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CONTROLLABILITY PROPERTIES FROM THE EXTERIOR UNDER POSITIVITY CONSTRAINTS FOR A 1-D FRACTIONAL HEAT EQUATION

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Abstract. We study the controllability of trajectories, under positivity constraints on the control or the state, of a one-dimensional heat equation involving the fractional Laplace operator $(-\partial_x^2)^s$ (with $0 < s < 1$) on the interval $(-1, 1)$. Our control function is localized in a bounded open set $\mathcal{O}$ in the exterior of $(-1, 1)$, that is, $\mathcal{O} \subset \mathbb{R} \setminus (-1, 1)$. We show that there exists a minimal (strictly positive) time $T_{\min}$ such that the fractional heat dynamics can be controlled from any initial datum in $L^2(-1, 1)$ to a positive trajectory through the action of an exterior positive control, if and only if $1/2 < s < 1$. In addition, we prove that at this minimal controllability time, the constrained controllability is achieved by means of a control that belongs to a certain space of Radon measures. Finally, we provide several numerical illustrations that confirm our theoretical results.

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1. **Introduction.** In this paper, we are concerned with the constrained controllability from the exterior of the one-dimensional non-local heat equation associated with the fractional Laplacian on $(-1,1)$. More precisely, we consider the system

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T), \\ u = g\chi_{\mathcal{O}} & \text{in } (-1,1)^c\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } (-1,1), \end{cases} \tag{1}$$

where $u = u(x,t)$ is the state to be controlled, $0 < s < 1$ is a real number, $(-\partial_x^2)^s$ denotes the fractional Laplace operator defined for a sufficiently smooth function v by the following singular integral (see Section 2 for more details):

$$(-\partial_x^2)^s v(x) := C_s \, \text{P.V.} \int_{\mathbb{R}} \frac{v(x)-v(y)}{|x-y|^{1+2s}}\, dy, \quad x \in \mathbb{R},$$

and $g = g(x,t)$ is the exterior control function which is localized in a nonempty bounded open subset $\mathcal{O}$ of $(-1,1)^c := \mathbb{R}\setminus(-1,1)$.

Our principal goal is to analyze whether the parabolic equation (1) can be driven from any given initial datum $u_0 \in L^2(-1,1)$ to a desired final target by means of the control action, but preserving some non-negativity constraints on the control and/or the state.

Fractional order operators (in particular the fractional Laplace operator) have recently emerged as a modeling alternative in various branches of science. They usually describe anomalous diffusion. A number of stochastic models for explaining anomalous diffusion have been introduced in the literature. Among them we quote the fractional Brownian motion, the continuous time random walk, the Lévy flights, the Schneider gray Brownian motion, and more generally, random walk models based on evolution equations of single and distributed fractional order in space (see e.g. [20, 27, 38, 46]). In general, a fractional diffusion operator corresponds to a diverging jump length variance in the random walk. See also [5, 56] for the relevance of fractional operators in geophysics and imaging science.

In many PDEs models some constraints need to be imposed when considering concrete applications. This is for instance the case of diffusion processes (heat conduction, population dynamics, etc.) where realistic models have to take into account that the state represents some physical quantity which must necessarily remain positive (see e.g. [14]). This topic is also related to some other relevant applications, like the optimal management of compressors in gas transportation networks which requires the preservation of severe safety constraints (see e.g. [17, 39, 49]). Finally, this issue is also important in other PDEs problems based on scalar conservation laws, including (but not limited to) the Lighthill-Whitham and Richards traffic flow models ([16, 34, 42]) or the isentropic compressible Euler equation ([25]).

The controllability theory for PDEs has been developed principally without taking into account eventual constraints associated to the phenomenon described by the model under analysis. Actually, to the authors' knowledge, the literature on constrained controllability is currently very limited and the majority of the available results do not guarantee that controlled trajectories fulfill the physical restrictions of the processes under consideration.

In the context of the local heat equation, the problem of constrained controllability has been addressed in [35, 40] for the linear and semi-linear cases. In particular, in the mentioned references, the authors proved that, provided the control time is long enough, the linear and semi-linear local heat equations are controllable to any positive steady state or trajectory through the action of non-negative boundary controls. Moreover, for a positive initial datum, as a consequence of the maximum principle, the positivity of the state is preserved as well. On the other hand, these references, also showed the failure of the constrained controllability if the time horizon is too short.

We mention that the existence of a minimal time for constrained controllability may appear non-intuitive with respect to the unconstrained case, in which linear and semi-linear local parabolic systems are known to be controllable at any positive time. However, this is actually not surprising. Indeed, often times, norm-optimal controls allowing to reach the desired target are characterized by large oscillations in the proximity of the final time, which are enhanced when the time horizon of the control is small. This is due to the fact that those controls are restrictions of solutions of the adjoint system, and eventually leads to control trajectories that go beyond the physical thresholds and fail to fulfill the positivity constraint (see [26]). On the other hand, when the time interval is long, controls of small amplitude are allowed and we may expect the control property to be achieved through small deformations of the state and, in particular, preserving its positivity.

In addition to the results for heat-like equations, constrained controllability properties have been also analyzed for other classes of parabolic models appearing in the context of population dynamics. In particular, in [30, 36], it has been shown that the controllability of Lotka-McKendrick type systems with age structuring can be obtained by preserving the positivity of the state, once again in a long enough time horizon. These results have been recently extended in [37] to general infinite-dimensional systems with age structure.

The study of the controllability properties under positivity constraints is a very reasonable question for scalar-valued parabolic equations, which are canonical examples where the positivity is preserved for the free dynamics. Therefore, the issue of whether the system can be controlled in between two states by means of positive controls, by possibly preserving also the positivity of the controlled solution, arises naturally.

For completeness, we remark that constrained controllability properties have been analyzed also for hyperbolic models in [41]. There, the authors obtained the controllability to steady states and trajectories of the wave equation through the action of a positive control, acting either in the interior or on the boundary of the considered domain. Nevertheless, in that case control and state positivity are not interlinked. Indeed, because of the lack of a maximum principle, the sign of the control does not determine the sign of the solution whose positivity is no longer guaranteed.

In the context of the fractional heat equation, the analysis of controllability problems is still in its infancy and, essentially, limited to one-dimensional models. The results currently available are as follows:

- **Interior control**: In the absence of constraints, the interior controllability properties of the fractional heat equation have been analyzed in [8], where it was proved that the model is null-controllable with an L^2-control localized in any open set $\omega \subset (-1,1)$, and in any time $T > 0$, provided that $1/2 < s < 1$. This has been extended to the constrained controllability case in [10], where the authors have shown that the equation is null controllable (hence, controllable to trajectories) with positive L^∞-controls, for any $1/2 < s < 1$

and any open set $\omega \subset (-1,1)$, provided that the time horizon T for the null-controllability is sufficiently large.

- **Exterior control**: The exterior unconstrained controllability properties of (1) have been analyzed in [53] where the authors obtained analogous results to the ones in the aforementioned papers (that is, null-controllability in any time $T > 0$ if and only if $1/2 < s < 1$), but this time by means of an L^2-control function acting from the exterior of the domain where the PDE is satisfied.

