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Not all that is β_0 is β -function: the DGLAP resummation and the running coupling in NLO JIMWLK

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ABSTRACT: We reanalyze the origin of the large transverse logarithms associated with the QCD one loop β function coefficient in the NLO JIMWLK Hamiltonian. We show that some of these terms are not associated with the running of the QCD coupling constant but rather with the DGLAP evolution. The DGLAP-like resummation of these logarithms is mandatory within the JIMWLK Hamiltonian, as long as the color correlation length in the projectile is larger than that in the target. This regime in fact covers the whole range of rapidities at which JIMWLK evolution is supposed to be applicable. We derive the RG equation that resums these logarithms to all orders in α_s in the JIMWLK Hamiltonian. This is a nonlinear equation for the eikonal scattering matrix $S(\mathbf{x})$. We solve this equation, and perform the DGLAP resummation in two simple cases: the dilute limit, where both the projectile and the target are far from saturation, and the saturated regime, where the target correlation length also determines its saturation momentum.

KEYWORDS: Deep Inelastic Scattering or Small-x Physics, The Strong Coupling, Algorithms and Theoretical Developments, Effective Field Theories of QCD

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1 Introduction

At high energy, the energy evolution of hadronic cross sections (see e.g. [1] for review) is governed by the Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) evolution equation [2–6]. The JIMWLK equation was derived at next-to-leading order (NLO) almost ten years ago [7–9], following the earlier derivation of the NLO dipole model (Balitsky-Kovchegov equation), see [10–13]. With the derivation, it became immediately apparent that improvements are necessary due to the presence of large transverse logarithms in the NLO kernel. Two types of terms contain large transverse logarithms in the NLO JIMWLK equation. The first type is proportional to the one loop coefficient of the QCD β -function, β_0 ; while the second does not involve the β -function and is related to the lifetime ordering of fluctuations.

Different recipes for the resummation of both types of logarithms have been extensively discussed in the literature in recent years [14–20]. In this paper, we will have nothing to say about the second type of logarithms, which we will ignore from now on, with the realization that the physics that goes into their resummation is very different from the one discussed here.

The consensus for the β_0 -dependent logarithms has been that they should all be resummed into the running of the QCD coupling constant [21, 22], see also [23]. Different recipes have been advanced for the optimal scale setting in the running coupling.¹

In this paper, we revisit the resummation of this type of logarithms. We show that not all of the logarithms that contain β_0 are of the same physical origin. While some of them are associated with the running of the coupling for the emission of a soft gluon, the correct understanding of an important subset of them is in terms of the *DGLAP evolution* [25–27] of the projectile wave function. The latter logarithms are large if there is a large disparity between the typical transverse momentum in the projectile and target wave functions, and have to be resummed via solving a DGLAP evolution equation just like in DIS. We explicitly set up the (transverse) evolution equation for the JIMWLK Hamiltonian that resums these logarithms into the modified JIMWLK kernel. We also provide an approximate solution to this equation in two simple cases: the dilute limit and the limit of dense saturated target. The main qualitative result of this resummation is the smearing of the emission point of the gluon in the JIMWLK Hamiltonian over the transverse resolution scale with a function decreasing as a power of a distance. Physically this transverse resolution scale is provided by the saturation momentum of the color fields in the target. Our analysis also suggests a simple scale setting procedure for the QCD running coupling, which does not require any elaborate considerations. Together with the DGLAP resummation, this simple scale setting procedure takes care of all the large transverse logarithms.²

Our discussion is set explicitly in the context of a “dense” target and “dilute” projectile in the following sense. We assume that the color fields in the target are correlated on a transverse distance scale Q_T^{-1} . In the projectile we also assume the existence of an analogous scale Q_P^{-1} , and we require that $Q_T \gg Q_P$, i.e. that the correlation length of the color fields in the target is much smaller than that in the projectile.

The existence of such correlation length is obvious in the saturation regime. In this case we identify Q_T with the saturation momentum in the target, and analogously for the projectile. The hierarchy of the correlation length then just means that the target has a larger saturation momentum, and is therefore the denser of the two objects. Physically it is clear that this situation must involve significant DGLAP evolution between the projectile and the target saturation scale. The large target saturation momentum plays the role of the transverse resolution scale Q^2 in DIS, while the smaller projectile saturation momentum is akin to Λ_{QCD} in the proton wave function which determines typical transverse momenta of the partons in the proton.

However our approach is also applicable when neither target nor projectile are saturated, i.e. the eikonal scattering factors are perturbatively close to unity. In this case the scale Q_s^{-1} has the meaning of the transverse scale on which the color neutralization occurs in the wave function. For example, for a single dipole projectile, Q_P^{-1} would be of the order of the dipole size. It then determines simultaneously the typical transverse momentum of the partons in the projectile and the typical transverse size of the projectile hadron. Similarly, Q_T^{-1} for a single dipole target determines both, the target size and the transverse scale on which the

¹As was investigated in ref. [24], these schemes violate the semi-positivity of the JIMWLK Hamiltonian.

²Again, in this statement, we do not include the logarithms associated with the lifetime ordering of fluctuations.

scattering resolves the transverse structure of the projectile. In this case the hierarchy of scales means that the target “dipole” has smaller transverse size than the projectile “dipole”, so that it is still true that the projectile is more dilute than the target on the relevant resolution scale Q_T^{-1} . Indeed we show that only in the case when there is a large separation of scales Q_P and Q_T , the DGLAP evolution is significant, and only in this case the transverse logarithms in question need to be resummed.

This paper is structured as follows. In section 2 we review the NLO JIMWLK Hamiltonian and the common lore about the resummation of β_0 dependent transverse logarithms. In section 3 we introduce the physical interpretation in terms of transverse resolution scale and formalize it by defining (in the JIMLWK framework) the analog of factorization scale in DIS. We derive the RG equation for the JIMWLK Hamiltonian which imposes the overall factorization scale independence of physical amplitudes, and discuss what is the best choice for initial and final scales in this evolution. In section 4 we provide an approximate solution for this equation and discuss its properties in the absence of quarks. In section 5 we generalize the discussion to include quarks. Finally, we conclude in section 6 with a discussion of our results.

2 JIMWLK at NLO and the large transverse logarithms

The JIMWLK Hamiltonian is the operator that generates rapidity evolution of the hadronic S -matrix at high energy. We take it here as acting on the projectile wave function (see ref. [28] for a comprehensive introduction), or rather probability distribution $\mathcal{W}_P$, so that the rapidity evolution of the scattering matrix of a hadronic projectile on a hadronic target is given by

$$\frac{d}{dY}\mathcal{S} = \langle H_{\text{JIMWLK}}[S, J]\mathcal{W}_P[S] \rangle_T \quad (2.1)$$

where $\mathcal{S}$ is the hadronic S -matrix, S -is an eikonal propagation factor (Wilson line) of a projectile parton on a target field, J is a rotation operator (see below) and the averaging on r.h.s. is performed over the ensemble of the target fields.

2.1 JIMWLK at leading order

At leading order the JIMWLK Hamiltonian (see e.g. ref. [9]) is given by

$$H_{\text{JIMWLK}}^{\text{LO}} = \frac{\alpha_s}{2\pi^2} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \left[J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x}) S^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right]. \quad (2.2)$$

Here $X_i \equiv \mathbf{x}_i - \mathbf{z}_i$ and $Y_i \equiv \mathbf{y}_i - \mathbf{z}_i$. The LO kernel is a product of two Weizsacker-Williams (WW) fields. S is the gluon’s eikonal scattering matrix, that is the Wilson line in adjoint representation. The left and right rotation operators $J_L^a(\mathbf{x})$ and $J_R^a(\mathbf{x})$ generate left and right rotations of the Wilson line $S(\mathbf{x})$ at the same transverse point $\mathbf{x}$.

$$J_R^a(\mathbf{x}) S^{eb}(\mathbf{y}) = [S(\mathbf{y}) T^a]^{eb} \delta^2(\mathbf{x} - \mathbf{y}); \quad J_R^a(\mathbf{x}) S^{Teb}(\mathbf{y}) = -[T^a S^T(\mathbf{y})]^{eb} \delta^2(\mathbf{x} - \mathbf{y}). \quad (2.3)$$

J_R and J_L are related by

$$J_R^a(\mathbf{x}) = S^{\dagger ab}(\mathbf{x}) J_L^b(\mathbf{x}). \quad (2.4)$$

2.2 JIMWLK Hamiltonian at NLO — the relevant terms

The NLO JIMWLK Hamiltonian deduced in refs. [7, 8] is:

$$\begin{aligned}
 H_{\text{JIMWLK}}^{\text{NLO}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} K_{JSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \left[J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2 J_L^a(\mathbf{x}) S^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \\
 & + \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \left[J_L^a(\mathbf{x}) D^{ad}(\mathbf{z}, \mathbf{z}') J_R^d(\mathbf{y}) - \frac{N_c}{2} J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - \frac{N_c}{2} J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) \right] \\
 & + \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') \left[2 J_L^a(\mathbf{x}) \text{tr}[V^\dagger(\mathbf{z}) t^a V(\mathbf{z}') t^b] J_R^b(\mathbf{y}) - \frac{1}{2} J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - \frac{1}{2} J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) \right] \\
 & + \dots
 \end{aligned} \tag{2.5}$$

where, to simplify notations, we introduced

$$D^{ab}(\mathbf{z}_1, \mathbf{z}_2) = \text{Tr}[T^a S(\mathbf{z}_1) T^b S^\dagger(\mathbf{z}_2)] \tag{2.6}$$

while ellipsis denotes terms that do not depend on β_0 and do not play any role in our discussion. In eq. (2.5), the rotation operators J_L and J_R do not act on the eikonal factors S in the Hamiltonian itself, but only on those in the wave function $\mathcal{W}_P[S]$ in eq. (2.1).

The kernels K in this equation are given by the following expressions.

$$\begin{aligned}
 K_{JSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv & \frac{\alpha_s^2(\mu) X \cdot Y}{4\pi^3 X^2 Y^2} \\
 & \times \left[\beta_0 \left(\frac{1}{2} \ln[X^2 \mu^2] + \frac{1}{2} \ln[Y^2 \mu^2] \right) + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} N_f \right].
 \end{aligned} \tag{2.7}$$

Here the UV cutoff μ^2 is related to the normalization point in the $\overline{MS}$ scheme $\mu_{\overline{MS}}^2$ (which we consider as being taken in the far UV) as $\mu^2 = \frac{\mu_{\overline{MS}}^2}{4} e^{2\gamma}$ and β_0 is the first coefficient of the QCD β -function:

$$\beta_0 = \beta_0^g + \beta_0^q \equiv \frac{11N_c - 2N_f}{3}. \tag{2.8}$$

Next:

$$\begin{aligned}
 K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = & \frac{\alpha_s^2(\mu)}{16\pi^4} \left[\frac{4}{Z^4} + \left\{ 2 \frac{X^2(Y')^2 + (X')^2 Y^2 - 4(X-Y)^2 Z^2}{Z^4(X^2(Y')^2 - (X')^2 Y^2)} + \frac{(X-Y)^4}{X^2(Y')^2 - (X')^2 Y^2} \right. \right. \\
 & \times \left(\frac{1}{X^2(Y')^2} + \frac{1}{Y^2(X')^2} \right) + \frac{(X-Y)^2}{Z^2} \left(\frac{1}{X^2(Y')^2} - \frac{1}{Y^2(X')^2} \right) \left. \right\} \ln \left(\frac{X^2(Y')^2}{(X')^2 Y^2} \right) \\
 & - \frac{2I(\mathbf{x}, \mathbf{z}, \mathbf{z}')}{Z^2} - \frac{2I(\mathbf{y}, \mathbf{z}, \mathbf{z}')}{Z^2} \Big] + \widetilde{K}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'),
 \end{aligned} \tag{2.9}$$

with $Z_i \equiv \mathbf{z}_i - \mathbf{z}'_i$ and

$$\begin{aligned} \widetilde{K}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = & \frac{\alpha_s^2(\mu)}{16\pi^4} \left(\frac{(Y')^2}{(X')^2 Z^2 Y^2} - \frac{Y^2}{Z^2 X^2 (Y')^2} + \frac{1}{Z^2 (Y')^2} - \frac{1}{Z^2 Y^2} + \frac{(X-Y)^2}{X^2 Z^2 Y^2} \right. \\ & \left. - \frac{(X-Y)^2}{(X')^2 Z^2 (Y')^2} + \frac{(X-Y)^2}{(X')^2 X^2 (Y')^2} - \frac{(X-Y)^2}{X^2 (X')^2 Y^2} \right) \ln \left(\frac{X^2}{(X')^2} \right) + (\mathbf{x} \leftrightarrow \mathbf{y}). \end{aligned} \quad (2.10)$$

The term $\widetilde{K}$ arises from a specific ordering of the rotation operators $J_{R(L)}$ in the terms we omitted in eq. (2.5). Similarly to those terms, $\widetilde{K}$ does not contain any physics related to the QCD β -function and will not play any role in our discussion.

