

Open-Loop Chance Constrained Stochastic Optimal Control via the One-Sided Vysochanskij–Petunin Inequality

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Abstract—While many techniques have been developed for chance constrained stochastic optimal control with Gaussian disturbance processes, far less is known about computationally efficient methods to handle non-Gaussian processes. In this article, we develop a method for solving chance constrained stochastic optimal control problems for linear time-invariant systems with general additive disturbances with finite moments and unimodal chance constraints. We propose an open-loop control scheme for multivehicle planning, with both target sets and collision avoidance constraints. Our method relies on the one-sided Vysochanskij–Petunin inequality, a tool from statistics used to bound tail probabilities of unimodal random variables. Using the one-sided Vysochanskij–Petunin inequality, we reformulate each chance constraint in terms of the expectation and standard deviation. While the reformulated bounds are conservative with respect to the original bounds, they have a simple and closed form, and are amenable to difference of convex optimization techniques. We demonstrate our approach on a multisatellite rendezvous problem.

Index Terms—Arbitrary disturbances, chance constrained stochastic optimal control, multivehicle motion planning, stochastic linear systems.

I. INTRODUCTION

AUTONOMOUS systems that are high risk, expensive, or safety critical require assurances they will not enter unsafe conditions that may lead to costly damage to property or loss of life. Satellite constellations and self-driving cars are just two examples where failure can be prohibitively expensive. Stochasticity, such as that due to modeling errors, external forces, or incomplete knowledge of the environment, complicates efforts to provide formal assurances in autonomous systems. Probabilistic assurances, while not as strong as those based in robust approaches that presume a worst-case scenario [1], [2], allow for assurances tailored to a desired level of confidence or risk.

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Code is available online at <https://github.com/unm-hscl/shawnpriore-moment-control>.

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However, although many stochastic effects are non-Gaussian (such as heavy tail phenomena in relative satellite dynamics), few methods exist that can accommodate non-Gaussian stochastic processes within a stochastic optimal control framework.

One of the primary challenges associated with stochastic optimal control with non-Gaussian processes is the lack of analytic expressions for the cumulative distribution function (CDF) of the state as it evolves over time. This deficiency is relevant for the evaluation of chance constraint probabilities, and typically requires high dimensional and often intractable integration. Characteristic function-based approaches utilize closed form expressions in the Fourier domain to approximate the CDF with numerical Fourier inversions, but are limited to evaluation of chance constraints for convex sets [3], [4], [5]. Simulation-based approaches [6], [7], [8] bypass the need for integration, but are reliant upon on the quality and size of the samples. Further, in practice, these approaches may be additionally limited by computational memory, necessary for large samples, as well as the need to sample the distribution. Sample reduction methods [8], [9], [10] decrease computational burden, by focusing on scenario characteristics and comparing them with previous solutions. However, the characteristic function approach requires numerical approximations of the CDF or its inverse [5], and the sampling approaches produce confidence bounds on chance constraint satisfaction [8], both of which weaken guarantees.

In contrast, methods that employ concentration inequalities provide almost surely assurances of chance constraint satisfaction through over-approximations. Chebyshev’s inequality [11] and Cantelli’s inequality [11] have been used to develop chance constraint reformulations that are an affine combination of a constraint’s expectation and standard deviation [12], [13], [14], [15]. These inequalities only require knowledge of the expectation and the standard deviation, which can be easily calculated for linear constraints. However, reliance on these inequalities typically provides quite conservative bounds [15].

Our approach also invokes concentration inequalities, and hence provides almost surely guarantees, but employs an inequality that is less conservative than those in [12], [13], [14], and [15]. We use the one-sided Vysochanskij–Petunin inequality [16], a refinement of Cantelli’s inequality that is tailored to unimodal distributions. Although it has less generality than Cantelli’s inequality, the one-sided Vysochanskij–Petunin inequality typically results in far less conservatism in the over-approximation. Indeed, its probabilistic bound is reduced by a factor of 5/9, as compared with the bound from Cantelli’s

inequality. Hence, we propose application of the one-sided Vysochanskij–Petunin inequality to chance constraint evaluation that arises in multivehicle planning problems: that is, in 1) reaching a terminal target set and 2) avoiding collision with obstacles in the environment as well as with other vehicles. The main drawback in our approach is the need for unimodality of each constraint, over the entire trajectory. Unimodality is assured for convex constraints in linear time-invariant (LTI) systems for certain classes of disturbance processes (such as Gaussian, Laplacian, or uniform on a convex interval), however for other disturbance processes, unimodality must be validated empirically. A secondary drawback in our approach is the use of an open-loop design. Open-loop controllers tend to be conservative with respect to their closed-loop counterparts, and often are resilient to stability assurances [17], [18]. However, strategies, such as prestabilization or augmented state vectors can be used to mitigate these downsides [18], [19].

The main contribution of this article is a *closed-form* reformulation of chance constraints, for polytopic target sets and collision avoidance constraints, that is amenable to difference of convex programming solutions. Our approach is relevant for LTI systems with arbitrary distributions with finite moments, and with chance constraints that are unimodal.

The rest of this article is organized as follows. Section II provides mathematical preliminaries and formulates the optimization problem. Section III derives the difference of convex functions optimization problem reformulation of the chance constraints. Section IV demonstrates our approach on two multi-satellite rendezvous problems. Finally, Section V concludes this article.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Mathematical Preliminaries

We denote the interval that enumerates all natural numbers from a to b , inclusively, as $\mathbb{N}_{[a,b]}$. We denote vectors with an arrow accent, as $\mathbf{x} \in \mathbb{R}^n$. Random variables are indicated with a bold case $\mathbf{X}$. For a random variable $\mathbf{X}$, we denote the expectation as $\mathbb{E}[\mathbf{X}]$, variance as $\text{Var}(\mathbf{X})$, and standard deviation as $\text{Std}(\mathbf{X})$. For a vector input, $\text{Va}(\cdot)$ will reference the variance–covariance matrix of the random vector. For two random variables $\mathbf{X}$ and $\mathbf{Y}$, $\text{Cov}(\mathbf{X}, \mathbf{Y})$ denotes the covariance between the two variables. We denote the 2-norm of a matrix or vector by $\|\cdot\|$. For a matrix $\mathbf{A}$, $\text{tr}(\mathbf{A})$ will denote the trace of $\mathbf{A}$. Last, we denote a block diagonal matrix with elements $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_q$ as $\text{diag}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_q)$.

B. Problem Formulation

Consider a scenario, such as the one shown in Fig. 1, in which three satellites rendezvous with a refueling station while avoiding each other, other spacecraft, and debris. With potentially non-Gaussian disturbances corrupting the satellite dynamics, we seek to synthesize a controller to construct an optimal rendezvous maneuver that meets probabilistic target set and collision avoidance constraints.

We presume the evolution of N_v vehicles are governed by the discrete-time LTI system

$$\mathbf{x}_i(k+1) = \mathbf{A}_i \mathbf{x}_i(k) + \mathbf{B}_i \mathbf{u}_i(k) + \mathbf{w}_i(k) \quad (1)$$

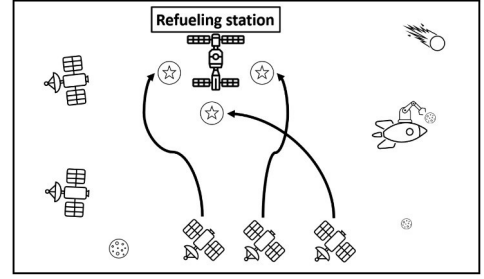


Fig. 1. Scenario in which three satellites need to rendezvous with a refueling station while avoiding each other, other spacecraft, scientific instruments, and debris.

with state $\mathbf{x}_i(k) \in \mathcal{X} \subseteq \mathbb{R}^n$, input $\mathbf{u}_i(k) \in \mathcal{U} \subseteq \mathbb{R}^m$, and $\mathbf{w}_i(k) \in \mathbb{R}^n$ that follows an arbitrary but known disturbance, and initial condition $\mathbf{x}(0)$. We presume the initial conditions, $\mathbf{x}(0)$, are known, the bounded control authority, $\mathcal{U}$, is a convex polytope, and that the system evolves over a finite time horizon of $N \in \mathbb{N}$ steps. We presume each disturbance, $\mathbf{w}_i(k)$, has probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P}_{\mathbf{w}_i(k)})$ with outcomes Ω , Borel σ -algebra $\mathcal{B}(\Omega)$, and probability measure $\mathbb{P}_{\mathbf{w}_i(k)}$ [20].