The concept of exterior controllability for fractional models has been introduced in the literature only recently. In this regard, we shall recall that, as it has been shown in [52], a boundary control (that is, the case where the control g is localized in a subset of the boundary) does not make sense for the fractional Laplacian. This is due to the non-locality of the operator and the fact that the fractional models with boundary conditions (Dirichlet, Neumann or Robin) are ill-posed. For this reason, for problems involving the fractional Laplacian the correct notion of a boundary controllability is actually the exterior one, requiring that the control function must be localized outside the domain where the PDE is satisfied, as in the system (1).

Let us also mention that exterior control problems also appear in many realistic applications. Some examples of problems where it *may be of relevance* to place the control outside the domain where a PDE is fulfilled, noticing that currently local models are used to capture these applications, are:

(a) Magnetic drug delivery: The drug with ferromagnetic particles is injected in the body and an external magnetic field is used to steer it to a desired location.
(b) Acoustic testing: The aerospace structures are subjected to the sound from the loudspeakers.

We refer to [4, 6] and their references for a further discussion and the derivation of the exterior control. Let us also mention that the present work is the only available one on constrained controllability properties from the exterior for fractional evolution equations.

For completeness, we also mention that the controllability properties of the fractional heat equation in open subsets of $\mathbb{R}^N$ $(N \geq 2)$ are still not fully understood by the mathematical community. The classical tools (see e.g. [57] and the references therein) like the Carleman estimates usually used to study the controllability for heat equations are still not available for the fractional Laplacian in bounded domains (except in the whole space $\mathbb{R}^N$). For this reason, in the multi-dimensional case, the best possible controllability result currently available for fractional evolution equations is the approximate controllability (from the interior or the exterior) recently obtained in [51, 52]. However, there are multidimensional results on the interior [7] and the exterior optimal control problems [4, 6].

As we said above, the main concern of the present paper is to investigate if it is possible to control from the exterior of $(-1,1)$, the fractional heat dynamics (1) from any initial datum $u_0 \in L^2(-1,1)$ to any positive trajectory $\widehat{u}$, under positivity constraints on the control and/or the state. This delicate question has been formulated in [10] as an open problem. A complete answer of this question is provided in the present paper. In more detail, the key novelties and the specific results we obtained are as follows:

(i) Firstly, we show in Theorem 3.7 that if $1/2 < s < 1$, then the system (1) is controllable from any given initial datum in $L^2(-1,1)$ to zero (and, by translation, to trajectories) in any time $T > 0$ by means of L^∞-controls supported in $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c := \mathbb{R} \setminus [-1,1]$. This extends considerably the analysis of [53], where only the classical case of L^2-controls was considered. The proof will use the canonical approach of reducing the question of controllability with an L^∞-control to a dual observability problem in L^1, and the use of Fourier series expansions to obtain a new result on the L^1-observation of linear combinations of real exponentials. Notice that, contrary to the case of interior controls, for the exterior control, the L^1-observability inequality involves the non-local normal derivative (see (7)) of solutions to the adjoint equation. This normal derivative being a non-local operator makes the problem investigated here more challenging.

(ii) Secondly, as a consequence of our first result, in Theorem 2.4, we establish the existence of a minimal (strictly positive) time $T_{\min}$ such that the fractional heat dynamics (1) can be controlled to positive trajectories through the action of a positive L^∞-control. Moreover, if the initial datum is assumed to be positive as well, then the maximum principle guarantees the positivity of the states too.

(iii) Thirdly, we prove in Theorem 2.5 that, in the minimal controllability time $T_{\min}$, the controllability to positive trajectories holds through the action of a positive control in a space of Radon measures.

(iv) Finally, we mention that we have not been successful to have an analytic lower bound of the minimal controllability time $T_{\min}$. We accomplish this with the help of some numerical simulations in Section 5. Notice also that the mentioned numerical simulations confirm all our theoretical results. We emphasize that we impose the exterior condition using the approach introduced in [4, 6].

The rest of the paper is organized as follows. In the first part of Section 2 we fix some notations and state the main results of the paper. The first one (Theorem 2.4) shows that under a positivity constraint on the control, the system (1) is controllable to trajectories, and in addition, if the initial datum is non-negative, then the state is also non-negative. Furthermore, our second main result (Theorem 2.5) shows that the constrained controllability to trajectories in minimal time is achieved by controls which belong to a certain space of Radon measures. In the second part of Section 2 we recall some known results on fractional parabolic problems as they are needed throughout the article. In Section 3 we prove that there is a control function in $L^\infty(\mathcal{O}\times(0,T))$ (without any positivity constraint) such that the system (1) is null controllable in any time $T>0$ provided that $1/2<s<1$. Section 4 is devoted to the proofs of our main results. In Section 5 we provide numerical examples that confirm our theoretical findings. Finally, Section 6 is devoted to some final comments and open problems.

2. Notations, main results and preliminaries.

In this section we give some notations, state our main results and recall some known results as they are needed throughout the paper. We start by introducing the fractional order Sobolev spaces and by giving a rigorous definition of the fractional Laplace operator.

2.1. Fractional order Sobolev spaces and the fractional Laplace operator.

Let us introduce the function spaces needed to investigate our problems, that is, the fractional order Sobolev spaces. In what follows, we will only provide the definitions and properties which are relevant for our results. More complete presentations can be found in several references, including but not limited to [18, 28, 50].

Let $\Omega \subset \mathbb{R}$ be an arbitrary open set. We denote by $C_c(\overline{\Omega})$ the space of all continuous functions with compact support in $\overline{\Omega}$, and for $0 < \gamma \leq 1$, we let

$$C_c^{0,\gamma}(\overline{\Omega}) := \left\{ u \in C_c(\overline{\Omega}) : \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma} < +\infty \right\}.$$

Given $0 < s < 1$ we define

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\, dxdy < +\infty \right\},$$

and we endow it with the norm given by

$$\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u(x)|^2\, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\, dxdy \right)^{\frac{1}{2}}.$$

We set

$$\widetilde{H}_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}) : \; u = 0 \;\text{ in }\; \Omega^c \right\}.$$

We notice that if $\Omega \subset \mathbb{R}$ is a bounded interval and $0 < s \neq 1/2 < 1$, then by [28, Chapter 1], $\widetilde{H}_0^s(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^s(\Omega)}$ with equivalent norms.

It is well-known (see e.g. [18]) that we have the following continuous embedding: If $1/2 < s < 1$, then

$$\widetilde{H}_0^s(\Omega) \hookrightarrow C_0^{0,s-\frac{1}{2}}(\overline{\Omega}) := \left\{ u \in C^{0,s-\frac{1}{2}}(\overline{\Omega}) : \; u = 0 \text{ on } \partial\Omega \right\}. \tag{2}$$

We shall denote by $\widetilde{H}^{-s}(\Omega) = (\widetilde{H}_0^s(\Omega))^\star$ the dual of $\widetilde{H}_0^s(\Omega)$ with respect to the pivot space $L^2(\Omega)$. In that case we have the following continuous embeddings: $\widetilde{H}_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \widetilde{H}^{-s}(\Omega)$. We shall also let $\langle \cdot, \cdot \rangle_{-s,s}$ denote their duality pairing.