The terms involving $I(\mathbf{x})$ and $I(\mathbf{y})$ do not depend on one of the variables that is integrated over, either $\mathbf{x}$ or $\mathbf{y}$. These terms therefore vanish when the Hamiltonian acts on globally color singlet states. The explicit form for them is given by

$$\begin{aligned} I(\mathbf{x}, \mathbf{z}, \mathbf{z}') & \equiv \frac{1}{X^2 - (X')^2} \left(\frac{X^2 + (X')^2}{Z^2} - \frac{X \cdot X'}{X^2} - \frac{X \cdot X'}{(X')^2} - 2 \right) \ln \left(\frac{X^2}{(X')^2} \right) \\ & = \frac{1}{X^2 - (X')^2} \left(\frac{X^2 + (X')^2}{Z^2} + \frac{Z^2 - X^2}{2(X')^2} + \frac{Z^2 - (X')^2}{2X^2} - 3 \right) \ln \left(\frac{X^2}{(X')^2} \right). \end{aligned} \quad (2.11)$$

For further reference we note that at small values of $|X| \ll |Y|$,

$$\int_{\mathbf{z}'} [K_{JSSJ} - \widetilde{K}] \approx \frac{\alpha_s^2}{4\pi^3} \frac{11}{3} \frac{X \cdot Y}{X^2 Y^2} \ln X^2 \mu^2. \quad (2.12)$$

Finally

$$\begin{aligned} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') = & \frac{\alpha_s^2 N_f}{8\pi^4} \left(\frac{2}{Z^4} - \frac{(X')^2 Y^2 + (Y')^2 X^2 - (X-Y)^2 Z^2}{Z^4 (X^2 (Y')^2 - (X')^2 Y^2)} \ln \left(\frac{X^2 (Y')^2}{(X')^2 Y^2} \right) \right. \\ & \left. - \frac{I_f(\mathbf{x}, \mathbf{z}, \mathbf{z}')}{Z^2} - \frac{I_f(\mathbf{y}, \mathbf{z}, \mathbf{z}')}{Z^2} \right), \end{aligned} \quad (2.13)$$

where the term that vanishes when acting on color singlet states is:

$$I_f(\mathbf{x}, \mathbf{z}, \mathbf{z}') \equiv \frac{2}{Z^2} - \frac{2X \cdot X'}{Z^2 (X^2 - (X')^2)} \ln \left(\frac{X^2}{(X')^2} \right). \quad (2.14)$$

2.3 The “subtraction” method

Importantly, the term involving the kernel K_{JSSJ} has the very same structure as the LO Hamiltonian. It has an explicit logarithmic UV divergence in the form of $\ln \mu_{\overline{MS}}$. The coefficient multiplying $\ln \mu_{\overline{MS}}$ is twice the value necessary to collect these logarithms into the QCD β -function (see e.g. ref. [29]). However the kernel $K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}')$ also leads to logarithmic UV divergence when the distance between the two emitted gluons vanishes, $Z = \mathbf{z} - \mathbf{z}' \rightarrow 0$ as was noted in refs. [21, 22]. The same is true for the $q\bar{q}$ kernel $K_{q\bar{q}}$. For simplicity of presentation we will discuss explicitly only the gluon emission for now. The $q\bar{q}$

emission is treated in an identical way. Since at short distances $S^\dagger(\mathbf{z}')S(\mathbf{z})\big|_{\mathbf{z}'\rightarrow\mathbf{z}} \rightarrow 1$, one can get rid of the divergence in the K_{JSSJ} term by subtracting from it a linear term in S , and adding the same term to K_{JSJ} term. This results in a UV finite term involving the second power of S at the expense of adding a piece to the K_{JSJ} kernel. The coefficient of the $\ln \mu_{\overline{MS}}^2$ in the modified K_{JSJ} is then exactly right to absorb this term into the β -function.

However there is arbitrariness in the way one can subtract the divergence from K_{JSSJ} . For example in ref. [21], the divergent subtraction term was multiplied by the Wilson line at position $S(\mathbf{z})$, while in ref. [22] the position of the Wilson line was determined by the kinematics of DGLAP splitting in coordinate space. Since the subtracted term is then added to K_{JSJ} , the result is not universal. For example, in ref. [21], the result³ of the subtraction (we write out explicitly only the β_0 -dependent terms) is⁴

$$K_{JSJ} \rightarrow K_{JSJ}^B = \frac{\alpha_s^2(\mu)\beta_0}{16\pi^3} \left\{ -\frac{(X-Y)^2}{X^2Y^2} \ln(X-Y)^2 \mu^2 + \frac{1}{X^2} \ln Y^2 \mu^2 + \frac{1}{Y^2} \ln X^2 \mu^2 \right\} \quad (2.15)$$

while in ref. [22]

$$K_{JSJ} \rightarrow K_{JSJ}^{KW} = \frac{\alpha_s^2(\mu)\beta_0}{8\pi^3} \frac{X \cdot Y}{X^2Y^2} \left\{ \frac{X^2 \ln X^2 \mu^2 - Y^2 \ln Y^2 \mu^2}{X^2 - Y^2} - \frac{X^2Y^2}{X \cdot Y} \frac{\ln \frac{X^2}{Y^2}}{X^2 - Y^2} \right\}. \quad (2.16)$$

As the next step both in refs. [21] and [22] one chooses to absorb *all logarithmic terms* in K_{JSJ} into renormalized coupling constant $\alpha_s(\Lambda)$. The scale in $\alpha_s(\Lambda)$ is chosen in such a way as to reproduce all the logarithms in K_{JSJ} *including their finite parts* when expanding $\alpha_s(\Lambda)$ to NLO in $\alpha_s(\mu)$. The two subtraction prescriptions therefore result in different resummations, corresponding to different finite NLO leftovers in K_{JSSJ} (below with a slight abuse of notations we use $\alpha_s(X^2)$ and $\alpha_s(X^{-2})$ interchangeably for the same quantity):

$$K_{JSJ}^B \rightarrow \frac{\alpha_s((X-Y)^2)}{2\pi^2} \frac{X \cdot Y}{X^2Y^2} + \frac{\alpha_s(X^2)}{4\pi^2} \frac{1}{X^2} \left(1 - \frac{\alpha_s((X-Y)^2)}{\alpha_s(Y^2)} \right) + \frac{\alpha_s(Y^2)}{4\pi^2} \frac{1}{Y^2} \left(1 - \frac{\alpha_s((X-Y)^2)}{\alpha_s(X)} \right), \quad (2.17)$$

$$K_{JSJ}^{KW} \rightarrow \frac{1}{2\pi^2} \frac{\alpha_s(X^2)\alpha_s(Y^2)}{\alpha_s(R^2)} \frac{X \cdot Y}{X^2Y^2}, \quad (2.18)$$

where, in the latter, a rather complex scale was introduced in the denominator

$$R^2 = \sqrt{X^2Y^2} \left(\frac{Y^2}{X^2} \right)^{\Theta/2}, \quad \Theta = \frac{X^2 + Y^2}{X^2 - Y^2} - 2 \frac{X^2Y^2}{X \cdot Y} \frac{1}{X^2 - Y^2}. \quad (2.19)$$

³Note that in ref. [21] and ref. [22], the dipole kernel was considered, i.e. the action of JIMWLK on a dipole color singlet. The kernels are trivially related through (see e.g. ref. [8])

$$K_{\text{dipole}}(X, Y) = K_{JSJ}(X, X) + K_{JSJ}(Y, Y) - 2K_{JSJ}(X, Y).$$

⁴In terms of the kernels, the subtraction is given by (see ref. [9])

$$K_{JSJ} \rightarrow K_{JSJ}^{\text{sub}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \equiv K_{JSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}) - \frac{N_c}{2} \int_{\mathbf{z}'} K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') - \frac{1}{2} \int_{\mathbf{z}'} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}').$$

The subtraction prescriptions described above are rather *ad hoc*. Although both render the kernels finite, one can devise many more subtraction prescriptions that achieve the same goal. Critically, there is no obvious reason for the resummation of the finite logarithmic parts of K_{JSJ} that do not contain the UV cutoff $\mu_{\overline{MS}}$ into the running coupling in this or that particular way.

As we explain below this ambiguity is rooted in the fact that the physics associated with the resummation of the finite large logarithmic terms is not related to the physics of running coupling. In the next section, we give a transparent physical argument that the correct way to resum the transverse logarithms is rather different, and in addition to the QCD running coupling it requires the resummation of DGLAP splittings in the JIMWLK Hamiltonian.

3 The dressed gluon and the DGLAP splittings

We start by reexamining the origin of the transverse logarithm in eq. (2.7).

As we alluded to earlier, in this term the coefficient of the UV divergent contribution $\ln \mu_{\overline{MS}}^2$ is $2\beta_0$ (relative to the LO), and thus cannot be simply resummed into the running QCD coupling in the LO term. The reason for this is easy to understand. Recall, that in QCD one defines the running coupling as the matrix element of the interaction Hamiltonian between physical *dressed* gluon states. On the other hand, K_{JSJ} is calculated as the amplitude for production of a *bare* gluon state from the valence charge [9]. To relate this amplitude to the coupling constant one needs to multiply the gluon state by the wave function renormalization factor $\mathcal{Z}^{1/2}(Q^2)$, which at one loop is given by

$$\mathcal{Z}^{1/2}(Q^2) = 1 + \frac{\alpha_s}{8\pi} \beta_0 \ln \frac{Q^2}{\mu^2}. \quad (3.1)$$

The scale Q^2 in eq. (3.1) is an arbitrary normalization scale at which one chooses to define the renormalized gluon field

$$A_\mu^Q(x) = \mathcal{Z}^{-1/2}(Q) A_\mu(x). \quad (3.2)$$

Thus to identify the UV logarithm associated with the running of the coupling one has to express the bare gluon field (or state) in terms of the “renormalized” gluon field via eq. (3.2). Technically within the JIMWLK Hamiltonian this procedure amounts to multiplying the LO kernel by the factor $\mathcal{Z}^{1/2}(Q)$, with the understanding that now the production and scattering refer to the dressed renormalized gluon. This multiplication of the LO kernel absorbs some of the NLO corrections and hence results in the modification of the NLO kernel⁵

$$K_{JSJ} \rightarrow \frac{\alpha_s^2 X \cdot Y}{4\pi^3 X^2 Y^2} \left(\beta_0 \left[\frac{1}{2} \ln[X^2 \mu^2] + \frac{1}{2} \ln[Y^2 \mu^2] - \frac{1}{2} \ln \frac{\mu^2}{Q^2} \right] + \left(\frac{67}{9} - \frac{\pi^2}{3} \right) N_c - \frac{10}{9} N_f \right). \quad (3.3)$$

This expression now clearly contains all the UV logs to be combined into the running coupling.

However now the question arises, what is the fate of the UV divergence in K_{JSSJ} discussed above? The answer is that this divergence should also cancel if one rewrites the JIMWLK

⁵In this discussion we only consider the coefficient of the real term (the JSJ term) for simplicity.

Hamiltonian in terms of the physical dressed gluon amplitudes. The reason this has not been apparent so far, is because in our discussion above we have only included the simple multiplicative wave function renormalization factor into the “redefinition” of the dressed gluon. However at NLO the physical dressed gluon state also contains a two gluon (and a quark-anti quark) component due to gluon splitting, which we are well familiar with as part of the DGLAP evolution. We have to include those additional components in the definition of the dressed gluon state and its scattering amplitude. This is the goal of the rest of this section.

Technically we take the following route. At NLO a gluon can split into a two-gluon (or quark-antiquark) pair. This splitting in collinear setting is described by the DGLAP splitting function. The splitting amplitude is of course well known. The square of this amplitude provides the real contribution to the $g \rightarrow gg$ ($\bar{q}q$) DGLAP splitting functions. The virtual part of the splitting function is given by the total probability that the initial bare gluon state decays into a gg (or $\bar{q}q$) state.

Our goal is to write the scattering amplitude of the dressed state which is resolved with transverse resolution Q . In terms of Wilson renormalization group that would correspond to “integrating out” modes with transverse momenta above the resolution scale Q which defines the “renormalized” dressed states, as in the discussion above. Note, that although we treat the kinematics of the DGLAP splitting exactly, we still assume that the individual bare partons produced as a result are energetic enough so that they scatter on the target eikonally. This is of course the only consistent way to include the scattering of these partons in the JIMWLK evolution.

3.1 The dressed gluon scattering matrix

We consider first the splitting into two gluons, and in this section disregard the possibility of splitting into $\bar{q}q$ for simplicity. It is straightforward to include the quarks, and we will do it in section 5.

Within the DGLAP collinear splitting approximation a gluon splits into a gg pair with the probability given by the real part of the gluon to gluon splitting function P_{gg} . The S -matrix of the gluon state dressed at order α_s with resolution Q then equals

$$\begin{aligned} \mathbb{S}_Q^{ab}(\mathbf{z}) = & S^{ab}(\mathbf{z}) + \frac{\alpha_s}{2\pi^2} \int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+} \left(\xi^2 + (1-\xi)^2 + \xi^2(1-\xi)^2 \right) \\ & \times \int_{\mu^{-1} < |Z| < Q^{-1}} \frac{d^2 Z}{Z^2} \left[D^{ab}(\mathbf{z} + (1-\xi)Z, \mathbf{z} - \xi Z) - N_c S^{ab}(\mathbf{z}) \right] \end{aligned} \quad (3.4)$$

The first term (D) in the second line in (3.4) represents the real emission, while the second term is the virtual correction which conserves the total normalization of the state. This expression for the S -matrix follows from the kinematics of gluon splitting in coordinate space, where the distance between the daughter gluons and the parent gluon is determined by the longitudinal momentum fraction ξ carried by each gluon. Note that when coordinates in D equal each other it simplifies to

$$D^{ab}(\mathbf{z}, \mathbf{z}) = \text{Tr} \left[\underbrace{S^+(\mathbf{z}) T^a S(\mathbf{z}) T^b}_{S_{aa'} T^{a'}} \right] = N_c S^{ab}(\mathbf{z}). \quad (3.5)$$

Importantly the plus prescription on the integral is defined with respect to both poles, at $\xi = 0$ and $\xi = 1$, that is

$$\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+} f(\xi) = \int_0^1 d\xi \left(\frac{1}{\xi(1-\xi)} f(\xi) - \frac{1}{\xi} f(0) - \frac{1}{1-\xi} f(1) \right). \quad (3.6)$$

This is in contradistinction to the usual DGLAP splitting function, where only the $\xi = 0$ pole is treated with the plus prescription. The reason for this double subtraction of both poles from the splitting function is that the pole at $\xi = 1$ is already included in the energy evolution generated by the LO JIMWLK equation (in ref. [9] these poles are referred to as LO^2 terms because they arise from iterations of the LO JIMWLK). It is only the finite remainder of the splitting function that has to be included in the “collinear splitting term” we currently consider. This remainder completes the low x truncation of the splitting function used in the standard JIMWLK formulation to the full splitting function necessary to properly account for the correct DGLAP evolution. It is crucial to recognize this point, and we will come back to it in the section 6. With this caveat in mind, in the rest of this paper we will refer to the splitting encoded in eq. (3.4) as the DGLAP splitting.

