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Characterizing finitely generated fields by a single field axiom

By Philip Dittmann and Florian Pop

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Abstract

We resolve the strong Elementary Equivalence versus Isomorphism Problem for finitely generated fields. That is, we show that for every field in this class, there is a first-order sentence that characterizes this field within the class up to isomorphism. Our solution is conditional on resolution of singularities in characteristic two and unconditional in all other characteristics.

1. Introduction

First-order logic naturally applies to the study of fields. Consequently, it is of interest to investigate the expressive power of first-order logic in natural classes of fields. This is well understood in the cases of algebraically closed fields, real-closed fields and p-adically closed fields. Namely, every such field K is elementary equivalent to its "constant field" κ, i.e., the relative algebraic closure of the prime field in K, and its first-order theory is decidable.

This article is concerned with fields that are at the center of (birational) arithmetic geometry, namely the finitely generated fields K, which are the function fields of integral $\mathbb{Z}$-schemes of finite type. The *Elementary Equivalence versus Isomorphism Problem*, EEIP for short, asks whether the elementary theory $\mathfrak{Th}(K)$ of a finitely generated field K (always in the language of rings) encodes the isomorphism type of K in the class of all finitely generated fields. This question goes back to the 1970s seems to have first been posed explicitly in [Pop02], after previous work, in particular by Rumely [Rum80], Duret [Dur92], and Pierce [Pie99].

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On the other hand, through the work of Rumely [Rum80], much more than the EEIP is known for global fields; namely, the existence of uniformly definable Gödel functions proved in that article implies that each global field K is axiomatizable by a single sentence θ_K^{Ru} in the class of global fields, i.e., θ_K^{Ru} holds in a global field L if and only if $L \cong K$. This was extended and sharpened by the second author in [Pop17], by showing that for every finitely generated field K of *Kronecker dimension* $\dim(K) \leqslant 2$, there exists a sentence θ_K such that θ_K holds in a finitely generated field L if and only if $L \cong K$ as fields. Here, for arbitrary fields F, the Kronecker dimension is $\dim(F) := \mathrm{td}(F) + 1$ if $\mathrm{char}(F) = 0$, respectively $\dim(F) := \mathrm{td}(F)$ if $\mathrm{char}(F) > 0$, where $\mathrm{td}(F)$ is the *absolute* transcendence degree of F.

In this note we establish the analogue of this stronger property for all finitely generated fields K thus, in particular, completely resolving the EEIP; in characteristic two, though, our proof is conditional, requiring a version of resolution of singularities in algebraic geometry, called "above $\mathbb{F}_2$." (See Section 2 for the version of resolution that we need.)

THEOREM 1.1. *Let K be a finitely generated field. If* $\mathrm{char}(K) = 2$ *and* $\dim(K) > 3$, *assume that resolution of singularities above $\mathbb{F}_2$ holds. Then there exists a sentence θ_K in the language of rings such that any finitely generated field L satisfies θ_K if and only if $L \cong K$.*

Our approach follows an idea of Scanlon in [Sca08], and thereby establishes an even stronger statement, giving information about the class of definable sets in finitely generated fields. Specifically, it shows that the class of definable sets is as rich as possible. One way of making this precise (cf. [AKNS20, Lemma 2.17]) is the following statement. (See [Sca08, §2] or [AKNS20, §2] for a discussion of the notion of bi-interpretability.)

THEOREM 1.2. *Let K be an infinite finitely generated field. If* $\mathrm{char}(K) = 2$ *and* $\dim(K) > 3$, *assume that resolution of singularities above $\mathbb{F}_2$ holds. Then K is bi-interpretable with $\mathbb{Z}$ (where both K and $\mathbb{Z}$ are considered as structures in the language of rings).*

Note that while this completely characterizes the definable sets in K, certain questions of uniformity across the class of finitely generated fields are left open; see, e.g., [Poo07, Question 1.8].

The chief technical result on which the theorems above build, and indeed the result that occupies the bulk of this article, concerns a definability statement regarding *prime divisors* of finitely generated fields. Recall that a *prime divisor* of an arbitrary field K with $\dim(K)$ finite is any discrete valuation v whose residue field Kv has $\dim(Kv) = \dim(K) - 1$. For finitely generated fields K, a valuation v is a prime divisor of K if and only if

$\dim(Kv) = \dim(K) - 1$; see, e.g., [EP05, Th. 3.4.3]. A prime divisor v is called *geometric* if $\mathrm{char}(K) = \mathrm{char}(Kv)$ and *arithmetic* otherwise. Throughout, we freely identify valuations v with their valuation rings $\mathcal{O}_v$ and, in particular, do not distinguish between equivalent valuations.

Since the cases $\dim(K) \leqslant 2$ were treated already in [Pop17] and [Rum80], we will consider the following family of hypotheses indexed by $d \geqslant 3$:

$$(\mathrm{H}_d)\quad \begin{cases} \text{–}K \text{ is finitely generated with } \dim(K) = d. \\ \text{–If } \mathrm{char}(K) = 2 \text{ and } d > 3, \text{ resolution of singularities holds above } \mathbb{F}_2. \end{cases}$$

THEOREM 1.3. *Let $d \geqslant 3$. The geometric prime divisors of fields satisfying (H_d) are uniformly first-order definable. In other words, there exists a formula $\mathsf{val}_d(X, \underline{Y})$ in the language of rings such that for every field K satisfying (H_d) and every geometric prime divisor $\mathcal{O}$ of K, there exists a tuple $\underline{y}$ in K such that*

$$\mathcal{O} = \{x \in K : K \models \mathsf{val}_d(x, \underline{y})\},$$

and conversely, for every tuple $\underline{y}$, the subset of K defined above is either a geometric prime divisor or empty.

1.1. *Short historical note and the genesis of this article*. The first step in the resolution of the strong form of the EEIP as mentioned in Theorem 1.1 above is Rumely's work [Rum80], which itself builds on previous ideas of J. Robinson. The next major step toward the resolution of the strong EEIP was the introduction of the "Pfister form machinery" in [Pop02], followed by the work of Poonen [Poo07], providing (among other things) uniform first-order formulas to define the maximal global subfields of finitely generated fields, and Scanlon [Sca08], which reduces the strong EEIP to first-order defining the geometric prime divisors of finitely generated fields, and finally the introduction of the cohomological higher local-global principles (LGPs) in [Pop17], as a tool for recovering prime divisors. The present paper is a synthesis of previous separate approaches to the problem by the authors and supersedes the manuscripts [Pop18b], [Pop18a], [Dit18], [Dit19], which are not intended for publication anymore. The proof builds on and expands the above ideas and tools, but it is not a straightforward extension of the methods of [Rum80], [Pop17], especially because the higher LGPs involved (cf. [KS12], [Jan16]) lead to some additional complications compared to the Brauer–Hasse–Noether LGP for global fields, respectively Kato's LGP in the case Kronecker dimension two. Finally, in this note the authors do not discuss the natural question of the complexity of the formulas describing prime divisors, thus the sentences characterizing the isomorphism type. It would also be interesting to treat the EEIP for fields that are finitely generated over natural base fields such as $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}_p$; cf. [PP08].

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2. Preliminaries: Cohomological Local-Global Principles (LGP)

The proof for the definability of prime divisors is based on local-global principles for certain cohomology groups over fields which were introduced in [Kat86]. These extend the well-known Brauer–Hasse–Noether LGP, in particular the injectivity of the canonical map

$$\imath_K : \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_{\hat{v}}),$$

where K is a global field, the sum is over all places v of K, and $K_{\hat{v}}$ is the completion at v.

Recall that for an arbitrary field K and $i \in \mathbb{Z}$, one defines the G_K-modules $\mathbb{Z}/n(i)$ as follows: First, if $\mathrm{char}(K)$ does not divide n, then $\mathbb{Z}/n(i) := \mu_n^{\otimes i}$ is $\mathbb{Z}/n$ endowed with the G_K-action via the i^{th}-power of the cyclotomic character of G_K. Second, if $p := \mathrm{char}(K) > 0$ and $n = mp^r$ with $(m, p) = 1$, then $\mathbb{Z}/n(i) := \mathbb{Z}/m(i) \oplus W_r\,\Omega^i_{\log}[-i]$, where $W_r\,\Omega_{\log}$ is the logarithmic part of the de Rham–Witt complex on the étale site over K; see Illusie [Ill79, Ch. I, 5.7]. (Note that these two definitions agree when $\mathrm{char}(K)$ is positive and does not divide n.) With these notations, one has (see [Kat86, Introduction])

$$\mathrm{H}^1\big(K, \mathbb{Z}/n(0)\big) = \mathrm{Hom}_{\mathrm{cont}}(G_K, \mathbb{Z}/n), \quad \mathrm{H}^2\big(K, \mathbb{Z}/n(1)\big) = {}_n\mathrm{Br}(K).$$

Noticing that K is a global field precisely if $\dim(K) = 1$, and the Brauer–Hasse–Noether local-global principle is an LGP for $\mathrm{H}^2\big(K, \mathbb{Z}/n(1)\big)$, Kato proposed that for "arithmetically significant" fields K with $\dim(K) = d$; e.g., for finitely generated fields, there should hold similar LGPs for $\mathrm{H}^{d+1}\big(K, \mathbb{Z}/n(d)\big)$; see Kato's seminal paper [Kat86], in particular, for how Milnor K-theory plays into the bigger picture. In the same paper, Kato proved several forms of such LGPs for finitely generated fields K with $\dim(K) = 2$. There was/is steady progress on Kato's conjectures, see Kerz-Saito [KS12] and Jannsen [Jan16], where both more literature and an account of previous results can be found.

We mention below three special instances of these (much more general) results that we will need in the sequel. We consider the following context:

- Throughout the paper $\boldsymbol{n=2}$, and to simplify notation set $\Lambda = \mathbb{Z}/2$.[1]
- For arbitrary fields F and $i \geqslant 0$, denote $\mathrm{H}^{i+1}(F) := \mathrm{H}^{i+1}(F, \Lambda(i))$.[2]

For a field F, recall the following general facts:

(a) For any extension $E|F$, one has the *restriction* map

$$\mathrm{res}_{E|F} : \mathrm{H}^{i+1}(F) \to \mathrm{H}^{i+1}(E), \;\; \boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}_E.$$

(b) Let w be a discrete valuation on F with residue field Fw. Under mild hypotheses, which are always satisfied in the sequel, there is a boundary homomorphism

$$\partial_w : \mathrm{H}^{i+1}(F) \to \mathrm{H}^{i}(Fw)$$

(see [Kat86, p. 149]). By construction, it factors through $\mathrm{H}^{i+1}(F_w)$, where F_w is the henselization of F with respect to w.

The first higher dimensional LGP proposed by Kato in [Kat86] is Jannsen [Jan16, Th. 0.4]. We consider and explain it in our notation for $n = 2$. Let K be finitely generated of Kronecker dimension $d \geqslant 1$ and $k_1 \subset K$ be a global subfield that is relatively algebraically closed in K. Then $K|k_1$ is a finitely generated field over k_1 with $\mathrm{td}(K|k_1) = d-1$. Let $\mathbb{P}(k_1)$ denote the set of places of k_1, and let $k_{1\widehat{v}}$ be the completion of k at $v \in \mathbb{P}(k_1)$. Then the relative algebraic closure $k_{1v} \subset k_{1\widehat{v}}$ of k_1 in $k_{1\widehat{v}}$ satisfies the following: k_{1v} is the real closure of k at v if v is a real place, and $k_{1v} = \overline{\mathbb{Q}}$ if v is a complex place, respectively k_{1v} is the henselization of k_1 at finite places $v \in \mathbb{P}_{\mathrm{fin}}(k_1)$. Since k_1 is relatively algebraically closed in K and k_{1v} is separable over k_1, $K \otimes_{k_1} k_{1v}$ is a domain, hence $K_{\widehat{v}} := K k_{1\widehat{v}} := \mathrm{Quot}(K \otimes_{k_1} k_{1v})$ is a well-defined field. In this notation (Jannsen [Jan16, Th. 0.4]) $n = 2$, and $\mathrm{char}(K) \neq 2$ shows that the canonical map $\imath_{k_1} = \oplus_{v\in\mathbb{P}(k_1)} \mathrm{res}_{K_{\widehat{v}}|K} : \mathrm{H}^{d+1}(K) \to \bigoplus_{v\in\mathbb{P}(k_1)} \mathrm{H}^{d+1}(K_{\widehat{v}})$ is well defined and injective. (Note that Jannsen writes F for our K, K for our k_1, and F_v for our $K_{\widehat{v}}$.) Hence if $K_v = K k_{1v} \subset K_{\widehat{v}}$ is the compositum of k_{1v} and K inside $K_{\widehat{v}}$, setting $\boldsymbol{\alpha}_v := \mathrm{res}_{K_v|K}(\boldsymbol{\alpha})$, one gets the following.