We notice that in most of our results, the open set Ω will be the bounded open interval $(-1, 1)$ or the control region $\mathcal{O}$.

Next, we give a rigorous definition of the fractional Laplace operator. Let

$$\mathcal{L}_s^1(\mathbb{R}) := \left\{ u : \mathbb{R} \to \mathbb{R} \;\text{ measurable and }\; \int_\mathbb{R} \frac{|u(x)|}{(1+|x|)^{1+2s}}\, dx < +\infty \right\}.$$

For $u \in \mathcal{L}_s^1(\mathbb{R})$ and $\varepsilon > 0$ we set

$$(-\partial_x^2)_\varepsilon^s u(x) := C_s \int_{\{y\in\mathbb{R}:\, |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dy, \quad x \in \mathbb{R},$$

where C_s is a normalization constant given by

$$C_s := \frac{s 2^{2s} \Gamma\left(\frac{2s+1}{2}\right)}{\pi^{\frac{1}{2}} \Gamma(1-s)}. \tag{3}$$

The *fractional Laplacian* $(-\partial_x^2)^s$ is defined by the following singular integral:

$$(-\partial_x^2)^s u(x) := C_s \,\text{P.V.} \int_\mathbb{R} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dy = \lim_{\varepsilon\downarrow 0} (-\partial_x^2)_\varepsilon^s u(x), \quad x \in \mathbb{R}, \tag{4}$$

provided that the limit exists for a.e. $x \in \mathbb{R}$.

For more details on the fractional Laplace operator we refer to [12, 18, 21, 22, 50] and their references.

Next, we consider the realization of $(-\partial_x^2)^s$ in $L^2(-1,1)$ with the zero Dirichlet exterior condition. That is, the operator (see e.g. [15, 48])

$$\begin{cases} D((-\partial_x^2)^s_D) := \Big\{ u \in \widetilde{H}^s_0(-1,1) : \ (-\partial_x^2)^s u \in L^2(-1,1) \Big\}, \\ (-\partial_x^2)^s_D u := ((-\partial_x^2)^s u)|_{(-1,1)}. \end{cases} \tag{5}$$

By [48], $(-\partial_x^2)^s_D$ has a compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. In addition, the eigenvalues are of finite multiplicity and are simple if $1/2 \le s < 1$ (see [31, Proposition 3]).

Let $(\varphi_n)_{n\in\mathbb{N}}$ be the orthonormal basis of eigenfunctions associated with $(\lambda_n)_{n\in\mathbb{N}}$. Then, $\varphi_n \in D((-\partial_x^2)^s_D)$ for every $n \in \mathbb{N}$, $(\varphi_n)_{n\in\mathbb{N}}$ is total in $L^2(-1,1)$ and solves the following Dirichlet problem:

$$\begin{cases} (-\partial_x^2)^s \varphi_n = \lambda_n \varphi_n & \text{in } (-1,1), \\ \varphi_n = 0 & \text{in } (-1,1)^c. \end{cases} \tag{6}$$

Next, for $u \in H^s(\mathbb{R})$ we introduce the *non-local normal derivative* $\mathcal{N}_s$ given by

$$\mathcal{N}_s u(x) := C_s \int_{-1}^{1} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dy, \quad x \in [-1,1]^c := \mathbb{R}\setminus[-1,1], \tag{7}$$

where C_s is the constant given in (3).

The following unique continuation property, which shall play an important role in the proof of our main results, has been recently obtained in [52, Theorem 16].

Lemma 2.1. *Let $\lambda > 0$ be a real number and $\mathcal{O} \subset (-1,1)^c$ an arbitrary nonempty open set. If $\varphi \in D((-\partial_x^2)^s_D)$ satisfies*

$$(-\partial_x^2)^s_D \varphi = \lambda\varphi \ \text{ in } (-1,1) \quad \text{and} \quad \mathcal{N}_s\varphi = 0 \ \text{ in } \mathcal{O},$$

then $\varphi = 0$ in $\mathbb{R}$.

For more details on the Dirichlet problem associated with the fractional Laplace operator we refer the interested reader to [9, 29, 43, 44, 52] and their references.

We conclude this section with the following integration by parts formula, whose proof may be found in [19, Lemma 3.3] (see also [52, 54]).

Lemma 2.2. *Let $u \in \widetilde{H}^s_0(-1,1)$ be such that $(-\partial_x^2)^s u \in L^2(-1,1)$ and $\mathcal{N}_s u \in L^2((-1,1)^c)$. Then, the identity*

$$\frac{C_s}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\, dxdy = \int_{-1}^{1} v(x)(-\partial_x^2)^s u(x)\, dx + \int_{(-1,1)^c} v(x)\mathcal{N}_s u(x)\, dx, \tag{8}$$

holds for every $v \in H^s(\mathbb{R})$.

2.2. **Main results.** In this section we state the main results of the paper. First, we introduce our notion of very weak solutions.

Definition 2.3. Let $g \in L^\infty((0,T);L^\infty(\mathcal{O}))$. We say that a function $u \in L^\infty((0,T);L^2(\mathbb{R}))$ is a very weak solution of (1) if $u = g$ a.e. in $\mathcal{O}\times(0,T)$ and the identity

$$0 = \int_0^T\int_{-1}^{1} \Big(-\partial_t\phi + (-\partial_x^2)^s\phi\Big) u\, dxdt + \int_{-1}^{1} u(x,T)\phi(x,T)dx - \int_{-1}^{1} u_0(x)\phi(x,0)dx$$

$$+\int_0^T\int_{\mathcal{O}} g\mathcal{N}_s\phi\,dxdt$$

holds for every

$$\phi\in L^2((0,T);D((-\partial_x^2)_D^s))\cap H^1((0,T);L^2(-1,1)).$$

Notice that if $\phi\in L^2((0,T);D((-\partial_x^2)_D^s))$, then, by Remark 3.3 below, $\mathcal{N}_s\phi$ exists and belongs to $L^2((0,T);L^2(\mathcal{O}))$, where we recall that $(-\partial_x^2)_D^s$ is the operator defined in (5). Additionally, in Theorem 2.6 below we guarantee the existence and uniqueness of solutions for (1) in the sense of Definition 2.3.

We start with our controllability to trajectories result of the system (1) with L^∞-controls and positivity constraints.