As we will show later, this double subtraction indeed correctly accounts for the large transverse logarithm in K_{JSSJ} .

With the definition eq. (3.6) we have

$$\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+} \left(\xi^2 + (1-\xi)^2 + \xi^2(1-\xi)^2 \right) = -\frac{11}{6} = -\frac{\beta_0^g}{2N_c} \quad (3.7)$$

which is proportional to the one loop gluon contribution to the β -function, $\beta_0^g = \frac{11N_c}{3}$.

The next step is now clear. We should perturbatively re-express all the eikonal factors in the Hamiltonian in terms of $\mathbb{S}_Q$, using

$$\begin{aligned} S^{ab}(\mathbf{z}) &= \mathbb{S}_Q^{ab}(\mathbf{z}) - \frac{\alpha_s}{2\pi^2} \int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+} \left(\xi^2 + (1-\xi)^2 + \xi^2(1-\xi)^2 \right) \\ &\quad \times \int_{|Z| < Q^{-1}} \frac{d^2 Z}{Z^2} \left(\mathbb{D}_Q^{ab}(\mathbf{z} + (1-\xi)Z, \mathbf{z} - \xi Z) - N_c \mathbb{S}_Q^{ab}(\mathbf{z}) \right), \end{aligned} \quad (3.8)$$

with

$$\mathbb{D}_Q^{ab}(\mathbf{z}_1, \mathbf{z}_2) = \text{Tr}[T^a \mathbb{S}_Q(\mathbf{z}_1) T^b \mathbb{S}_Q^\dagger(\mathbf{z}_2)]. \quad (3.9)$$

We note that

$$\frac{\alpha_s N_c}{2\pi^2} \int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+} \left(\xi^2 + (1-\xi)^2 + \xi^2(1-\xi)^2 \right) \int_{|Z| < Q^{-1}} \frac{d^2 Z}{Z^2} \mathbb{S}_{ab}(\mathbf{z}) = -\frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{Q^2} \mathbb{S}^{ab}(\mathbf{z}). \quad (3.10)$$

At this point, we will simplify our derivations in the following by neglecting the ξ dependence in the coordinates of the Wilson lines in eq. (3.4). Recall that $\xi = 0$ and $\xi = 1$ do not play any special role in the integration over ξ due to the double plus prescription. It is clear that with leading logarithmic accuracy in Q the exact location of the two points $\mathbf{z}_1$ and $\mathbf{z}_2$ within the integration region in eq. (3.4) does not matter as long as the distance

between them remains fixed. Since the distance between the two points in eq. (3.4) does not depend on ξ , and since in this paper we only concern ourselves with leading logarithmic contributions, we will simply set ξ to a fixed value in the arguments of the Wilson lines. To preserve the symmetry between ξ and $1 - \xi$ we substitute $\xi = 1/2$ in the coordinates of D in eq. (3.4) and onward. This allows us to perform explicitly the integral over ξ using eq. (3.7). We stress that this approximation is not mandatory within our approach, and can be easily relaxed without altering logarithmic in Q terms.

Thus from now on we use

$$\mathbb{S}_Q^{ab}(\mathbf{z}) = \left[1 + \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{Q^2} \right] S^{ab}(\mathbf{z}) - \frac{\alpha_s \beta_0^g}{4\pi^2 N_c} \int_{|Z| < Q^{-1}} \frac{d^2 Z}{Z^2} D^{ab}(\mathbf{z} + Z/2, \mathbf{z} - Z/2) \quad (3.11)$$

and

$$S^{ab}(\mathbf{z}) = \left[1 - \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{Q^2} \right] \mathbb{S}_Q^{ab}(\mathbf{z}) + \frac{\alpha_s \beta_0^g}{4\pi^2 N_c} \mathbb{D}_Q^{ab}(\mathbf{z}),$$

where we have introduced the notation

$$\mathbb{D}_Q^{ab}(\mathbf{z}) \equiv \int_{|Z| < Q^{-1}} \frac{d^2 Z}{Z^2} \mathbb{D}^{ab}(\mathbf{z} + Z/2, \mathbf{z} - Z/2). \quad (3.12)$$

Since at the moment, we work at fixed order α_s^2 , the substitution of $\mathbb{S}_Q$ should be done perturbatively, which in practice means that the $O(\alpha_s)$ term in eq. (3.8) should only be kept at $H_{\text{JIMWLK}}^{\text{LO}}$, while in $H_{\text{JIMWLK}}^{\text{NLO}}$ all factors of S should be substituted by $\mathbb{S}_Q$.

Let us first consider the real term in $H_{\text{JIMWLK}}^{\text{LO}}$. Substituting eq. (3.8) we obtain

$$\begin{aligned} & \frac{\alpha_s}{2\pi^2} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} 2J_L^a(\mathbf{x}) S^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) = \\ & \frac{\alpha_s}{2\pi^2} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} 2J_L^a(\mathbf{x}) \mathbb{S}_Q^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \left[1 - \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{Q^2} \right] \\ & + \frac{\alpha_s^2 \beta_0^g}{8\pi^4 N_c} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} 2J_L^a(\mathbf{x}) J_R^b(\mathbf{y}) \mathbb{D}_Q^{ab}(\mathbf{z}). \end{aligned} \quad (3.13)$$

We observe that the logarithmic term linear in $\mathbb{S}$ in eq. (3.13) combines with the real part of the kernel K_{JSJ} to yield precisely the transverse logarithm in eq. (3.3). In addition the quadratic in $\mathbb{S}$ term in eq. (3.13) combines with the real term of K_{JSJ} in eq. (2.9), so that the UV divergence at $\mathbf{z} - \mathbf{z}' \rightarrow 0$ cancels. This is precisely what we set out to achieve by introducing the dressed gluon state.

It is thus very convenient to re-express the JIMWLK Hamiltonian in terms of the dressed gluon operators, as it achieves the cancellation of the UV divergence just like various subtractions discussed above, but with clear physical justification.

There is one subtlety in the procedure just described. What we did is to introduce a gluon state “dressed” by the DGLAP emissions with resolution Q^2 irrespective of whether the gluon in question is far away or close to the valence charge that it was emitted from. Physically this is not right. Proximity to the emitter provides an infrared cutoff on possible

DGLAP splittings. The collinear logarithm in the standard DGLAP cascade is formally collinearly divergent for large distance or small transverse momentum emissions. This collinear divergence however is there only as long as one assumes that the parton splits completely independently of all other partons present in the hadronic wave function. The moment one accounts for the fact that the gluon is not isolated in space, but is a part of hadronic wave function with finite size R , the collinear divergence disappears. This size R can be a small perturbative scale for a heavy quarkonium, or a nonperturbative scale of order $\Lambda_{\text{QCD}}^{-1}$ for a nucleon etc. Either way the presence of this scale regulates the DGLAP emissions in the IR by cutting off the phase space for splitting into pairs of the size larger than R .

The situation is very similar in the case at hand. To see this we should examine K_{JSSJ} which reflects the probability of DGLAP splittings present in JIMWLK at NLO. The probability of emission from a single source at point $\mathbf{x}$ can be read off the integrand of K_{JSSJ} by setting $\mathbf{y} = \mathbf{x}$. Setting $\mathbf{y} = \mathbf{x}$ and examining K_{JSSJ} we find that for $Z^2 \ll X^2$, the kernel behaves as $1/Z^2$. Integration over Z in this range produces the UV divergence we discussed. On the other hand, when the size of the pair is large relative to the distance between the emitter and the center of mass of the pair, $Z^2 \gg (X + \frac{Z}{2})^2$ we find that the kernel (probability of splitting) behaves as $(X + \frac{Z}{2})^2/Z^6$. One can see this also from eq. (2.12) which makes it explicit that the integration over the size of the gluon pair is cut off by the distance to the closest source in the emission amplitude. This tells us that K_{JSSJ} does not allow DGLAP splittings of a gluon into pairs of transverse size larger than the distance between the gluon in question and its “valence” emitter.

In this discussion, the hadronic size R is directly analogous to the distance $|X - Z/2|$ between the gluon and the closest parton in the wave function, most of the time, this closest parton being the emitter of the gluon in question.

The procedure outlined in eq. (3.13) does not differentiate between the two regimes $Q^{-2} < (X - Z/2)^2$ and $Q^{-2} > (X - Z/2)^2$ and therefore does not properly take into account the suppression discussed above. Mathematically the consequence is that, although when the last term in eq. (3.13) is combined with the K_{JSSJ} term in NLO JIMWLK it does get rid of the $Z \rightarrow 0$ divergence, it produces a logarithmic term $\ln(X^2 Q^2)$, for all values of X , rather than only for $X^2 > Q^{-2}$. This spurious logarithm at small X may be large and its introduction should be avoided.

It is not hard to modify our strategy slightly in order to avoid this difficulty. Let us define the resolution scale that depends on the position of the gluon relative to the closest valence charge. In the context of the real term in the LO JIMWLK, in terms of the coordinates $\mathbf{x}, \mathbf{y}, \mathbf{z}$ we introduce:

$$\bar{Q}^2 = \max \left\{ Q^2, \frac{1}{X^2}, \frac{1}{Y^2} \right\}. \quad (3.14)$$

We now express $S(\mathbf{z})$ in terms of $\mathbb{S}_{\bar{Q}}(\mathbf{z})$ in $H_{\text{JIMWLK}}^{\text{LO}}$.

3.2 The reorganized Hamiltonian

The resulting Hamiltonian can be organized in the convenient way into two parts H_Q^{JSJ} and H_Q^{JSSJ} . To do so we first introduce an auxiliary scale

$$\tilde{Q}^2 = \max \left\{ \frac{1}{X^2}, \frac{1}{Y^2} \right\}. \quad (3.15)$$

We now split the virtual term in $H_{\text{JIMWLK}}^{\text{LO}}$ as follows

$$\begin{aligned} & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] = \\ & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \left\{ \left[1 - \frac{\alpha_s \beta_0^g}{8\pi} (\ln X^2 \mu^2 + \ln Y^2 \mu^2) + \frac{\alpha_s \beta_0^g}{8\pi} (\ln X^2 \tilde{Q}^2 + \ln Y^2 \tilde{Q}^2) \right] [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] \right. \\ & \left. + \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{\tilde{Q}^2} [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] \right\}. \end{aligned} \quad (3.16)$$

The logic of this splitting will become clear in a short while.⁶

It is convenient to reorganize the Hamiltonian as

$$H_{\text{JIMWLK}} = H_Q^{\text{JSJ}} + H_Q^{\text{JSSJ}} + \dots \quad (3.17)$$

where

$$\begin{aligned} H_Q^{\text{JSJ}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{\alpha_s}{2\pi^2} \frac{X \cdot Y}{X^2 Y^2} \left\{ \left(1 + \frac{\alpha_s \beta_0^g}{8\pi} [\ln X^2 \mu^2 + \ln Y^2 \mu^2] \right) [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2J_L^a(\mathbf{x}) \mathbb{S}_{\tilde{Q}}^{ab}(\mathbf{z}) J_R^b(\mathbf{y})] \right. \\ & + \frac{\alpha_s \beta_0^g}{8\pi} [(\ln X^2 \tilde{Q}^2 + \ln Y^2 \tilde{Q}^2) [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] \\ & \left. - 2(\ln X^2 \tilde{Q}^2 + \ln Y^2 \tilde{Q}^2) J_L^a(\mathbf{x}) \mathbb{S}_{\tilde{Q}}^{ab}(\mathbf{z}) J_R^b(\mathbf{y})] \right\}. \end{aligned} \quad (3.18)$$

$$\begin{aligned} H_Q^{\text{JSSJ}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{\text{JSSJ}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') J_L^a(\mathbf{x}) \mathbb{D}_{\tilde{Q}}^{ad}(\mathbf{z}, \mathbf{z}') J_R^d(\mathbf{y}) - \frac{\alpha_s^2 \beta_0^g}{4\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \mathbb{D}_{\tilde{Q}}^{ab}(\mathbf{z}) J_L^a(\mathbf{x}) J_R^b(\mathbf{y}) \\ & - \left[\frac{N_c}{2} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{\text{JSSJ}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') - \frac{\alpha_s^2 \beta_0^g}{8\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \ln \frac{\mu^2}{\tilde{Q}^2} \right] [J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) + J_L^a(\mathbf{x}) J_L^a(\mathbf{y})]. \end{aligned} \quad (3.19)$$

Note that eq. (3.18) exhibits explicitly real-virtual cancellations. As $X \rightarrow 0$, we have $\tilde{Q} = \bar{Q} = 1/|X|$, $\mathbb{S}_{\tilde{Q}}(\mathbf{z}) = S(\mathbf{z}) \simeq S(\mathbf{x})$, and therefore the real and virtual terms cancel for points $\mathbf{z}$ such that $|X| < Q_T^{-1}$.

Eq. (3.19) has some appealing properties. First, it is UV finite. The integral over Z in the definition of $\mathbb{D}_{\tilde{Q}}(\mathbf{z})$ cancels the UV divergence in the first term. The UV cancellation also occurs between the two virtual terms in the second line of eq. (3.19).

Moreover, eq. (3.19) contains no large finite logarithms if the target is dense, i.e. if Q_T is in fact a saturation momentum.