We write the dynamics at time k as an affine sum of the initial condition and the concatenated control sequence and disturbance

$$\mathbf{x}_i(k) = \mathbf{A}^k \mathbf{x}_i(0) + \mathbf{C}(k) \mathbf{U}_i + \mathbf{D}(k) \mathbf{W}_i \quad (2)$$

with

$$\mathbf{U}_i = \mathbf{u}_i(0) \dots \mathbf{u}_i(N-1) \in \mathcal{U}^N \quad (3a)$$

$$\mathbf{W}_i = \mathbf{w}_i(0) \dots \mathbf{w}_i(N-1) \in \mathbb{R}^{Nn} \quad (3b)$$

$$\mathbf{C}(k) = \mathbf{A}^{k-1} \mathbf{B} \dots \mathbf{A} \mathbf{B} \mathbf{B} \mathbf{0}_{n \times (N-k)m} \in \mathbb{R}^{n \times Nm} \quad (3c)$$

$$\mathbf{D}(k) = \mathbf{A}^{k-1} \dots \mathbf{A} \mathbf{I}_n \mathbf{0}_{n \times (N-k)n} \in \mathbb{R}^{n \times Nn} \quad (3d)$$

We seek to minimize a convex performance objective $J : \mathcal{X}^{N \times N_v} \times \mathcal{U}^{N \times N_v} \rightarrow \mathbb{R}$. We presume desired polytopic target sets that each vehicle must reach, known and static obstacles that each vehicle must avoid, as well as the need for collision avoidance between each pair of vehicles, all with desired likelihoods

$$\mathbb{P} \left(\bigwedge_{i=1}^{N_v} \bigwedge_{k=1}^N \mathbf{x}_i(k) \in \mathcal{T}_i(k) \right) \geq 1 - \alpha \quad (4a)$$

$$\mathbb{P} \left(\bigwedge_{i=1}^{N_v} \bigwedge_{k=1}^N S(\mathbf{x}_i(k) - \mathbf{o}(k)) \geq r \right) \geq 1 - \beta \quad (4b)$$

$$\mathbb{P} \left(\bigwedge_{i=1}^{N_v-1} \bigwedge_{j=i+1}^{N_v} \bigwedge_{k=1}^N S(\mathbf{x}_i(k) - \mathbf{x}_j(k)) \geq r \right) \geq 1 - \gamma. \quad (4c)$$

We presume convex, compact, and polytopic sets $\mathcal{T}_i(k) \subseteq \mathbb{R}^n$, positive semidefinite and diagonal matrix $\mathbf{S} \in \mathbb{R}^{q \times n}$, positive scalar $r \in \mathbb{R}_+$, nonrandom object locations $\mathbf{o}(k) \in \mathbb{R}^n$, and probabilistic violation thresholds $\alpha, \beta, \gamma \in (0, 1/6)$. The probabilistic violation thresholds are restricted as a condition for

optimally of the solutions. Here, S is designed to extract the position of the vehicle from the state vector.

Definition 1 (Reverse convex constraint): A reverse convex constraint is the complement of a convex constraint, that is, $f(x) \geq c$ for a convex function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and a scalar $c \in \mathbb{R}$.

Note that the collision avoidance constraints inside the probability functions are reverse-convex as per Definition 1.

We seek to solve the following optimization problem:

$$\text{minimize } J(X_1, \dots, X_{N_v}, U_1, \dots, U_{N_v}) \quad (5a)$$

$$\text{subject to } U_1, \dots, U_{N_v} \in \mathcal{U}^N \quad (5b)$$

$$\text{Dynamics (2) with } x_1(0), \dots, x_{N_v}(0) \quad (5c)$$

$$\text{Probabilistic constraints (4)} \quad (5d)$$

where $X_i = [x_i(1) \dots x_i(N)]$ is the concatenated state vector for vehicle i .

For this problem to be tractable for arbitrary disturbances, we make several key assumptions about the disturbance and its resulting impact on the constraints.

Assumption 1: Disturbance vectors, W_i , are all pairwise independent. Hence, for any i and j , where $i \neq j$, the joint CDF, $\Phi_{W_i, W_j}(a, b)$ can be factored into the product of the marginal CDFs, $\Phi_{W_i}(a)$ and $\Phi_{W_j}(b)$. So, $\Phi_{W_i, W_j}(a, b) = \Phi_{W_i}(a)\Phi_{W_j}(b)$.

Assumption 2: All components of the disturbance vector, $W_i = [w_{i1} \ w_{i2} \ \dots \ w_{iN_n}]$, are mutually independent. Hence, for any set of unique integers $S \subseteq \mathbb{N}_{1, N_n}$, the subset $\{w_{ij} \mid j \in S\}$ has a joint CDF $\Phi_{\{w_{ij} \mid j \in S\}}(\cdot, \dots, \cdot)$ can be factored into the product of the marginal CDFs, $\Phi_{w_{ij}}(\cdot)$ for $j \in S$. So, $\Phi_{\{w_{ij} \mid j \in S\}}(\cdot, \dots, \cdot) = \prod_{j \in S} \Phi_{w_{ij}}(\cdot)$.

Assumption 3: Each component of the disturbance vector, $W_i = [w_{i1} \ w_{i2} \ \dots \ w_{iN_n}]$, has finite and well defined moments at least up to the fourth order, $E[w_{ij}^p] < \infty$ for $p \in \mathbb{N}_{1, 4}$.

Statistically, pairwise and mutual independence can be assumed in many cases without much consequence as most multivariate distributions can be constructed in this manner. However, the multivariate Cauchy and the multivariate t are the most prominent examples that cannot meet Assumption 2 as elements are not independent by construction. In many ways, Assumption 2 is the most restrictive of these assumptions as it restricts our approach to physical phenomena that disturb each state independently. Assumption 3 is easily met as most distributions have analytic expressions for moments.

Lastly, we consider the impact of W_i on the chance constraints in (4).

Definition 2 (Unimodal distribution [21]): A unimodal distribution is a distribution whose CDF is convex in the region $(-\infty, a)$ and concave in the region (a, ∞) for some $a \in \mathbb{R}$.

Definition 3 (Strong unimodal distribution [21]): A strong unimodal distribution is one in which unimodality is preserved by convolution. That is, for two independent strong unimodal random variables, Y and Z , the random variable $Y + Z$ is also unimodal.

Assumption 4: The distribution that describes each probabilistic constraint in (4) is marginally unimodal.

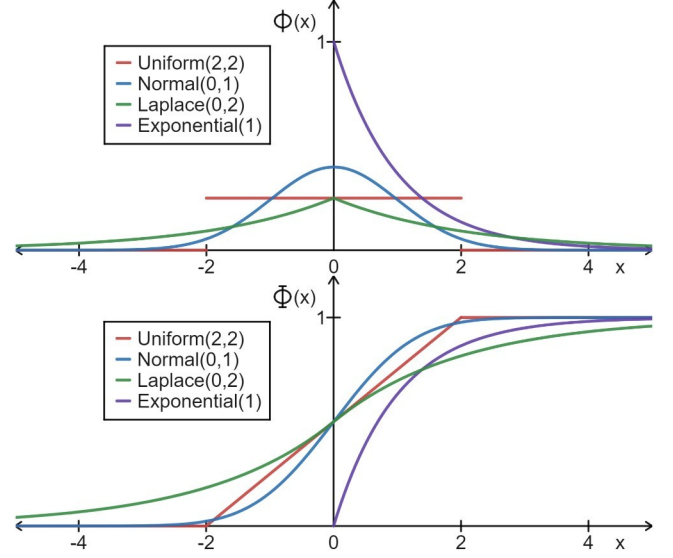


Fig. 2. PDFs (top) and CDFs (bottom) of unimodal distributions as per Definition 2 with the mode at 0. Each of the distributions shown here have a log concave PDF, which in turn implies a log concave CDF. Log concavity of the PDF assures strong unimodality as per Definition 3.

Assumption 4 is required such that we can develop bounds on the chance constraint probabilities. In rare cases, unimodality can be verified analytically by properties of strong unimodality. For example, Gaussian or exponential random variables are strong unimodal and any affine summation of these random variables will always be unimodal. One method to check for strong unimodality is to establish that the probability density function (PDF) is log concave as all distributions that are strong unimodal also have a log concave PDF per the theorem of Ibragimov [22]. Fig. 2 graphs the PDF and CDF of several common strong unimodal distributions with PDFs that are easy to show are log concave. As unimodality can be challenging to show analytically, the easiest method to validate unimodality is empirically. By numerically evaluating the empirical CDF with a large enough sample size (we recommend at least on the order of 10^4 samples), one can validate unimodality in terms of Definition 2 via Algorithm 1 in Appendix A.