Theorem 2.4. *Let $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]^c$ be an arbitrary nonempty bounded open set and s a real number such that $1/2<s<1$. Let $0<\widehat{u}_0\in L^2(-1,1)$ and an exterior control $\widehat{g}\in L^\infty(\mathcal{O}\times(0,T))$ for which there is a positive constant α such that $\widehat{g}\geq\alpha$ a.e. in $\mathcal{O}\times(0,T)$. Consider a positive trajectory $\widehat{u}$ of (1) with initial datum $\widehat{u}_0$ and exterior condition $\widehat{g}$. Then, for every $u_0\in L^2(-1,1)$ there exist $T>0$ large enough and a non-negative control $g\in L^\infty(\mathcal{O}\times(0,T))$ such that the corresponding very weak solution u of (1) satisfies $u(\cdot,T)=\widehat{u}(\cdot,T)$ a.e. in $(-1,1)$. In addition, if $u_0\geq 0$ a.e. in $(-1,1)$, then $u\geq 0$ a.e. in $(-1,1)\times(0,T)$.*

Next, let $\mathcal{O}\subset(-1,1)^c$ be an arbitrary nonempty bounded open and $\mathcal{M}(\mathcal{O}\times(0,T))$ be the space of Radon measures on $\mathcal{O}\times(0,T)$. Then $\mathcal{M}(\mathcal{O}\times(0,T))$ endowed with the norm

$$\|\mu\|_{\mathcal{M}(\mathcal{O}\times(0,T))}:=\sup\left\{\int_{\mathcal{O}\times(0,T)}\xi(x,t)d\mu(x,t):\ \xi\in C(\overline{\mathcal{O}}\times[0,T]),\ \max_{\overline{\mathcal{O}}\times[0,T]}|\xi|=1\right\},$$

is a Banach space.

Moreover, since according to Theorem 2.4 the constrained controllability to trajectories holds true if the time horizon is large enough, let us define the minimal controllability time $T_{\min}$ by

$$T_{\min}:=\inf\Big\{T>0:\exists\, 0\leq g\in L^\infty((0,T);L^\infty(\mathcal{O}))\text{ such that }\ u(\cdot,T)=\widehat{u}(\cdot,T)\Big\}. \tag{9}$$

Our second main result shows that at $T=T_{\min}$, the controllability to trajectories of the system (1) is achieved with controls in $\mathcal{M}(\mathcal{O}\times(0,T))$.

Theorem 2.5. *Let the hypothesis of Theorem 2.4 hold with $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]^c$ an arbitrary nonempty bounded open set, and let $T:=T_{\min}$ be the minimal controllability time given by (9). Then, there exists a non–negative control $g\in\mathcal{M}(\mathcal{O}\times(0,T))$ such that the corresponding solution u of (1) satisfies $u(\cdot,T)=\widehat{u}(\cdot,T)$ a.e. in $(-1,1)$.*

2.3. **Well-posedness of the parabolic problems.** In this section we collect some well-known results contained in [52, 53, 55] regarding the well-posedness and the series representation of solutions to the system (1) and the associated dual system. In addition, we shall recall the maximum principle for fractional heat equations.

Throughout the remainder of the article, without any mention, $(\varphi_n)_{n\in\mathbb{N}}$ denotes the orthonormal basis of eigenfunctions of the operator $(-\partial_x^2)_D^s$ associated with the eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. If $u \in L^2(-1,1)$, then we shall let $u_n := (u, \varphi_n)_{L^2(-1,1)}$. Furthermore, for a given measurable set $E \subseteq \mathbb{R}^N$ $(N \geq 1)$, we shall denote by $(\cdot,\cdot)_{L^2(E)}$ the scalar product in $L^2(E)$.

With respect to the system (1), we have the following existence result and the explicit representation of solutions in terms of series. The proof can be found in [52, 53, 55] where they have used the notion of admissible control and observation operators.

Theorem 2.6. *Let $\mathcal{O} \subset (-1,1)^c$ be an arbitrary non-empty bounded open set and $0 < s < 1$. Then, for every $u_0 \in L^2(-1,1)$ and $g \in L^\infty((0,T); L^\infty(\mathcal{O}))$, the system (1) has a unique very weak solution u given by*

$$u(x,t) = \sum_{n=1}^{\infty} u_{0,n} e^{-\lambda_n t} \varphi_n(x) + \sum_{n=1}^{\infty} \left(\int_0^t (g(\cdot,\tau), \mathcal{N}_s \varphi_n)_{L^2(\mathcal{O})} e^{-\lambda_n (t-\tau)} d\tau \right) \varphi_n(x). \tag{10}$$

Remark 2.7. *Let us notice that following the strategy used for the local case $s = 1$ in the monograph [33], we can deduce that the time regularity of weak solutions of (1) can be improved, but one cannot get continuous solutions on the closed interval $[0,T]$ with value in $L^2(\Omega)$. We refer to [53, 55] for further discussions on this topic.*

Using the classical integration by parts formula, we have that the following backward system

$$\begin{cases} -\partial_t \psi + (-\partial_x^2)^s \psi = 0 & \text{in } (-1,1) \times (0,T), \\ \psi = 0 & \text{in } (-1,1)^c \times (0,T), \\ \psi(\cdot,T) = \psi_T & \text{in } (-1,1), \end{cases} \tag{11}$$

can be viewed as the dual system associated with (1).

Definition 2.8. Let $\psi_T \in L^2(-1,1)$. By a weak solution to (11), we mean a function

$$\psi \in C([0,T]; L^2(-1,1)) \cap L^2((0,T); \widetilde{H}_0^s(-1,1)) \cap H^1((0,T); \widetilde{H}^{-s}(-1,1)),$$

such that $\psi(\cdot,T) = \psi_T$ a.e. in $(-1,1)$, and the identify

$$-\langle \psi_t(\cdot,t), v \rangle_{-s,s} + \frac{C_s}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\psi(x,t) - \psi(y,t))(v(x) - v(y))}{|x-y|^{1+2s}} dx dy = 0$$

holds for every $v \in \widetilde{H}_0^s(-1,1)$ and almost every $t \in (0,T)$.

We have the following existence result (see e.g. [52, 53, 55]).

Theorem 2.9. *Let $\psi_T \in L^2(-1,1)$ and $0 < s < 1$. Then, the dual system (11) has a unique weak solution ψ which is given by*

$$\psi(x,t) = \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n (T-t)} \varphi_n(x). \tag{12}$$

In addition the following assertions hold.

1. *There is a constant $C > 0$ such that for all $t \in [0,T]$,*

$$\|\psi(\cdot,t)\|_{L^2(-1,1)} \leq C \|\psi_T\|_{L^2(-1,1)}.$$

2. *For every $t \in [0,T)$ fixed, $\mathcal{N}_s\psi(\cdot,t)$ exists, belongs to $L^2((-1,1)^c)$ and is given by*

$$\mathcal{N}_s\psi(x,t) = \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n(T-t)} \mathcal{N}_s\varphi_n(x). \tag{13}$$

In (12) *and* (13) *we have set $\psi_{T,n} := (\psi_T, \varphi_n)_{L^2(-1,1)}$.*

We conclude this section with the comparison principle taken from [3, Corollary 2.11]. This will be used in the proof of our main results.

Theorem 2.10. *Let u_0 and v_0 be such that $u_0 \geq v_0$ a.e. in $(-1,1)$ and let g, h be such that $g \geq h$ a.e. in $(-1,1)^c \times (0,T)$. Let u be the weak solution of* (1) *with initial datum u_0 and exterior datum g. Let v be the weak solution of* (1) *with initial datum v_0 and exterior datum h. Then $u \geq v$ a.e. in $(-1,1) \times (0,T)$.*

3. **Null controllability with L^∞-controls without constraints.** In this section we analyze the null controllability properties of (1) with control functions belonging to $L^\infty((0,T);L^\infty(\mathcal{O}))$, but without imposing any positivity constraint on the control and/or the state. These results shall play a crucial role in the proofs of our main theorem.