For the real term we argue as follows. The JIMWLK approach is valid for scattering of a dilute projectile on a dense target. Let us assume that the target field distribution is characterized by a saturation momentum Q_T . Two eikonal scattering factors within a

⁶One may wonder if one needs also to express the right and left rotation operator in terms of rotations associated with the dressed S -matrices $\mathbb{S}_Q$. It turns out, however, that even if done, this procedure is purely perturbative and does not contribute to the resummation of large logarithms, see appendix A. It is therefore optional, and in a sense can be regarded as simply “fixing the factorization scheme” for DGLAP resummation. For simplicity, we choose not to do it here.

transverse distance $|Z| \ll (Q_T)^{-1}$ are practically the same, while if $|Z| \gg (Q_T)^{-1}$, the two eikonal factors fluctuate independently and therefore the product averages to zero. As we will explain shortly, we are interested in $Q < Q_T$. Under these assumptions, the integral over the region $|Z| < (Q_T)^{-1}$ cancels between the first two terms in the first line by the same token as does the UV cutoff μ . For the distances $Q^{-1} > |Z| > (Q_T)^{-1}$ there is no algebraic cancellation, however each one of the terms is small since it contains two eikonal factors separated by a large distance, and this product averages to zero. This range of Z therefore does not give a logarithmic contribution to the hadronic S -matrix. Thus the contribution only comes from a small region in the phase space $|Z| \sim (Q_T)^{-1}$ and cannot be logarithmically large for any $Q < Q_T$.

As for the virtual term, integrating K_{JSSJ} over Z , for $X^2 < Y^2$ one obtains $\ln \mu^2 X^2$, since the integration is cut off in the IR by $|X|$.⁷ The coefficient of this logarithm is exactly the same as in the second virtual term, and the two logarithms cancel between the two virtual terms.

One could worry that eq. (3.18) contains $\ln \frac{Y^2}{X^2}$, which for $Y^2 \gg X^2$ is large and may require special attention. However this potential logarithm is harmless. It arises in the term responsible for emission of a virtual gluon very close to one of the valence charges in the wave function. The probability of emission of a gluon at any point $\mathbf{z}$ is given by the square of the sum of the emission amplitudes from all the valence charge. If $\mathbf{z}$ is much closer to one valence charge than to any of the others, the emission amplitude is dominated by the amplitude of emission from this particular valence charge, and the contribution of the other charges can be neglected. The same of course is also true for the conjugate amplitude. Thus the conjugate amplitude of such (virtual or real) emission is dominated by the emission from the same charge as in the amplitude. In other words, if $\mathbf{z}$ is very close to a given charge $\mathbf{x}$, the integral over Y in the conjugate amplitude will be dominated by $Y \approx X$ simply due to the structure of the WW field in the kernel. Thus one does not expect the region of $Y^2 \gg X^2$ to be of any practical importance and $\ln \left(\frac{Y^2}{X^2} \right)$ should not be considered large.

So far we have just rewritten the NLO JIMWLK Hamiltonian in a convenient way. As argued above, the term H_Q^{JSSJ} is UV finite and does not contain any large logarithms. All the large logarithms are now explicit in H_Q^{JSJ} . These logarithms come in two types. The first one is the UV divergent term in the first term of H_Q^{JSSJ} , and the second one is the finite log of the type $\ln(Y^2 Q^2)$ in the last term.

4 The resummation

The two large logarithms, $\ln(X^2 \mu^2)$ and $\ln(X^2 Q^2)$ need to be resummed. The physical origin of the two is quite clear: the UV logarithm is associated with setting the scale of the running coupling, while the Q^2 dependent logarithm is associated with setting the resolution scale for DGLAP splittings.

4.1 The running coupling

The resummation of the UV logarithm is equivalent to choosing the scale in the running coupling. The simplest and the most natural scale choice for a Coulomb field at the distance

⁷The cutoff is actually $|Y|$ if $|Y| < |X|$, but these are mapped into each other by a simple change of variables $\mathbf{x} \rightarrow \mathbf{y}$.

X from the source that creates it, is of course just X itself. That is how things work in the textbook examples in QED and QCD. Indeed, eq. (3.18) suggests precisely this simple scale choice. In principle of course there are infinite number of scale choices that reproduce the NLO expression upon expansion. For example, any combination of running couplings of the form

$$\frac{\alpha_s^\lambda(X^{-2}) \alpha_s^\lambda(Y^{-2}) \alpha_s^{1-2\lambda}((XY)^{-1})}{2\pi^2} \quad (4.1)$$

with λ — an arbitrary real number, yields the same $O(g^4)$ expansion in the bare coupling. The choice $\lambda = 1$ leads to the “triumvirate” form, see refs. [30–32]. The most natural choice that we adopt here is $\lambda = 1/2$. Given that the LO JIMWLK describes a simple emission of a gluon at a fixed transverse distance from the valence source, we do not see any physical reason to choose anything but this distance as a physical renormalization scale.

Defining as usual, the running QCD coupling constant $\alpha_s(\mu^2)$ as the solution of the one loop renormalization group equation, we can rewrite eq. (3.18) in terms of the effective QCD charge $g(\mu^2)$, defined as always via $\alpha_s(\mu^2) = g^2(\mu^2)/4\pi$,

$$\begin{aligned} H_Q^{\mathbb{S}J} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{g(X^{-2})g(Y^{-2})}{8\pi^3} \frac{X \cdot Y}{X^2 Y^2} \\ & \times \left\{ \left(1 + \frac{\alpha_s \beta_0^g}{8\pi} \left[\ln X^2 \tilde{Q}^2 + \ln Y^2 \tilde{Q}^2 \right] \right) [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] \right. \\ & \left. - 2 \left(1 + \frac{\alpha_s \beta_0^g}{8\pi} \left[\ln X^2 \tilde{Q}^2 + \ln Y^2 \tilde{Q}^2 \right] \right) J_L^a(\mathbf{x}) \mathbb{S}_Q^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right\}. \end{aligned} \quad (4.2)$$

4.2 The DGLAP corrections

We rewrote the Hamiltonian in terms of $\mathbb{S}_Q$ in the fixed order in perturbation theory. This eliminated the original UV divergences, but introduced dependence on an arbitrary scale Q . This scale plays the role identical to that of the factorization scale in the standard collinear treatment of DIS. The physics (the JIMWLK Hamiltonian) eventually does not depend on this scale, but choosing a convenient value for it has the potential of simplifying calculations considerably.

Varying the scale Q does not change physics, but shifts large perturbative corrections between the Q dependence of the Hamiltonian and the Q dependence of the S -matrix of the dressed gluon. In particular, if we can choose the scale Q such that the logarithm in eq. (4.2) is not large, the Hamiltonian has a simple form that only perturbatively differs from the LO Hamiltonian (modulo the running coupling effects). The price we pay is that at this scale the dressed gluon is very different from the bare gluon state, as its S -matrix, $\mathbb{S}_Q(\mathbf{z})$ is very different from the original eikonal factor $S(\mathbf{z})$.

This is closely analogous to DIS or jet production, where changing the resolution scale shifts large corrections between the PDF and the hard part of the scattering. In fact, the Hamiltonian H_{JIMWLK} is very similar to TMD PDF: in the leading order, the square of the WW field is precisely the number of soft gluons at point $\mathbf{z}$ emitted from the valence charges. The scattering matrix, on the other hand, is similar to the hard part. With high resolution $\mathbb{S}_{Q=\mu} = S$, which is analogous to the situation in DIS where choosing the factorization scale close to UV cutoff eliminates large corrections from the hard scattering part. While for lower resolution scale $Q \ll \mu$, the dressed gluon scattering matrix, similarly to the hard scattering part, acquires large corrections.

In DIS and similar processes, the optimal choice of factorization scale is determined by the external scale furnished by the observable, e.g. the momentum transfer Q^2 . In the case of JIMWLK evolution, as discussed in the introduction the role of this scale is played by the saturation momentum of the target, Q_T . This is perfectly natural since the target saturation momentum is a scale external to the evolution of the projectile wave function resulting in the JIMWLK equation. Since the eikonal scattering factors $S(\mathbf{z}_1)$ and $S(\mathbf{z}_2)$ do not differ from each other as long as $|\mathbf{z}_1 - \mathbf{z}_2| < Q_T^{-1}$, for any choice of factorization scale $Q^2 \geq Q_T^2$ the scattering matrix of a dressed gluon (which in coordinate space has a transverse size $\sim Q^{-1}$) is the same as the bare eikonal factor S up to small perturbative corrections. The most convenient choice of Q is, therefore, $Q^2 = Q_T^2$, since it eliminates the necessity to evolve the dressed gluon density matrix. The Hamiltonian of course, has to be evolved to this scale, in a complete analogy to the DGLAP evolution of parton PDF's. Calculating this evolution is our next goal.

Before doing so, however, we must generalize the evolution of the dressed gluon scattering matrix to account for possible further gluon emissions, which are essential if the evolution interval is logarithmically large.

Including additional DGLAP emissions in the dressed gluon scattering matrix is quite straightforward. We go back to eq. (3.11) and write it in the differential form. We also use the fact that for an infinitesimal variation of Q , the change in the scattering matrix on the right-hand side of the equation can be neglected. We thus arrive at the DGLAP equation⁸

$$\frac{\partial}{\partial \ln Q^2} \mathbb{S}_Q^{ab}(\mathbf{z}) = -\frac{\alpha_s \beta_0^g}{4\pi} \left[\mathbb{S}_Q^{ab}(\mathbf{z}) - \frac{1}{N_c} \int \frac{d\phi}{2\pi} \left(\mathbb{D}_Q^{ab} \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi, \mathbf{z} - \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) \right) \right] \quad (4.3)$$

where $\mathbf{e}_\phi$ is the unit vector in the tangential direction ϕ . The solution of this equation, in principle, sums all DGLAP emissions of the initial gluon within the transverse radius Q^{-1} .

The resummation of the DGLAP logarithms is achieved by requiring that the resummed Hamiltonian does not depend on the arbitrary scale Q that we have introduced above:

$$\frac{d}{d \ln Q^2} H_{\text{JIMWLK}} = \frac{\partial H}{\partial \ln Q^2} + \int_u \left[\frac{\delta H}{\delta \mathbb{S}_Q(u)} \frac{\partial \mathbb{S}_Q(u)}{\partial \ln Q^2} \right] = 0. \quad (4.4)$$

To resum the DGLAP logarithms we need to solve the RG equation for the Hamiltonian. We will do this in two simple case below. But first, we have to understand the initial condition and the interval of the evolution in Q over which we need to evolve.

As discussed above, the target is characterized by the correlation length Q_T^{-1} .⁹ Clearly, if $Q > Q_T$, the DGLAP cascade we resum in $\mathbb{S}_Q$ is localized inside the single correlation length, the scattering amplitude of all gluons (and quarks) inside the cascade are the same, the cascade is not resolved by scattering on the target, and its scattering amplitude is equal

⁸In eq. (4.3) the object for which the equation is solved does not depend on the longitudinal momentum fraction ξ . This of course is the consequence of the high energy limit, in which the scattering matrix of a gluon in eikonal approximation does not depend on its longitudinal momentum. This is also the reason why only the ξ -integral of the splitting function, and not the splitting function itself enters eq. (4.3). To relax this approximation one would have to go beyond eikonal, and assign additional longitudinal momentum label to the S -matrix. This would lead to a modification of eq. (4.3) where S itself would now depend on the longitudinal momentum fraction, and would make the equation visually more similar to the usual form of the DGLAP equation. This would be the right step in the direction of including subeikonal corrections, but is far beyond the purpose of the present work.

⁹Solution of the LO JIMWLK shows that in the saturated regime, the color correlation length is about 2.5 times larger than the inverse saturation momentum [33].

to the single eikonal factor S . Thus it is useless to consider Q greater than Q_T . Therefore the evolution in Q starts from $Q = Q_T$ with the initial condition

$$\mathbb{S}_{Q_T}(\mathbf{x}) = S(\mathbf{x}). \quad (4.5)$$

As for the interval of the evolution, clearly we should stop the evolution at the value of Q for which the Hamiltonian, when expressed in terms of $\mathbb{S}_Q$ does not contain large logarithms. At this point H_{JIMWLK} will be a nice function of $\mathbb{S}_Q$ with coefficients in the expansion in powers of α_s being all numbers of order unity not enhanced by large transverse logs.

With this in mind let us examine the expressions eq. (3.18), (3.19). The large logarithms present there are of the type $\ln X^2 Q^2$ or $\ln Y^2 Q^2$. Typically the distance between the emitter and the emission point of the gluon, $|X|$ (and $|Y|$) is of order of the average distance between the charges in the projectile wave function. This in turn is precisely the scale Q_P that we have introduced previously. Thus clearly if we choose $Q = Q_P$, no large transverse logarithms will be present in the Hamiltonian. Notice that the scale Q_P is dynamical, that is it evolves with rapidity.

We thus conclude that we should evolve from $Q = Q_T$ down to $Q = Q_P$. In fact, we can do a little better than that. We can in principle, choose the interval of the evolution for $\mathbb{S}_Q(\mathbf{z})$ to depend on X and Y . This makes sense, especially for points $\mathbf{z}$ which are significantly closer to some valence charges than the average distance. Again, examining eqs. (3.18) and (3.19) we see that the optimal choice for the end point of the evolution is simply $Q(\mathbf{z}) = \tilde{Q}$. This choice is naturally of order Q_P for most of the emissions but avoids generating spurious logarithms for emission points too close to one of the emitters.

Thus the resummation of DGLAP logs is achieved by the following straightforward procedure. We solve the RG equation (4.3) for $\mathbb{S}(\mathbf{z})$ with the initial condition eq. (4.5) to find $\mathbb{S}_{\tilde{Q}}(\mathbf{z})$ (for fixed $\mathbf{x}, \mathbf{y}$). We then substitute the solution into eq. (3.18), (3.19). With $\mathbb{S}_{\tilde{Q}}$ expressed in terms of S via the solution of RG equations, this yields the resummed Hamiltonian which satisfies eq. (4.4) and is free of large logarithms. Keeping all terms in eqs. (3.18) and (3.19) gives, in addition to the resummation, some genuine small $O(\alpha_s^2)$ corrections. In the following, we will not keep these corrections for simplicity, which is then equivalent to only working with the (running coupling) LO Hamiltonian where $S(\mathbf{z})$ is substituted by $\mathbb{S}_{\tilde{Q}}(\mathbf{z})$.