Problem 1: Under Assumptions 1–4, solve the stochastic optimization problem (5) with probabilistic violation thresholds α, β , and γ for open loop controllers $U_1, \dots, U_{N_v} \in \mathcal{U}^N$.

The main challenge in solving Problem 1 is assuring (5d). In this form, assuring (5d) requires the evaluation of high dimensional and frequently intractable integrals. In addition, even if these integrals could be evaluated and closed forms could be found, the collision avoidance constraints (4b) and (4c) would still be reverse convex.

III. METHODS

Our approach to solve Problem 1 involves reformulating each chance constraint as an affine summation of the constraint's expectation and standard deviation, i.e., $E[S(x_i(k) - x_j(k))]$ and $\text{Std}(S(x_i(k) - x_j(k)))$, respectively, for the collision avoidance constraint. This form is amenable to the one-sided

Vysochanskij–Petunin inequality [16], which allows for almost surely guarantees of chance constraint satisfaction.

Theorem 1 (One-sided Vysochanskij–Petunin inequality [16]): Let x be a real valued unimodal random variable with finite expectation $E[x]$ and finite, nonzero standard deviation $\text{Std}(x)$. Then, for $\lambda > \sqrt{5/3}$

$$P(x - E[x] \geq \lambda \text{Std}(x)) \leq \frac{4}{9(\lambda^2 + 1)} \quad (6)$$

The one-sided Vysochanskij–Petunin inequality is applicable only to unimodal distributions. It is based on Gauss's inequality, which provides a bound for one sided tail probabilities of a unimodal random variable to be sufficiently far away from the expectation. Specifically, the bound encompasses values at least λ standard deviations away from the mean.

By applying (6) to the random variable $-x$, we get the lower tail bound

$$P(x - E[x] \leq -\lambda \text{Std}(x)) \leq \frac{4}{9(\lambda^2 + 1)} \quad (7)$$

We first make use of (6) and (7) to bound the chance constraint probabilities based on an affine summation of the expectation and standard deviation.

A. Polytopic Target Set Constraint

First, consider the reformulation of (4a). Without loss of generality, we presume $N_v = 1$ and $N = 1$ for brevity. The polytope $T_i(k)$ can be written as the intersection of N_{Tik} half-space inequalities

$$P(x_i(k) \in T_i(k)) = P\left(\bigcap_{j=1}^{N_{Tik}} G_{ijk} x_i(k) \leq h_{ijk}\right) \quad (8)$$

where $G_{ijk} \in \mathbb{R}^n$ and $h_{ijk} \in \mathbb{R}$ define a half-space constraint within the polytopic target set. We take the complement and employ Boole's inequality to separate the combined chance constraints into a series of individual chance constraints

$$P(x_i(k) \in T_i(k)) = P\left(\bigcap_{j=1}^{N_{Tik}} G_{ijk} x_i(k) \geq h_{ijk}\right) \quad (9a)$$

$$\leq \bigcap_{j=1}^{N_{Tik}} P(G_{ijk} x_i(k) \geq h_{ijk}) \quad (9b)$$

Using the approach in [6], we introduce variable ω_{ijk} to allocate risk to each of the individual chance constraints

$$P(G_{ijk} x_i(k) \geq h_{ijk}) \leq \omega_{ijk} \quad (10a)$$

$$\bigcap_{j=1}^{N_{Tik}} \omega_{ijk} \leq \alpha \quad (10b)$$

$$\omega_{ijk} \geq 0. \quad (10c)$$

To find a solution to (10), we need to find an appropriate value for ω_{ijk} . As the probabilistic function in (10a) often lacks a closed form, is intractable or nonconvex, we will need to

bypass computing this constraint directly. In order to bypass this computation, we observe that

$$\begin{aligned} P(G_{ijk} x_i(k) \geq h_{ijk}) \\ \leq P(G_{ijk} x_i(k) \geq E[G_{ijk} x_i(k)] + \lambda_{ijk} \text{Std}(G_{ijk} x_i(k))) \end{aligned} \quad (11)$$

can be enforced by the linear constraint

$$E[G_{ijk} x_i(k)] + \lambda_{ijk} \text{Std}(G_{ijk} x_i(k)) \leq h_{ijk}. \quad (12)$$

Then, (12), in conjunction with Theorem 1, allows us to upper bound (11), and by association evaluate (10a). We can substitute ω_{ijk} with $\frac{4}{9(\lambda_{ijk}^2 + 1)}$ and change the risk allocation variable from ω_{ijk} to λ_{ijk} . Further, enforcement of (12) makes (10a) an unnecessary intermediary step between (12) and (10c). Hence, we can remove (10a) from the system of equations to solve and write (10)–(12) as

$$E[G_{ijk} x_i(k)] + \lambda_{ijk} \text{Std}(G_{ijk} x_i(k)) \leq h_{ijk} \quad (13a)$$

$$\bigcap_{j=1}^{N_{Tik}} \frac{4}{9(\lambda_{ijk}^2 + 1)} \leq \alpha \quad (13b)$$

$$\lambda_{ijk} \geq \sqrt{\frac{5}{3}} \quad (13c)$$

which is enumerated over the indices, i, j , and k .

Lemma 1: For the controllers $U_1, \dots, U_{N_v}$, if there exists risk allocation variables λ_{ijk} satisfying (13) for constraints (4a), then $U_1, \dots, U_{N_v}$ satisfy (5d).

Proof: Satisfaction of (13a) implies (11) holds. The Vysochanskij–Petunin inequality upper bounds (11). Boole's inequality and De Morgan's law guarantee that if (13b) holds then (5d) is satisfied.

Lastly, we show that the constraint reformulation (13) will always be convex.

Lemma 2: The constraint (13) is convex in U_i and in $(\lambda_{1ik}, \dots, \lambda_{N_{ik}})$

Proof: We start by exploiting the properties of the expectation and variance operator to write (13a) as

$$\begin{aligned} G_{ijk} A^k x_i(0) + C(k)U_i + D(k)E W_i \\ + \lambda_{ijk} \sqrt{G_{ijk} D(k) \text{Var}(W_i) D(k) G_{ijk}} \leq h_{ijk} \end{aligned} \quad (14)$$

which is affine, and hence convex, in U_i and λ_{ijk} . Then

$$\frac{\partial^2}{\partial \lambda_{ijk}^2} \frac{4}{9(\lambda_{ijk}^2 + 1)} = -\frac{8 - 3\lambda_{ijk}^2 + 1}{9\lambda_{ijk}^2 + 1}^3 \quad (15)$$

which is positive, and hence convex, when $\lambda_{ijk} \geq 3^{-1/2}$. Hence, with the restriction (13c), (13b) is a convex constraint. Thus, the set over which $\lambda_{1ik}, \dots, \lambda_{N_{ik}}$ is optimized is convex. Further, in the problem formulation we defined the control authority to be a closed and convex set. Hence, we can conclude the chance constraint reformulation (13) is convex.

B. 2-Norm-Based Collision Avoidance Constraints

Next, consider the reformulation of the constraints (4b) and (4c). Here, we will derive the reformulation for (4c), but the reformulation of (4b) is nearly identical. Without loss of generality, let

$$z = SA^k(x_i(0) - x_j(0)) + SC(k)(U_i - U_j) \quad (16a)$$

$$z = SD(k)(W_i - W_j) \quad (16b)$$

be the nonstochastic and stochastic element of $S(x_i(k) - x_j(k))$ from (4c), respectively. Then, we can write the norm as

$$S(x_i(k) - x_j(k)) = z + z. \quad (17)$$

We start by observing

$$\begin{aligned} & P \left(\bigcap_{i=1}^{N_v-1} \bigcap_{j=i+1}^{N_v} \bigcap_{k=1}^N (z + z \geq r) \right) \\ &= P \left(\bigcap_{i=1}^{N_v-1} \bigcap_{j=i+1}^{N_v} \bigcap_{k=1}^N (z + z^2 \geq r^2) \right) \end{aligned} \quad (18)$$

as the norm is nonnegative. Thus, we can write the 2-norm constraint as

$$P \left(\bigcap_{i=1}^{N_v-1} \bigcap_{j=i+1}^{N_v} \bigcap_{k=1}^N (z + z^2 \geq r^2) \right) \geq 1 - \gamma. \quad (19)$$