We start by introducing our notion of null controllability of the system (1) and an L^1-observability inequality for the associated dual system (11).

Definition 3.1. We say that the system (1) is null controllable in time $T > 0$, if for every $u_0 \in L^2(-1,1)$, there exists a control function $g \in L^\infty((0,T);L^\infty(\mathcal{O}))$ such that the associated unique very weak solution u satisfies

$$u(x,T) = 0 \quad \text{for a.e. } x \in (-1,1). \tag{14}$$

Definition 3.2. The system (11) is said to be L^1-observable in time $T > 0$, if there exists a constant $C = C(T) > 0$ such that the following observability inequality

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)} \leq C \left(\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|\, dxdt \right)^2 \tag{15}$$

holds for every $\psi_T \in L^2(-1,1)$, where ψ is the unique weak solution of (11) with final datum ψ_T, and $\mathcal{N}_s\psi$ is the non-local normal derivative of ψ given in (13).

Remark 3.3. We observe the following facts.

1. Firstly, it is worth mentioning that the weak solution ψ (see (12)) of the system (5) is indeed a strong solution of (5). In fact, ψ is given by

$$\psi(x,t) = \left(e^{-(T-t)(-\partial_x^2)_D^s} \psi_T \right)(x), \quad x \in (-1,1),\ t \in (0,T),$$

where $(e^{-t(-\partial_x^2)_D^s})_{t\geq 0}$ is the strongly continuous, analytic, and submarkovian semigroup on $L^2(-1,1)$ generated by the operator $-(-\partial_x^2)_D^s$ (see e.g. [15] for the result of generation of a submarkovian semigroup). Thus, it follows from semigroups theory ([13, Theorem 3.2.1]) that ψ enjoys the following regularity:

$$\psi \in C([0,T];L^2(-1,1)) \cap H^1((0,T);\widetilde{H}^{-s}(-1,1)) \cap C([0,T);D((-\partial_x^2)_D^s)).$$

In addition, if the final datum $\psi^T \in D((-\partial_x^2)_D^s)$, then ψ is a classical solution in the sense that

$$\psi \in C^1([0,T];L^2(-1,1)) \cap C([0,T];D((-\partial_x^2)_D^s))$$

and the equation is satisfied for every $t \in [0,T]$.

2. Secondly, let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c$, where $\mathcal{O}$ is an arbitrary nonempty bounded open set. Notice that the observability inequality (15) makes sense. In fact, since $\psi(t,\cdot) \in \widetilde{H}_0^s(-1,1)$ for a.e. $t \in (0,T)$, it follows from [24, Lemma 3.2] (see also [23]) that $\mathcal{N}_s\psi(t,\cdot) \in H^s_{\text{loc}}((-1,1)^c)$, and hence, belongs to $L^2(\mathcal{O})$ for a.e. $t \in (0,T)$. We shall show in Lemma 4.3 below that $\mathcal{N}_s\psi \in L^\infty((0,T);L^\infty(\mathcal{O}))$.

We have the following result.

Theorem 3.4. *Let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c$ be an arbitrary nonempty bounded open set and $1/2 < s < 1$. Then the following assertions are equivalent.*

1. *For every $u_0 \in L^2(-1,1)$ and $T > 0$, the system (1) is null controllable in time $T > 0$. Moreover, there is a constant $C_1 > 0$ (independent of u_0) such that the control g satisfies the following estimate:*
$$\|g\|_{L^\infty((0,T);L^\infty(\mathcal{O}))} \le C_1\|u_0\|_{L^2(-1,1)}. \tag{16}$$
2. *For every $T > 0$ and $\psi_T \in L^2(-1,1)$ the dual system (11) is L^1-observable.*

Proof. (a) $\Rightarrow$ (b): Assume that (1) is null controllable in time $T > 0$. Then there exists a control function $g \in L^\infty((0,T);L^\infty(\mathcal{O}))$ such that (14) holds. Let ψ be the unique weak solution of (11) with $\psi_T \in L^2(-1,1)$. Taking $\phi = \psi$ as a test function in Definition 2.3 of solutions to (1), we get that

$$\int_{-1}^1 u_0(x)\psi(x,0)dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)\,dxdt. \tag{17}$$

Letting $u_0(x) := \psi(x,0)$ in (17) and using the Hölder inequality we obtain that

$$\begin{aligned}\int_{-1}^1 |\psi(x,0)|^2dx &= \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)\,dxdt\\ &\le \|g\|_{L^\infty((0,T);L^\infty(\mathcal{O}))}\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|\,dxdt. \end{aligned}\tag{18}$$

Using Young's inequality and (16) we get from (18) that

$$\begin{aligned}\int_{-1}^1 |\psi(x,0)|^2dx &\le \frac{1}{2\varepsilon}\|g\|^2_{L^\infty((0,T);L^\infty(\mathcal{O}))} + \frac{\varepsilon}{2}\left(\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|\,dxdt\right)^2\\ &\le \frac{C_1}{2\varepsilon}\|u_0\|^2_{L^2(-1,1)} + \frac{\varepsilon}{2}\left(\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|\,dxdt\right)^2\\ &\le \frac{C_1}{2\varepsilon}\|\psi(\cdot,0)\|^2_{L^2(-1,1)} + \frac{\varepsilon}{2}\left(\int_0^T \int_{\mathcal{O}} |\mathcal{N}_s\psi(x,t)|\,dxdt\right)^2\end{aligned}$$

for every $\varepsilon > 0$, where the last estimate follows from the fact that $u_0(x) = \psi(x,0)$. Taking $\varepsilon := C_1$ in the preceding estimate, we can deduce that (15) holds.

(b) $\Rightarrow$ (a): We have to show that (15) implies the null controllability of (1). Let $\psi_T \in L^2(-1,1)$ and ψ the associated unique solution of the dual system (11). Taking $\phi = \psi$ in Definition 2.3 of solutions to (1), we get that for every $\psi_T \in L^2(-1,1)$ and $u_0 \in L^2(-1,1)$,

$$\int_{-1}^1 u_0(x)\psi(x,0)dx - \int_{-1}^1 u(x,T)\psi_T(x)dx = \int_0^T \int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)\,dxdt. \tag{19}$$

Let us consider the linear subspace Λ of $L^1((0,T);L^1(\mathcal{O}))$ given by:

$$\Lambda := \left\{ \mathcal{N}_s\psi \Big|_{\mathcal{O}\times(0,T)} \; : \; \psi \text{ solves (11) with } \psi_T \in L^2(-1,1) \right\}.$$

Let $u_0 \in L^2(-1,1)$ and consider the linear functional $F : \Lambda \to \mathbb{R}$ defined by

$$F(\mathcal{N}_s\psi) := (u_0, \psi(\cdot,0))_{L^2(-1,1)}.$$

Since $\mathcal{O}$ is bounded and $\mathcal{N}_s\psi \in L^2(\mathcal{O}\times(0,T)) \hookrightarrow L^1(\mathcal{O}\times(0,T))$, it follows from the observability inequality (15) that there is a constant $C > 0$ such that

$$\begin{aligned} |F(\mathcal{N}_s\psi)| &\le \|u_0\|_{L^2(-1,1)}\|\psi(\cdot,0)\|_{L^2(-1,1)} \le C\|u_0\|_{L^2(-1,1)}\|\mathcal{N}_s\psi\|_{L^1(\mathcal{O}\times(0,T))} \\ &\le C\|u_0\|_{L^2(-1,1)}\|\mathcal{N}_s\psi\|_{\Lambda}. \end{aligned}$$

We have shown that F is well defined and bounded on the set Λ.