The solution of the RG equation eq. (4.3) in general is not trivial, but under reasonable assumptions, the equation can be solved analytically in two limits: the dilute limit and the saturation limit.

4.3 Resummation in the dilute limit — the BFKL

Let us first consider the situation where the projectile and the target are both dilute. In this case, the S matrix is close to unity

$$S(\mathbf{x}) = 1 + iT^a \alpha^a(\mathbf{x}), \quad \mathbb{S}_Q(\mathbf{x}) = 1 + iT^a \alpha_Q^a(\mathbf{x}). \quad (4.6)$$

To linear order in α_Q the RG equation (4.3) becomes

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{z}) = \frac{\alpha_s \beta_0^g}{8\pi^2} \int d\phi \left(\alpha_Q^c \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) - \alpha_Q^c(\mathbf{z}) \right). \quad (4.7)$$

Fourier transforming this into momentum space, we obtain

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{p}) = \frac{\alpha_s \beta_0^g}{8\pi^2} \int d\phi \left(e^{i\frac{1}{2}Q^{-1}\mathbf{p} \cdot \mathbf{e}_\phi} - 1 \right) \alpha_Q^c(\mathbf{p}) = R(p, Q) \alpha_Q^c(\mathbf{p}) \quad (4.8)$$

with

$$R(p, Q) = \frac{\alpha_s \beta_0^g}{4\pi} \left[J_0 \left(\frac{p}{2Q} \right) - 1 \right]. \quad (4.9)$$

The solution is

$$\alpha_Q^c(\mathbf{p}) = \exp \left[- \int_Q^{Q_T} \frac{dQ^2}{Q^2} R(p, Q) \right] \alpha^c(\mathbf{p}). \quad (4.10)$$

The result of the integral is qualitatively easy to understand. Let us rewrite the exponential factor in eq. (4.10) in the form

$$\int_Q^{Q_T} \frac{dQ^2}{Q^2} R(Q, p) = \frac{\alpha_s \beta_0^g}{4\pi^2} \int_{|\mathbf{x}_Q|=\frac{1}{2Q_T}}^{|\mathbf{x}_Q|=\frac{1}{2Q}} \frac{d^2 \mathbf{x}_Q}{\mathbf{x}_Q^2} \left(e^{i\mathbf{p} \cdot \mathbf{x}_Q} - 1 \right) \quad (4.11)$$

where we introduced a two dimensional vector $\mathbf{x}_Q \equiv \frac{1}{2Q} \mathbf{e}_\phi$.

For p very small, $p \ll Q$, the two terms in eq. (4.11) cancel, and the integral vanishes. For p very large, $p \gg Q_T$ the first term vanishes upon integration. Thus for such p the integral is given by the second term

$$- \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{Q_T^2}{Q^2}. \quad (4.12)$$

For intermediate values $2Q < p < 2Q_T$ the two terms essentially cancel in the range $|\mathbf{x}_Q| < 1/p$, whereas for $|\mathbf{x}_Q| > 1/p$ only the unity contributes. A good approximation to the result is then

$$- \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{p^2}{4Q^2}. \quad (4.13)$$

With the logarithmic accuracy, the factor of 4 can be ignored under the logarithm, which we will consistently do in what follows.

Thus qualitatively, we have

$$\int_Q^{Q_T} \frac{dQ^2}{Q^2} R(Q, p) \approx - \frac{\alpha_s \beta_0^g}{4\pi} \left[\ln \frac{p^2}{Q^2} \theta(Q_T^2 - p^2) \theta(p^2 - Q^2) + \ln \frac{Q_T^2}{Q^2} \theta(p^2 - Q_T^2) \right]. \quad (4.14)$$

In momentum space the solution for the field α_Q is approximated by

$$\alpha_Q^a(\mathbf{p}) = \left[\theta(Q^2 - p^2) + \left(\frac{p^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \theta(Q_T^2 - p^2) \theta(p^2 - Q^2) + \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \theta(p^2 - Q_T^2) \right] \times \alpha^a(\mathbf{p}). \quad (4.15)$$

The above is obtained at a fixed coupling constant. Of course it is clear that higher order corrections to JIMWLK, in addition to the contribution resummed by eq. (4.15), will also bring in terms associated with the running of the coupling at the DGLAP splitting vertices. This is in addition to the running of the coupling in the LO JIMWLK discussed in the beginning of this section. Physically, it is obvious that the scale in the coupling that enters the splitting must be taken as Q . The argument here is standard and is identical to the one that we used to determine the scale of the coupling constant in eq. (4.2): the DGLAP splitting at resolution Q is emission of a gluon at the distance $1/Q$ from the emitter and this distance determines the scale of the coupling constant. This is of course what is routinely done in the standard solution of DGLAP equation. Thus to include these running coupling corrections we should take

$$\alpha_s(Q) = \frac{4\pi}{\beta_0^g} \frac{1}{\log \frac{Q^2}{\Lambda_{\text{QCD}}^2}} \quad (4.16)$$

in eqs. (4.8), (4.9). The r.h.s. of eq. (4.14) is then modified to

$$\approx - \left[\ln \frac{\frac{p^2}{\Lambda_{\text{QCD}}^2}}{\ln \frac{Q^2}{\Lambda_{\text{QCD}}^2}} \theta(Q_T^2 - p^2) \theta(p^2 - Q^2) + \ln \frac{\frac{Q_T^2}{\Lambda_{\text{QCD}}^2}}{\ln \frac{Q^2}{\Lambda_{\text{QCD}}^2}} \theta(p^2 - Q_T^2) \right]. \quad (4.17)$$

This modification is formally of the higher order in α_s and we will not consider it further here. We only note that including running α_s as in eq. (4.17) does not present any fundamental difficulties and follows the same route as accounting for the running of the coupling in the standard DGLAP evolution.

Since the JIMWLK Hamiltonian is customarily written in the coordinate space, we need to Fourier transform eq. (4.15) back into coordinate space. To that end we need to compute the 2-d Fourier transform

$$G_Q(\mathbf{x}) \equiv \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}} \int_Q^{Q_T} \frac{dQ^2}{Q^2} R(Q, p) \simeq \int \frac{d^2p}{(2\pi)^2} e^{i\mathbf{p}\cdot\mathbf{x}} \times \left[\theta(Q^2 - p^2) + \left(\frac{p^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \theta(Q_T^2 - p^2) \theta(p^2 - Q^2) + \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \theta(p^2 - Q_T^2) \right]. \quad (4.18)$$

This is relatively straightforward. The Fourier transform can be calculated exactly in terms of a hyper geometric function. However, given the fact that our calculation in the momentum space itself was approximate, it is much more instructive to write the result in the approximate form, which however exhibits all the important properties of the solution. The details of this derivation are present in appendix B. The final result is (we partially expand to order α_s , but keep all powers of $\alpha_s \ln \frac{Q_T^2}{Q^2}$)

- For small $|\mathbf{x}|$: $|\mathbf{x}| < \frac{1}{Q_T}$

$$G_Q(\mathbf{x}) = \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \delta^{(2)}(\mathbf{x}) + \frac{\alpha_s \beta_0^g}{4\pi} \frac{Q^2}{\pi} \left[1 - \left(\frac{Q_T^2}{Q^2} \right)^{1 + \frac{\alpha_s \beta_0^g}{4\pi}} \right], \quad (4.19)$$

- For intermediate $|\mathbf{x}|$: $\frac{1}{Q_T} < |\mathbf{x}| < \frac{1}{Q}$

$$G_Q(\mathbf{x}) = \frac{\alpha_s \beta_0}{4\pi} \frac{Q^2}{\pi} \left[1 - \left(\frac{1}{Q^2 x^2} \right)^{1 + \frac{\alpha_s \beta_0^g}{4\pi}} \right], \quad (4.20)$$

- At large $|\mathbf{x}|$, $|\mathbf{x}| > \frac{1}{Q}$ the function behaves as

$$G_Q(\mathbf{x}) \propto \frac{\alpha_s \beta_0^g}{4\pi} \frac{1}{(Q|\mathbf{x}|)^{5/2}} \rightarrow 0. \quad (4.21)$$

One can check that the function $G_Q(\mathbf{x})$ satisfies (up to corrections of order $O(\alpha_s^2)$) the “sum rule” $\int d^2\mathbf{x} G_Q(\mathbf{x}) = 1$ as required by eq. (4.18). An additional consistency check is that $G_Q(\mathbf{x}) \rightarrow \delta^2(\mathbf{x})$ when either $\alpha_s = 0$ or $Q = Q_T$.

We can now write the solution for α_Q as

$$\begin{aligned} \alpha_Q(\mathbf{z}) &= \int d^2\mathbf{x} G_Q(\mathbf{z} - \mathbf{x}) \alpha(\mathbf{x}) \\ &\approx \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0^g}{4\pi}} \alpha(\mathbf{z}) + \int_{1/Q_T < |\mathbf{x}-\mathbf{z}| < 1/Q} d^2\mathbf{x} \frac{\alpha_s \beta_0^g}{4\pi} \frac{Q^2}{\pi} \left[1 - \left(\frac{1}{Q^2(\mathbf{x}-\mathbf{z})^2} \right)^{1 + \frac{\alpha_s \beta_0^g}{4\pi}} \right] \alpha(\mathbf{x}). \end{aligned} \quad (4.22)$$

Here in the second line, we neglected terms of order α_s not enhanced by transverse logarithms. Note, that we keep the second term in the second line, even though formally it has an explicit prefactor of α_s . The reason is that the integrand is a very slowly decreasing function of $|\mathbf{x} - \mathbf{z}|$ so that for slowly varying $\alpha(\mathbf{x})$ it integrates to a quantity of order unity. In particular, this term is crucial to satisfy the sum rule $\int d^2\mathbf{z} G_Q(\mathbf{z}) = 1$.

To write the resummed Hamiltonian in the dilute limit we need only to keep the LO term, since the NLO terms now become genuinely small corrections. For consistency, in the LO term we have to expand the left and right charge operators as well as the S -matrix $S(\mathbf{z})$ in $\alpha(\mathbf{x})$. This procedure is well known and it leads to the BFKL limit. The only modification in our case is that all the factors $\alpha(\mathbf{z})$ have to be substituted by the resummed amplitude $\alpha_{\tilde{Q}}(\mathbf{z})$. Thus in the dilute limit the DGLAP resummation results in the following Hamiltonian

$$\begin{aligned} H_{\text{BFKL}}^{\text{resummed}} &= \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{g(X^{-2})g(Y^{-2})}{8\pi^3} \frac{X \cdot Y}{X^2 Y^2} \\ &\times \left[(\alpha(\mathbf{x}) - \alpha_{\tilde{Q}}(\mathbf{z})) (\alpha(\mathbf{y}) - \alpha_{\tilde{Q}}(\mathbf{z})) \right]^{ab} \frac{\delta^2}{\delta \alpha^a(\mathbf{x}) \delta \alpha^b(\mathbf{y})}, \end{aligned} \quad (4.23)$$

with $\alpha_{\tilde{Q}}(\mathbf{z})$ given by eq. (4.22) with $\tilde{Q}(\mathbf{z})$ defined in eq. (3.15).

4.4 The saturation regime

Now let us consider the saturated regime. Here the momentum Q_T of the target plays a dual role. First, it provides the correlation length for the Wilson lines like before. But now it also provides the saturation scale. This means that a Wilson line separated from any object composed of other Wilson lines by a distance greater than $1/Q_T$, is small. Although this smallness does not hold configuration by configuration, but only in the sense of averaging over the target ensemble, in practice we can neglect terms that involve products of Wilson lines separated by distances larger than $1/Q_T$.

If we do this, the evolution simplifies greatly. Recall, that we are evolving from $1/|X| \gg Q_T$ to Q_T , so that the distance between the Wilson lines in the nonlinear term in the evolution equation (4.3) is always greater than the correlation length, and therefore the quadratic term is much smaller than the linear term. We can therefore neglect the quadratic term in eq. (4.3) altogether. Then the equation becomes linear, and is dominated entirely by the virtual term:

$$\frac{\partial \mathbb{S}_Q(\mathbf{z})}{\partial \ln Q^2} = -\frac{\alpha_s \beta_0^g}{4\pi} \mathbb{S}_Q(\mathbf{z}) \quad (4.24)$$

with the solution

$$\mathbb{S}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s \beta_0^g}{4\pi}} \mathbb{S}_{Q_T}(\mathbf{z}) \approx \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s \beta_0^g}{4\pi}} S(\mathbf{z}). \quad (4.25)$$

The resummed Hamiltonian in this regime is

$$\begin{aligned} H_{\text{JIMWLK}}^{\text{resummed}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{g(X^{-2})g(Y^{-2})}{8\pi^3} \frac{X \cdot Y}{X^2 Y^2} \\ & \times \left\{ J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) \right. \\ & \left. - 2J_L^a(\mathbf{x}) \left[\theta(Q_T - \tilde{Q}(\mathbf{z})) \left[\frac{Q_T^2}{\tilde{Q}^2(\mathbf{z})} \right]^{\frac{\alpha_s \beta_0^g}{4\pi}} + \theta(\tilde{Q}(\mathbf{z}) - Q_T) \right] S^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right\} \end{aligned} \quad (4.26)$$

with $\tilde{Q}$ defined as before eq. (3.15), $\tilde{Q}(\mathbf{z}) = \max \left\{ \frac{1}{X^2}, \frac{1}{Y^2} \right\}$.

