} \times \cdots \times S_{e_s}$ and the second is $\prod_{i=1}^{s} C_{\lambda_i}^{e_i}$. It is easy to check that $C_{S_n}(x) = \prod_{i=1}^{s}(C_{\lambda_i}^{e_i} \rtimes S_{e_i})$ where S_{e_i} acts on the direct product $C_{\lambda_i}^{e_i}$ by permutation.

The next step is to study the abelianisation of $C_{S_n}(x)$. Note that the derived subgroup of $C_{S_n(x)}$ is given by

$$\prod_{i=1}^{s}[C_{\lambda_i} \wr S_{e_i}, C_{\lambda_i} \wr S_{e_i}].$$

In general, the derived subgroup of a semi-direct product $N \rtimes H$ is equal to $([N,N][N,H]) \rtimes [H,H]$. In our case $H = S_{e_i}$ and $N = C_{\lambda_i}^{e_i}$. So, $[S_{e_i}, S_{e_i}] = A_{e_i}$ and $[N,N] = 1$. It is easy to check that $[C_{\lambda_i}^{e_i}, S_{e_i}]$ is isomorphic to $C_{\lambda_i}^{e_i-1}$. Hence,

$$[C_{\lambda_i} \wr S_{e_i}, C_{\lambda_i} \wr S_{e_i}] \cong C_{\lambda_i}^{e_i-1} \rtimes A_{e_i}.$$

Consequently the abelianisation of $C_{S_n}(x)$ is isomorphic to

$$\prod_{i=1}^{s} (C_{\lambda_i} \times S_{e_i}/A_{e_i}).$$

The statement follows. ☐

Lemma 3. *Let p be a prime, and n a non-negative integer. The number of parts of length divisible by p in all partitions of n, counted without multiplicity, is equal to the number of parts of length p in all partitions of n, counted with multiplicity.*

Proof. Using the notation $\lambda = (\lambda_1^{e_1}, \dots, \lambda_s^{e_s})$ for a partition of n, we consider the set $\mathcal{S}_1$ of pairs $\{(\lambda, e)$ with some $\lambda_i = p$ and $1 \leqslant e \leqslant e_i\}$, and the set $\mathcal{S}_2$ of pairs $\{(\lambda, \lambda_i)$ with $p|\lambda_i\}$. We define a function $\mathcal{S}_1 \to \mathcal{S}_2$ by the rule $(\lambda, e) \mapsto (\lambda', ep)$, where $\lambda' = (\lambda_1^{e_1}, \dots, (ep)^{e'}, \dots, p^{e_i - e}, \dots, \lambda_s^{e_s})$ and where $e' = e_j + 1$ if $\lambda_j = ep$ was already a part of λ, or $e' = 1$ if not (that is, take e parts of size p and make them one part of size ep). One can define an inverse $\mathcal{S}_2 \to \mathcal{S}_1$ that takes (λ, λ_i) to $(\lambda', e = \lambda_i/p)$, where $\lambda' = (\lambda_1^{e_1}, \dots, \lambda_i^{e_i - 1}, \dots, p^{u+e}, \dots, \lambda_s^{e_s})$, where $u = e_j$ if $p = \lambda_j$ was already a part of λ, otherwise $u = 0$ (that is, take one part of size ep and make it e parts of size p). As $|\mathcal{S}_1|$ is the number of parts of length p in all partitions of n, and $|\mathcal{S}_2|$ is the number of parts of length divisible by p in all partitions of n (without multiplicity), the lemma is established by these inverse bijections. ☐

Theorem 6. *If the characteristic of the field k is different from 2, then*

$$\mathrm{HH}^1(kS_n) \cong \bigoplus_{\lambda \in \mathcal{P}(n)} \mathrm{Hom}\left(\prod_{p|\lambda_i} C_{\lambda_i}, k^+ \right).$$

Therefore, $\dim_k(\mathrm{HH}^1(kS_n))$ *is equal to the total number of parts, counted without multiplicity, divisible by p in all partitions of n. It has generating series*

$$\sum_{n \geqslant 0} \dim_k(\mathrm{HH}^1(kS_n)) t^n = \frac{t^p}{1 - t^p} \prod_{n \geqslant 1} \frac{1}{(1 - t^n)}.$$

Proof. In Theorem 5, the term S_{e_i}/A_{e_i} will not contribute because the characteristic of the field is greater than 2 and S_{e_i}/A_{e_i} is either trivial or isomorphic to C_2. This yields the first statement.