By taking the complement and applying Boole's inequality

$$\begin{aligned} & P \left(\bigcap_{i=1}^{N_v-1} \bigcap_{j=i+1}^{N_v} \bigcap_{k=1}^N (z + z^2 \leq r^2) \right) \\ & \leq \sum_{i=1}^{N_v-1} \sum_{j=i+1}^{N_v} \sum_{k=1}^N P(z + z^2 \leq r^2). \end{aligned} \quad (20)$$

Using the approach in [6], we introduce risk variables ω_{jk} to allocate risk to each of the individual probabilities

$$P(z + z^2 \leq r^2) \leq \omega_{jk} \quad (21a)$$

$$\sum_{i=1}^{N_v-1} \sum_{j=i+1}^{N_v} \sum_{k=1}^N \omega_{jk} \leq \gamma \quad (21b)$$

$$\omega_{jk} \geq 0. \quad (21c)$$

In a similar fashion to Section III-A, we add an additional reverse convex constraint based on the expectation and standard deviation of $z + z^2$ to (21) such that the constraint becomes

$$P(z + z^2 \leq r^2) \leq \omega_{jk} \quad (22a)$$

$$E(z + z^2) - \lambda_{ijk} \text{Std}(z + z^2) \geq r^2 \quad (22b)$$

$$\sum_{i=1}^{N_v-1} \sum_{j=i+1}^{N_v} \sum_{k=1}^N \omega_{jk} \leq \gamma \quad (22c)$$

$$\omega_{jk} \geq 0. \quad (22d)$$

By adding constraint (22b), we can bypass the need to compute the (22a) directly. We do so by observing that satisfaction of (22b), also implies

$$\begin{aligned} & P(z + z^2 \leq r^2) \\ & \leq P(z + z^2 \leq E(z + z^2) - \lambda_{ijk} \text{Std}(z + z^2)). \end{aligned} \quad (23)$$

From Assumption 4 and Theorem 1, we know that the right-hand side of (23) is upper bounded as per (7). Hence, we can use the substitution

$$\omega_{jk} = \frac{4}{9(\lambda_{ijk}^2 + 1)} \quad (24)$$

and determine the value for λ_{ijk} in terms of ω_{jk}

$$\lambda_{ijk} = \frac{4}{9\omega_{jk}} - 1 \quad (25)$$

so long as $\lambda \geq \sqrt{5/3}$. This implies $\omega_{jk} \leq 1/6$ is a necessary restriction on ω_{jk} . As $\gamma < 1/6$, any solution will require that $\omega_{jk} < 1/6$. Then, we can write (22) as

$$P(z + z^2 \leq r^2) \leq \omega_{ijk} \quad (26a)$$

$$E(z + z^2) - \frac{4}{9\omega_{jk}} - 1 \cdot \text{Std}(z + z^2) \geq r^2 \quad (26b)$$

$$\sum_{i=1}^{N_v-1} \sum_{j=i+1}^{N_v} \sum_{k=1}^N \omega_{jk} \leq \gamma \quad (26c)$$

$$\omega_{jk} \in (0, 1/6). \quad (26d)$$

Since Theorem 1 guarantees that satisfaction of (26b) also satisfies (26a) for any value $\gamma \in (0, 1/6)$ (26a) is redundant and can be removed. The constraint is then

$$E(z + z^2) - \frac{4}{9\omega_{jk}} - 1 \cdot \text{Std}(z + z^2) \geq r^2 \quad (27a)$$

$$\sum_{i=1}^{N_v-1} \sum_{j=i+1}^{N_v} \sum_{k=1}^N \omega_{jk} \leq \gamma \quad (27b)$$

$$\omega_{jk} \in (0, 1/6). \quad (27c)$$

Note that (27a) is a biconvex constraint [23]. For known risk allocation values of ω_{jk} , written as $\tilde{\omega}_{jk}$, the final constraint is

$$E(z + z^2) - \frac{4}{9\tilde{\omega}_{jk}} - 1 \cdot \text{Std}(z + z^2) \geq r^2. \quad (28)$$

Lemma 3: If the controller $U_1, \dots, U_\ell$ satisfies (28) for constraints (4b) and (4c), then $U_1, \dots, U_\ell$ satisfy (4).

Proof: Satisfaction of (28) implies (23) is satisfied for $\lambda_{ijk} = \frac{4}{9\tilde{\omega}_{jk}} - 1$. Theorem 1 guarantees satisfaction of (4).

Next, we find the expanded form of (28) and show that the constraint is always a difference of convex function constraint.

Definition 4 (Difference of convex functions constraint): A difference of convex functions constraint has the form

$$f(x) - g(x) \leq 0 \quad (29)$$

in which $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ are convex functions for $x \in \mathbb{R}^n$.

Lemma 4: The constraint (28) is a difference of convex function constraint in U_i for the constraint (4b) in $U_i - U_j$ for the constraint (4c).

Proof: We first find the expectation and variance of the norm. To find the expectation, we expand the norm

$$\mathbb{E} \|z + z^2\|^2 = \mathbb{E} \|z\|^2 + 2\mathbb{E} z^T z + \mathbb{E} \|z\|^2 \quad (30a)$$

$$= \mathbb{E} \|z\|^2 + 2\mathbb{E} z^T \mathbb{E}[z] + \mathbb{E} \|z\|^2 \quad (30b)$$

$$= \begin{bmatrix} I_q & \mathbb{E}[z] \\ \mathbb{E}[z]^T & \mathbb{E} \|z\|^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} z \\ 1 \end{bmatrix}^2. \quad (30c)$$

Remember q is the dimension of the matrix S designed to extract the position elements of the state. Here, $\mathbb{E} \|z + z^2\|^2$ is the squared norm of a vector matrix product. Hence, the expectation is convex. We compute the variance in a similar manner to the expectation

$$\text{Var} \|z + z^2\|^2 \quad (31a)$$

$$= \text{Var} \|z\|^2 + 2\mathbb{E} z^T z + \mathbb{E} \|z\|^2 \quad (31b)$$

$$= \text{Var} 2z^T z + 2\text{Cov} 2z^T z, z^T z + \text{Var} \|z\|^2 \quad (31c)$$

$$= 4z^T \text{Var}(z) z + 4z^T \text{Cov} z, z^T z + \text{Var} \|z\|^2 \quad (31d)$$

where

$$\text{Cov} z, z^T z = \mathbb{E} z z^T z - \mathbb{E}[z] \mathbb{E} z^T z. \quad (32)$$

Assumptions 1–3 guarantees a closed form for (31). Thus, we can write the standard deviation as the 2-norm

$$\begin{aligned} \text{Std} \|z + z^2\|^2 &= \left[\begin{array}{cc} 4\text{Var}(z) & 2\text{Cov} z, z^T z \\ 2\text{Cov} z, z^T z & \text{Var} \|z\|^2 \end{array} \right]^{\frac{1}{2}} z \\ &= \left[\begin{array}{cc} 4\text{Var}(z) & 2\text{Cov} z, z^T z \\ 2\text{Cov} z, z^T z & \text{Var} \|z\|^2 \end{array} \right]^{\frac{1}{2}} z \end{aligned} \quad (33)$$

Since the standard deviation is the 2-norm of an affine function, the standard deviation is convex [24]. We can now write (28) as (34) shown at the bottom of this page. By multiplying both sides of (34) by -1 , (34) is the difference of two convex functions in the control as per Definition 4.

C. Difference of Convex Functions Framework

Combining the results from Sections III-A and III-B, we obtain a new optimization problem

$$\underset{\substack{U_1, \dots, U_{N_v} \\ \lambda_{ijk}}}{\text{minimize}} \quad J(X_1, \dots, X_{N_v}, U_1, \dots, U_{N_v}) \quad (35a)$$

$$\text{subject to } U_1, \dots, U_{N_v} \in \mathcal{U}^N \quad (35b)$$

$$\text{Moments defined by dynamics (2)} \quad (35c)$$

$$\text{with initial conditions } x_1(0), \dots, x_{N_v}(0) \quad (35d)$$

$$\text{Constraints (13) and (34)}. \quad (35d)$$

Reformulation 1: Under Assumptions 1–4, solve the stochastic optimization problem (35) with probabilistic violation thresholds α, β , and γ for open loop controllers $U_1, \dots, U_{N_v} \in \mathcal{U}^N$ and optimization parameters λ_{ijk} .

Lemma 5: Solutions to Reformulation 1 are conservative solutions to Problem 1.

Proof: Lemmas 1 and 3 guarantee the probabilistic constraints (4) are satisfied. The (6)–(7) are always conservative. Hence, the reformulated constraints will be conservative with respect to the chance constraint. The expectation and variance terms in Reformulation 1 encompass and replace the dynamics used in Problem 1. The cost function and input constraints remain unchanged.