By the Hahn-Banach Theorem, the functional F can be extended to a bounded linear functional $\widetilde{F} : L^1((0,T);L^1(\mathcal{O})) \to \mathbb{R}$ such that

$$|\widetilde{F}v| \le C_1\|u_0\|_{L^2(-1,1)}\|v\|_{L^1((0,T);L^1(\mathcal{O}))}, \quad \forall\, v \in L^1((0,T);L^1(\mathcal{O})).$$

Moreover, by the Riesz representation Theorem in L^p-spaces ([11, Chapter 4, Theorem 4.14]), there is a function $g \in L^\infty((0,T);L^\infty(\mathcal{O})) = \Big(L^1((0,T);L^1(\mathcal{O}))\Big)^\star = \Big(L^1(\mathcal{O}\times(0,T))\Big)^\star$ such that

$$\|g\|_{L^\infty((0,T);L^\infty(\mathcal{O}))} \le C\|u_0\|_{L^2(-1,1)}$$

and

$$\widetilde{F}(\xi) = \int_0^T\int_{\mathcal{O}} g(x,t)\xi(x,t)dxdt, \quad \forall\, \xi \in L^1((0,T);L^1(\mathcal{O})). \tag{20}$$

Notice that $\mathcal{N}_s\psi \in \Lambda$. Thus, using the definition of F we get from (20) that

$$F(\mathcal{N}_s\psi) = \int_0^T\int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)dxdt = (u_0,\psi(\cdot,0))_{L^2(-1,1)},$$

for every $\psi_T \in L^2(-1,1)$. We have shown that there is a control $g \in L^\infty((0,T); L^\infty(\mathcal{O}))$ such that (16) is satisfied and

$$\int_{-1}^1 u_0(x)\psi(x,0)dx = \int_0^T\int_{\mathcal{O}} g(x,t)\mathcal{N}_s\psi(x,t)dxdt \tag{21}$$

for every $\psi_T \in L^2(-1,1)$. It follows from (19) and (21) that

$$\int_{-1}^1 u(x,T)\psi_T(x)dx = 0$$

for every $\psi_T \in L^2(-1,1)$. Thus, $u(x,T) = 0$ for a.e. $x \in (-1,1)$. The proof is finished. □

The results in Theorem 3.4 show that, in order to obtain the null controllability of the system (1), it is enough to prove the L^1-observability inequality (15). To do this, we need first to establish some auxiliaries results.

We start with the following Ingham-type result recently obtained in [10, Theorem 2.4].

Theorem 3.5. *Let $(\mu_n)_{n\ge1} \subset [0,\infty)$ be a sequence satisfying the following conditions:*

1. *There exists $\gamma > 0$ such that $\mu_{n+1} - \mu_n \ge \gamma$ for all $n \ge 1$.*

2. $\displaystyle\sum_{n\geq 1}\frac{1}{\mu_n}<\infty.$

Then, for any $T>0$, there is a constant $C(T)>0$ such that, for any sequence $(c_n)_{n\geq 1}$ of numbers it holds the inequality:

$$\sum_{n\geq 1}|c_n|e^{-\mu_n T}\leq C(T)\left\|\sum_{n\geq 1}c_n e^{-\mu_n t}\right\|_{L^1(0,T)}. \tag{22}$$

Moreover, $C(T)$ is uniformly bounded away from $T=0$ and blows-up exponentially as $T\downarrow 0^+$.

The second auxiliary and technical result we shall need is adapted from the results contained in [53]. In fact, by [53], $\|\mathcal{N}_s\varphi_n\|_{L^2(\mathcal{O})}$ is uniformly bounded from below, where $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]$ is an arbitrary bounded open set. In the settings of the present paper, we shall need a similar estimate but for the L^1-norm.

Lemma 3.6. *Let $1/2<s<1$. Then, for every nonempty bounded open set $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]^c$, there exists a constant $\eta>0$ such that for every $k\in\mathbb{N}$, $\mathcal{N}_s\varphi_k$ is uniformly bounded from below by η in $L^1(\mathcal{O})$. Namely,*

$$\exists\,\eta>0 \text{ such that } \forall\, k\in\mathbb{N},\ \|\mathcal{N}_s\varphi_k\|_{L^1(\mathcal{O})}\geq\eta. \tag{23}$$

Proof. We divide the proof in several steps. Let $1/2<s<1$.

Step 1: Firstly, since $\varphi_k=0$ in $(-1,1)^c$ for every $k\in\mathbb{N}$, it follows from the definition of $(-\partial_x^2)^s$ and $\mathcal{N}_s$ that for almost every $x\in\mathcal{O}\subseteq(-1,1)^c$, we have

$$\begin{aligned}(-\partial_x^2)^s\varphi_k(x)&=C_s\text{P.V.}\int_{\mathbb{R}}\frac{\varphi_k(x)-\varphi_k(y)}{|x-y|^{1+2s}}\,dy=C_s\int_{-1}^{1}\frac{\varphi_k(x)-\varphi_k(y)}{|x-y|^{1+2s}}\,dy\\&=\mathcal{N}_s\varphi_k(x).\end{aligned} \tag{24}$$

We have shown that $(\mathcal{N}_s\varphi_k)|_{\mathcal{O}}=((-\partial_x^2)^s\varphi_k)|_{\mathcal{O}}$ for every $k\in\mathbb{N}$.

Secondly, let us introduce the auxiliary function $q:\mathbb{R}\to[0,\infty)$ defined by:

$$q(x):=\begin{cases}0 & x\in\left(-\infty,-\frac{1}{3}\right),\\ \dfrac{9}{2}\left(x+\dfrac{1}{3}\right)^2 & x\in\left(-\frac{1}{3},0\right),\\ 1-\dfrac{9}{2}\left(x-\dfrac{1}{3}\right)^2 & x\in\left(0,\frac{1}{3}\right),\\ 1 & x\in\left(\frac{1}{3},+\infty\right).\end{cases} \tag{25}$$

For any $\alpha>0$, we define the function $F_\alpha:\mathbb{R}\to\mathbb{R}$ as follows:

$$F_\alpha(x)=F(\alpha x):=\sin\left(\alpha x+\frac{(1-s)\pi}{4}\right)-G(\alpha x),$$

where G is the Laplace transform of the function

$$\gamma(y):=\frac{\sqrt{4s}\sin(s\pi)}{2\pi}\frac{y^{2s}}{1+y^{4s}-2y^{2s}\cos(s\pi)}\exp\left(\frac{1}{\pi}\int_0^{+\infty}\frac{1}{1+r^2}\log\left(\frac{1-r^{2s}y^{2s}}{1-r^2y^2}\right)dr\right).$$