We also learn that $\dim_k(\mathrm{HH}^1(kS_n))$ is equal to the total number of parts divisible by p in all partitions of n, counted without multiplicity. By Lemma 3, this the number of times p occurs as a part in a partition of n. The generating series for this sequence can be found in [24]. More precisely, we can associate to any sequence (a_i) the function on partitions $L(\lambda) := \sum a_i \kappa_i$, where κ_i is the number of parts of size i in λ; then [24, eq. 23, p. 185] reads

$$\sum_{n\geqslant 0} t^n \sum_{\lambda\in\mathcal{P}(n)} L(\lambda) = \left(\sum_{n\geqslant 1} \frac{a_n t^n}{1-t^n}\right) \prod_{n\geqslant 1} \frac{1}{(1-t^n)}.$$

If we take $a_p = 1$ and $a_i = 0$ for $i \neq p$ then we obtain the desired series. □

Theorem 7. *If k is a field of characteristic 2 then*

$$\sum_{n\geqslant 0} \dim_k(\mathrm{HH}^1(kS_n))t^n = \frac{2t^2}{1-t^2} \prod_{n\geqslant 1} \frac{1}{(1-t^n)}.$$

Proof. By Theorem 5,

$$\mathrm{HH}^1(kS_n) \cong \bigoplus_{\lambda\in\mathcal{P}(n)} \mathrm{Hom}\left(\prod_{2|\lambda_i} C_{\lambda_i}, k^+\right) \oplus \mathrm{Hom}\left(\prod_{e_i\geqslant 2} C_2, k^+\right).$$

So, the computation of $\dim_k(\mathrm{HH}^1(kS_n))$ splits into two parts. For the first summand we count the number of parts in partitions of n that are divisible by 2; as in the proof of Theorem 6 this is given by the generating series

$$\frac{t^2}{1-t^2} \prod_{n\geqslant 1} \frac{1}{(1-t^n)}.$$

For the second summand we must count the number of parts with multiplicity 2 or more in all partitions of n. In the usual formula for the total number of partitions

$$\sum_{n\geqslant 0, \lambda\in\mathcal{P}(n)} t^n = \prod_{n\geqslant 1} \frac{1}{(1-t^n)}$$

(cf. [24]), the factor $1/(1-t^i) = (1 + t^i + t^{2i} + \cdots)$ corresponds to parts of length i, with a term t^{ie} contributing 1 to the coefficient of λ if λ contains a part of length i with multiplicity e. To modify this formula to count partitions with a chosen part of multiplicity $e \geqslant 2$, we simply replace this factor with $t^{2i}/(1-t^i) = (t^{2i} + t^{3i} + t^{4i} + \cdots)$. In total we get

$$\sum_{i\geqslant 1} \left(t^{2i} \prod_{n\geqslant 1} \frac{1}{(1-t^n)}\right) = \frac{t^2}{1-t^2} \prod_{n\geqslant 1} \frac{1}{(1-t^n)}.$$

The statement of the theorem follows. □

An element x in a finite group is *p-regular* if its order is coprime to p, and otherwise it is called *p-singular*. In the case of S_n, the p-singular elements are those containing at least one cycle of length divisible by p. In other words, the corresponding partition contains a part divisible by p. We write $\mathcal{SP}(n)$ for the set of all partitions of n corresponding to conjugacy classes of p-singular elements.

Corollary 3. *If k is a field of characteristic p then* $\dim_k(\mathrm{HH}^1(kS_n)) \geqslant |\mathcal{SP}(n)|$.

Finally, in the next section we will need the following fact.

Corollary 4. *If k is a field of characteristic* $p > 2$ *then* $\dim_k(\mathrm{HH}^1(kS_p)) = 1$.

Proof. By Theorem 5, we just need to count the number of parts of length p in all partitions of p, and there is clearly just one. □

Remark 6. Recently, the authors of [4] have also computed $\dim_k(\mathrm{HH}^1(kS_n))$ in terms of generating functions.

5 | COUNTER-EXAMPLES TO THE EXISTENCE OF NON-INTEGRABLE DERIVATIONS

In this section, we answer a question considered by Farkas, Geiss and Marcos.

Question 2 [8]. Let G be a finite group and let k be a field such that char(k) divides the order of G, must kG admit a non-integrable derivation?