Fact 2.1 (cf. Jannsen [Jan16, Th. 0.4] for $n = 2$). *Suppose* $\mathrm{char}(K) \neq 2$. *Then one has*

$\boldsymbol{\alpha} \in \mathrm{H}^{d+1}(K)$ equals 0 if and only if $\boldsymbol{\alpha}_v \in \mathrm{H}^{d+1}(K_v)$ equals 0 for all $v \in \mathbb{P}(k_1)$.

We next briefly recall the higher dimensional generalizations of the Brauer–Hasse–Noether LGP as proposed by Kato. These involve so-called *arithmetical*

[1] The facts in the remainder of this section hold for $\Lambda = \mathbb{Z}/\ell^e$, provided $\ell \neq \mathrm{char}(K)$ and $\mu_{\ell^e} \subset K$.

[2] Note that in [EKM08] one denotes $\mathrm{H}^{i+1}(F) := \mathrm{H}^{i+1}(F, \Lambda(i))$ in Section 16, and $\mathrm{H}^{i}(F) := \mathrm{H}^{i}(F, \Lambda(i))$ in Section 101.

Bloch–Ogus complexes; see Kato [Kat86, §1] for details. Namely, for an excellent normal integral scheme X with $\dim(X) = d$ and function field $K = \kappa(X)$, let $X_i = X^{d-i}$ be the set of points $x \in X$ with $\dim(x) := \dim \overline{\{x\}} = i$, or equivalently, $\operatorname{codim}(x) = d - i$. Under mild hypotheses on X, which are always satisfied in the situations we consider, Kato shows (see [Kat86, Prop. 1.7]) that one has a complex (with the first term placed in degree d):[3]

$$C_n^0(X): \qquad \mathrm{H}^{d+1}\big(K\big) \xrightarrow{\partial_d} \bigoplus_{x \in X_{d-1}} \mathrm{H}^d\big(\kappa(x)\big) \to \cdots \to \bigoplus_{x \in X_0} \mathrm{H}^1\big(\kappa(x)\big).$$

The first map ∂_d is defined in terms of the discrete valuations of K defined by the points x in $X_{d-1} = X^1$ as follows: Since X is normal, the local ring $\mathcal{O}_x$ is a DVR, say, with canonical valuation w_x and residue field $Kw_x = \kappa(x)$. Hence every $x \in X^1$ gives rise to a residue map $\partial_x : \mathrm{H}^{i+1}\big(K\big) \to \mathrm{H}^d\big(\kappa(x)\big)$ as indicated at b) above, and one has $\partial_d := \oplus_{x \in X^1} \partial_x$.

Let K_{w_x} be the henselization of K at w_x, so $K_{w_x} w_x = K w_x = \kappa(x)$. For $\boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big)$, recall its image $\boldsymbol{\alpha}_{w_x} \in \mathrm{H}^{d+1}\big(K_{w_x}\big)$ as defined in item (a) above. Then by definition, one has $\partial_x(\boldsymbol{\alpha}) = \partial_{w_x}(\boldsymbol{\alpha}_{w_x})$ in $\mathrm{H}^d\big(\kappa(x)\big)$. Hence if $\mathrm{H}_d\big(C_n^0(X)\big) = 0$ (i.e., if the first map of the complex is injective), one has

$$(2.1) \quad \boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big) \text{ is trivial iff } \boldsymbol{\alpha}_{w_x} \in \mathrm{H}^{d+1}\big(K_{w_x}\big) \text{ is trivial for all } x \in X^1.$$

Among other things, in [Kat86, Cor., p. 145] Kato proves $\mathrm{H}_2\big(C_n^0(X)\big) = 0$ for a two-dimensional projective regular integral $\mathbb{Z}$-scheme X such that $K = \kappa(X)$ has no orderings.

The generalization of Kato's result above to higher dimensions suitable for our purposes is given by (some special form of more general) results by Jannsen [Jan16] and Kerz–Saito [KS12]; see Facts 2.2 and 2.3 below.

Let R be either a finite field with char $\neq 2$, or the valuation ring of a henselization of a global field k at some $v \in \mathbb{P}_{\mathrm{fin}}(k)$ such that $\operatorname{char}(kv) \neq 2$. Let X be a proper regular integral flat R-scheme, let $K = \kappa(X)$ be its field of rational functions, let $d = \dim X = \dim K > 0$, and notice that X is excellent and $n = 2$ is invertible on X. Kerz–Saito [KS12] denote the Kato complex $C_n^0(X)$ introduced above by $\mathrm{KC}(X, \mathbf{Z}/n\mathbf{Z})$ and its homology by $\mathrm{KH}_a(X, \mathbf{Z}/n\mathbf{Z})$. This being said, Theorem 8.1 of loc. cit. asserts for $a = d$ and $\Lambda = \mathbb{Z}/2$ that $\mathrm{KH}_a(X, \Lambda)\big) = 0$; that is, $\mathrm{H}_d\big(C_2^0(X)\big) = 0$ in the notation of Kato. Hence by (2.1) above one has the following.

FACT 2.2 (cf. Kerz–Saito [KS12, Th. 8.1] for $a = d$, $l = 2$, $\Lambda = \mathbb{Z}/2\mathbb{Z}$). *Let R, X and $K = \kappa(X)$ be as above. Then for $\boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big)$, one has*

> *$\boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big)$ is trivial if and only if $\boldsymbol{\alpha}_{w_x} \in \mathrm{H}^{d+1}\big(K_{w_x}\big)$ is trivial for all $x \in X^1$.*

[3] Actually, this is a special case of the more general context in [Kat86].

Finally, we consider the case char $= 2 = n$. Following Jannsen (see [Jan16, Def. 4.18]) we say that *resolution of singularities holds above* $\mathbb{F}_2$ if the following hold:

(i) For any proper integral $\mathbb{F}_2$-variety X, there is a proper birational morphism $\tilde{X} \to X$, where $\tilde{X}$ is a smooth (or equivalently regular) $\mathbb{F}_2$-variety.

(ii) Every affine smooth $\mathbb{F}_2$-variety U has an open immersion $U \hookrightarrow X$, where X is a projective smooth $\mathbb{F}_2$-variety, and $X \backslash U$ is a simple normal crossings divisor.

Resolution of singularities is well known for surfaces and holds in dimension three (in general) by Cossart–Piltant [CP09]. Further, if resolution of singularities above $\mathbb{F}_2$ holds, then any finitely generated field of characteristic two has a smooth proper model over $\mathbb{F}_2$.

This being said, Fact 2.3 below follows from results by several authors; e.g., Kato [Kat86] for $\dim(K) = 2$, Suwa [Suw95, p. 270] for $\dim(K) = 3$, and (conditionally) Jannsen [Jan16, Th. 0.10] for $\dim(K)$ arbitrary. Namely, let K be a finitely generated field with $\mathrm{char}(K) = 2$, and if $d = \dim(K) > 3$, suppose that resolution of singularities holds above $\mathbb{F}_2$. Let X be a projective smooth $\mathbb{F}_2$-model for K. Noting that Jannsen denotes Kato's complex $C_n^0(X)$ introduced above by $C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})$, [Jan16, Th. 0.10] (for $a = d$ and $n = 2$) asserts that $\mathrm{H}_a\big(C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})\big) = 0$; that is, $\mathrm{H}_d\big(C_2^0(X)\big) = 0$ in the notation of Kato. Hence by (2.1) above one has the following.

FACT 2.3 (cf. Jannsen [Jan16, Th. 0.10] for $a = d$ and $n = 2$). *In the above notation and hypothesis, for all* $\boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big)$, *the following holds*:

> $\boldsymbol{\alpha} \in \mathrm{H}^{d+1}\big(K\big)$ *is trivial if and only if* $\boldsymbol{\alpha}_{w_x} \in \mathrm{H}^{d+1}\big(K_{w_x}\big)$ *is trivial for all* $x \in X^1$.

3. Consequences/applications of the local-global principles

We begin by recalling a few basic facts about *Pfister forms*, which are at the core of first-order definability of prime divisors. For a field F and $a \in F^\times$, set $\langle\langle a\rangle\rangle = x_1^2 - ax_2^2$,[4] respectively $\langle\langle a]] := x_1^2 + x_1x_2 + ax_2^2$. For an $(i+1)$-tuple $\boldsymbol{a} = (a_i, \ldots, a_0)$ with $a_i, \ldots, a_0 \in F^\times$, the $(i+1)$-fold Pfister form $q_{\boldsymbol{a}}$ is defined as follows (see [EKM08, 9.B] for details):

- If $\mathrm{char}(F) \neq 2$, then $q_{\boldsymbol{a}} := q_{a_i,\ldots,a_0} := \langle\langle a_i\rangle\rangle \otimes \cdots \otimes \langle\langle a_0\rangle\rangle$.
- If $\mathrm{char}(F) = 2$, then $q_{\boldsymbol{a}} := q_{a_i,\ldots,a_0} := \langle\langle a_i\rangle\rangle \otimes \ldots \langle\langle a_1\rangle\rangle \otimes \langle\langle a_0]]$.[5]

[4] Some other sources prefer the convention $\langle\langle a\rangle\rangle = x_1^2 + ax_2^2$ in the case $\mathrm{char}(F) \neq 2$.

[5] In this case, one could allow $a_0 = 0$ without harm, but we prefer to require all $a_i \neq 0$ for uniformity.

It is well known (see [EKM08, Cor. 9.10]) that a form $q_{\boldsymbol{a}}$ as defined above is isotropic if and only if it is hyperbolic. Further, recalling that $\mathrm{H}^{i+1}\big(F\big) := \mathrm{H}^{i+1}\big(F, \Lambda(i)\big)$ with $\Lambda = \mathbb{Z}/2$ as introduced above, by [EKM08, §16],[6] to every Pfister form $q_{\boldsymbol{a}} = \langle\langle \boldsymbol{a} \rangle\rangle$ or $q_{\boldsymbol{a}} = \langle\langle \boldsymbol{a}]]$, one can attach in a canonical way a cohomological invariant

$$e(q_{\boldsymbol{a}}) \in \mathrm{H}^{i+1}\big(F\big).$$

Let $N := 2^{i+1} - 1$. Then $q_{\boldsymbol{a}}$ is a quadratic form in $N+1$ variables $\boldsymbol{x} = (x_1, \dots, x_{N+1})$, and

the associated variety $V_{q_{\boldsymbol{a}}} := V_F(q_{\boldsymbol{a}}) \hookrightarrow \mathbb{P}^N_F$ *is a smooth* F*-subvariety of* $\mathbb{P}^N_F$.

Fact 3.1. *In the above notation, the following hold*:

(1) *The Pfister form* $q_{\boldsymbol{a}}$ *is isotropic over* F *if and only if* $e(q_{\boldsymbol{a}}) = 0$ *in* $\mathrm{H}^{i+1}\big(F\big)$.

(2) *Let* $E|F$ *be a field extension, and let* $q_{\boldsymbol{a},E}$ *be* $q_{\boldsymbol{a}}$ *viewed over* E. *One has* $e(q_{\boldsymbol{a},E}) = \mathrm{res}\,\big(e(q_{\boldsymbol{a}})\big)$ *under* $\mathrm{res}_{E|F} : \mathrm{H}^{i+1}\big(F\big) \to \mathrm{H}^{i+1}\big(E\big)$.

Concerning the proofs, assertion(1) is implied by the Milnor Conjecture (although previous weaker results would suffice (see [EL72], [Kat82]); to be precise, use [EKM08, Fact 16.2] together with the fact that the $(i+1)$-fold Pfister form $q_{\boldsymbol{a}}$ is isotropic if and only if it is hyperbolic, which is the case if and only if its class in the Witt ring of F lies in $I_q^{i+2}(F)$ [EKM08, Th. 23.7(1)]). Assertion (2) follows by definition.

We conclude this preparation with the following facts scattered throughout the literature (although some of them might be new in the generality presented here); variants of these will be used later. For the reader's sake, we give the (straightforward) full proofs.

Proposition 3.2. *Let* F *be henselian with respect to a non-trivial non-dyadic valuation* w, *i.e.*, $\big(\mathrm{char}(F), \mathrm{char}(Fw)\big) \neq (0, 2)$. *Let* $k \subset F$ *be its constant subfield, i.e., the relative algebraic closure of the prime subfield in* F. *Let* $\boldsymbol{\varepsilon} = (\varepsilon_r, \dots, \varepsilon_0)$ *be* w*-units in* F.

(1) *Suppose that* $w(\varepsilon_1 - 1) > 0$. *Then* $q_{\varepsilon_1, \varepsilon_0}$ *is isotropic over* F. *Hence* $q_{\boldsymbol{\varepsilon}}$ *is isotropic over* F.