We note that (35) is a difference of convex functions optimization problem. A difference of convex functions optimization problem has the form

$$\underset{x}{\text{minimize}} \quad f_0(x) - g_0(x)$$

$$\text{subject to } f_i(x) - g_i(x) \leq 0 \quad \text{for } i \in \mathcal{N} \quad (36)$$

in which $f_0, f_i(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_0, g_i(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}$ for $x \in \mathbb{R}^n$ are convex. While (35a)–(35c) are convex, (35d) is difference of convex due to the constraint (34).

We employ the convex–concave procedure [25] to solve (35). By taking a first order approximation of the expectation of the 2-norm in (34), we can solve the difference of convex function optimization problem iteratively as a convex optimization problem. By updating the first order approximation at each iteration, the convex–concave procedure solves to a local optimum. Here, the first order approximation transforms the difference of convex function constraint (34) into the convex constraint (37) shown at the bottom of the next page, where the superscript p indicated the value from the previous iteration's solution. The main benefit of solving this problem with the convex–concave procedure is the first order approximation makes the constraint convex while maintaining the probabilistic assurances. Since feasibility of (36) is dependent on the feasibility of the initial conditions, we use slack variables to accommodate potentially infeasible initial conditions that can occur during the iterative

$$\begin{bmatrix} I_q & \mathbb{E}[z] \\ \mathbb{E}[z]^T & \mathbb{E} \|z\|^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} z \\ 1 \end{bmatrix}^2 - \frac{4}{9\tilde{\omega}_{ijk}} - 1 \left[\begin{array}{cc} 4\text{Var}(z) & 2\text{Cov} z, z^T z \\ 2\text{Cov} z, z^T z & \text{Var} \|z\|^2 \end{array} \right]^{\frac{1}{2}} z \geq r^2. \quad (34)$$

process [25], [26]. As we use a difference of convex functions optimization framework, Lemma 3 guarantees that any solution that is synthesized during iterative process will be a feasible but locally optimal solution.

D. Generalizability

The work presented in this section can easily be generalized beyond motion planning problem. The work presented in Sections III-A and III-B rely on constraint forms that are not unique to the motion planning problem. For example, Section II-I-A only relies on the linear form of the constraint to provide a reformulation that can meet safety violation thresholds. Linear chance constraints are common in many controls applications including finance [16], [27], power generation [28], [29], and resource management [30]. However, the work in Section III-B relies on the 2-norm and reverse convex form, making it less broadly applicable outside of motion planning problems. For any application that can use a generalization of this method will require Assumptions 1–4, as is appropriate for the problem in question.

IV. RESULTS

We demonstrate our method on a multisatellite rendezvous problem with two different disturbances that impact the relative satellite dynamics. All computations were done on a 1.80 GHz i7 processor with 16 GB of RAM, using MATLAB, CVX [31], Gurobi [32], and Mosek [33]. Polytopic construction and plotting was done with MPT3 [34].

Consider a scenario, in which N_v satellites, called the deputies, are stationed in geostationary Earth orbit, and tasked to rendezvous with a refueling spacecraft, called the chief. The satellites are tasked with reaching a new configuration represented by polytopic target sets. Each deputy must avoid other deputies while navigating to their respective target sets. The relative planar dynamics of each deputy, with respect to the position of the chief are described by the Clohessy–Wiltshire–Hill (CWH) equations [35]

$$\ddot{x} - 3\omega^2 x - 2\omega \dot{y} = \frac{F_x}{m_c} \quad (38a)$$

$$\ddot{y} + 2\omega \dot{x} = \frac{F_y}{m_c} \quad (38b)$$

with input $u_i = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$, and orbital rate $\omega = \sqrt{\frac{\mu}{R_0^3}}$, Earth's gravitational parameter μ , orbital radius R_0 , and mass of the deputy m_c . For demonstration purposes, we assume $R_0 =$

35 622 km and $m_c = 1500$ kg. This approximates the size of a Boeing's 702SP communication satellite. We discretize (38) under impulsive thrust assumptions, with sampling time $\Delta t = 60$ s. The sampling time is meant to reflect the speed at which a satellite can perform a maneuver with an electric propulsion system, such as an ion thruster [36], [37]. We insert a disturbance process that captures uncertainties in the model specification, so that dynamics of each deputy are described by

$$x_i(k+1) = Ax_i(k) + Bu_i(k) + w_i(k). \quad (39)$$

We assume that the disturbances adhere to Assumptions 1–2.

A. Exponential Disturbance

Exponential disturbances are the type of distribution that is a big motivator for our approach. They exist in real systems but very few methods can handle them. We presume, for the purpose of demonstration, that we have an exponential disturbance. This could occur because of inaccuracies in the impulsive thrust model, drag forces in low Earth orbit, or third body gravity.

The exponential distribution is defined as follows.

Definition 5 (Exponential distribution): An exponential distribution is one which elicits the PDF

$$\phi(x) = \lambda e^{-\lambda x} \quad (40)$$

with rate parameter $\lambda > 0$ and $x \geq 0$.

The exponential distribution presents several challenges for existing methods. We define the following two distributions to analyze these challenges.

Definition 6 (Hypoexponential distribution): A hypoexponential distribution is one which elicits the PDF

$$\phi(x) = -a e^{x\Theta} \Theta \mathbf{1} \quad (41)$$

with probability row vector $\mathbf{a}$, subgenerator matrix Θ , and $x \geq 0$.

Definition 7 (Weibull distribution): A Weibull distribution is one which elicits the PDF

$$\phi(x) = \frac{k}{\lambda} \frac{x}{\lambda}^{k-1} e^{-(x/\lambda)^k} \quad (42)$$

with scale parameter $\lambda > 0$, shape parameter $k > 0$, and $x \geq 0$.

First, a linear sum of independent but not identically distributed exponential random variables, as the case for the polytopic target set constraint, results in a hypoexponential distribution. While a closed form expression of the CDF exists, a closed form expression of the constraint would result in a reverse convex constraint. Further, as the CDF is not invertible, quantile methods cannot be used.

$$\frac{4}{9\omega_{jk}^2} - 1 \left[\begin{array}{cc} 4\text{Var}(z) & 2\text{Cov}(z, z) \\ 2\text{Cov}(z, z) & \text{Var}(z) \end{array} \right]^{\frac{1}{2}} z \\ - \left(\begin{array}{cc} I_q & E[z]^{\frac{1}{2}} z^p \\ E[z] & E[z] z^p \end{array} + 2(E[z] + E[z]) SC(k) U_i - U_j - U_i^p + U_j^p \right) \leq -r^2. \quad (37)$$

First order approximation of $E[z + z^2]$ based on previous iteration's solution.

Second, the squared difference of exponential random variables, as is the case with the collision avoidance constraint, results in the sum of Weibull random variables. The PDF, CDF, and characteristic function of a sum of Weibull random variables can only be expressed as an infinite summation [38], [39]. Thus, closed form evaluations of the chance constraint probabilities are practically impossible. At present, methods that create bounds based on moments are the only methods that allow for almost surely satisfaction of each chance constraint.

1) Experimental Setup: In this experiment, we demonstrate the proposed methodology for N_v taking values of both 3 and 4. We presume the admissible control set for each deputy is $U_i = [-0.2, 0.2] \frac{m}{s\Delta t}$, corresponding to an 5 N ion thruster, and time horizon $N = 8$, corresponding to 8 min of operation. The performance objective is based on fuel consumption

$$J(U_1, \dots, U_{N_v}) = \sum_{i=1}^{N_v} U_i U_i. \quad (43)$$

The terminal sets $T_i(N)$ are $5\text{ m} \times 5\text{ m}$ boxes centered around desired terminal locations in x, y coordinates with velocity bounded in both directions by $[-0.5, 0.5]\text{ m/s}$. For collision avoidance, we presume that each deputy must remain at least $r = 10\text{ m}$ away from each other, hence $S = I_2 - O_2$ to extract the positions. Violation thresholds for terminal sets and collision avoidance are α and γ , respectively. The chance constraints are defined as

$$P \left(\bigcap_{i=1}^{N_v} x_i(N) \in T_i(N) \right) \geq 1 - \alpha \quad (44)$$

$$P \left(\bigcap_{k=1}^8 \bigcap_{i,j=1}^{N_v} S(x_i(k) - x_j(k)) \geq r \right) \geq 1 - \gamma. \quad (45)$$

We will demonstrate the proposed method with $\alpha = \gamma$ taking values of 0.05, 0.075, and 0.1. Fig. 1 provides a graphic representation of the demonstration presented.