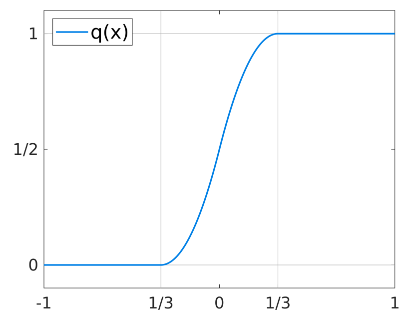

FIGURE 1. Graphic of the function $q(x)$

Next, we define the sequence of real numbers

$$\mu_k := \frac{k\pi}{2} - \frac{(1-s)\pi}{4}, \quad k \geq 1.$$

It has been shown in [32, Example 6.1] that F_{μ_k} is the solution of the system

$$\begin{cases} (-\partial_x^2)^s F_{\mu_k}(x) = \mu_k F_{\mu_k}(x) & x > 0, \\ F_{\mu_k}(x) = 0 & x \leq 0. \end{cases}$$

In other words, $\{F_{\mu_k}\}_{k\geq 1}$ are the eigenfunctions of $(-\partial_x^2)^s$ on the interval $(0,\infty)$ with the zero Dirichlet exterior condition, and $\{\mu_k\}_{k\geq 1}$ are the corresponding eigenvalues. Let us now define

$$\varrho_k(x) := q(-x)F_{\mu_k}(1+x) + (-1)^k F_{\mu_k}(1-x), \quad x \in \mathbb{R}, \quad k \geq 1.$$

Notice that $F_{\mu_k}(1+x) = 0$ for $x \leq -1$ and $F_{\mu_k}(1-x) = 0$ for $x \geq 1$. This fact, together with the definition (25) of the function q imply that, for all $k \geq 1$, $\varrho_k(x) = 0$ for $x \in \mathbb{R} \setminus (-1,1)$. In addition, it follows from [31, Lemma 1] that $\{\varrho_k\}_{k\in\mathbb{N}} \subset D((-\partial_x^2)_D^s)$ and there is a constant $C > 0$ such that

$$\left|(-\partial_x^2)^s \varrho_k(x) - \mu_k^{2s} \varrho_k(x)\right| \leq \frac{C(1-s)}{\sqrt{2s}} \mu_k^{-1}, \quad \text{for all} \quad x \in (-1,1),\ k \geq 1.$$

Furthermore, by [31, Proposition 1], there is a constant $C > 0$ such that for every $k \geq 1$, we have

$$\|\varrho_k - \varphi_k\|_{L^2(-1,1)} \leq \frac{C(1-s)}{k}.$$

Step 2: Now, let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]$ be an arbitrary nonempty bounded open set and assume that for every $\eta > 0$ there exists $k \in \mathbb{N}$ such that

$$\|\mathcal{N}_s \varphi_k\|_{L^1(\mathcal{O})} < \eta. \tag{26}$$

Since $1/2 < s < 1$, from (2), we have the continuous and dense embedding

$$\widetilde{H}_0^s(\mathcal{O}) \hookrightarrow C_0(\overline{\mathcal{O}}) := \left\{u \in C(\overline{\mathcal{O}}) : \ u = 0 \text{ on } \ \partial\mathcal{O}\right\},$$

which by duality, implies that $\mathcal{M}(\mathcal{O}) \hookrightarrow H^{-s}(\mathcal{O})$. Since $L^1(\mathcal{O}) \hookrightarrow \mathcal{M}(\mathcal{O})$, we can deduce from this and (26), that there are a constant $C > 0$ (independent of n) and a subsequence $(k_n)_{n\in\mathbb{N}}$ such that for n large enough,

$$\|\mathcal{N}_s \varphi_{k_n}\|_{\widetilde{H}^{-s}(\mathcal{O})} \leq \frac{C}{n}. \tag{27}$$

Step 3: Using the triangle inequality and the fact that the fractional Laplacian defines an isomorphism between $\widetilde{H}_0^s(-1,1)$ and $\widetilde{H}^{-s}(-1,1)$, we get that there is a constant $C>0$ such that

$$\begin{aligned}\|\varrho_{k_n}-\varphi_{k_n}\|^2_{\widetilde{H}_0^s(-1,1)} &\leq C\left\|(-\partial_x^2)^s\varrho_{k_n}-(-\partial_x^2)^s\varphi_{k_n}\right\|^2_{H^{-s}(-1,1)}\\ &\leq C\left\|(-\partial_x^2)^s\varrho_{k_n}-(-\partial_x^2)^s\varphi_{k_n}\right\|^2_{L^2(-1,1)}\\ &\leq C\Big(\left\|(-\partial_x^2)^s\varrho_{k_n}-\mu_{k_n}^{2s}\varrho_{k_n}\right\|^2_{L^2(-1,1)}\\ &\quad+\left\|\varrho_{k_n}(\mu_{k_n}^{2s}-\lambda_{k_n})\right\|^2_{L^2(-1,1)}\\ &\quad+\left\|\lambda_{k_n}\varrho_{k_n}-(-\partial_x^2)^s\varphi_{k_n}\right\|^2_{L^2(-1,1)}\Big). \end{aligned}\tag{28}$$

Then, by (28) and Step 1, we have that there is a constant $C_{k_n}(s)>0$ which converges to zero as $n\to+\infty$, such that

$$\|\varrho_{k_n}-\varphi_{k_n}\|^2_{\widetilde{H}_0^s(-1,1)}\leq C_{k_n}(s).$$

Let the operator L be defined by

$$L:\widetilde{H}_0^s(-1,1)\to\widetilde{H}^{-s}(\mathcal{O}),\ v\mapsto Lv:=((-\partial_x^2)^s v)|_{\mathcal{O}}=(\mathcal{N}_s v)|_{\mathcal{O}}.$$

By [23, Lemma 2.2], the operator L is compact, injective with dense range. Let $B_1:=\overline{B}\left(\varrho_{k_n},C_{k_n}(s)\right)$ be the closed ball in $\widetilde{H}_0^s(-1,1)$ with center in ϱ_{k_n} and radius $C_{k_n}(s)$. Since L is a compact operator, we have that the image of B_1, namely $L(B_1)$, is totally bounded in $\widetilde{H}^{-s}(\mathcal{O})$. Therefore, there exists $N\in\mathbb{N}$ and $\{\psi_1,\dots,\psi_N\}\subseteq B_1$ such that for every $\varepsilon>0$ we have

$$L(B_1)\subseteq\bigcup_{j=1}^{N}\overline{B}_{\widetilde{H}^{-s}(\mathcal{O})}(L(\psi_j),\varepsilon).$$

We notice that φ_{k_n} belongs to B_1. Thus, there exists $j\in\{1,\dots,N\}$ such that

$$L(\varphi_{k_n})\in\overline{B}_{\widetilde{H}^{-s}(\mathcal{O})}(L(\psi_j),\varepsilon).$$

We have shown that for n large enough,

$$\|L(\varphi_{k_n})-L(\psi_j)\|_{\widetilde{H}^{-s}(\mathcal{O})}\leq\varepsilon.$$

Since $\psi_j\in B_1$, firstly we obtain that $\varphi_{k_n}\to\psi_j$, as $n\to\infty$ in $\widetilde{H}_0^s(-1,1)$ and secondly, we have that ψ_j is an element of the spectrum $\{(\varphi_k,\lambda_k)\}_{k\geq1}$. That is, ψ_j is a solution of (6). Finally, as $L(\varphi_{k_n})$ converges to zero in $\widetilde{H}^{-s}(\mathcal{O})$ (by (27)), we can deduce that $L(\psi_j)=\mathcal{N}_s\psi_j=(-\partial_x^2)^s\psi_j=0$ a.e. in $\mathcal{O}$. It follows from Lemma 2.1 (see also [23]) that $\psi_j=0$ a.e. in $\mathbb{R}$, which is a contradiction. The proof is finished. □

Now we can state and prove the main result of this section.