(2) *Let* $\overline{\boldsymbol{\varepsilon}}$ *be the image of* $\boldsymbol{\varepsilon}$ *under the residue map* $\mathcal{O}_w^\times \to Fw$, *and let* $\boldsymbol{\pi} = (\pi_s, \dots, \pi_1)$, $\pi_i \in F^\times$ *be such that* $w(\pi_s), \dots, w(\pi_1)$ *are* $\mathbb{F}_2$*-independent in* $wF/2$. *The following are equivalent*:
   - (i) $q_{\overline{\boldsymbol{\varepsilon}}}$ *is isotropic over* Fw;
   - (ii) $q_{\boldsymbol{\varepsilon}}$ *is isotropic over* F;
   - (iii) $q_{(\boldsymbol{\pi}, \boldsymbol{\varepsilon})}$ *is isotropic over* F.

[6] Be aware of the inconsistency of notation in [EKM08]; see footnote 2 of this article.

(iv) *Suppose that* $\dim(F) = r$ *and* $q_{\boldsymbol{\varepsilon}}$ *is isotropic over the compositum* $F_v = k_vF$ *for each real closure* k_v *of* k *(if there are any such* k_v*). Then* $q_{\boldsymbol{\varepsilon}}$ *is isotropic over* F.

Proof. Let $N := 2^{r+1}-1$, and recall the $(r+1)$-fold Pfister form $q_{\boldsymbol{\varepsilon}} = q_{\boldsymbol{\varepsilon}}(\boldsymbol{x})$ in variables $\boldsymbol{x} = (x_1, \ldots, x_{N+1})$. Since w is non-dyadic, $V_{q_{\boldsymbol{\varepsilon}}} \hookrightarrow \mathbb{P}^N_{\mathcal{O}_w}$ is a smooth $\mathcal{O}_w$-subvariety of $\mathbb{P}^N_{\mathcal{O}_w}$, with special fiber the projective smooth Kw-variety $V_{q_{\overline{\boldsymbol{\varepsilon}}}} \hookrightarrow \mathbb{P}^N_{Fw}$. Hence by Hensel's Lemma, the specialization map on rational points $V_{q_{\boldsymbol{\varepsilon}}}(F) \to V_{q_{\overline{\boldsymbol{\varepsilon}}}}(Fw)$ is surjective, implying

(3.1) $\quad$ $q_{\boldsymbol{\varepsilon}}$ *is isotropic over* F *if and only if* $q_{\overline{\boldsymbol{\varepsilon}}}$ *is isotropic over* Fw.

To (1): Since $\overline{\varepsilon}_1 = 1$, $q_{\overline{\varepsilon}_1,\overline{\varepsilon}_0} = q_{1,\overline{\varepsilon}_0}$ is isotropic over Fw, thus so are $q_{\varepsilon_1,\varepsilon_0}$ and $q_{\boldsymbol{\varepsilon}}$ over F by (3.1).

To (2): Setting $\boldsymbol{x}_\chi = (x_{\chi,i})_{i\leqslant N}$, $\pi_\chi = \prod_i \pi_i^{\chi(i)}$ for $\chi : \{1, \ldots, s\} \to \{0{,}1\}$, $\boldsymbol{y} = (\boldsymbol{x}_\chi)_\chi$, one has $q_{\boldsymbol{\pi},\boldsymbol{\varepsilon}}(\boldsymbol{y}) = \sum_\chi \pi_\chi q_{\boldsymbol{\varepsilon}}(\boldsymbol{x}_\chi)$. By (3.1) above, $q_{\overline{\boldsymbol{\varepsilon}}}$ is isotropic over Fw if and only if $q_{\boldsymbol{\varepsilon}}$ is isotropic over F, and if so, $q_{\boldsymbol{\pi},\boldsymbol{\varepsilon}}$ is isotropic over F. For the converse, let $q_{\overline{\boldsymbol{\varepsilon}}}$ be anisotropic. Then $w\big(q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})\big) \in 2\cdot wF$ for all $\boldsymbol{\nu} \neq 0$ in F^N, and for $\boldsymbol{\mu} = (\boldsymbol{\nu}_\chi)_\chi \neq \boldsymbol{0}$, one has Since $w(\pi_\chi) = \sum_i \chi(i)w(\pi_i)$, and $w\big(q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}_\chi)\big) \in 2\cdot wF$, and $\big(w(\pi_i)\big)_i$ are independent in $wF/2$, it follows that the summands in $q_{\boldsymbol{\pi},\boldsymbol{\varepsilon}}(\boldsymbol{\mu}) = \sum_\chi \pi_\chi q_{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}_\chi)$ have distinct values. Hence $q_{\boldsymbol{\pi},\boldsymbol{\varepsilon}}(\boldsymbol{\mu}) \neq 0$, thus $q_{\boldsymbol{\pi},\boldsymbol{\varepsilon}}$ is anisotropic.

To (3): We first claim that $e := \dim(Fw) < \dim(F) = r$. Indeed, by the Abhyankar Inequality (see, e.g., [EP05, Th. 3.4.3]) one has $\mathrm{td}(F) - \mathrm{td}(Fw) \geqslant r(w)$, where $\mathrm{td}(\bullet)$ is the absolute transcendence degree and $r(w) := \dim_{\mathbb{Q}}((wF/wk)\otimes\mathbb{Q})$ is the rational rank of the abelian group wF/wk. First, if $w|_k$ is non-trivial, then $\mathrm{char}(k) = 0$ and kw is algebraic over a finite field and therefore $\dim(F) - \dim(Fw) = 1 + \mathrm{td}(F) - \mathrm{td}(Fw) \geqslant 1 + r(w) > 0$. Second, if $w|_k$ is trivial, then $r(w) > 0$, hence $\dim(F) - \dim(Fw) = \mathrm{td}(F) - \mathrm{td}(Fw) \geqslant r(w) > 0$.

Case 1: $\mathrm{char}(Fw) = p > 0$. Then $e = \dim(Fw) = \mathrm{td}(Fw)$, and $q_{\overline{\boldsymbol{\varepsilon}}}$ is a quadratic form in 2^{r+1} variables over Fw. Since $e < r$, and Fw is a C_{e+1}-field, $q_{\overline{\boldsymbol{\varepsilon}}}(\boldsymbol{x}) = 0$ has non-trivial solutions in Fw, i.e., $q_{\overline{\boldsymbol{\varepsilon}}}$ is isotropic over Fw. Hence so is $q_{\boldsymbol{\varepsilon}}$ over F by (3.1).

Case 2: $\mathrm{char}(Fw) = 0$. Then w is trivial on the constant field k of F, and by Hensel's Lemma, there is a field of representatives $E \subset F$ for Fw. Further, E is relatively algebraically closed in F, so $k \subset E$ is relatively closed in E, and $\dim(E) = \dim(Fw) = e < r$. Let $\boldsymbol{\eta} = (\eta_r, \ldots, \eta_0) \in E^{r+1}$ be the lifting of $\overline{\boldsymbol{\varepsilon}} = (\overline{\varepsilon}_r, \ldots, \overline{\varepsilon}_0)$. Then $\varepsilon_i = \eta_i\delta_i$ with $\delta_i \in F$ and $w(\delta_i - 1) > 0$. Since $\mathrm{char}(Fw) = 0$, by Hensel's Lemma, each δ_i is a square in F, thus $q_{\boldsymbol{\varepsilon}} \approx q_{\boldsymbol{\eta}}$ over F, and $q_{\boldsymbol{\eta}}$ is defined over $E \subset F$. We consider the following condition on subfields $E' \subset E$:

(3.2) $\qquad E'$ *is finitely generated*, $\eta_r, \dots, \eta_0 \in E'$.

For every E' satisfying (3.2), consider $k' := k \cap E'$, and for $v' \in \mathbb{P}(k')$, let $E'_{v'} = k'_{v'}E'$ be the compositum of E' and $k'_{v'}$ and $e(q_{\boldsymbol{\eta},v'})$ let be the cohomological invariant of $q_{\boldsymbol{\eta}}$ in $\mathrm{H}^{r+1}(E'_{v'})$.

Claim. *There is* $E' \subset E$ *satisfying* (3.2) *such that* $e(q_{\boldsymbol{\eta},v'}) = 0$ *for all* $v' \in \mathbb{P}(k')$.

Proof of the claim. Let E' satisfy (3.2), $k' = E' \cap k$. First, if $v' \in \mathbb{P}(k')$ is not a real place, then $E'_{v'}$ has no orderings; hence by the well-known behavior of cohomological dimension in field extensions we have $\mathrm{cd}(E'_{v'}) \leqslant \mathrm{cd}(E') \leqslant \dim(E') + 1 \leqslant \dim(Fw) + 1 < r + 1$. Thus $\mathrm{H}^{r+1}(E'_{v'}) = 0$, implying that $e(q_{\boldsymbol{\eta},v'}) = 0$. Second, concerning real places of k', let $\Sigma_{E'} \subset \mathbb{P}(k')$ be the (possibly empty) set of all real places v' such that $e(q_{\boldsymbol{\eta},v'}) \neq 0$. By contradiction, suppose that $\Sigma_{E'}$ is non-empty for all E' satisfying (3.2). Considering all $E' \subset E'' \subset E$ satisfying (3.2), the restriction maps $\Sigma_{E''} \to \Sigma_{E'}$ make $(\Sigma_{E'})_{E'}$ into a projective system of finite non-empty sets, having as projective limit the non-empty set $\Sigma_E \subset \mathbb{P}(k)$ of all $v \in \mathbb{P}(k)$ that satisfy $v' := v|_{k'} \in \Sigma_{E'}$ for all E' (where as always $k' = E' \cap k$). For $v \in \Sigma_E$, let k_v be the real closure of k at v, and let $w_v|w$ be the unique prolongation of the Henselian valuation w of F to the algebraic extension $F_v = k_vF$ of F. Since w is trivial on k, the residue field F_vw_v is the compositum k_vFw, and further, $E_v := k_vE \subset F_v$ is a field of representatives for k_vFw. Since $q_{\boldsymbol{\eta}}$ is isotropic over F_v, it is so over k_vFw, hence over $E_v = k_vE$. Equivalently, by Fact 3.1, $e(q_{\boldsymbol{\eta},v}) = 0$ in $\mathrm{H}^{r+1}(E_v)$. On the other hand, since cohomology is compatible with inductive limits, $e(q_{\boldsymbol{\eta},v}) = \varinjlim e(q_{\boldsymbol{\eta},v'}) \neq 0$, because $e(q_{\boldsymbol{\eta},v'}) \neq 0$ for all E', contradiction! The claim is proved. □

Back to the proof in Case (2), let $E' \subset E$ satisfy the claim. Set $F' = E'(\boldsymbol{t})$ for $\boldsymbol{t}$ a transcendence basis of $F|E'$. Then $F' \subset F$ is finitely generated, $E' \cap k = k' = F' \cap k$ and $E'_{v'} \subset F'_{v'} := k'_{v'}F'$ for all $v' \in \mathbb{P}(k')$. Since $e(q_{\boldsymbol{\eta},v'}) = 0$ in $\mathrm{H}^{r+1}(E'_{v'})$, $q_{\boldsymbol{\eta}}$ is isotropic over $E'_{v'}$, hence over $F'_{v'}$ for each $v' \in \mathbb{P}(k')$. Hence by Fact 3.1, $e(q_{\boldsymbol{\eta},v'}) = 0$ in $\mathrm{H}^{r+1}(F'_{v'})$ for all $v' \in \mathbb{P}(k')$, and therefore, by Fact 2.1, $e(q_{\boldsymbol{\eta}}) = 0$ in $\mathrm{H}^{r+1}(F')$. Equivalently, $q_{\boldsymbol{\eta}}$ is isotropic over F' and thus over F. Finally, $q_{\boldsymbol{\varepsilon}} \approx q_{\boldsymbol{\eta}}$ is isotropic over F. □

3.1. *Prime divisors via anisotropic* k_1*-nice Pfister forms.* We now state a technical condition for the Pfister forms we are going to work with. This technical condition in particular serves to ensure that orderings and dyadic places can always be eliminated from our subsequent considerations.

Definition 3.3. Let K be a field satisfying Hypothesis (H_d) from the introduction, and let $q_{\boldsymbol{a}}$ be a Pfister form defined by $\boldsymbol{a} := (a_d, \dots, a_1, a_0)$ with all $a_i \in K^\times$.

(1) Let $k_1 \subset K$ be a global subfield. We say that $q_{\boldsymbol{a}}$ is k_1-*nice* if $a_1, a_0 \in k_1$, and the two-fold Pfister form q_{a_1,a_0} satisfies

(3.3) If $v \in \mathbb{P}(k_1)$ is real, or dyadic, or $v(a_0) \neq 0$, or $v(a_1) < 0$, then q_{a_1,a_0} is isotropic over k_{1v}.