As has been established [6], [23], biconvexity associated with having both risk allocation and control variables can be addressed in an iterative fashion, by alternately solving for the risk allocation variables, then for the control. However, for our demonstration, to isolate the impact of the one-sided Vysochanskij–Petunin inequality, we presume a fixed risk allocation. We uniformly allocate risk such that

$$P(S(x_i(k) - x_j(k)) \geq r) \geq 1 - \hat{\gamma} \quad \forall i, j, k \quad (46)$$

where $\hat{\gamma} = \frac{\gamma}{24 \text{ constraints}} = \frac{.15}{24} = 3.125 \times 10^{-3}$ when $N_v = 3$ and $\hat{\gamma} = \frac{\gamma}{48 \text{ constraints}} = \frac{.15}{48} = 1.5625 \times 10^{-3}$ when $N_v = 4$. These values remain constraint throughout the iterative solution finding process.

We define the solution convergence thresholds for the convex-concave procedure as both the difference of sequential performance objectives as less than 10^{-6} and the sum of slack variables as less than 10^{-8} . Difference of convex programs were limited to 100 iterations. The first order approximations of the reverse convex constraints were initially computed assuming no system input.

For a random variable $x \sim \text{Exp}(\lambda)$, where λ is the rate parameter

$$E[x^n] = \frac{n!}{\lambda^n} \quad \forall n \in \mathbb{N}. \quad (47)$$

Hence, Assumption 3 is valid. For brevity, the derivation of expectation, variance, and covariance terms for the collision avoidance constraint (37) can be found in Appendix B.

For the target set constraint, we can determine that the chance constraint is unimodal as the exponential distribution is a strongly unimodal distribution, as per Definition 3. Hence, the affine constraint is unimodal. However, the Weibull random variables that result in the collision avoidance constraint are not strong unimodal. Here, unimodality of the constraint was validated numerically via Algorithm 1 for each vehicle pair and each time step after computing the solution. Validation was completed with randomly sampled 50 000 disturbances.

2) Comparison Methodology: We compare our method against the method in [15], the predecessor of the method proposed in this work based on Cantelli's inequality. This approach is effective for and has been demonstrated on systems, which have target constraints and can be solved via convex optimization. We extend this method to accommodate 2-norm-based collision constraints (as in Section III-B) for the purpose of comparison with our own approach. We do not consider methods based on Chebyshev's inequality because they have shown to be less effective than [15] in a target constraint problem [13].

Theorem 2 (Cantelli's inequality [11]): Let x be a real valued random variable with finite expectation $E[x]$ and finite, nonzero standard deviation $\text{Std}(x)$. Then, for any $\lambda > 0$

$$P(x - E[x] \geq \lambda \text{Std}(x)) \leq \frac{1}{\lambda^2 + 1}. \quad (48)$$

3) Experimental Results: We see in Table I that the proposed method was able to find a solution in all six cases, where as the stochastic optimal control approach with Cantelli's inequality was only able to find a solution in three cases. This is one of the key benefits of the proposed methodology. The proposed method uses a significantly less conservative inequality in comparison to Cantelli's inequality. This allows us to solve more challenging problems, such as tighter safety thresholds or more vehicles, while maintaining the almost surely guarantees of chance constraint satisfaction.

Further, when both methods did find a solution, the solution cost of the proposed method was about 10% less, on average. This can also be observed as for all cases, the post-hoc satisfaction of the target set constraint was consistently lower for the proposed method. Again, these results were expected, as the method benefits from a less conservative concentration inequality.

Further, the lower solution cost of the proposed method's solution is directly attributable to the decrease in conservativeness of the one-sided Vysochanskij–Petunin inequality over Cantelli's inequality. This can easily be seen in Fig. 3, where we have plotted the expectation of the trajectory when there are four vehicles and the probability violation threshold is $\alpha = \gamma = 0.1$. In the left subfigure, we see that the expected position of the proposed methodology is closer to its nearest hyperplane constraint within each target set. Since the one-sided

TABLE I

COMPARISON OF SOLUTION, COMPUTATION TIME, AND POST-HOC EMPIRICAL ESTIMATES OF CHANCE CONSTRAINT PROBABILITIES BETWEEN PROPOSED METHOD AND THE STOCHASTIC OPTIMAL CONTROL METHOD UTILIZING CANTELLI'S INEQUALITY (SOC WITH CANTELLI'S INEQUALITY)

N_v	Metric	$\alpha = \gamma = 0.05$		$\alpha = \gamma = 0.075$		$\alpha = \gamma = 0.1$	
		Proposed Method	SOC with Cantelli's Inequality [15]	Proposed Method	SOC with Cantelli's Inequality [15]	Proposed Method	SOC with Cantelli's Inequality [15]
3	Solve Time (s)	8.2337		6.9137	6.8533	6.8555	6.3414
	Iterations	12	No	9	10	9	9
	Solution Cost in $(\frac{m}{s\Delta t})^2$	0.0443	Solution	0.0431	0.0460	0.0425	0.0446
	Satisfaction of (44)	0.9999	Found	0.9994	1.0000	0.9971	1.0000
	Satisfaction of (45)	1.0000		1.0000	1.0000	1.0000	1.0000
4	Solve Time (sec)	30.6113		21.1406		17.9908	30.7663
	Iterations	25	No	17	No	15	26
	Solution Cost in $(\frac{m}{s\Delta t})^2$	0.0813	Solution	0.0755	Solution	0.0726	0.0833
	Satisfaction of (44)	1.0000	Found	0.9999	Found	0.9992	1.0000
	Satisfaction of (45)	1.0000		1.0000		1.0000	1.0000

Problem formulation includes CWH dynamics with exponential disturbance for number of vehicles (N_v) equal to 4 and 5 and probability violation thresholds (α and γ) taking values 0.05, 0.75, and 0.1. Post-hoc empirical estimates of chance constraint probabilities is measured as the ratio of 10^4 samples that did not satisfy the constraint.

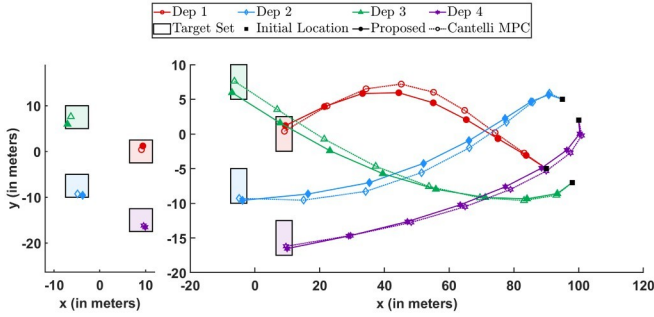


Fig. 3. Plot of the expectation of the trajectories between the proposed method (solid line with filled in markers), and the stochastic optimal control method using Cantelli's inequality [15] (dotted line with white markers). Trajectories were generated with four satellites with planar CWH dynamics disturbed by an exponential disturbance, and probabilistic violation thresholds $\alpha = \gamma = 0.1$. The full trajectory is displayed on the right and the terminal state is on the left. We see the two methods had similar trajectories but notice the stochastic optimal control method using Cantelli's inequality is further from its nearest hyperplane constraint of the target set for each vehicle. This is a direct consequence of the additional conservativeness of Cantelli's inequality over the one-sided Vysochanskij–Petunin inequality used in the proposed method. While the trajectories may look similar, the proposed method has an approximately 13% lower solution cost than the comparison method.

Vysochanskij–Petunin inequality is less conservative, we are able to find a lower solution cost with a trajectory nearer to the target set hyperplane boundaries while still satisfying the probabilistic violation thresholds.

As this example demonstrates, we can find solutions and make probabilistic guarantees for disturbances that may arise in common circumstances, despite tight safety thresholds. As discussed earlier, these distributional assumptions made in this example result in complicated distributions that lack analytical form. It is in examples like this, where the model assumptions result in complex but unimodal distributions, we believe this method provides a novel means to find a solution.

B. Gaussian Disturbance

In this example, we make two simplifying assumptions. First, we only consider a convex joint chance constraint with a time-varying target set, as in Section III-A. Second, we assume

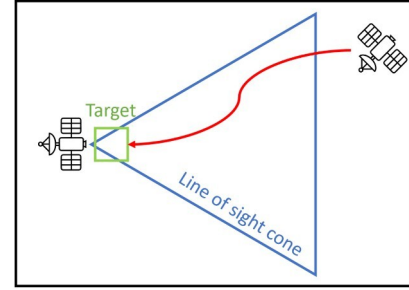


Fig. 4. Graphic representation of the problem posed in Section IV-B. Here, the dynamics of the deputy is perturbed by additive Gaussian noise. We attempt to find a control sequence that allows the deputy to rendezvous with the chief while meeting probabilistic time varying target set requirements.

the disturbance is Gaussian. Combined, these two simplifying assumptions allow us to compare the proposed approach with more traditional approaches, including sample-based approaches, quantile-based approaches, and moment-based approaches, while preserving theoretical guarantees of chance constraint satisfaction.