As all inner derivations are integrable, a necessary condition that should hold in order to state the previous question is the following: let G be a finite group and assume the characteristic of the field k divides the order of G. Then $\mathrm{HH}^1(kG) \neq 0$. This has been shown in [9] using the classification of finite simple groups.

The authors state their question in terms of the automorphism group scheme, writing that *It is tempting to conjecture that kG does not have a smooth automorphism group scheme* [8, below Theorem 2.2]. Their question is equivalent to Question 2 by [8, Theorem 1.2]: the automorphism group scheme of a finite-dimensional algebra A is smooth if and only if every derivation on A is integrable.

In the following theorem, we exhibit a family of counter-examples for any algebraically closed field of prime characteristic greater than 2.

Theorem 8. *Let k a field of characteristic* $p \geqslant 3$ *and let* kS_p *be the group algebra of the symmetric group on p letters. Then* $\mathrm{HH}^1(kS_p)$ *has a k-basis given by a single integrable derivation.*

The first part of Theorem 8 will follow from Corollary 4. To prove that the only outer derivation in $\mathrm{HH}^1(kS_p)$ is integrable, we will use the fact that the only non-semisimple block of kS_p is derived equivalent to a symmetric Nakayama algebra.

We recall some basic results about blocks of symmetric groups.

A *node* (i, j) in the Young diagram $[\lambda]$ of λ forms part of the *rim* if $(i+1, j+1) \notin [\lambda]$. A *p-hook* in λ is a connected part of the rim of $[\lambda]$ consisting of exactly p nodes, whose removal leaves the Young diagram a partition. The *p-core* of λ, usually denoted by $\gamma(\lambda)$, is the partition obtained by repeatedly removing all p-hooks from λ. The number of p-hooks we remove is the *p-weight* of λ, usually denoted by w. It is easy to note that the p-core of a partition is well-defined, that is, is independent of the way in which we remove the p-hooks. The blocks of group algebras of symmetric groups are determined by p-cores and weights:

Theorem 9 (Nakayama Conjecture). *The blocks of the symmetric group S_n are labelled by pairs (γ, w), where γ is a p-core and w is the associated p-weight such that $n = |\gamma| + pw$. Hence, the Specht module S^λ lies in the block labelled by (γ, w) of kS_n if and only if λ has p-core γ and weight w.*

Note that the statement above holds also for the simple modules, see the paragraph after [7, Theorem 8.3.1]. It is easy to see that blocks of weight 0 are matrix algebras and blocks of weight 1 have cyclic defect group.

Background in Brauer graph algebras can be found, for example, in [28]. The details about the importance of Brauer tree algebras in modular representation can be found in [3]. For further background in the modular representation theory of finite groups, see [19] and [20].

A particularly nice class of self-injective algebras are the self-injective Nakayama algebras. For background on Nakayama algebras, see, for example, [2, Chapter V] and [1, Chapter IV.2]. The symmetric Nakayama algebra having e simple modules and Loewy length $em + 1$ is a Brauer tree algebra with respect to the Brauer star with e vertices and with exceptional multiplicity em. It is worth noting that not every Brauer tree algebra is isomorphic is a symmetric Nakayama algebra, however, a result due to Rickard [23] shows that every Brauer tree algebra is derived equivalent to a symmetric Nakayama algebra, see also [30, Theorem 6.10.1].

Theorem 10. *Let k be a field and let A be a Brauer tree algebra associated to a Brauer tree with e edges and with exceptional multiplicity m. Then A is derived equivalent to the symmetric Nakayama algebra N_e^{me}.*

We have all the ingredients in hand needed to prove Theorem 8:

Proof of Theorem 8. The principal block of kS_p, denoted by B_0, has cyclic defect C_p and the number of simple modules of B_0 is $p-1$. This follows by Nakayama Conjecture because there are $p-1$ partitions having the same p-core of the partition representing the trivial module. The weight of B_0 is 1 hence B_0 has cyclic defect C_{p^d} for some d. In this case B_0 is a Brauer tree algebra for a Brauer tree with $p-1$ edges and exceptional multiplicity 1, see after [7, Example 5.1.4]. Hence, B_0 has cyclic defect C_p. By Theorem 10 we have that B_0 is derived equivalent to the Nakayama algebra N_{p-1}^{p-1}. The Gabriel quiver associated with N_{p-1}^{p-1} has a set of vertices given by $\{e_i\}_{i=1}^{p-1}$ and it has $p-1$ arrows $\{a_i\}_{i=1}^{p-1}$ such that $t(a_i) = s(a_{i+1}) = e_{i+1}$ for $i \neq p-1$ and $t(a_{p-1}) = s(a_1)$. The rest of the blocks of kS_p are matrix algebras because they have weight 0. Therefore, $\mathrm{HH}^1(kS_p) \cong \mathrm{HH}^1(N_{p-1}^{p-1})$, and this restricts to an isomorphism $\mathrm{HH}^1_{\mathrm{int}}(kS_p) \cong \mathrm{HH}^1_{\mathrm{int}}(N_{p-1}^{p-1})$ by the derived invariance of integrable derivations.