(2) We say that $q_{\boldsymbol{a}}$ is *nice* if there is there is a global subfield $k_1 \subset K$ such that $q_{\boldsymbol{a}}$ is k_1-nice.

Note that being nice is not an isometry invariant of Pfister forms, so strictly speaking it is a property of the concrete presentation; this should not lead to confusion.

Due to the results of the previous section, we now have the following local-global principle for isotropy of nice Pfister forms.

Proposition 3.4. *Let K satisfy Hypothesis* (H_d), *let $k_1 \subset K$ be a global subfield, and let $q_{\boldsymbol{a}}$ be an anisotropic k_1-nice Pfister form over K. The following hold*:

(1) *There is a prime divisor w of K such that $q_{\boldsymbol{a}}$ is anisotropic over the w-henselization K_w.*

(2) *If w is a prime divisor of K such that $q_{\boldsymbol{a}}$ is anisotropic over the w-henselization K_w, then w is non-dyadic, $w(a_0) = 0$, $w(a_1) \geqslant 0$, and $w(a_i)$ is odd for some $i = 1, \dots, d$.*

Proof. To (1): By Fact 3.1(1) and (2) above, proving that $q_{\boldsymbol{a}}$ is anisotropic over K_w is equivalent to proving that the image of $e(q_{\boldsymbol{a}})$ under the restriction map $\mathrm{res}_w : \mathrm{H}^{d+1}(K) \to \mathrm{H}^{d+1}(K_w)$ does not vanish. Noticing that $e(q_{\boldsymbol{a}}) \neq 0$ in $\mathrm{H}^{d+1}(K)$, proceed as follows:

Case 1. If $\mathrm{char}(K) = 2$, then choosing a smooth projective $\mathbb{F}_2$-model X for K, by Fact 2.3 above, there is a prime divisor w of K, say $w = w_x$ for some point $x \in X^1$, such that $\mathrm{res}_w\big(e(q_{\boldsymbol{a}})\big) \neq 0$ in $\mathrm{H}^{d+1}(K_w)$, and therefore $q_{\boldsymbol{a}}$ is anisotropic over K_w.

Case 2. If $\mathrm{char}(K) \neq 2$, we apply Fact 2.1 above, so there is $v \in \mathbb{P}(k_1)$ such that $\mathrm{res}_v(e(q_{\boldsymbol{a}})) \neq 0$ in $\mathrm{H}^{d+1}(K_v)$. Hence if $q_{\boldsymbol{a},v}$ is the Pfister form $q_{\boldsymbol{a}}$ viewed over K_v, then $e(q_{\boldsymbol{a},v}) = \mathrm{res}_v\big(e(q_{\boldsymbol{a}})\big) \neq 0$. Equivalently, $q_{\boldsymbol{a},v}$ is anisotropic over K_v, hence its Pfister subform q_{a_1,a_0} is anisotropic over $k_{1v} \subset K_v$. Thus by condition (3.3) above, v is a finite non-dyadic place of k_1. In particular, letting $R \subset k_{1v}$ be the henselization of $\mathcal{O}_v$, it follows that $\mathrm{char}(kv) \neq 2$. Let X_v be any projective R-model of K_v. Then using *prime to ℓ-alterations with* $\ell = 2$ (see [ILO14, Exp. X, Th. 2.4]), there are a projective regular irreducible R-scheme

$\tilde{X}$ and a projective surjective R-morphism $\tilde{X} \to X_v$ defining a finite field extension $\tilde{K} \mid K_v$ of degree prime to 2. In particular, the restriction of $e(q_{\boldsymbol{a},v})$ in $\mathrm{H}^{d+1}(\tilde{K})$ is non-zero. Hence by Fact 2.2 above, there exists $\tilde{x} \in \tilde{X}^1$ such that setting $\tilde{w} := w_{\tilde{x}}$, for the $\tilde{w}$-henselization $\tilde{K}_{\tilde{w}}$ of $\tilde{K}$ one has $\mathrm{res}_{\tilde{w}}\left(e(q_{\boldsymbol{a},v})\right) \neq 0$ in $\mathrm{H}^{d+1}(\tilde{K}_{\tilde{w}})$. Hence letting $q_{\boldsymbol{a},\tilde{w}}$ be the Pfister form $q_{\boldsymbol{a}}$ viewed over $\tilde{K}_{\tilde{w}}$, one has $e(q_{\boldsymbol{a},\tilde{w}}) = \mathrm{res}_{\tilde{w}}\left(e(q_{\boldsymbol{a},v})\right) \neq 0$ in $\mathrm{H}^{d+1}(\tilde{K}_{\tilde{w}})$, concluding by Fact 3.1 that $q_{\boldsymbol{a},\tilde{w}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$. Let $w := \tilde{w}|_K$. Then since $\tilde{K} \mid K$ is an algebraic extension, and $\tilde{w}$ is a prime divisor of $\tilde{K}$, it follows that $w = \tilde{w}|_K$ is a prime divisor of K, and the w-henselization K_w is contained in $\tilde{K}_{\tilde{w}}$. Since $q_{\boldsymbol{a},\tilde{w}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$, it follows that $q_{\boldsymbol{a}}$ is anisotropic over K_w.

To (2): Let $v := w|_{k_1}$ be the restriction of w to k_1 (which might be the trivial valuation). Then k_{1v} is contained in K_w. Hence since $q_{\boldsymbol{a}}$ is anisotropic over K_w, its subform q_{a_1,a_0} (which is defined over k_1) is anisotropic over k_{1v}. Since q_{a_1,a_0} is k_1-nice, either v is trivial or $v \in \mathbb{P}(k_1)$ must be finite non-dyadic and $v(a_0) = 0$, $v(a_1) \geqslant 0$. Hence w is non-dyadic, and further, $w(a_0) = v(a_0) = 0$, $w(a_1) = v(a_1) \geqslant 0$. It remains to show that $w(a_i)$ is odd for some $i = 1, \ldots, d$. If not, for all such i we may write $a_i = b_i c_i^2$ for some $b_i, c_i \in K^\times$ with $w(b_i) = 0$. But then $q_{\boldsymbol{a}} \approx q_{b_d,\ldots,b_1,a_0}$, and the latter form is isotropic over K_w by Proposition 3.2(3) (where the hypothesis on real places is satisfied by niceness of $q_{\boldsymbol{a}}$). Therefore $q_{\boldsymbol{a}}$ is also isotropic over K_w in contradiction to the hypothesis. □

3.2. *Abundance of anisotropic k_1-nice Pfister forms*. In Section 3.1 above we saw that anisotropic nice Pfister forms over a finitely generated field K remain anisotropic over some henselization of K with respect to some non-dyadic prime divisors of K. In this subsection, we prove that given any geometric prime divisor w of K, and a global subfield $k_1 \subset K$ with w trivial on k_1, there are "many" k_1-nice Pfister forms that remain anisotropic over the w-henselization K_w. Our actual result, Proposition 3.8 below, is more complicated to state, because we want to realize additional restrictions on the Pfister forms.

LEMMA 3.5. *Let l_1/k_1 be a finite separable extension of global fields, and let $\Sigma \subset \mathbb{P}_{\mathrm{fin}}(k_1)$ be a finite set of finite places of k_1. Then there exists a k_1-nice Pfister form q_{a_1,a_0} over k_1 such that $v(a_1) = v(a_0) = 0$ for all $v \in \Sigma$ and q_{a_1,a_0} is anisotropic over l_1.*

Proof. We may enlarge Σ to contain all dyadic places of k_1. There are infinitely many finite places of k_1 that split completely in l_1. Pick one such place v_1 that is not in Σ. Using weak approximation, choose $a_0 \in k_1^\times$ such that $v(a_0) = 0$ for all $v \in \Sigma$, and $v_1(a_0) = 0$, and furthermore the reduction of the polynomial $X^2 - X - a_0$ in $k_1 v_1[X]$ is irreducible if the characteristic of

k_1v_1 is 2, respectively the reduction of the polynomial $X^2 - a_0$ in $k_1v_1[X]$ is irreducible if the characteristic of k_1v_1 is not 2. (The case distinction here arises from the different definition of the form q_{a_0} depending on the characteristic.)

Let $l' = k_1(\alpha_0)$, with α_0 a root of $X^2 - X - a_0$ respectively $X^2 - a_0$. Pick a place $v_0 \in \mathbb{P}_{\rm fin}(k_1)\backslash\Sigma$ that splits completely in l', hence $v_0 \neq v_1$ because v_1 is inert in l'. Using the Strong Approximation Theorem, choose $a_1 \in k_1^\times$ satisfying the following four conditions:

- $v_1(a_1) = 1$;
- a_1 is a norm of the local extension $k_{1v}(\alpha_0)|k_{1v}$ for all the finitely many $v \in \mathbb{P}_{\rm fin}(k_1)$ for which $v(a_0) \neq 0$, all dyadic v and all real v;
- $v(a_1) = 0$ for all $v \in \Sigma$;
- $v(a_1) \geqslant 0$ at all $v \in \mathbb{P}_{\rm fin}(k_1)\backslash\{v_0\}$.

(The condition at dyadic $v \in \Sigma$ is thus that $v(a_1) = 0$ and a_1 is a local norm, both of which are open conditions satisfied in a v-neighbourhood of 1.) The following hold: First, q_{a_1,a_0} is anisotropic over k_{1,v_1} by the definitions of v_1, a_0, a_1 and Proposition 3.2(2). Hence q_{a_1,a_0} is anisotropic over $l_1 \subset k_{1,v_1}$. Second, we claim that q_{a_1,a_0} is k_1-nice. Indeed, by the choice of a_1, one has the following: If $v(a_0) \neq 0$ or v is dyadic or v is real, then a_1 is a norm of $k_{1v}(\alpha_0)/k_{1v}$. Hence in these cases, q_{a_1,a_0} is isotropic over k_{1v}. Finally, if $v(a_1) < 0$, then $v = v_0$, hence v is totally split in $l' = k_1(\alpha_0)$, implying that $\alpha_0 \in k_{1v}$. Hence q_{a_1,a_0} is isotropic over k_{1v}. □

LEMMA 3.6. *Let K satisfy Hypothesis* (H_d), *and let w be a geometric prime divisor of K. There is a global subfield $k_1 \subset K$, and k_1-algebraically independent elements $\boldsymbol{u} = (u_i)_{d>i>1}$ of K such that w is trivial on $k_1(\boldsymbol{u})$ and Kw is finite separable over $k_1(\boldsymbol{u})$. Moreover, if $u_d \in K$ has $w(u_d) = 1$, then $(u_d, \boldsymbol{u})$ is a separating transcendence basis of $K|k_1$.*

Proof. Since w is geometric, K and Kw have the same prime field κ_0, and are separably generated over κ_0. Proceed as follows:

(i) If $\mathrm{char}(K) = 0$, let $(u_i)_{d>i>1}$ be any w-units that lift a transcendence basis of Kw.

(ii) If $\mathrm{char}(K) > 0$, let $(u_i)_{d>i>0}$ be w-units that lift a separating transcendence basis of Kw.

Let $k_1 \subset K$ be the constant field in case (i) and the relative algebraic closure of $\kappa_0(u_1)$ in K in case (ii), and set $\boldsymbol{u} = (u_i)_{d>i>1}$ in both cases. Then w is trivial on $k_1(\boldsymbol{u})$, and the residue of $\boldsymbol{u}$ in Kw is a separating transcendence basis of Kw over k_1. Assume now that $w(u_d) = 1$; thus, in particular, w is not trivial on $k_1(u_d, \boldsymbol{u})$. Since w is trivial on $k_1(\boldsymbol{u})$ and non-trivial on $k_1(u_d, \boldsymbol{u})$, u_d cannot be algebraic over $k_1(\boldsymbol{u})$. Hence since $\mathrm{td}\big(K \,|\, k_1(\boldsymbol{u})\big) = 1$, $(u_d, \boldsymbol{u})$ is a transcendence basis of K over k_1, and $K|k_1(u_d, \boldsymbol{u})$ is a finite field extension. We claim that $K \,|\, k_1(u_d, \boldsymbol{u})$

is separable. Indeed, let K_s be the separable closure of $k_1(u_d, \boldsymbol{u})$ in K, and set $w_s := w|_{K_s}$. Since $K|K_s$ is purely inseparable, w is the only prolongation of w_s to K, and the following hold: First, $w(u_d) = 1 = w_s(u_d)$, hence $e(w|w_s) = 1$. Second, $Kw \,|\, K_s w_s$ is purely inseparable, and since $Kw \,|\, k_1(\boldsymbol{u})$ is separable and $k_1(\boldsymbol{u}) \subset K_s w_s$, one must have $Kw = K_s w_s$, hence $f(w|w_s) = 1$. Third, since $K \supset K_s$ are function fields in one variable over $k_1(\boldsymbol{u})$, the fundamental equality for w_s and its unique prolongation w to K holds, see; e.g., [Che51, Ch. 2IV,§1, Th. 1]. Hence $[K : K_s] = e(w|w_s)f(w|w_s) = 1$, and thus $K = K_s$ is separable over $k_1(u_d, \boldsymbol{u})$. □

Definition 3.7. Let K satisfy Hypothesis (H$_d$), let $k_1 \subset K$ be a global subfield, and let $\boldsymbol{t} = (t_i)_{d>i>1}$ be k_1-algebraically independent in K. A *$k_1,\boldsymbol{t}$-test form* for an element $a_d \in K^\times$ is any k_1-nice Pfister form $q_{\boldsymbol{a}}$ defined by $\boldsymbol{a} = (a_d, a_{d-1}, \ldots, a_1, a_0)$, where $(a_i)_{d>i>1} = \boldsymbol{t} - \boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon} = (\epsilon_i)_{d>i>1}$ are such that $\epsilon_i \in k_1$ for $1<i<d$ are v-units for all $v \in \mathbb{P}_{\mathrm{fin}}(k_1)$ with $v(a_1) > 0$.