1) Experimental Setup: For this experiment, we presume there is a single deputy, $N_v = 1$ that must stay within a pre-defined line of sight cone and reach a terminal target set, as shown in Fig. 4. We presume the admissible control set is $U_i = [-0.1, 0.1] \frac{m}{s\Delta t}$, corresponding to a 2.5 N ion thruster, and time horizon $N = 5$, corresponding to 5 min of operation. The performance objective is based on fuel consumption

$$J(U_1) = U_1 U_1. \quad (49)$$

The line-of-sight cone is defined by the inequalities

$$-x + 2y \leq 0$$

$$-x - 2y \leq 0$$

$$x \leq 10. \quad (50)$$

The terminal sets $T(N)$ is a $2 \times 1 \text{ m}^2$ near the origin with velocity bounded in both directions by $[-0.1, 0.1] \text{ m/s}$. The violation thresholds for joint target set constraint is $\alpha = 0.05$

TABLE II
COMPARISON OF SOLUTION, COMPUTATION TIME, AND POST-HOC EMPIRICAL ESTIMATES OF CHANCE CONSTRAINT PROBABILITIES FOR CWH DYNAMICS WITH MULTIVARIATE GAUSSIAN DISTURBANCE FOR VARYING VALUES OF α

α	Metric	Proposed Method	SOC with Cantelli's Inequality [15]	Quantile Method [40], [41]	Scenario Approach [7], [10]	Particle Control [42]
0.05	Solve Time (s)	0.1794	No	1.0414	3.6287	14.9423
	Solution Cost in $(\frac{m}{s\Delta t})^2$	6.0075×10^{-4}	Solution	5.1885×10^{-4}	5.3572×10^{-4}	5.1597×10^{-4}
	Satisfaction of (51)	1.0000	Found	0.9505	0.9954	0.8785
0.1	Solve Time (s)	0.1694	0.1611	2.3789	1.2991	58.5369
	Solution Cost in $(\frac{m}{s\Delta t})^2$	5.4625×10^{-4}	6.3025×10^{-4}	5.1420×10^{-4}	5.3807×10^{-4}	5.1375×10^{-4}
	Satisfaction of (51)	0.9996	1.0000	0.9053	0.9982	0.9025

Post-hoc empirical estimates of chance constraint probabilities is measured as the ratio of 10^4 samples that did not satisfy the constraint.

The chance constraint is defined as

$$\prod_{k=1}^5 \mathbb{P}(x_1(k) \in T_1(k)) \geq 1 - \alpha \quad (51)$$

where α will take values of 0.05 and 0.1.

We presume the disturbance is Gaussian

$$w_1(k) \sim N(0, \text{diag}(10^{-3}, 10^3, 10^8, 10^8)). \quad (52)$$

Using the properties of the Gaussian disturbance, we know that all moments exist such that Assumption 3 is valid. Further, affine summations of Gaussian disturbances are still Gaussian. Hence, each target set constraint is unimodal, validating Assumption 4.

2) Comparison Methodologies: Here, we compare the proposed methodology against a broader field of chance constrained stochastic optimal control methods. Several methods exist to solve convex chance constraints in a Gaussian regime. Hence, we select comparison methodologies that have been used extensively to solve chance constrained problems with Gaussian disturbances but can also handle non-Gaussian disturbances. Specifically, we compare the proposed method with the moment-based approach with Cantelli's inequality [15] outlined in Section IV-A, quantile approach in [40] and [41], the scenario approach in [7] and [10], and the particle control approach in [42].

The quantile approach results in a reformulation that is a convex in the input and the Gaussian quantile function. The quantile method allows for almost surely guarantees of chance constraint satisfaction as the disturbance is Gaussian. The particle control approach relies on sample disturbances and the chance constraint reformulation results in a mixed integer linear program. The particle control approach can only guarantee chance constraint satisfaction asymptotically as the number of samples goes to infinity. To minimize computational complexity, we select 100 samples as the sample size of our disturbance to compute the optimal control trajectory with the particle control approach. Like the particle control approach, the scenario approach relies on samples to compute an optimal controller. The reformulation of the scenario approach results in a linear program. The scenario approach can guarantee chance constraints up to a probabilistic confidence bound δ . By setting the confidence bound to a sufficiently small value, the probabilistic guarantees of the scenario approach closely resemble that of the proposed method. We compute the number of samples required for the scenario

approach with the formula [43]

$$N_s \geq \frac{2}{\alpha} \ln \frac{1}{\delta} + N_o \quad (53)$$

where N_s is the number of samples required and N_o is the number of optimization variables. Here, $N_o = 10$ and we choose $\delta = 10^{-16}$. Hence, $N_s = 937$ when $\alpha = 0.05$ and $N_s = 469$ when $\alpha = 0.1$.

We expect the proposed method to result in more conservative solutions compared with the quantile-based and sample-based approaches. This stems from the conservative nature of the one-sided Vysochanskij–Petunin inequality [16]. However, we also expect to see the proposed method compute solutions in less time than the comparison methods. We expect this as the proposed method does not rely on samples as the scenario and particle control method, and the simplicity of the proposed reformulation in comparison to the quantile approach. In comparison to the stochastic optimal control method with Cantelli's inequality, we expect to see similar results to Section IV-A, where the proposed approach resulted in less conservative solutions while maintaining similar solve times and post-hoc estimates of chance constraint probabilities.

3) Experimental Results: As Table II tabulates, the proposed method performed as expected in comparison to the panel of methods we tested. In comparison to the stochastic optimal control method with Cantelli's inequality, we saw similar results to Section IV-A, whereas the proposed method was able to find a solution in both cases, and provided a less conservative solution when $\alpha = 0.1$. In comparison to the quantile approach, we see the proposed solution resulted in a more conservative and costly solution but up to an order of magnitude faster to compute. However, the need to know or estimate the quantile is often a barrier for many distributions that the proposed method does not have. Finally, in comparison to the two sample-based methods, we see significantly lower solution times for the proposed methodology, up to two order of magnitude. Both the proposed method and the scenario approach were able to consistently meet chance constraint safety thresholds, whereas the particle control method was only able to meet the safety threshold in one case.

In Fig. 5, we plot the expectation of the solution trajectories from each method and a subplot of each time step zoomed in to the local region around the solutions. We see that as the trajectories evolve, there are three distinct groupings of solutions. The important takeaway of this graph is that the trajectory of the proposed method lies approximately in the middle of the other methodologies. We expect the proposed method to be less

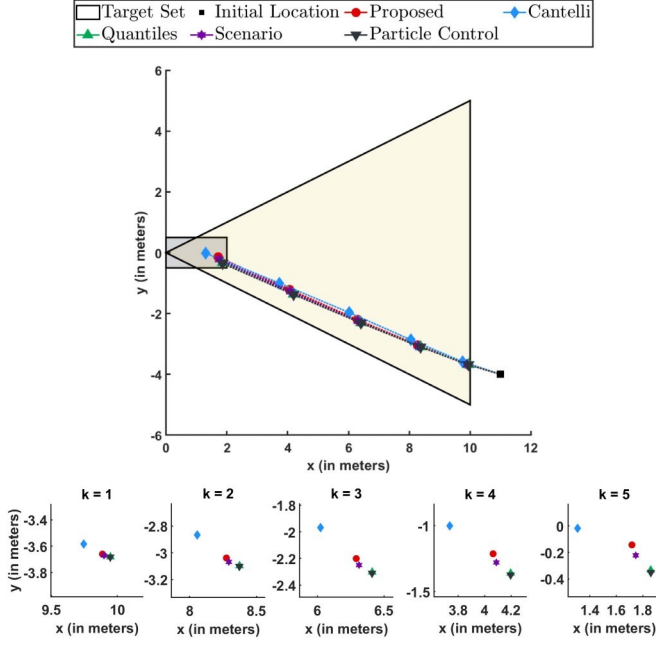


Fig. 5. Plot of the expectation of trajectories between the proposed method (red, circle), stochastic optimal control method utilizing Cantelli's inequality [15] (blue, diamond), quantile-based approach [40], [41] (green, triangle pointing up), the scenario approach [7], [10] (purple, 6-pointed star), and the particle control approach [42] (black, triangle pointing down). Trajectories were generated for one satellite with CWH dynamics disturbed by a multivariate Gaussian disturbance, and probabilistic violation threshold $\alpha = 0.1$. Here, we observe the trajectories are very similar. Below the primary plot, a subplot is included for each time step to better view the dispersion of the methods at that time step. In the time step plots, we see that the trajectories of the proposed approach is similar to that of the scenario approach but distinct from the rest of the comparison methods.

conservative than the method employing Cantelli's inequality but more conservative than the quantile approach. Conversely, since our method does not rely on sample datasets, such as the scenario approach and particle control, whose trajectories would change given different sample sets, we know we can expect consistent and reproducible results.