The first Betti number $\beta(Q)$ of the underying graph of N_{p-1}^{p-1} is 1. This is because the Gabriel quiver is connected, the number of edges is $p-1$, the number of vertices is $p-1$ and consequently $\beta(Q) = (p-1) - (p-1) + 1 = 1$. As N_{p-1}^{p-1} is monomial, by [6, Theorem C] we have that the maximal total rank is 1 and it is easy to see that the map f sending a_1 to a_1 and sending any other arrow to zero is a diagonal outer derivation. From Corollary 4, we have $\dim_k(\mathrm{HH}^1(kS_p)) = 1$. We deduce that there are no other outer derivations. Recall that N_{p-1}^{p-1} is the bound quiver algebra $kC_{p-1}/J_{p-1,p-1}$ where $J_{p-1,p-1}$ is the ideal in the path algebra kC_{p-1} generated by the composition of p consecutive arrows. Let $\rho_1 = (a_1 \dots a_{p-1}a_1 = 0)$. Then any other relation that generates the ideal $J_{p-1,p-1}$ is given by the path that starts and ends at a_i for every $2 \leqslant i \leqslant n$. We construct the automorphism $\alpha = \mathrm{id} + ft \in \mathrm{Aut}(A[\![t]\!])$. This $k[\![t]\!]$-automorphism preserves the relations. We check for ρ_1 because for the rest of generating relations the proof is similar. We have

$$\alpha(a_1 \dots a_{p-1}a_1) = \alpha(a_1)\dots\alpha(a_{p-1})\alpha(a_1) = (a_1 + a_1t)a_2 \dots a_{p-1}(a_1 + a_1t)$$
$$= (a_1 \dots a_{p-1} + a_1 \dots a_{p-1}t)(a_1 + a_1t) = 0.$$

Therefore, f is integrable and the statement follows. □

Remark 7. Note that in order to construct the previous counter-example we have considered a Gabriel quiver without loops. In [8], the authors consider p-groups, which are local algebras, hence all the arrows are loops.

APPENDIX: GERSTENHABER'S COMPOSITION COMPLEXES

In Section 2, we used results of Gerstenhaber [10] to establish facts about the Lie algebra of integral derivations on an algebra. In this Appendix, we survey the more general definitions in [10] and compare them with the context of Section 2.

Definition A.1 [10]. Let k be a commutative ring. A composition complex C over k is a sequence $C^0, C^1, \dots$ of k-modules, and for each m, n and $0 \leqslant i \leqslant m-1$ a bilinear composition operation

$$C^m \times C^n \to C^{m+n-1} \quad (f, g) \mapsto f \circ_i g,$$

as well as for each m, n a bilinear cup product operation

$$C^m \times C^n \to C^{m+n} \quad (f, g) \mapsto f \smile g,$$

satisfying, for any $f \in C^n$, $g \in C^m$ and $h \in C^l$, the conditions

$$(f \circ_i g) \circ_j h = \begin{cases} (f \circ_j h) \circ_{i+l-1} h & \text{if } 0 \leqslant j \leqslant i-1 \\ f \circ_i (g \circ_{j-i} h) & \text{if } i \leqslant j \leqslant n-1, \end{cases}$$

and

$$(f \smile g) \circ_j h = \begin{cases} (f \circ_i h) \smile g & \text{if } 0 \leqslant i \leqslant m-1 \\ f \smile (g \circ_{i-m} h) & \text{if } m \leqslant i \leqslant m+n-1. \end{cases}$$

We further assume that the cup product of C is associative, and that there is unit element $1 \in C^1$ such that $1 \circ_0 f = f \circ_i 1 = f$ for any $f \in C^m$ and $0 \leqslant i \leqslant m-1$.