Proposition 3.8. *Let K satisfy Hypothesis* (H$_d$), *and let w be a geometric prime divisor of K. Let $k_1 \subset K$ be a global subfield, let $\boldsymbol{t} = (t_i)_{d>i>1}$ be k_1-algebraically independent elements of K such that w is trivial on $k_1(\boldsymbol{t})$ and $Kw\,|\,k_1(\boldsymbol{t})$ is finite separable. Then there is a Zariski open dense subset $U \subset k_1^{\times\, d-2}$ satisfying the following*: *For every $\boldsymbol{\epsilon} = (\epsilon_i)_{d>i>1} \in U$, there is a k_1-nice Pfister form q_{a_1,a_0}, such that for arbitrary $a_d \in K^\times$ with $w(a_d)$ odd, setting $(a_i)_{d>i>1} = \boldsymbol{t} - \boldsymbol{\epsilon}$ and $\boldsymbol{a} = (a_d, \ldots, a_1, a_0)$, one has that $q_{\boldsymbol{a}}$ is a $k_1, \boldsymbol{t}$-test form for a_d that is anisotropic over K_w.*

Proof. The normalization morphism $S \to S_{\boldsymbol{t}}$ of $S_{\boldsymbol{t}} := \operatorname{Spec} k_1[\boldsymbol{t}, \boldsymbol{t}^{-1}]$ in the finite separable field extension $l := Kw \hookleftarrow k_1(\boldsymbol{t})$ is a finite generically separable cover, thus étale above a Zariski open dense subset $U_l \subset S_{\boldsymbol{t}}$. Hence for $\boldsymbol{\epsilon} := (\epsilon_i)_{d>i>1} \in U := U_l(k_1)$, any preimage $s_{\boldsymbol{\epsilon}} \mapsto \boldsymbol{\epsilon}$ of $\boldsymbol{\epsilon}$ under the morphism $S \to S_{\boldsymbol{t}}$ is a smooth point of S, $\boldsymbol{\pi} := (a_i)_{d>i>1} := (t_i - \epsilon_i)_{d>i>1}$ is a regular system of parameters at $s_{\boldsymbol{\epsilon}} \mapsto \boldsymbol{\epsilon}$, and the residue field extension $k_1 = \kappa(\boldsymbol{\epsilon}) \hookrightarrow \kappa(s_{\boldsymbol{\epsilon}}) =: k_{\boldsymbol{\epsilon}}$ is finite separable. In particular, the completion of the local ring $\mathcal{O}_{s_{\boldsymbol{\epsilon}}}$ is the ring of formal power series $\widehat{\mathcal{O}}_{s_{\boldsymbol{\epsilon}}} = k_{\boldsymbol{\epsilon}}[\![\boldsymbol{\pi}]\!]$ in the variables $\boldsymbol{\pi} = (a_i)_{d>i>1}$ over $k_{\boldsymbol{\epsilon}}$. Hence one has $k_1(\boldsymbol{\pi})$-embeddings

$$\begin{aligned} l = Kw = \operatorname{Quot}(\mathcal{O}_{s_{\boldsymbol{\epsilon}}}) &\hookrightarrow \operatorname{Quot}(\widehat{\mathcal{O}}_{s_{\boldsymbol{\epsilon}}}) \\ &= \operatorname{Quot}\big(k_{\boldsymbol{\epsilon}}[\![a_2, \ldots, a_{d-1}]\!]\big) \hookrightarrow k_{\boldsymbol{\epsilon}}((a_2)) \cdots ((a_{d-1})) =: \widehat{l}. \end{aligned}$$

Let $\Sigma \subset \mathbb{P}(k_1)$ be any finite set of finite places such that all $(\epsilon_i)_{d>i>1}$ are Σ-units, and for $l_1 := k_{\boldsymbol{\epsilon}}$, consider $a_1, a_0 \in k_1$ as in Lemma 3.5. Then for $a_d \in K^\times$ with $w(a_d)$ odd, setting $\boldsymbol{a} := (a_d, \ldots, a_0)$ with $(a_i)_{d>i\geqslant 0}$ as introduced above, we claim that $q_{\boldsymbol{a}}$ is a $k_1, \boldsymbol{t}$-test form that satisfies the requirements of Proposition 3.8. Indeed, q_{a_1,a_0} is anisotropic over $l_1 = k_{\boldsymbol{\epsilon}}$, by the choice of

$a_1, a_0 \in k_1$. Hence $q_{\boldsymbol{\pi},a_1,a_0}$ is anisotropic over $\widehat{l}$, by Proposition 3.2(2) (applied with the natural valuation on $\widehat{l}$ with value group $\mathbb{Z}^{d-2}$), thus anisotropic over $Kw \subset \widehat{l}$. In particular, since $\boldsymbol{\pi}, a_1, a_0$ is a system of w-units, and $w(a_d)$ is odd, one gets that $q_{\boldsymbol{a}} = q_{a_d,\boldsymbol{\pi},a_1,a_0}$ is anisotropic over K_w, by Proposition 3.2(2). □

3.3. *A strengthening of Proposition* 3.4. In this subsection, we prove a strengthening of Proposition 3.4 under refined hypotheses.

For an arbitrary field F, we let Val_F be the Riemann–Zariski space (of equivalence classes of valuations) of F. We endow Val_F with the *patch topology*, which is the coarsest topology such that the sets of the form $\{v \in \mathrm{Val}_F \mid v(a) \geqslant 0\}$, $a \in F$ are open and closed. It follows that the sets $\{v \in \mathrm{Val}_F \mid v(b) > 0\}$, $\{v \in \mathrm{Val}_F \mid v(c) = 0\}$ are open and closed for all $b, c \in F$.

The patch topology makes Val_F a compact Hausdorff space; see, for instance, the discussion in [ZS75, Ch. VI, §17, proof of Th. 40].

LEMMA 3.9. *In the above notation, let F_w be the w-henselization at $w \in \mathrm{Val}_F$. One has the following*:

(1) *Let $E|F$ a finite extension. Then the set*

$$\mathcal{V}_{E|F} := \{w \in \mathrm{Val}_F \mid E \text{ is } F\text{-embeddable in } F_w\}$$

is open in the patch topology.

(2) *Let $q_{\boldsymbol{a}}$ be a quadratic form over F. Then the set*

$$\mathcal{V}_{\boldsymbol{a}} := \{w \in \mathrm{Val}_F \mid q_{\boldsymbol{a}} \text{ is isotropic over } F_w\}$$

is open in the patch topology.

Proof. To (1): Recall that the henselization $F_w|F$ is a separable algebraic extension, hence if $\mathcal{V}_{E|F}$ is non-empty, $E|F$ is separable. Let $w \in \mathcal{V}_{E|F}$. We also write w for the (canonical) prolongation of w to F_w and its restriction to E. By Hilbert decomposition theory (see, e.g., [KN14, Th. 1.2]), $E = F[\eta]$ with η satisfying $w(\eta) = w(p'(\eta)) = 0$ and η having minimal polynomial $p(t) = t^n + \sum_{i<n} a_i t^i \in F[t]$ such that $w(a_i) \geqslant 0$. Since $F_w|F$ is an immediate extension, there is $x \in F$ with $w(x-\eta) > 0$, hence $w\big(p(x)\big) > 0$, and $w\big(p'(x)\big) = 0$. The set

$$\mathcal{V}_w = \{\tilde{w} \in \mathrm{Val}_F \mid \tilde{w}(a_i) \geqslant 0 \text{ for all } i < n, \tilde{w}(p(x)) > 0, \tilde{w}(p'(x)) = 0\}$$

is open (and closed) in the patch topology and $w \in \mathcal{V}_w$. On the other hand, if $\tilde{w} \in \mathcal{V}_w$, then the polynomial $p(t)$ has a zero in the henselization $F_{\tilde{w}}$, thus E is F-embeddable into $F_{\tilde{w}}$. Conclude that $\mathcal{V}_w \subset \mathcal{V}_{E|F}$, hence the latter is open in the patch topology, as claimed.

To (2): Let $w \in \mathcal{V}_{\boldsymbol{a}}$; that is, $q_{\boldsymbol{a}}$ is isotropic over F_w. Then there is a finite subextension $E|F$ of $F_w|F$ such that $q_{\boldsymbol{a}}$ is isotropic over E. Then $\mathcal{V}_{\boldsymbol{a}}$ contains the neighborhood $\mathcal{V}_{E|F}$ of $q_{\boldsymbol{a}}$. Thus $\mathcal{V}_{\boldsymbol{a}}$ is open. □

Proposition 3.10. *Suppose that K satisfies Hypothesis* (H_d). *Let $L|K$ be finite separable, and let $a_d \in K^\times$. Suppose that there are a global subfield $k_1 \subset K$ and k_1-algebraically independent elements $\boldsymbol{u} = (u_i)_{d>i>1}$ of K, such that setting $\boldsymbol{t} := (t_i)_i := (u_i^2 - u_i)_i$, there is a $k_1,\boldsymbol{t}$-test form $q_{\boldsymbol{a}}$ for a_d that is anisotropic over the fields $L(\alpha)$ with $\alpha^2 - \alpha = a_d/\theta^2$, $\theta = (a_{d-1}\cdots a_1)^N$ for all $N > 0$. Then there exists a prime divisor w_L of L that is trivial on $k_1(\boldsymbol{t})$ such that $w_L(a_d) > 0$ is odd, and $q_{\boldsymbol{a}}$ is anisotropic over L_{w_L}.*

Proof. First, let $N > 0$ be fixed, and for $\theta = (a_{d-1}\cdots a_1)^N$ and $\alpha^2 - \alpha = a_d/\theta^2$, set $\tilde{K} := L(\alpha)$. Then $q_{\boldsymbol{a}}$ is an anisotropic k_1-nice Pfister form over $\tilde{K}$, hence Proposition 3.4 implies that there is a non-dyadic prime divisor $\tilde{w} = \tilde{w}_N$ of $\tilde{K}$ such that $q_{\boldsymbol{a}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$. Recalling that $\boldsymbol{a} = \big(a_d, (a_i)_{d>i>1}, a_1, a_0\big) = \big(a_d, (t_i - \epsilon_i)_{d>i>1}, a_1, a_0\big)$, we claim

Claim 1. *One has $\tilde{w}(a_i) \geqslant 0$ for $i < d$.*

Proof of Claim 1. Let $v := \tilde{w}|_{k_1}$ be the restriction of $\tilde{w}$ to k_1. First, suppose that v is non-trivial. Then $k_{1v} \subset \tilde{K}_{\tilde{w}}$, hence the fact that $q_{\boldsymbol{a}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$ implies that q_{a_1,a_0} is anisotropic over k_{1v}. Since q_{a_1,a_0} is k_1-nice and anisotropic over k_{1v}, Proposition 3.2(3) applied to $F = k_{1v}$ and q_{a_1,a_0} implies that a_1 and a_0 cannot both be v-units, and so Proposition 3.4(2) implies that $v(a_0) = 0$ and $v(a_1) > 0$. Since $q_{\boldsymbol{a}}$ is a $k_1,\boldsymbol{t}$-test form for a_d and $v(a_1) > 0$, one has $v(\epsilon_i) = 0$ by definition, thus $\tilde{w}(\epsilon_i) = v(\epsilon_i) = 0$ for $1 < i < d$. Second, if v is trivial, then $\tilde{w}(\epsilon_i) = v(\epsilon_i) = 0$ for all $i < d$ as well. Hence independently on whether v is trivial or not, one has $\tilde{w}(a_0) = 0$, $\tilde{w}(a_1) \geqslant 0$, and $\tilde{w}(\epsilon_i) = 0$ for all $1 < i < d$. Next, by contradiction, suppose that $\tilde{w}(a_i) < 0$ for some $i < d$. Then $1 < i < d$, and since $a_i = t_i - \epsilon_i$, we must have $\tilde{w}(t_i) < 0$. Hence $t_i = u_i^2 - u_i$ in K implies that $\tilde{w}(u_i) < 0$. Therefore, $a_i = u_i^2 - u_i - \epsilon_i = u_i^2 a_i'$ with $a_i' = 1 - 1/u_i + \epsilon_i/u_i^2$ a principal $\tilde{w}$-unit. Hence by Proposition 3.2(1) it follows that q_{a_i',a_0} is isotropic over $\tilde{K}_{\tilde{w}}$, thus so are q_{a_i,a_0} and $q_{\boldsymbol{a}}$ — contradiction!