This example illustrates that the proposed method has merits when comparing to a panel of commonly used stochastic optimal control methods. The proposed method offers computational efficiency, interpretability, and almost surely guarantees of chance constraint satisfaction, while only requiring validation of unimodality, and knowledge of moments. This is particularly true when moment-based methods are the only option, as the one-sided Vysochanskij–Petunin inequality is currently the least conservative concentration inequality that will result in a convex, tractable reformulation [16].

V. CONCLUSIONS AND FUTURE WORK

We proposed a framework to solve chance-constrained stochastic optimal control problems for LTI systems subject to arbitrary disturbances under moment and unimodality assumptions. This work focuses on probabilistic requirements for polytopic target sets and 2-norm-based collision avoidance

constraints. Our approach relies on the one-sided Vysochanskij–Petunin inequality to reformulate joint chance constraints into a series of inequalities that can be readily solved as a difference of convex functions optimization problem. We demonstrated our method on a multisatellite rendezvous scenario under exponential and Gaussian disturbance assumptions and compare with an stochastic optimal control approach using Cantelli's inequality (the predecessor of this work), a quantile-based approach, the scenario approach, and the particle control approach. We showed that this approach is amenable to disturbances that prove challenging or impossible to solve with other methods and demonstrated the proposed method has computational benefits in comparison to other commonly used methods.

Methodologically, we are interested in exploring probabilistic inequalities that result in less conservative bounds. For cases in which the disturbance is unknown and samples are available, we are currently exploring moment-based approaches that rely on sample approximations of moments and provide probabilistic guarantees.

APPENDIX A

A. Numerical Evaluation of Unimodality

Algorithm 1 is constructing an affine approximation of empirical CDF then testing whether there is a single inflection point by comparing the slopes of the affine segments.

B. Derivation of Norm Expectation and Variance in Exponential Case

We keep with the notation used in Section III-B. Since we assumed the disturbances are independent and identically distributed, from (47) we find

$$\begin{aligned} E[\mathbf{z}] &= \mathbf{0}_{q \times 1} \\ \text{Var}(\mathbf{z}) &= 2SD(k) \text{Var}(\mathbf{W}_i) D(k)S \end{aligned} \quad (54)$$

where

$$\begin{aligned} \text{Var}(\mathbf{W}_i) &= \text{diag}(2\sigma^2 \cdot I_2, 10^8 \cdot I_2, \dots, 2\sigma^2 \cdot I_2, 10^8 \cdot I_2) \end{aligned} \quad (55)$$

Next, from (54)

$$E[\mathbf{z} \mathbf{z}^T] = \text{tr}(\text{Var}(\mathbf{z})) \quad (56)$$

Next, we find $\text{Var}(\mathbf{z} \mathbf{z}^T)$. For brevity, we denote $\mathbf{W}_i - \mathbf{W}_j$ as $\mathbf{W}$. Then

$$\begin{aligned} \text{Var}(\mathbf{z} \mathbf{z}^T) &= \text{Var}(\mathbf{W} D(k)S D(k)W) \end{aligned} \quad (57a)$$

$$= \text{Var}_{p=1, q=1}^{Nn, Nn} a_{pq} W_p W_q \quad (57b)$$

$$= \sum_{p=1}^{Nn} \sum_{q=1}^{Nn} \sum_{r=1}^{Nn} \sum_{s=1}^{Nn} \text{Cov}(a_{pq} W_p W_q, a_{rs} W_r W_s) \quad (57c)$$

Algorithm 1: Numerical Check for Unimodality.

Input: Empirical cumulative distribution function points $(x_i, \hat{F}(x_i))$ for samples x_i with $i \in \mathbb{N}_{[1, N_s]}$, and maximum error threshold ξ .

Output: 1 if unimodal or 0 if not unimodal

$i \leftarrow 0$

$\mathbb{S} \leftarrow \emptyset$

while $i < N_s$ **do**

for $j = N_s$ **to** $i + 1$ **by** -1 **do**

$\underline{m} \leftarrow \frac{\hat{F}(x_j) - \hat{F}(x_i)}{x_j - x_i}$

$\underline{b} \leftarrow \hat{F}(x_j) - x_j \times \underline{m}$

for $k = i + 1$ **to** $j - 1$ **by** 1 **do**

$\epsilon_k = \hat{F}(x_k) - (p_y \times \underline{m} + \underline{b})$

if $\epsilon_k > \xi$ **then**

next j

end

end

$\mathbb{S} \leftarrow \mathbb{S} \cup \{\underline{m}\}$

break

end

$i \leftarrow j$

end

$w \leftarrow 0$

$N_c \leftarrow \text{card}(\mathbb{S})$ # cardinality of set $\mathbb{S}$

for $i = 2$ **to** N_c **by** 1 **do**

 # $\mathbb{S}_i$ is the i^{th} element of $\mathbb{S}$

if $\mathbb{S}_i \geq \mathbb{S}_{i-1}$ **then**

if $w = 1$ **then**

return 0

end

else

$w \leftarrow 1$

end

end

return 1

where a_{pq} is the (p, q) th element of $D(k)S SD(k)$. Then

$$\begin{aligned} & \text{Cov}_{p=1, q=1, r=1, s=1}^{Nn, Nn, Nn, Nn} a_{pq} W_p W_q, a_{rs} W_r W_s \\ &= \sum_{p=1}^{Nn} \text{Var}_{pp} a_{pp} W_p^2 + 4 \sum_{1 \leq p < q \leq Nn} \text{Var}_{pq} a_{pq} W_p W_q. \end{aligned} \quad (58)$$

Here, all remaining covariance terms is zero as each element is mutually independent by Assumptions 1 and 2, and the first and third moments being zero. So

$$\begin{aligned} & \sum_{p=1}^{Nn} \text{Var}_{pp} a_{pp} W_p^2 + 4 \sum_{1 \leq p < q \leq Nn} \text{Var}_{pq} a_{pq} W_p W_q \\ &= \sum_{p=1}^{Nn} a_{pp}^2 E W_p^4 - E W_p^2^2 \quad \& \end{aligned} \quad (59a)$$

$$\begin{aligned} & + 4 \sum_{1 \leq p < q \leq Nn} a_{pq}^2 E W_p^2 E W_q^2 \\ &= 3 \sum_{p=1}^{Nn} a_{pp}^2 E W_p^2^2 + 2 \sum_{p=1}^{Nn} \sum_{q=1}^{Nn} a_{pq}^2 E W_p^2 E W_q^2 \end{aligned} \quad (59b)$$

as in this example $E[W_p^4] = 6E[W_p^2]^2$. Then, let $\mathbf{a}$ be a vector consisting of the diagonal elements of $D(k)S SD(k)$. So

$$3 \sum_{p=1}^{Nn} a_{pp}^2 E W_p^2^2 + 2 \sum_{p=1}^{Nn} \sum_{q=1}^{Nn} a_{pq}^2 E W_p^2 E W_q^2 \quad (60a)$$

$$\begin{aligned} &= 12 \mathbf{a}^T \text{Var} W_i^2 \mathbf{a} \\ &\quad + 8 \mathbf{r}^T D(k)S \text{Var} W_i SD(k)^T. \end{aligned} \quad (60b)$$

Finally, we find $\text{Cov}(\mathbf{z}, \mathbf{z})$

$$\text{Cov}(\mathbf{z}, \mathbf{z}) = E \mathbf{z} \mathbf{z}^T - E[\mathbf{z}] E \mathbf{z}^T. \quad (61)$$

The second term is zero by (54). For a random vector, the expectation is a vector of the expectations of each element. Then, for the i th element

$$E z_i z_i = E z_i^3 + \sum_{j=i}^{Nn} E[z_i] E z_j^2. \quad (62)$$

Since the first and third moments of $\mathbf{z}$ are zero, then the sum is zero. Thus, $\text{Cov}(\mathbf{z}, \mathbf{z}) = \mathbf{0}$.

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