Equations (4.26) and (4.23) conclude our derivation in the two simple limits. Our remaining task is to include the DGLAP logarithms due to the presence of the quarks and antiquarks. We will do this in the following section.

5 Including quarks

Including quarks into our framework is fairly straightforward. We need to add the process of a gluon splitting into a $q\bar{q}$ pair, and additionally to allow a quark to split into a quark and a gluon. The first process already appears in the NLO JIMWLK. The second process appear only at NNLO, because at leading order we do not include quarks into the hadronic wave function. The quarks are generated only at NLO via emission from gluons, and thus can split only at the NNL order. Nevertheless this process is important and affects the DGLAP resummation at leading order.

5.1 Dressed quarks and gluons

The bare quark (antiquark) at the transverse point $\mathbf{x}$ scatters with the eikonal scattering matrix $V(\mathbf{x})$ ($V^\dagger(\mathbf{x})$) in the (anti) fundamental representation of the $SU(N_c)$ group. As a unitary matrix, it of course, satisfies $V^\dagger V = 1$. In addition the gluon and the quark eikonal factors are related by

$$S^{ab}(\mathbf{x}) = 2 \text{Tr}[V^\dagger(\mathbf{x}) \tau^a V(\mathbf{x}) \tau^b] \quad (5.1)$$

where τ^a are generators of $SU(N_c)$ in the fundamental representation.

An important thing we have to be aware of, is that these simple properties do not hold for the S matrices of dressed quarks and gluons. Thus for the DGLAP dressed quark states, which we will define in a short while analogously to dressed gluon states we have discussed so far

$$\mathbb{V}_Q^\dagger(\mathbf{x})\mathbb{V}_Q(\mathbf{x}) \neq 1; \quad \mathbb{S}_Q^{ab}(\mathbf{x}) \neq 2 \text{Tr}[\mathbb{V}_Q^\dagger(\mathbf{x})\tau^a\mathbb{V}_Q(\mathbf{x})\tau^b]. \quad (5.2)$$

This is important since the dressed quark S -matrix cannot be inferred automatically from the dressed gluon S -matrix, and thus can (and does) satisfy an independent DGLAP evolution equation.

Keeping this in mind, we can immediately generalize the equation for the dressed gluon scattering matrix. To leading perturbative order it reads

$$\begin{aligned} \mathbb{S}_Q^{ab}(\mathbf{z}) = & \left[1 + \frac{\alpha_s\beta_0}{4\pi} \ln \frac{\mu^2}{Q^2} \right] S^{ab}(\mathbf{z}) - \frac{\alpha_s\beta_0^g}{4\pi^2 N_c} \int_{|Z|<Q^{-1}} \frac{d^2 Z}{Z^2} D^{ab}(\mathbf{z} + \frac{Z}{2}, \mathbf{z} - \frac{Z}{2}) \\ & - \frac{\alpha_s\beta_0^q}{4\pi^2 N_c} \int_{|Z|<Q^{-1}} \frac{d^2 Z}{Z^2} D^{F\,ab}(\mathbf{z} + \frac{Z}{2}, \mathbf{z} - \frac{Z}{2}) \end{aligned} \quad (5.3)$$

where $\beta_0^q = -\frac{2N_F}{3}$ is the quark contribution to the one loop QCD β -function, and we have defined

$$D^{F\,ab}(\mathbf{z} + Z/2, \mathbf{z} - Z/2) = 2 \text{Tr}[\tau^a V(\mathbf{z} + Z/2) \tau^b V^+(\mathbf{z} - Z/2)]. \quad (5.4)$$

As before, we can rewrite the complete NLO JIMWLK Hamiltonian as

$$H_{\text{JIMWLK}} = H_Q^{\text{JSJ}} + H_Q^{\text{JSSJ}} + H_Q^{\bar{q}q} + \dots \quad (5.5)$$

where now

$$\begin{aligned} H_Q^{\text{JSJ}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{\alpha_s X \cdot Y}{2\pi^2 X^2 Y^2} \left\{ \left(1 + \frac{\alpha_s\beta_0}{8\pi} [\ln[X^2\mu^2] + \ln[Y^2\mu^2]] \right) \right. \\ & \times \left[J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) - 2 J_L^a(\mathbf{x}) \mathbb{S}_Q^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \\ & + \frac{\alpha_s\beta_0}{8\pi} [(\ln X^2 \bar{Q}^2 + \ln Y^2 \bar{Q}^2) [J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y})] \\ & \left. - 2(\ln X^2 \bar{Q}^2 + \ln Y^2 \bar{Q}^2) J_L^a(\mathbf{x}) \mathbb{S}_Q^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right] \Big\}, \end{aligned} \quad (5.6)$$

$$\begin{aligned} H_Q^{\text{JSSJ}} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') J_L^a(\mathbf{x}) \mathbb{D}_Q^{ad}(\mathbf{z}, \mathbf{z}') J_R^d(\mathbf{y}) - \frac{\alpha_s^2 \beta_0^g}{4\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \mathbb{D}_Q^{ab}(\mathbf{z}) J_L^a(\mathbf{x}) J_R^b(\mathbf{y}) \\ & - \left[\int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} \frac{N_c}{2} K_{JSSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') - \frac{\alpha_s^2 \beta_0^g}{8\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \ln \frac{\mu^2}{\bar{Q}^2} \right] [J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) + J_L^a(\mathbf{x}) J_L^a(\mathbf{y})] \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} H_Q^{\bar{q}q} = & \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') J_L^a(\mathbf{x}) \mathbb{D}_Q^{F\,ab}(\mathbf{z}, \mathbf{z}') J_R^b(\mathbf{y}) - \frac{\alpha_s^2 \beta_0^q}{4\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \mathbb{D}_Q^{F\,ab}(\mathbf{z}) J_L^a(\mathbf{x}) J_R^b(\mathbf{y}) \\ & - \left[\int_{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}'} \frac{1}{2} K_{q\bar{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}') - \frac{\alpha_s^2 \beta_0^q}{4\pi^3} \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{X \cdot Y}{X^2 Y^2} \ln \frac{\mu^2}{\bar{Q}^2} \right] [J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) + J_L^a(\mathbf{x}) J_L^a(\mathbf{y})]. \end{aligned} \quad (5.8)$$

For the dressed quark scattering matrix guided by exactly the same considerations, we have at the leading perturbative order

$$\mathbb{V}_Q^{\alpha\beta}(\mathbf{z}) = \left[1 + \frac{3}{2} \frac{\alpha_s C_F}{4\pi} \ln \frac{\mu^2}{Q^2} \right] V^{\alpha\beta}(\mathbf{z}) - \frac{3}{2} \frac{\alpha_s}{4\pi^2} \int_{|Z| < Q^{-1}} \frac{d^2 Z}{Z^2} \left[\tau^a S^{ab}(\mathbf{z} + Z/2) V(\mathbf{z} - Z/2) \tau^b \right]^{\alpha\beta} \quad (5.9)$$

where we have used the fact that for the quark-gluon splitting function

$$\int_0^1 d\xi \left[\frac{1 + (1 - \xi)^2}{\xi} \right]_+ = -\frac{3}{2} \quad (5.10)$$

and

$$C_F = \frac{N_c^2 - 1}{2N_c}. \quad (5.11)$$

Given these perturbative expressions we can write down the DGLAP equations which now include quarks. For the adjoint scattering matrix we have

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} \mathbb{S}_Q^{ab}(\mathbf{z}) = & -\frac{\alpha_s}{4\pi} \left[\beta_0 \mathbb{S}_Q^{ab}(\mathbf{z}) \right. \\ & \left. - \frac{1}{N_c} \int \frac{d\phi}{2\pi} \left(\beta_0^g \mathbb{D}_Q^{ab}(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi, \mathbf{z} - \frac{1}{2} Q^{-1} \mathbf{e}_\phi) + \beta_0^q \mathbb{D}_Q^{F ab}(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi, \mathbf{z} - \frac{1}{2} Q^{-1} \mathbf{e}_\phi) \right) \right] \end{aligned} \quad (5.12)$$

and for the fundamental scattering matrix

$$\frac{\partial}{\partial \ln Q^2} \mathbb{V}_Q^{\alpha\beta}(\mathbf{z}) = -\frac{3}{2} \frac{\alpha_s}{2\pi} \left[C_F \mathbb{V}_Q^{\alpha\beta}(\mathbf{z}) - \int \frac{d\phi}{2\pi} \mathbb{D}_Q^{(FA)\alpha\beta}(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi, \mathbf{z} - \frac{1}{2} Q^{-1} \mathbf{e}_\phi) \right] \quad (5.13)$$

where

$$\begin{aligned} \mathbb{D}_Q^{F ab}(\mathbf{z} + Z/2, \mathbf{z} - Z/2) & \equiv 2 \text{Tr} [\tau^a \mathbb{V}_Q(\mathbf{z} + Z/2) \tau^b \mathbb{V}_Q^+(\mathbf{z} - Z/2)], \\ \mathbb{D}_Q^{(FA)\alpha\beta}(\mathbf{z} + Z/2, \mathbf{z} - Z/2) & \equiv \left[\tau^a \mathbb{S}_Q^{ab}(\mathbf{z} + Z/2) \mathbb{V}_Q(\mathbf{z} - Z/2) \tau^b \right]^{\alpha\beta}. \end{aligned} \quad (5.14)$$

The interesting albeit not unexpected property of these equations is that the evolution of the adjoint and fundamental scattering matrices mix. This means that, although we only need to find the adjoint $\mathbb{S}_Q$ in order to perform the DGLAP resummation of the JIMWLK Hamiltonian at NLO, we have to consider also the Q^2 evolution of $\mathbb{V}_Q$. This is so even though the terms in the JIMWLK Hamiltonian in which we would have to use the solution of eq. (5.13) appear only starting at NNLO.

5.2 The resummation — saturated regime

As just mentioned, to resum the transverse logarithms at leading order we do not need to know $\mathbb{V}_Q$, but only $\mathbb{S}_Q$. This makes the resummation in saturation regime very straightforward. In eq. (5.12) we can neglect, as before the “real” $\mathbb{D}_Q$ terms which involve the product of two scattering matrices. The solution is then exactly the same as eq. (4.25) but with the complete one loop β -function

$$\mathbb{S}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s \beta_0}{4\pi}} \mathbb{S}_{Q_T}(\mathbf{z}) \approx \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s \beta_0}{4\pi}} S(\mathbf{z}). \quad (5.15)$$

For completeness, we can also straightforwardly solve for $\mathbb{V}_Q$, even though it is not needed to obtain the resummed Hamiltonian

$$\mathbb{V}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2} \right]^{\frac{3\alpha_s(N_c^2-1)}{8\pi N_c}} \mathbb{V}_{Q_T}(\mathbf{z}) \approx \left[\frac{Q_T^2}{Q^2} \right]^{\frac{3\alpha_s(N_c^2-1)}{8\pi N_c}} V(\mathbf{z}). \quad (5.16)$$

The resummed JIMWLK Hamiltonian is then

$$H_{\text{JIMWLK}}^{\text{resummed}} = \int_{\mathbf{x}, \mathbf{y}, \mathbf{z}} \frac{g(X^{-2})g(Y^{-2})}{8\pi^3} \frac{X \cdot Y}{X^2 Y^2} \left\{ J_L^a(\mathbf{x}) J_L^a(\mathbf{y}) + J_R^a(\mathbf{x}) J_R^a(\mathbf{y}) \right. \\ \left. - 2 J_L^a(\mathbf{x}) \left[\theta(Q_T - \tilde{Q}(\mathbf{z})) \left[\frac{Q_T^2}{\tilde{Q}^2(\mathbf{z})} \right]^{\frac{\alpha_s \beta_0}{4\pi}} + \theta(\tilde{Q}(\mathbf{z}) - Q_T) \right] S^{ab}(\mathbf{z}) J_R^b(\mathbf{y}) \right\}. \quad (5.17)$$

If we wanted to keep nonlogarithmic $O(\alpha_s^2)$ terms in the resummed Hamiltonian, we would need to use the solutions eqs. (5.15), (5.16) and substitute them into the $O(\alpha_s^2)$ terms in the Hamiltonian, but we will not concern ourselves with this here.

5.3 The resummation: the dilute regime

Here we again assume that both, $\mathbb{S}_Q$ and $\mathbb{V}_Q$ are close to unity:

$$\mathbb{S}_Q(\mathbf{x}) = 1 + iT^a \alpha_Q^a(\mathbf{x}); \quad \mathbb{V}_Q(\mathbf{x}) = 1 + i\tau^a \alpha_Q^{Fa}(\mathbf{x}). \quad (5.18)$$

Note, that due to eq. (5.2), we are not assuming that α_Q is equal to α_Q^F . The evolution equation now becomes a little more complicated, since it mixes α_Q with α_Q^F .