Claim 2. *One has $\tilde{w}(a_d) > N\tilde{w}(a_i)$ for $i < d$.*

Proof of Claim 2. We first prove that $\tilde{w}(a_d) \geqslant \tilde{w}(\theta^2)$. By contradiction, suppose that $\tilde{w}(a_d) < \tilde{w}(\theta^2)$. Then $\alpha^2 - \alpha = a_d/\theta^2$ in $\tilde{K}$ implies $\tilde{w}(\alpha) < 0$; hence $\eta := 1 - 1/\alpha$ is a principal $\tilde{w}$-unit. Thus $a_d = (\alpha\theta)^2(1 - 1/\alpha) = u^2\eta$ with $u = \alpha\theta$, and we get a contradiction as above in the proof of Claim 1. Second, by Claim 1, one has $\tilde{w}(a_i) \geqslant 0$ for all $i < d$, and therefore $\tilde{w}(\theta) = N\sum_{0<i<d}\tilde{w}(a_i) \geqslant 0$. Hence $\tilde{w}(a_d) \geqslant 2\tilde{w}(\theta) \geqslant 2N\tilde{w}(a_i)$ for all $i < d$. On the other hand, since $q_{\boldsymbol{a}}$ is anisotropic over $\tilde{K}_{\tilde{w}}$, it follows by Proposition 3.2(3) that $\tilde{w}(a_i) \neq 0$ for some $i \leqslant d$, and for such an i, we have $\tilde{w}(a_i) > 0$ because $\tilde{w}(a_i) \geqslant 0$ by Claim 1. Therefore, $\tilde{w}(a_d) \geqslant 2N\tilde{w}(a_i)$ for $i < d$ implies both $\tilde{w}(a_d) > 0$ and $\tilde{w}(a_d) > N\tilde{w}(a_i)$ for $i < d$. Claim 2 is proved. □

Coming back to the proof of Proposition 3.10, for each integer $N > 0$, let $\mathcal{V}_{\boldsymbol{a},N}$ be the set of valuations w on L satisfying the following conditions:

(i) $q_{\boldsymbol{a}}$ is anisotropic over the henselization L_w;
(ii) $w(a_i) \geqslant 0$ and $w(a_d) > Nw(a_i)$ for all $i < d$.

We notice that $\mathcal{V}_{\boldsymbol{a},N}$ is closed, hence compact, in the patch topology. Indeed, the set of all w satisfying condition (ii) is open and closed by definition. Second, the complement of the set of valuations satisfying condition (i) is open by Lemma 3.9(2). Finally, each $\mathcal{V}_{\boldsymbol{a},N}$ is non-empty by Claims 1 and 2, because the valuation $\tilde{w} = \tilde{w}_N$ considered there lies in $\mathcal{V}_{\boldsymbol{a},N}$.

Since $\mathcal{V}_{\boldsymbol{a},N+1} \subset \mathcal{V}_{\boldsymbol{a},N}$, it follows by compactness that $\mathcal{V}_{\boldsymbol{a}} := \cap_N \mathcal{V}_{\boldsymbol{a},N}$ is non-empty, so let us fix $w_{\boldsymbol{a}} \in \mathcal{V}_{\boldsymbol{a}}$. Then $q_{\boldsymbol{a}}$ is anisotropic over $L_{w_{\boldsymbol{a}}}$, and $w_{\boldsymbol{a}}(a_i) \geqslant 0$, $w_{\boldsymbol{a}}(a_d) > Nw_{\boldsymbol{a}}(a_i)$ for all $N > 0$ and $i < d$. Set $\mathfrak{p} := \{x \in L \,|\, w_{\boldsymbol{a}}(a_d) \leq N\, w_{\boldsymbol{a}}(x)$ for some $N > 0\}$. Then $\mathfrak{p} \subset \mathcal{O}_{w_{\boldsymbol{a}}}$ is obviously a prime ideal such that $a_d \in \mathfrak{p}$, $a_i \notin \mathfrak{p}$ for all $i < d$. Let w_L be the valuation with valuation ring $\mathcal{O}_{w_L} = (\mathcal{O}_{w_{\boldsymbol{a}}})_{\mathfrak{p}}$. Then $\mathfrak{m}_{w_L} = \mathfrak{p}$, and the following hold:

(a) One has an inclusion of henselizations $L_{w_L} \subset L_{w_{\boldsymbol{a}}}$, so $q_{\boldsymbol{a}}$ is anisotropic over L_{w_L}.
(b) Since $a_i \notin \mathfrak{p} = \mathfrak{m}_{w_L}$, the a_i are w_L-units for $i < d$.

CLAIM 3. *w_L is trivial on $k_1(\boldsymbol{t})$, and hence w_L is a prime divisor of $L|k_1(\boldsymbol{t})$.*

Proof of Claim 3. We first claim that $v := (w_L)|_{k_1}$ is trivial. By contradiction, suppose that v is non-trivial, and let $k_{1v} \subset L_{w_L}$ be the Henselization of k_1 with respect to v inside L_{w_L}. Since $q_{\boldsymbol{a}}$ is a $k_1,\boldsymbol{t}$-test form for a_d that is anisotropic over L_{w_L}, it follows that q_{a_1,a_0} is a k_1-nice form that is anisotropic over k_{1v}. Hence v is not dyadic. On the other hand, since a_i, $i < d$ are w_L-units, one has $v(a_i) = w_L(a_i) = 0$ for $i = 0, 1$; hence by Proposition 3.2(3) applied to q_{a_1,a_0} over k_{1v} it follows that q_{a_1,a_0} is isotropic over k_{1v} — contradiction! Next suppose, by contradiction, that w_L is not trivial on $k_1(\boldsymbol{t})$. Let $F \subset L_{w_L}$ be the relative algebraic closure of $k_1(\boldsymbol{t})$ in L_{w_L}, and set $w := (w_L)|_F$, $\boldsymbol{\varepsilon} := (a_{d-1}, \ldots, a_1, a_0)$. Then $q_{\boldsymbol{\varepsilon}}$ is defined over F, and w is a non-trivial henselian valuation of F such that all entries a_i of $\boldsymbol{\varepsilon}$ are w-units. Further, since w is trivial on k_1, it follows that w is non-dyadic. Finally, since q_{a_1,a_0} is isotropic over k_{1v} for all archimedean places of k_1, it follows that $q_{\boldsymbol{a}}$ is isotropic over $F_v := Fk_{1v}$ for all archimedean places v of k_1. Proposition 3.2(3) implies that $q_{\boldsymbol{\varepsilon}}$ is isotropic over F, hence over L_{w_L}, because $F \subset L_{w_L}$. Since $q_{\boldsymbol{\varepsilon}}$ is a Pfister subform of $q_{\boldsymbol{a}}$, it follows that $q_{\boldsymbol{a}}$ is isotropic over L_{w_L} — contradiction! Claim 3 is proved.

It is left to prove that $w_L(a_d)$ is positive and odd. First, $w_L(a_d) > 0$ by the definition of w_L. Finally, $w_L(a_d)$ is odd by Proposition 3.4(2). □

4. Uniform definability of the geometric prime divisors of K

In this section we show that geometric prime divisors of finitely generated fields are uniformly first-order definable. This relies in an essential way on the consequences of the cohomological principles presented in the previous section, and on the (obvious) fact that for an n-fold Pfister form $q_{\boldsymbol{a}}$, whether that $q_{\boldsymbol{a}}$ is (an)isotropic, or universal, over K and/or $\tilde{K} = K[\sqrt{-1}\,]$ is expressed by formulas in which the n entries in $\boldsymbol{a} = (a_n, \dots, a_1)$ are the only free variables. The Kronecker dimension $\dim(K)$ can be detected in a first-order way; see Pop [Pop02, Fact 1.1(3), Th. 1.5(3)]. Further, the relatively algebraically closed global subfields $k_1 \subset K$ of finitely generated fields K, and algebraic independence over such fields k_1 are uniformly first-order definable by Poonen [Poo07, Th. 1.4].

Notation/Remarks 4.1. Let K satisfy Hypothesis (H_d).

(1) For $a_d \in K^\times$ consider
   (a) relatively algebraically closed global subfields $k_1 \subset K$;
   (b) k_1-algebraically independent elements $\boldsymbol{u} = (u_i)_{d>i>1}$ of K;
   (c) systems $\boldsymbol{\epsilon} = (\epsilon_i)_{d>i>1}$ of elements of $k_1^\times$ and $a_1, a_0 \in k_1^\times$ such that q_{a_1,a_0} is a k_1-nice Pfister form and all ϵ_i are v-units for all finite places $v \in \mathbb{P}(k_1)$ satisfying $v(a_1) > 0$;
   (d) Set $\boldsymbol{t} := (t_i)_{d>i>1} = (u_i^2 - u_i)_{d>i>1} = \boldsymbol{u}^2 - \boldsymbol{u}$, and $a_i := t_i - \epsilon_i$ for $1 < i < d$, and consider the resulting $k_1, \boldsymbol{t}$-test form $q_{\boldsymbol{a}}$ for a_d defined by $\boldsymbol{a} = (a_d, \dots, a_1, a_0)$.

(2) For $\boldsymbol{t}, \boldsymbol{u}$ as above, let $k_{\boldsymbol{t}} = k_{\boldsymbol{u}}$ be the relative algebraic closure of $k_1(\boldsymbol{t})$ in K, and $\mathcal{D}_{K|k_{\boldsymbol{t}}}$ denote the set of prime divisors w of $K|k_{\boldsymbol{t}}$. Then $K = k_{\boldsymbol{t}}(C)$ for a unique projective normal $k_{\boldsymbol{t}}$-curve C, and $w \in \mathcal{D}_{K|k_{\boldsymbol{a}}}$ are in bijection with the closed points $P \in C$ via $\mathcal{O}_w = \mathcal{O}_P$.

(3) For $\theta, \tau \in K$ with $\theta \neq 0$, set $K_\theta := K(\alpha)$ and $K_\tau := K(\beta)$, where $\alpha^2 - \alpha = a_d/\theta^2$ and $\beta^2 - \beta = \tau^2/a_d$. Let $K_{\theta,\tau} := K_\theta(\beta) = K_\tau(\alpha) = K_\theta K_\tau$ be the compositum of K_θ and K_τ over K.

Finally, for the $k_1, \boldsymbol{t}$-test form $q_{\boldsymbol{a}}$ for a_d introduced above, we define

(4) $\mathfrak{b}_{\boldsymbol{a}} := \{\tau \in K \mid q_{\boldsymbol{a}}$ is anisotropic over $K_{\theta,\tau}$ for all $\theta \in k_{\boldsymbol{t}}^\times\}$, $\mathcal{O}_{\boldsymbol{a}} := \{a \in K \mid a \cdot \mathfrak{b}_{\boldsymbol{a}} \subset \mathfrak{b}_{\boldsymbol{a}}\}$.

(5) $\mathcal{V}_{\boldsymbol{a}} := \{w \in \mathcal{D}_{K|k_{\boldsymbol{t}}} \mid w(a_d) > 0$ and $q_{\boldsymbol{a}}$ is anisotropic over $K_w\}$, and for $w \in \mathcal{V}_{\boldsymbol{a}}$, set
$$\mathfrak{b}_w := \{\tau \in K \mid w(\tau^2) > w(a_d)\}.$$
Therefore the valuation ring $\mathcal{O}_w$ is equal to $\{a \in K \mid a \cdot \mathfrak{b}_w \subset \mathfrak{b}_w\}$.