The key example of a composition complex is the Hochschild cochain complex of a k-algebra A:

$$C^n(A) = \mathrm{Hom}(A^{\otimes n}, A) \quad \text{with} \quad f \circ_i g = f \circ (1^{\otimes i} \otimes g \otimes 1^{\otimes m-i-1})$$

for $f \in C^n$ and $g \in C^m$, and with the usual cup product

$$(f \smile g)(x_1 \otimes \cdots \otimes x_{n+m}) = f(x_1 \otimes \cdots \otimes x_n)g(x_{n+1} \otimes \cdots \otimes x_{n+m}).$$

Other examples of composition complexes given in [10] are the singular cochain complex of a topological space, and the cobar construction on a Hopf algebra. In general, one can work with any composition complex mimicking constructions that are standard for the Hochschild cochain complex, as the next definition shows.

Definition A.2 [10]. We provide a brief dictionary between the context of this paper and that of composition complexes.

(1) For $f \in C^m$ and $g \in C^n$ we define the circle product and the bracket

$$f \circ g = \sum_{i=0}^{m-1} (-1)^{i(n-1)} f \circ_i g, \quad [f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f$$

in C^{m+n-1}. These correspond to the usual circle product and Gerstenhaber bracket when $C = C^*(A)$.

(2) We call $m = 1 \smile 1 \in C^2$ the multiplication element of C, in the case of the Hochschild cochain complex this is the multiplication map $A \otimes A \to A$. We then define a differential on C by the rule $\partial(f) = [m, f]$, and C becomes a complex $C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} \cdots$ In particular, we obtain cohomology groups $\mathrm{H}^i(C)$. When $C = C^*(A)$, these yield the usual Hochschild differential and Hochschild cohomology groups $\mathrm{HH}^i(A)$.

(3) A derivation in C is a degree one cycle $D \in \mathrm{Der}(A) = Z^1(C) = \ker(C^1 \to C^2)$. An automorphism in C is an element $\alpha \in C^1$, invertible with respect to the circle product, such that $\alpha \circ m = \alpha \smile \alpha$. When $C = C^*(A)$, these correspond to derivations and automorphisms of the k-algebra A.

(4) By base change, C gives rise to a composition complex $C[\![t]\!]$ over the ring $k[\![t]\!]$. A one-parameter family of automorphisms in C is an automorphism in $C[\![t]\!]$, and we write $\mathrm{Aut}_1(C[\![t]\!])$ for the set of one-parameter families of automorphisms of the form $\alpha = 1 + \alpha_1 t + \alpha_2 t^2 + \cdots$ A derivation $D \in \mathrm{Der}(C)$ is called integrable if $D = \alpha_1$ for some $\alpha \in \mathrm{Aut}_1(C[\![t]\!])$. One can similarly define $[m, n)$-integrable derivations for any m, n by considering automorphisms in $C[t]/(t^n)$. Once again, if $C = C^*(A)$ this yields the usual notion of integrable derivation.

(5) Finally, if $\alpha \in \mathrm{Aut}_1(C[t]/(t^n))$, the obstruction theory of Subsection 2.1 can be generalised by setting $\mathrm{obs}(\alpha) = [\alpha_1 \smile \alpha_{n-1} + \cdots + \alpha_{n-1} \smile \alpha_1] \in \mathrm{H}^2(C)$.

With these definitions in place, the results stated in Section 2 all hold at the generality of any composition complex. As Gerstenhaber was primarily concerned with automorphisms and derivations, which can be understood from the first few degrees, the results of [10] are stated even more generally for composition complexes *truncated in degree 2*. That is, k-modules C^0, C^1, C^2 having the structure and properties of Definition A.1 to the extent that they are meaningful.

Remark A.1. In modern terminology, a composition complex is the same thing as a non-symmetric operad with multiplication [11]. For example, the composition complex $C^*(A)$ is the endomorphism operad of A. In [11], Gerstenhaber and Voronov explain how any non-symmetric operad with multiplication inherits the structure of a B_∞-algebra. This construction mirrors some of the ideas from Definition A.2; in particular, they construct the bracket (1) and differential (2) exactly as was done originally in [10]. Conversely there are many interesting examples of operads with multiplication (for example, the Kontsevich operad used in [11]), and each can be considered as a composition complex, which thereby obtains a notion of integrable derivation and an obstruction theory as in Definition A.2.

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