The equations for the Q^2 evolution now are

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{z}) = \frac{\alpha_s \beta_0^g}{8\pi^2} \int d\phi \left(\alpha_Q^c \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) - \alpha_Q^c(\mathbf{z}) \right) \\ + \frac{\alpha_s \beta_0^q}{8\pi^2} \int d\phi \left(\alpha_Q^{Fc} \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) - \alpha_Q^c(\mathbf{z}) \right) \quad (5.19)$$

and

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^{Fc}(\mathbf{z}) = \frac{3}{2} \frac{\alpha_s}{4\pi^2} \int d\phi \left(\frac{N_c}{2} \alpha_Q^c \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) - \frac{1}{2N_c} \alpha_Q^{Fc} \left(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi \right) - C_F \alpha_Q^{Fc}(\mathbf{z}) \right). \quad (5.20)$$

To solve these equations we follow the same route as in the previous section, i.e. transform into momentum space. Introducing the function

$$\tilde{R}(p, Q) = J_0 \left(\frac{p}{2Q} \right) - 1 \quad (5.21)$$

we obtain local equations in momentum space

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{p}) = \frac{\alpha_s \beta_0^g}{4\pi} \tilde{R}(p, Q) \alpha_Q^c(\mathbf{p}) + \frac{\alpha_s \beta_0^q}{4\pi} \left\{ \left[\tilde{R}(p, Q) + 1 \right] \alpha_Q^{Fc}(\mathbf{p}) - \alpha_Q^c(\mathbf{p}) \right\}, \quad (5.22)$$

$$\frac{\partial}{\partial \ln Q^2} \alpha_Q^{Fc}(\mathbf{p}) = \frac{3}{2} \frac{\alpha_s}{2\pi} \left\{ \left[\tilde{R}(p, Q) + 1 \right] \left(\frac{N_c}{2} \alpha_Q^c(\mathbf{p}) - \frac{1}{2N_c} \alpha_Q^{Fc}(\mathbf{p}) \right) - C_F \alpha_Q^{Fc}(\mathbf{p}) \right\}. \quad (5.23)$$

These are still pretty complicated. We now however employ the same simplification we used to analyze the no-quark case. Our calculation in eq. (4.14) amounts to the following simple approximation for $\tilde{R}$

$$\tilde{R}(p, Q) \approx 0; \quad p < Q; \quad \tilde{R}(p, Q) = -1; \quad p > Q. \quad (5.24)$$

Then for $p < Q$ we have

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{p}) &= \frac{\alpha_s \beta_0^q}{4\pi} \left[\alpha_Q^{Fc}(\mathbf{p}) - \alpha_Q^c(\mathbf{p}) \right], \\ \frac{\partial}{\partial \ln Q^2} \alpha_Q^{Fc}(\mathbf{p}) &= \frac{3}{2} \frac{\alpha_s N_c}{4\pi} \left[\alpha_Q^c(\mathbf{p}) - \alpha_Q^{Fc}(\mathbf{p}) \right]. \end{aligned} \quad (5.25)$$

While for $p > Q$

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} \alpha_Q^c(\mathbf{p}) &= -\frac{\alpha_s}{4\pi} \beta_0 \alpha_Q^c(\mathbf{p}), \\ \frac{\partial}{\partial \ln Q^2} \alpha_Q^{Fc}(\mathbf{p}) &= -\frac{3}{2} \frac{\alpha_s}{2\pi} C_F \alpha_Q^{Fc}(\mathbf{p}). \end{aligned} \quad (5.26)$$

These equations are easily solved. Recall that the interval of the evolution is from $\tilde{Q}$ to Q_T . Thus for $p < \tilde{Q}$ we only need to solve the first set of equations. Their diagonalized form is

$$\begin{aligned} \frac{\partial}{\partial \ln Q^2} \left[\alpha_Q^c(\mathbf{p}) - \alpha_Q^{Fc}(\mathbf{p}) \right] &= -\frac{\alpha_s}{4\pi} \left(\beta_0^q + \frac{3N_c}{2} \right) \left[\alpha_Q^c(\mathbf{p}) - \alpha_Q^{Fc}(\mathbf{p}) \right], \\ \frac{\partial}{\partial \ln Q^2} \left[\frac{3N_c}{2} \alpha_Q^c(\mathbf{p}) + \beta_0^q \alpha_Q^{Fc}(\mathbf{p}) \right] &= 0. \end{aligned} \quad (5.27)$$

Our initial condition is $\alpha_{Q_T} = \alpha_{Q_T}^F = \alpha$. The solution of eq. (5.27) with this initial condition is simply

$$\alpha_Q(p < Q) = \alpha_Q^F(p < Q) = \alpha(p < Q). \quad (5.28)$$

It is equally straightforward to solve for $p > Q_T$. For that, we need to solve eq. (5.26), which is straightforward. The solution is

$$\alpha_Q^c(p > Q_T) = \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \alpha^c(p > Q_T); \quad \alpha_Q^{Fc}(p > Q_T) = \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \alpha^c(p > Q_T). \quad (5.29)$$

Finally, we consider momenta $Q_T > p > Q$. For these momenta while evolving from Q to Q_T , we should evolve α according to eq. (5.26) between Q and p , and then according to eq. (5.27) between p and Q_T . The first part of the evolution results in the same expressions as eq. (5.29) but with Q_T substituted by p

$$\begin{aligned} \tilde{\alpha}_Q^c(Q_T > p > Q) &= \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \alpha^c(Q_T > p > Q); \\ \tilde{\alpha}_Q^{Fc}(Q_T > p > Q) &= \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \alpha^c(Q_T > p > Q) \end{aligned} \quad (5.30)$$

where $\tilde{\alpha}$ denotes the solution evolved from Q to p . This $\tilde{\alpha}$ now serves as the initial condition for the evolution from p to Q_T according to eq. (5.27). Eventually, we find the solution:

$$\begin{aligned}\alpha_Q^c(p) - \alpha_Q^{Fc}(p) &= \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \left\{ \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} - \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(p) \\ \frac{3N_c}{2} \alpha_Q^c(p) + \beta_0^q \alpha_Q^{Fc}(p) &= \left\{ \frac{3N_c}{2} \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} + \beta_0^q \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(p)\end{aligned}\quad (5.31)$$

from which it follows

$$\begin{aligned}\alpha_Q^c(p) &= \frac{1}{\beta_0^q + \frac{3N_c}{2}} \left\{ \left(\frac{3N_c}{2} + \beta_0^q \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \right. \\ &\quad \left. + \beta_0^q \left(1 - \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(p),\end{aligned}\quad (5.32)$$

$$\begin{aligned}\alpha_Q^{Fc}(p) &= \frac{1}{\beta_0^q + \frac{3N_c}{2}} \left\{ \frac{3N_c}{2} \left(1 - \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \right. \\ &\quad \left. + \left(\beta_0^q + \frac{3N_c}{2} \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(p).\end{aligned}\quad (5.33)$$

The last expression we present for completeness, since as mentioned above we only need α_Q to perform the resummation in JIMWLK at leading order. Thus for all values of momentum, we find

$$\alpha_Q^c(\mathbf{p}) = \alpha^c(\mathbf{p}) \theta(Q - p) \quad (5.34)$$

$$\begin{aligned}&+ \frac{1}{\beta_0^q + \frac{3N_c}{2}} \left\{ \left(\frac{3N_c}{2} + \beta_0^q \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \right. \\ &\quad \left. + \beta_0^q \left(1 - \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(\mathbf{p}) \theta(p - Q) \theta(Q_T - p) \\ &+ \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \alpha^c(\mathbf{p}) \theta(p - Q_T),\end{aligned}$$

$$\alpha_Q^{Fc}(\mathbf{p}) = \alpha^c(\mathbf{p}) \theta(Q - p) \quad (5.35)$$

$$\begin{aligned}&+ \frac{1}{\beta_0^q + \frac{3N_c}{2}} \left\{ \frac{3N_c}{2} \left(1 - \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0} \right. \\ &\quad \left. + \left(\beta_0^q + \frac{3N_c}{2} \left[\frac{p^2}{Q_T^2} \right]^{\frac{\alpha_s}{4\pi} [\beta_0^q + \frac{3N_c}{2}]} \right) \left[\frac{p^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \right\} \alpha^c(\mathbf{p}) \theta(p - Q) \theta(Q_T - p) \\ &+ \left[\frac{Q_T^2}{Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F} \alpha^c(\mathbf{p}) \theta(p - Q_T).\end{aligned}$$

Fourier transformation of these expressions into coordinate space does not present any additional difficulties and can be performed in the same way as in the previous section. We obtain

$$\begin{aligned} \alpha_Q(\mathbf{z}) \approx & \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \alpha(\mathbf{z}) \\ & + \int_{\frac{1}{Q_T} < |\mathbf{x}-\mathbf{z}| < \frac{1}{Q}} \frac{d^2 \mathbf{x}}{\beta_0^q + \frac{3N_c}{2}} \left\{ \left(\frac{3N_c}{2} + \beta_0^q Q_T^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q_T^2} \right]^{\frac{\alpha_s}{4\pi} \left[\beta_0^q + \frac{3N_c}{2} \right] + 1} \right) Q^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0 + 1} \right. \\ & \left. + \beta_0^q \left(1 - Q_T^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q_T^2} \right]^{\frac{\alpha_s}{4\pi} \left[\beta_0^q + \frac{3N_c}{2} \right] + 1} \right) Q^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q^2} \right]^{\frac{\alpha_s}{4\pi} 3C_F + 1} \right\} \alpha(\mathbf{x}) \end{aligned} \quad (5.36)$$

and

$$\begin{aligned} \alpha_Q^F(\mathbf{z}) \approx & \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s}{4\pi} 3C_F} \alpha^F(\mathbf{z}) \\ & + \int_{\frac{1}{Q_T} < |\mathbf{x}-\mathbf{z}| < \frac{1}{Q}} \frac{d^2 \mathbf{x}}{\beta_0^q + \frac{3N_c}{2}} \left\{ \left(\beta_0^q + \frac{3N_c}{2} Q_T^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q_T^2} \right]^{\frac{\alpha_s}{4\pi} \left[\beta_0^q + \frac{3N_c}{2} \right] + 1} \right) Q^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q^2} \right]^{\frac{3\alpha_s}{4\pi} C_F + 1} \right. \\ & \left. + \frac{3N_c}{2} \left(1 - Q_T^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q_T^2} \right]^{\frac{\alpha_s}{4\pi} \left[\beta_0^q + \frac{3N_c}{2} \right] + 1} \right) Q^2 \left[\frac{1}{(\mathbf{x}-\mathbf{z})^2 Q^2} \right]^{\frac{\alpha_s}{4\pi} \beta_0 + 1} \right\} \alpha^F(\mathbf{x}). \end{aligned} \quad (5.37)$$

The resummed Hamiltonian is now given by eq. (4.23) with $\alpha_{\tilde{Q}}(\mathbf{z})$ given by eq. (5.36) with $\tilde{Q}(\mathbf{z})$ defined in eq. (3.15).

6 Discussion

In this paper we considered a partial resummation of transverse logarithms in the NLO JIMWLK equation. We have shown that some of these logs, which are proportional to the one-loop coefficient of the QCD β -function are not associated with the renormalization of the coupling constant, but instead encode DGLAP corrections to the low x evolution. These DGLAP corrections are large whenever there is a large disparity between the correlation lengths (or saturation momenta) in the projectile and the target.

The small target correlation length Q_T^{-1} plays the role of the external scale on which the projectile structure is resolved. This is in direct analogy to the hard scale Q^2 in DIS. In the presence of large Q_T the scattering is dominated by the projectile configurations with typical distance between partons of order Q_T^{-1} , and therefore the projectile wave function needs to be evolved up to this scale. This is so even though JIMWLK describes the evolution of the total cross section, and not a small fraction of it, as in DIS at high Q . The DGLAP evolution inside JIMWLK resums the powers of $\alpha_s \ln \frac{Q_T^2}{Q_p^2}$. If the ratio of the two scales is not large, there is in principle no need in this resummation. However as is well known, JIMWLK equation is applicable for the scattering of a dilute object on a dense one, and therefore, resumming the DGLAP corrections is indeed essential in the region of validity of JIMWLK equation.

We showed how to resum these corrections by solving DGLAP equation that describes the scale evolution of the scattering matrix of dressed gluons and quarks. The S matrix needs to be evolved from the scale of the target’s correlation length (saturation momentum) to that of the projectile. We solved these equations explicitly in two limiting cases. The first is when the target is dilute, and the correlation length is not associated with the saturation momentum. In this case, both real and virtual corrections are important in the DGLAP evolution. The second case is when the target is saturated with large saturation momentum. In this case, the evolution is dominated by the virtual term and the solution has a simple scaling form.

The running coupling corrections of course are also there and also need to be resummed. However the physics of these corrections is simple and there is no need for an elaborate choice for the scale of the coupling constant. Although, of course, strictly speaking, one needs higher order corrections to make sure that the scale of the coupling is set properly, we observe that the simplest physically motivated choice, i.e., the distance between the emitter and the emitted gluon, is a perfectly good scale for the running strong charge in the emission amplitude. Choosing this scale and including the DGLAP resummation takes care of all large logarithms associated with the one loop β -function in the NLO JIMWLK.

There is one important point we need to realize about the DGLAP corrections discussed here. Although we have been referring throughout this paper to “dressed gluons” and “dressed quarks”, these are actually not the same normalized dressed states one encounters usually in QCD calculations. The dressed parton states are created by DGLAP emission with the complete splitting function. In the context of JIMWLK evolution, the situation is different. The low x part of the splitting function is associated with the JIMWLK evolution in rapidity, rather than the DGLAP splittings. As we noted earlier, this is the reason for the plus prescription that subtracts the contribution of both poles at $\xi = 0$ and $\xi = 1$ in eqs. (3.4), (3.6) and those following. This subtraction compensates for the inclusion of the low x part of the splitting function, which the standard low x evolution (be it JIMWLK or BK) uses formally at all values of x . As a result, whereas the (real contribution to the) usual splitting function is positive since it has the meaning of emission probability, in our case, the analogous quantity is negative, e.g., the coefficient in front of the second (“real”) term in eq. (5.3).

This negativity has a very stark consequence on our final results. Consider for example, eq. (4.25) which gives the solution for the scattering matrix of our “dressed gluon” at the resolution scale Q . If this were the physical dressed gluon state, one would, of course, expect that for $Q < Q_T$, the spatial extent of this state is large, and the scattering matrix is smaller than S since the scattering amplitude should be larger for a spatially extended state. However eq. (4.25) behaves in exactly the opposite way. As Q gets smaller, S_Q *increases*, i.e. becomes *less* saturated. This is precisely the effect of subtracting the low x part from the DGLAP splitting function. The low x evolution, therefore, partitions the DGLAP logarithms in a rather amusing way. In the Hamiltonian itself, only a single gluon emission with the low x part of the splitting function appears since the Hamiltonian is extracted from terms in the S -matrix, which are linear in the longitudinal phase space ΔY . However, calculating H_{JIMWLK} at higher orders in α_s brings an arbitrary number of emissions with the finite part of the splitting function (with the low x poles subtracted), which look like emissions with *negative* probability. These are the logarithms that we showed how to resum in the present paper. If we were to look at the second order in ΔY contribution to the scattering matrix,

i.e. the second iteration of the JIMWLK Hamiltonian in the evolution, we would find two eikonal emissions, i.e. two emissions with the low x part of the splitting function. Thus at this order we should be able to reconstruct the full splitting function for two emissions, but the rest would still look like emissions with negative probability. And so it goes.