THEOREM 4.2. *Let K satisfy Hypothesis* (H_d). *The following hold*:

(1) *For k_1, $\boldsymbol{u}$, $a_d \in K$ and $q_{\boldsymbol{a}}$ as in Notation/Remarks 4.1 above,*
$$\mathfrak{b}_{\boldsymbol{a}} = \bigcup_{w \in \mathcal{V}_{\boldsymbol{a}}} \mathfrak{b}_w, \quad \mathcal{O}_{\boldsymbol{a}} = \bigcap_{w \in \mathcal{V}_{\boldsymbol{a}}} \mathcal{O}_w.$$

(2) *For every geometric prime divisor w of K, there are k_1, $\boldsymbol{u}$, $a_d \in K$ as in Notation/Remarks 4.1 above such that $\mathcal{V}_{\boldsymbol{a}} = \{w\}$, and therefore,*

$$\mathcal{O}_w = \{a \in K \mid a \cdot \mathfrak{b}_{\boldsymbol{a}} \subset \mathfrak{b}_{\boldsymbol{a}}\}.$$

Proof. To (1): Let us first argue that $\mathfrak{b}_{\boldsymbol{a}} = \cup_{w\in\mathcal{V}_{\boldsymbol{a}}}\mathfrak{b}_w$.

"$\subset$": Let $\tau \in \mathfrak{b}_{\boldsymbol{a}}$. Set $L := K_\tau$. Then $q_{\boldsymbol{a}}$ is anisotropic over $K_{\theta,\tau} = K_\tau(\alpha) = L(\alpha)$ for all $\theta \in k_{\boldsymbol{t}}^\times$ and $\alpha^2 - \alpha = a_d/\theta^2$ and thus, in particular, for $\theta = (a_{d-1}\dots a_1)^N$ for all $N > 0$. Hence by Proposition 3.10, there is a prime divisor w_L of L that is trivial on $k_1(\boldsymbol{t})$, hence on its relative algebraic closure $k_{\boldsymbol{t}}$ inside K, such that $w_L(a_d) > 0$ is odd, and $q_{\boldsymbol{a}}$ is anisotropic over the henselization L_{w_L}. By contradiction, assume that $w_L(\tau^2) \leqslant w_L(a_d)$, hence $w_L(\tau^2) < w_L(a_d)$, because $w_L(a_d)$ is odd. Then $w_L(\tau^2/a_d) < 0$, hence $w_L(\beta) < 0$, so $a'_d := 1 - 1/\beta$ is a principal w_L-unit, thus $q_{a'_d,a_0}$ is isotropic over L_{w_L} by Proposition 3.2(1). Since $a_d = (a_d\beta/\tau)^2(1-1/\beta)$, one has $q_{a_d,a_0} \approx q_{a'_d,a_0}$ over L_{w_L}, hence q_{a_d,a_0} is isotropic over L_{w_L}. Thus $q_{\boldsymbol{a}}$ is isotropic over L_{w_L} as well — contradiction! Therefore $w_L(\tau^2) > w_L(a_d)$. Setting $w := (w_L)|_K$, we see that $w \in \mathcal{V}_{\boldsymbol{a}}$ and $\tau \in \mathfrak{b}_w$.

"$\supset$": Let $w \in \mathcal{V}_{\boldsymbol{a}}$ and $\tau \in \mathfrak{b}_w$ be given, i.e., $w(\tau^2) > w(a_d)$. Let $\theta \in k_{\boldsymbol{t}}^\times$ be arbitrary. By definitions, w is trivial on $k_{\boldsymbol{t}}$, $w(a_d) > 0$, and $q_{\boldsymbol{a}}$ is anisotropic over the henselization K_w. As $a_i \in k_{\boldsymbol{t}}$ and therefore $w(a_i) = 0$ for $i < d$, by Proposition 3.4(2) it follows that $w(a_d)$ is odd. Therefore one has $w(a_d/\theta^2) = w(a_d) > 0$ and $w(\tau^2/a_d) > 0$. Hence if $\alpha^2 - \alpha = a_d/\theta^2$ and $\beta^2 - \beta = \tau^2/a_d$, then $\alpha, \beta \in K_w$ by Hensel's Lemma. Thus $K_{\theta,\tau} \subset K_w$, and this implies that $q_{\boldsymbol{a}}$ is anisotropic over $K_{\theta,\tau}$. Therefore, $\tau \in \mathfrak{b}_{\boldsymbol{a}}$.

We have shown that $\mathfrak{b}_{\boldsymbol{a}} = \bigcup_{w\in\mathcal{V}_{\boldsymbol{a}}}\mathfrak{b}_w$. It follows immediately that $\mathcal{O}_{\boldsymbol{a}} \supset \bigcap_{w\in\mathcal{V}_{\boldsymbol{a}}}\mathcal{O}_w$. For the other inclusion, let $w \in \mathcal{V}_{\boldsymbol{a}}$ and set $\mu_w := \min\{w(y') \mid y' \in \mathfrak{b}_w\}$. Here the minimum exists since $\mathfrak{b}_w \subseteq \mathcal{O}_w$. For $x \in K \setminus \mathcal{O}_w$, set

$$\Sigma_{w,x} := \{y \in \mathfrak{b}_w \mid w(y) = \mu_w, w'\big((xy)^2\big) < w'(a_d)\ \forall\ w' \in \mathcal{V}_{\boldsymbol{a}} \setminus \{w\}\}.$$

Since $\mathcal{V}_{\boldsymbol{a}} \subset \mathcal{D}_{K|k_{\boldsymbol{a}}}$ is finite, the set $\Sigma_{w,x}$ is non-empty by weak approximation. (It is defined by an open condition for every $w' \in \mathcal{V}_{\boldsymbol{a}}$ including w.) Let $y_0 \in \Sigma_{w,x}$. Then $y_0 \in \mathfrak{b}_w \subseteq \mathfrak{b}_{\boldsymbol{a}}$, but $xy_0 \notin \mathfrak{b}_w$ by minimality of $w(y_0)$ since $w(x) < 0$, and $xy_0 \notin \bigcup_{w'\in\mathcal{V}_{\boldsymbol{a}}\setminus\{w\}}\mathfrak{b}_{w'}$ by definition of $\Sigma_{w,x}$. Hence $xy_0 \notin \mathfrak{b}_{\boldsymbol{a}}$, and thus $x\cdot\mathfrak{b}_{\boldsymbol{a}} \not\subset \mathfrak{b}_{\boldsymbol{a}}$. This shows $\mathcal{O}_{\boldsymbol{a}} \subset \mathcal{O}_w$ for all w, and therefore $\mathcal{O}_{\boldsymbol{a}} = \bigcap_{w\in\mathcal{V}_{\boldsymbol{a}}}\mathcal{O}_w$.

To (2): Let w be a geometric prime divisor of K. Then by Lemma 3.6, there is a (maximal) global subfield $k_1 \subset K$ and $\boldsymbol{u} = (u_i)_{d>i>1}$ algebraically independent over k_1 such that w is trivial on $k_1(\boldsymbol{u})$, and $Kw|k_1(\boldsymbol{u})$ is finite separable. Set $\boldsymbol{t} := \boldsymbol{u}^2 - \boldsymbol{u}$. Then $k_1(\boldsymbol{u})|k_1(\boldsymbol{t})$ is a finite abelian extension, hence $Kw|k_1(\boldsymbol{t})$ is finite separable, and $k_{\boldsymbol{t}} = k_{\boldsymbol{u}}$ inside K. Further recall that $K = k_{\boldsymbol{t}}(C)$ for a (unique) projective normal $k_{\boldsymbol{t}}$-curve C, and there is a unique closed point $P \in C$ with local ring $\mathcal{O}_P = \mathcal{O}_w$. By Riemann–Roch for the projective normal $k_{\boldsymbol{t}}$-curve C, for every sufficiently large $m \gg 0$, there is a function $f \in k_{\boldsymbol{t}}(C)^{\times}$ with $(f)_{\infty} = mP$. Let us fix such an $m \gg 0$ that is odd and such an f. Then the element $a_d := 1/f$ of the function field $K = k_{\boldsymbol{t}}(C)$ has $P \in C$ as its unique zero, and $w(a_d) = m$.

Applying Proposition 3.8, we find $\boldsymbol{\epsilon} = (\epsilon_i)_{d>i>1} \in k_1^{\times\, d-2}$ and $a_1, a_0 \in k_1^{\times}$ such that setting $\boldsymbol{a} = (a_d, \ldots, a_0)$ with $a_i = t_i - \epsilon_i$, $1 < i < d$, the resulting $q_{\boldsymbol{a}}$ is a $k_1, \boldsymbol{t}$-test form for a_d that is anisotropic over K_w. Moreover, since w is the unique prime divisor of $K|k_{\boldsymbol{t}}$ with $w(a_d) > 0$, it follows that $\mathcal{V}_{\boldsymbol{a}} = \{w\}$. Hence by assertion(1) above, $\mathcal{O}_w = \{a \in K \mid a \cdot \mathfrak{b}_{\boldsymbol{a}} \subset \mathfrak{b}_{\boldsymbol{a}}\}$. □

Recipe 4.3. One gets a uniform first-order description of the valuation rings $\mathcal{O}_w$ of all the geometric prime divisors w of K along the following steps:

(1) Consider the uniformly first-order definable k_1, $\boldsymbol{u} = (u_i)_{d>i>1}$, $k_1 \subset k_{\boldsymbol{t}} \subset K$, and further $\boldsymbol{a} := (a_d, \ldots, a_1, a_0)$ and $q_{\boldsymbol{a}}$ as in Notation/Remarks 4.1.

(2) Check whether $\mathcal{O}_{\boldsymbol{a}}$ as defined above is a non-trivial valuation ring of K. If so, $\mathcal{O}_{\boldsymbol{a}}$ is a geometric prime divisor of $K|k_{\boldsymbol{t}}$ by Theorem 4.2(1).

(3) By Theorem 4.2(2), the valuation ring $\mathcal{O}_w$ of any geometric prime divisor w of K arises as above.

This concludes the proof of Theorem 1.3.

Remark 4.4. Theorem 1.3 was stated and proved for finitely generated fields K with $d = \dim(K) > 2$. As we now explain, for finitely generated fields K of Kronecker dimension $d = 1, 2$, there are formulas val_1 and val_2 which uniformly describe the prime divisors in case $d = 1$, respectively the geometric prime divisors in case $d = 2$. For $d = 1$ (i.e., for global fields), all prime divisors are uniformly definable by Rumely [Rum80], Introduction, I. The prime divisors are geometric if and only if K is a global function field, which is a definable condition by II loc. cit. For $d = 2$, uniform definability of geometric prime divisors is one of the main results of Pop [Pop17]: Use that for every geometric prime divisor v of K we can find a global subfield $k_1 \subseteq K$ with v trivial on k_1 such that K is the function field of a smooth curve over k_1, and then apply [Pop17] Theorem 1.2 (cf. Conclusion 5.2).

5. Proof of the Main Theorem

We will now prove that every field satisfying Hypothesis (H_d) is bi-interpretable with the ring $\mathbb{Z}$, building on the uniform definability of the geometric prime divisors. The insight that this is possible is due to Scanlon [Sca08]. (More precisely one can use [Sca08, Th. 4.1], because the part of the proof needed here is not affected by the gap in the recipe of the definability of prime divisors in that paper.) For the convenience of the reader, we instead build on the later [AKNS20], where the bi-interpretability result is established for finitely generated integral domains (as well as some other rings).

PROPOSITION 5.1. *Let K satisfy Hypothesis* (H_d), *let $\mathcal{T}$ denote a transcendence basis of K, and let $R_{\mathcal{T}}$ be the integral closure in K of the subring generated by $\mathcal{T}$. Then the ring $R_{\mathcal{T}}$ is a finitely generated domain that is first-order definable* (*with parameters*).

Proof. Let $\kappa \subset K$ be the constant field of K. By Poonen [Poo07, Th. 1.3], κ is first-order definable. In characteristic zero, i.e., if κ is a number field, by Rumely [Rum80, Introduction, III], the ring of integers $\mathcal{O}_\kappa$ is first-order definable. To fix notation, we set $A := \kappa$ if $\mathrm{char}(K) > 0$ and $A := \mathcal{O}_\kappa$ otherwise. Hence $A \subset K$ is first-order definable, and $R := R_{\mathcal{T}}$ is the integral closure of $A[\mathcal{T}]$ in the field extension $K|K_0$, where $K_0 := \kappa(\mathcal{T})$. Further, R is a finite $A[\mathcal{T}]$-module (see, e.g., [Eis95, Cor. 213.13, Prop. 13.14]), hence R is a finitely generated ring. Hence it is left to prove that $R = R_{\mathcal{T}}$ is first-order definable.

Let $S = S_{\mathcal{T}} \subset K$ be the integral closure of $\kappa[\mathcal{T}]$ in K, and let $\mathcal{W}_{\mathcal{T}}$ be the set of geometric prime divisors w of K such that $\mathcal{T} \subset \mathcal{O}_w$. Since the geometric prime divisors of finitely generated fields K with $\dim(K) = d$ are a first-order definable family (by Theorem 1.3), it follows that $\mathcal{W}_{\mathcal{T}}$ is a first-order definable family.