By what time do we recover the complete positive probability (splitting function) at all orders in the DGLAP part of the evolution? It looks natural that one would need to evolve in Y by the interval that is of the same order as the interval of the DGLAP evolution, i.e. $\Delta Y \approx \ln \frac{Q_T^2}{Q_P^2}$ to restore the positivity of probabilities for all splittings. Interestingly, parametrically by the end of this evolution in Y we expect $Q_P^2(Y) \approx Q_T^2(Y)$. In principle, to evolve the system in Y any further, we need to go beyond the JIMWLK approximation since our projectile is not anymore dilute relative to the target. It is not clear to us at this point whether this observation has any significance, apart from strengthening the point that within the JIMWLK evolution per se, resumming the DGLAP corrections is a must.

For the future there are two obvious directions in which one should extend our results. First, it would be interesting to solve the full RG equation beyond the dilute and dense limits. The equation itself has a form somewhat similar to the BK equation, with the obvious difference that the basic object is a matrix rather than a scalar amplitude. It may not be possible to solve this equation analytically, but it is certainly worth a try.

Second, in the present paper we have only dealt with the resummation of DGLAP logarithms at leading order. Correspondingly, we did not keep genuinely perturbative $O(\alpha_s^2)$ corrections in the resummed Hamiltonian. For precision applications one certainly would need to do that. In principle it is a straightforward matter, as one would just need to keep $O(\alpha_s^2)$ in H_{JIMWLK} after substitution of $\mathbb{S}_{\bar{Q}}$. However, as always in going beyond leading order, one has to be careful about details and implement a consistent factorization scheme throughout the calculation, including more careful solution for α_Q and $\mathbb{S}_Q$. At this level of accuracy one also needs to include the running of the QCD coupling constant in the DGLAP evolution along the lines indicated in eq. (4.17).

In the regime where neither projectile nor target are saturated, the JIMWLK equation reduces to the BFKL equation. Various schemes that include both BFKL and DGLAP logarithms have been considered in the literature [34–47]. None of these approaches however are applied in coordinate space which is natural for the JIMWLK equation. Consequently at this stage we were unable to relate our procedure to those in the above papers, although we note that the “restoration” of the full DGLAP splitting function within the energy evolution is the central point of refs. [34–39].

We note that in recent years attempts have been, and are being made to unify the low and intermediate x physics including the saturation ideas [48–50]. Although technically our approach here is very different, the physics problem we address has the same origin and we hope these various approaches will converge in future.

A The right and left charges

We may want to express H_{JIMWLK} entirely in terms of the dressed gluons scattering matrix $\mathbb{S}_Q$. This would require also expressing the left- and right rotation operators as operators

acting on functions of $\mathbb{S}_Q$ rather than S . We consider the right charges first. Recall that

$$J_R^a(\mathbf{x})S^{eb}(\mathbf{y}) = [S(\mathbf{y})T^a]^{eb}\delta^2(\mathbf{x} - \mathbf{y}); \quad J_R^a(\mathbf{x})S^{Teb}(\mathbf{y}) = -[T^aS^T(\mathbf{y})]^{eb}\delta^2(\mathbf{x} - \mathbf{y}). \quad (\text{A.1})$$

Thus

$$J_R^a(\mathbf{x}) = -\text{Tr} \left[\frac{\partial}{\partial S(\mathbf{x})} T^a S^T(\mathbf{x}) \right]. \quad (\text{A.2})$$

In principle the problem is algebraic. We should just use the chain rule to express the charges in terms of $\mathbb{S}_Q$

$$\begin{aligned} J_R^a(\mathbf{x}) &= -\text{Tr} \left[\frac{\partial}{\partial S(\mathbf{x})} T^a S^T(\mathbf{x}) \right] = -\int_{\mathbf{y}} \text{Tr} \left[\frac{\partial \mathbb{S}_Q^{eb}(\mathbf{y})}{\partial S(\mathbf{x})} \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{y})} T^a S^T(\mathbf{x}) \right] \\ &= \int_{\mathbf{y}} \left[J_R^a(\mathbf{x}) \mathbb{S}_Q^{eb}(\mathbf{y}) \right] \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{y})}. \end{aligned} \quad (\text{A.3})$$

Using the definition we have

$$\begin{aligned} \frac{\partial \mathbb{S}_Q^{eb}(\mathbf{y})}{\partial S^{cd}(\mathbf{x})} &= \left[1 + \frac{\alpha_s \beta_0^g}{4\pi} \ln \frac{\mu^2}{Q^2} \right] \delta^{ec} \delta^{bd} \delta^2(\mathbf{y} - \mathbf{x}) \\ &\quad - 2 \frac{\alpha_s \beta_0^g}{4\pi^2 N_c} \frac{1}{(\mathbf{x} - \mathbf{y})^2} \left[T^e S(2\mathbf{y} - \mathbf{x}) T^b \right]_{cd} \Theta[(2Q)^{-2} - (\mathbf{x} - \mathbf{y})^2]. \end{aligned} \quad (\text{A.4})$$

So that

$$\begin{aligned} J_R^a(\mathbf{x}) &= -\text{Tr} \left[\frac{\partial}{\partial \mathbb{S}_Q(\mathbf{x})} T^a \mathbb{S}_Q^T(\mathbf{x}) \right] + \frac{\alpha_s \beta_0^g}{4\pi^2 N_c} \int_{\mathbf{y}-\mathbf{x}}^{\frac{1}{2Q}} \frac{1}{(\mathbf{x} - \mathbf{y})^2} \\ &\quad \times \left[\text{Tr} \left[T^e \mathbb{S}_Q(2\mathbf{y} - \mathbf{x}) T^b T^a \mathbb{S}_Q^T(\mathbf{x}) - T^e \mathbb{S}_Q(\mathbf{x}) T^a T^b \mathbb{S}_Q^T(2\mathbf{y} - \mathbf{x}) \right] \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{y})} \right. \\ &\quad \left. - \text{Tr} \left[T^e \mathbb{S}_Q(2\mathbf{x} - \mathbf{y}) T^c \mathbb{S}_Q^T(\mathbf{y}) \right] \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{x})} T_{bc}^a \right]. \end{aligned} \quad (\text{A.5})$$

We can easily check that this expression is UV finite. To see this we need to show that the sum of the integrands in the last two terms vanish for $\mathbf{y} \rightarrow \mathbf{x}$. This is obviously the case, since the $\text{SU}(N)$ generators in the penultimate line in this limit combine into $[T^a, T^b]$, and exactly cancel the term in the last line.

Now for the left charge

$$J_L^a(\mathbf{x}) = -\text{Tr} \left[\frac{\partial}{\partial S(\mathbf{x})} S^T(\mathbf{x}) T^a \right]. \quad (\text{A.6})$$

The same algebra as above yields

$$\begin{aligned} J_L^a(\mathbf{x}) &= -\text{Tr} \left[\frac{\partial}{\partial \mathbb{S}_Q(\mathbf{x})} \mathbb{S}_Q^T(\mathbf{x}) T^a \right] + \frac{\alpha_s \beta_0^g}{4\pi^2 N_c} \int_{\mathbf{y}-\mathbf{x}}^{\frac{1}{2Q}} \frac{1}{(\mathbf{x} - \mathbf{y})^2} \\ &\quad \times \left[\text{Tr} \left[T^a T^e \mathbb{S}_Q(2\mathbf{y} - \mathbf{x}) T^b \mathbb{S}_Q^T(\mathbf{x}) - T^e T^a \mathbb{S}_Q(\mathbf{x}) T^b \mathbb{S}_Q^T(2\mathbf{y} - \mathbf{x}) \right] \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{y})} \right. \\ &\quad \left. - \text{Tr} \left[T^c \mathbb{S}_Q(2\mathbf{x} - \mathbf{y}) T^b \mathbb{S}_Q^T(\mathbf{y}) \right] \frac{\partial}{\partial \mathbb{S}_Q^{eb}(\mathbf{x})} T_{ce}^a \right]. \end{aligned} \quad (\text{A.7})$$

Again, it is easy to check that this expression is UV finite.

It is natural to define the “dressed color charge” operators that measure the charge of the dressed gluons:

$$J_{Q,R}^a(\mathbf{x}) = -\text{Tr} \left[\frac{\partial}{\partial \mathbb{S}_Q(\mathbf{x})} T^a \mathbb{S}_Q^T(\mathbf{x}) \right]; \quad J_{Q,L}^a(\mathbf{x}) = -\text{Tr} \left[\frac{\partial}{\partial \mathbb{S}_Q(\mathbf{x})} \mathbb{S}_Q^T(\mathbf{x}) T^a \right]. \quad (\text{A.8})$$

As alluded to in the text, one can use these operators, rather than the original $J_{R(L)}^a$ to write the resummed form of H_{JIMWLK} . Although this may look natural in a certain sense, it does not make any difference in perturbation theory. The fact that eqs. (A.5) and (A.7) are free from the UV divergences means that the relation between the “bare charge” operators and the “dressed charge” operators in purely perturbative and does not contain large logarithms. Thus the choice of which charge operators to use in the perturbatively resummed Hamiltonian is equivalent to the choice of the factorisation scheme. At the leading order, at which we work in this paper, the choice is immaterial. However if one would want to keep all $O(\alpha_s^2)$ terms, including those that are not enhanced by the transverse logarithms, this choice has to be explicitly specified.

B Fourier transforming α_Q

To calculate $\alpha_Q(\mathbf{z})$ we need to calculate the following two dimensional Fourier transform:

$$G_Q(x) = \int \frac{d^2 p}{(2\pi)^2} e^{i\mathbf{p}\mathbf{x}} \times \left[\theta(4Q^2 - p^2) + \left(\frac{p^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \theta(Q_T^2 - p^2) \theta(p^2 - Q^2) + \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \theta(p^2 - Q_T^2) \right]. \quad (\text{B.1})$$

We split the integral into three contributions corresponding to the order of terms in eq. (B.1).

The first integral can be analytically computed

$$I_1 = \int \frac{d^2 p}{(2\pi)^2} e^{i\mathbf{p}\mathbf{x}} \theta(Q^2 - p^2) = \frac{Q J_1(Qx)}{\pi x}. \quad (\text{B.2})$$

Its asymptotics for small and large argument Qx are

$$I_1 \left(x \ll \frac{1}{Q} \right) \approx \frac{Q^2}{\pi}, \quad (\text{B.3})$$

$$I_1 \left(x \gg \frac{1}{Q} \right) \approx -\frac{Q^{1/2} \cos(Qx + \frac{\pi}{4})}{\pi^{3/2} x^{3/2}}. \quad (\text{B.4})$$

The corrections to the latter are of order $\mathcal{O}(x^{-5/2})$.

The third integral reduces to delta function and the same analytic integral as in I_1

$$I_3 = \int \frac{d^2 p}{(2\pi)^2} e^{i\mathbf{p}\mathbf{x}} \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \theta(p^2 - Q_T^2) = \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \delta^{(2)}(\mathbf{x}) - \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \frac{Q_T J_1(Q_T x)}{\pi x}. \quad (\text{B.5})$$

The integral of the second term in eq. (B.1) leads to hypergeometric functions and is not very useful. Here we will only consider its assymptics. For small argument,

$$I_2 \left(x \ll \frac{1}{Q_T} \right) \approx \frac{Q^2}{\pi} \int_1^{(Q_T/Q)^2} dy y^{\frac{\alpha_s \beta_0}{4\pi}} = \frac{Q^2}{\pi} \frac{1}{1 + \frac{\alpha_s \beta_0}{4\pi}} \left(\left[\frac{Q_T^2}{Q^2} \right]^{1 + \frac{\alpha_s \beta_0}{4\pi}} - 1 \right) \\ \approx \frac{Q^2}{\pi} \left(\left[\frac{Q_T^2}{Q^2} \right]^{1 + \frac{\alpha_s \beta_0}{4\pi}} - 1 \right). \quad (\text{B.6})$$

For large argument $Qx \gg 1$,

$$I_2 \left(x \gg \frac{1}{Q} \right) \approx \frac{Q^{1/2} \cos(Qx + \frac{\pi}{4})}{\pi^{3/2} x^{3/2}} - \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \frac{Q_T^{1/2} \cos(Q_T x + \frac{\pi}{4})}{\pi^{3/2} x^{3/2}} \quad (\text{B.7})$$

with corrections of order $\mathcal{O}(x^{-5/2})$. And finally, for the intermediate values, $\frac{1}{Q_T} \ll x \ll \frac{1}{Q}$,

$$I_2 \left(x \gg \frac{1}{Q} \right) \approx -\frac{Q^2}{\pi} - \frac{\alpha_s \beta_0}{4\pi} \frac{Q^2}{\pi (Q^2 x^2)^{1 + \frac{\alpha_s \beta_0}{4\pi}}} - \left(\frac{Q_T^2}{Q^2} \right)^{\frac{\alpha_s \beta_0}{4\pi}} \frac{Q_T^{1/2} \cos(Q_T x + \frac{\pi}{4})}{\pi^{3/2} x^{3/2}}. \quad (\text{B.8})$$

From this set of results, one obtains $G(x)$ in different regions as presented in eqs. (4.19), (4.20), and (4.21).

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