We claim that $S = \bigcap_{w\in\mathcal{W}_{\mathcal{T}}} \mathcal{O}_w$. First, "$\subset$" is clear, because $\mathcal{T} \subset \mathcal{O}_w$ implies that $S \subset \mathcal{O}_w$, hence $S \subset \bigcap_{w\in\mathcal{W}_{\mathcal{T}}} \mathcal{O}_w$. Second, for "$\supset$", let $\mathcal{X}^1 \subset \mathrm{Spec}(S)$ be the set of minimal non-zero prime ideals $\mathfrak{p}$. Then the local rings $S_\mathfrak{p}$, $\mathfrak{p} \in \mathcal{X}^1$ are valuation rings of geometric prime divisors of K, and $S = \bigcap_{\mathfrak{p}\in\mathcal{X}^1} S_\mathfrak{p}$; see, e.g., [Mat89, Th. 11.5(ii)]. Hence $S = \bigcap_{\mathfrak{p}\in\mathcal{X}^1} S_\mathfrak{p} \supset \bigcap_{w\in\mathcal{W}_{\mathcal{T}}} \mathcal{O}_w$.

In particular, the ring $S = \bigcap_{w\in\mathcal{W}_{\mathcal{T}}} \mathcal{O}_w$ is a definable subset of K.

Case 1: $\mathrm{char}(K) > 0$. Then $A = \kappa$ is a finite field, hence $R_{\mathcal{T}} = S_{\mathcal{T}}$ is first-order definable, and there is nothing left to prove.

Case 2: $\mathrm{char}(K) = 0$. Set $e = \mathrm{td}(K|\kappa)$. The *geometric prime e-divisors* of K are the valuations $\mathfrak{w}$ of K that are trivial on κ and have $\mathfrak{w}K = \mathbb{Z}^e$ lexicographically ordered. By general valuation theory, a valuation $\mathfrak{w}$ of K is a geometric prime e-divisor of K if and only if $\mathfrak{w}$ is of the form $\mathfrak{w} = w_1 \circ \cdots \circ w_e$ (as composition of places) such that w_e is a discrete valuation of K, and w_i is a discrete valuation of the residue field $\kappa(w_{i+1})$ of w_{i+1} for $i < e$. Since $\dim K = e + \dim\kappa$, each w_i must in fact be a geometric prime divisor of $\kappa(w_{i+1})$.

By uniform definability of geometric prime divisors of fields of fixed finite Kronecker dimension (Theorem 1.3 and Remark 4.4), the set $\mathcal{D}^e_{K|\kappa}$ of geometric prime e-divisors is a first-order definable family, using induction on Kronecker dimension and the following easy observation:

FACT 5.2. *If $\mathcal{O}_{w'} \subset F$ and $\mathcal{O}_{w''} \subset Fw'$ are first-order definable valuation rings, then the residue map $\mathcal{O}_{w'} \to Fw'$ is first-order definable, hence so is $\mathcal{O}_{w''\circ w'} \subset F$, as it is the preimage of the first-order definable set $\mathcal{O}_{w''}$ under the first-order definable map $\mathcal{O}_{w'} \to Fw'$.*

Further, the residue fields $\kappa_{\mathfrak{w}} := K\mathfrak{w}$ are finite extensions of κ, hence $\mathbb{P}_{\text{fin}}(\kappa_{\mathfrak{w}})$ and the integral closures $A_{\mathfrak{w}}|A$ of A in $\kappa_{\mathfrak{w}}$ are uniformly first-order definable; see [Rum80, Introduction, I, II, III]. For $\mathfrak{w} \in \mathcal{D}^e_{K|\kappa}$ and a prime divisor $v \in \mathbb{P}_{\text{fin}}(\kappa_{\mathfrak{w}})$, we set $\mathfrak{w}_v := v \circ \mathfrak{w}$, and for the given transcendence basis $\mathcal{T} = (t_1, \ldots, t_e)$ of $K|\kappa$, we denote

$$\mathcal{V}_{\mathcal{T}} = \{\, \mathfrak{w}_v \mid \mathfrak{w} \in \mathcal{W}_{\mathcal{T}}, v \in \mathbb{P}_{\text{fin}}(\kappa_{\mathfrak{w}}) \text{ such that } \mathfrak{w}_v(t_i) \geqslant 0 \text{ for } i = 1, \ldots, e \,\}.$$

Note that $\mathcal{V}_{\mathcal{T}}$ is a definable family by the fact that $\mathcal{W}_{\mathcal{T}}$ and $\mathbb{P}_{\text{fin}}(\kappa_w)$ are so. Hence the definability of $R_{\mathcal{T}}$ follows from Lemma 5.3 below. □

LEMMA 5.3. *One has $R_{\mathcal{T}} = \bigcap_{\mathfrak{w}_v \in \mathcal{V}_{\mathcal{T}}} \mathcal{O}_{\mathfrak{w}_v}$. Thus $R_{\mathcal{T}}$ is first-order definable.*

Proof. For every $\mathfrak{w}_v = v \circ \mathfrak{w} \in \mathcal{V}_{\mathcal{T}}$, one has $\mathcal{O}_{\mathfrak{w}_v} \subset \mathcal{O}_{\mathfrak{w}}$. Hence setting $R'_{\mathcal{T}} := \bigcap_{\mathfrak{w}_v \in \mathcal{V}_{\mathcal{T}}} \mathcal{O}_{\mathfrak{w}_v}$ and reasoning as above in the case of $S_{\mathcal{T}}$, one gets $R_{\mathcal{T}} \subset R'_{\mathcal{T}} \subset S_{\mathcal{T}}$. Hence to complete the proof of Lemma 5.3, it is left to prove the converse inclusion $R_{\mathcal{T}} \supset R'_{\mathcal{T}}$.

First, setting $K_0 := \kappa(\mathcal{T})$, one has that $K|K_0$ is a finite field extension, and $R_{\mathcal{T}} \subset S_{\mathcal{T}}$ are the integral closures of $R_{0,\mathcal{T}} := A[\mathcal{T}] \subset \kappa[\mathcal{T}] =: S_{0,\mathcal{T}}$ in the field extension $K|K_0$. Define $\mathcal{W}_{0,\mathcal{T}}$ and $\mathcal{V}_{0,\mathcal{T}}$ correspondingly for K_0 instead of K, and notice that $\mathcal{W}_{\mathcal{T}}$ and $\mathcal{V}_{\mathcal{T}}$ are the prolongations of $\mathcal{W}_{0,\mathcal{T}}$ and $\mathcal{V}_{0,\mathcal{T}}$ to K under the finite field extension $K|K_0$. Then by the characterization of integral closure using valuations, $R'_{\mathcal{T}}$ is the integral closure of $R'_{0,\mathcal{T}} := \bigcap_{\mathfrak{w}_v \in \mathcal{V}_{0,\mathcal{T}}} \mathcal{O}_{\mathfrak{w}_v}$ in the field extension $K|K_0$. Therefore, it is sufficient to prove that $R_{0,\mathcal{T}} = R'_{0,\mathcal{T}}$, or equivalently, to prove Lemma 5.3 in the special case $K = K_0 = \kappa(\mathcal{T})$, $R_{\mathcal{T}} = R_{0,\mathcal{T}} = A[\mathcal{T}]$, and that will be assumed from now on.

We already proved that $A[\mathcal{T}] = R_{\mathcal{T}}$ is contained in $R'_{\mathcal{T}}$, hence it is left to prove that $R'_{\mathcal{T}} \subset A[\mathcal{T}]$. Recalling that $R'_{\mathcal{T}} \subset S_{\mathcal{T}} = \kappa[\mathcal{T}]$ and $A[\mathcal{T}] = \bigcap_{v \in \mathbb{P}_{\text{fin}}(\kappa)} \mathcal{O}_v[\mathcal{T}]$, we have to prove the following:

CLAIM. *Every $f \in R'_{\mathcal{T}}$ is in $\mathcal{O}_v[\mathcal{T}]$ for all $v \in \mathbb{P}_{\text{fin}}(\kappa)$.*

Proof of Claim. Let $f \in R'_{\mathcal{T}}$ be given, and let $v \in \mathbb{P}_{\text{fin}}(\kappa)$ be fixed, say with residue field $\kappa_v = \kappa v$. Since $R'_{\mathcal{T}} \subset \kappa[\mathcal{T}]$, we can set $f = c \cdot g$ with $c \in \kappa$ and $g \in \mathcal{O}_v[\mathcal{T}]$ such that the reduction $\overline{g} \in \kappa_v[\mathcal{T}]$ is non-zero; e.g., $c = 0$ and $g = 1$ if $f = 0$. Hence in order to prove the claim, it is sufficient to prove that $v(c) \geqslant 0$. Since $\overline{g} \neq 0$, there is an e-tuple $\boldsymbol{\zeta}$ in the algebraic closure of κ_v such that $\overline{g}(\boldsymbol{\zeta}) \neq 0$. Then $\boldsymbol{\zeta}$ is an e-tuple of roots of unity of order prime to $\text{char}(\kappa_v)$, and we identify $\boldsymbol{\zeta}$ with its lift in the algebraic closure of κ. Let $\mathfrak{w} \in \mathcal{W}_{\mathcal{T}}$ be such that $\mathcal{T} \mapsto \boldsymbol{\zeta}$ under $\mathcal{O}_{\mathfrak{w}} \to K\mathfrak{w}$. Then $K\mathfrak{w} = \kappa[\boldsymbol{\zeta}] =: \kappa'$, and if v' prolongs v to κ', then the valuation $\mathfrak{w}_{v'} := v' \circ \mathfrak{w}$ lies in $\mathcal{V}_{\mathcal{T}}$ and satisfies

$$g \mapsto g(\boldsymbol{\zeta}) \mapsto \overline{g}(\boldsymbol{\zeta}) \neq 0$$

under $\mathcal{O}_{\mathfrak{w}_{v'}} \to \mathcal{O}_{v'} \to \kappa' v' = K_0 \mathfrak{w}_{v'}$. Hence g is a $\mathfrak{w}_{v'}$-unit, implying that $\mathfrak{w}_{v'}(f) = \mathfrak{w}_{v'}(c)$. Finally, since $f \in R'_{\mathcal{T}} \subset \mathcal{O}_{\mathfrak{w}_{v'}}$, one has $\mathfrak{w}_{v'}(f) \geqslant 0$, hence

$v(c) = v'(c) = \mathfrak{w}_{v'}(c) = \mathfrak{w}_{v'}(f) \geqslant 0$, concluding that $v(c) \geqslant 0$, thus $f = c \cdot g \in \mathcal{O}_v[\mathcal{T}]$, as claimed. □

Remark 5.4. The first-order definition from the proof of Proposition 5.1 can be seen to be uniform for fixed d; i.e., allowing for variables for the elements of $\mathcal{T}$, the defining formula can be chosen not to vary for all fields K satisfying Hypothesis (H_d).

We are now ready to prove the bi-interpretability theorem: a field K satisfying Hypothesis (H_d) is bi-interpretable with $\mathbb{Z}$, where both K and $\mathbb{Z}$ are considered as structures in the language of rings. We refer the reader to [AKNS20, §2] for a brief introduction to the notion of bi-interpretability.

Proof of the bi-interpretability theorem. Let K be a field satisfying (H_d), and let $R_{\mathcal{T}} \subseteq K$ be the definable subring from Proposition 5.1. Since $R = R_{\mathcal{T}}$ is a finitely generated integral domain, it is bi-interpretable with the ring $\mathbb{Z}$ by [AKNS20, Th. 3.1].

The field K is interpretable in R as a localization; cf. [AKNS20, Examples 2.9(4)]. Then K is definably isomorphic to the interpreted copy of K in the definable subset $R \subseteq K$, namely by assigning to each $x \in K$ the class of pairs $(a, b) \in R \times (R \setminus \{0\})$ with $x = a/b$, and likewise R is definably isomorphic to the copy of R defined in the interpreted copy of K, namely by identifying $r \in R$ with the pair $(r, 1)$ (thought of as standing for $\frac{r}{1}$ in $\mathrm{Frac}(R) = K$). Thus K is bi-interpretable with R, and therefore, by transitivity, bi-interpretable with $\mathbb{Z}$. □

The resolution of the strong form of the EEIP now follows from [AKNS20, Prop. 2.28].

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Technische Universität Dresden, Institut für Algebra, Dresden, Germany
E-mail: philip.dittmann@tu-dresden.de

Department of Mathematics, University of Pennsylvania,
Philadelphia, PA, USA
E-mail: pop@math.upenn.edu