Theorem 3.7. *Let $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]$ be an arbitrary nonempty bounded open set. Then, for every $u_0\in L^2(-1,1)$, $1/2<s<1$ and $T>0$, there exists a control function $g\in L^\infty(\mathcal{O}\times(0,T))$ such that the corresponding unique very weak solution u of (1) satisfies $u(x,T)=0$ for a.e. $x\in(-1,1)$. In addition, there is a constant $C=C(T)>0$ such that the control function g satisfies*

$$\|g\|_{L^\infty((0,T);L^\infty(\mathcal{O}))}\leq C\|u_0\|_{L^2(-1,1)}.\tag{29}$$

Proof. Recall that by Theorem 3.4, the null controllability of (1) together with (29), is equivalent to the L^1-observability inequality (15). Therefore, we shall prove that (15) holds.

Let $T > 0$, $\psi_T \in L^2(-1,1)$ and let $\psi \in C([0,T]; L^2(-1,1))$ be the associated unique weak solution of the dual system (11). It follows from Theorem 2.9 that

$$\psi(x,t) = \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n (T-t)} \varphi_n(x) \quad \text{and} \quad \mathcal{N}_s \psi(x,t) = \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n (T-t)} \mathcal{N}_s \varphi_n(x),$$

where we recall that $\psi_{T,n} := (\psi_T, \varphi_n)_{L^2(-1,1)}$.

Using the fact that $(\varphi_n)_{n\geq 1}$ is an orthonormal basis in $L^2(-1,1)$, we have that the L^1-observability inequality (15) becomes

$$\sum_{n=1}^{\infty} |\psi_{T,n}|^2 e^{-2\lambda_n T} \leq C(T) \left(\int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n (T-t)} \mathcal{N}_s \varphi_n(x) \right| dxdt \right)^2. \tag{30}$$

Using the change of variable $T - t \mapsto t$, we get from (30) that

$$\sum_{n=1}^{\infty} |\psi_{T,n}|^2 e^{-2\lambda_n T} \leq C(T) \left(\int_0^T \int_{\mathcal{O}} \left| \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n t} \mathcal{N}_s \varphi_n(x) \right| dxdt \right)^2. \tag{31}$$

We observe that $(\lambda_n)_{n\in\mathbb{N}}$ are simple (since we have assumed that $1/2 < s < 1$) and the following asymptotics hold (see e.g. [31]):

$$\lambda_n = \left(\frac{n\pi}{2} - \frac{(2-2s)\pi}{8} \right)^{2s} + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{32}$$

Therefore, letting $\mu_n := \lambda_n$ we have that the conditions (a) and (b) in Theorem 3.5 are both satisfied. Thus, we can deduce that (22) holds with c_n replaced with $\psi_{T,n}$.

Now, by [47, Section 8, page 28, Equation (8.i)] and [47, Section 9, page 33, Theorem I], we have that for almost every fixed $x \in \mathcal{O}$, there exists a constant $C(T) > 0$ which is uniformly bounded away from $T = 0$, such that

$$\sum_{n=1}^{\infty} |\psi_{T,n} \mathcal{N}_s \varphi_n(x)| e^{-\lambda_n T} \leq C(T) \int_0^T \left| \sum_{n=1}^{\infty} \psi_{T,n} \mathcal{N}_s \varphi_n(x) e^{-\lambda_n t} \right| dt. \tag{33}$$

By Lemma 3.6, $\|\mathcal{N}_s \varphi_n\|_{L^1(\mathcal{O})} \geq \eta > 0$. Thus, integrating (33) over $\mathcal{O}$ and using (23) we can deduce that

$$\eta \sum_{n=1}^{\infty} |\psi_{T,n}| e^{-\lambda_n T} \leq C(T) \int_{\mathcal{O}} \int_0^T \left| \sum_{n=1}^{\infty} \psi_{T,n} \mathcal{N}_s \varphi_n(x) e^{-\lambda_n t} \right| dtdx. \tag{34}$$

Since

$$\sum_{n=1}^{\infty} |\psi_{T,n}|^2 e^{-2\lambda_n T} \leq \left(\sum_{n=1}^{\infty} |\psi_{T,n}| e^{-\lambda_n T} \right)^2,$$

it follows from (34) that

$$\begin{aligned} \eta^2 \sum_{n=1}^{\infty} |\psi_{T,n}|^2 e^{-2\lambda_n T} &\leq \eta^2 \left(\sum_{n=1}^{\infty} |\psi_{T,n}| e^{-\lambda_n T} \right)^2 \\ &\leq C(T)^2 \left(\int_{\mathcal{O}} \int_0^T \left| \sum_{n=1}^{\infty} \psi_{T,n} \mathcal{N}_s \varphi_n(x) e^{-\lambda_n t} \right| dtdx \right)^2. \end{aligned}$$

Finally, using Fubini's theorem we get that

$$\sum_{n=1}^{\infty}|\psi_{T,n}|^2e^{-2\lambda_nT}\le\frac{C(T)^2}{\eta^2}\left(\int_0^T\int_{\mathcal{O}}\left|\sum_{n=1}^{\infty}\psi_{T,n}e^{-\lambda_nt}\mathcal{N}_s\varphi_n(x)\right|dxdt\right)^2.$$

We have shown that the L^1-observability inequality (15) holds. The proof is finished.

We conclude this section with the following observation. □

Remark 3.8. We mention the following facts.

1. We observe that since the constant $C(T)$ in (22) blows up exponentially as $T\downarrow 0^+$, we have that the constant in the L^1-observability inequality (15) also blows up exponentially as $T\downarrow 0^+$. This is consistent with the case of the interior control and the classical local case $s=1$, where the same phenomena occurs.
2. If $0<s\le 1/2$, then the eigenvalues $(\lambda_n)_{n\ge1}$ do not satisfy the conditions (a) and (b) in Theorem 3.5. Thus, in this case, one can deduce that the null-controllability result in Theorem 3.7 does not hold.

4. **Proofs of the main results.** In this section we give the proofs of the main results stated in Section 2.2.

Proof of Theorem 2.4. Due to the linearity of (1), and considering $z:=u-\widehat{u}$ a solution of

$$\begin{cases}\partial_tz+(-\partial_x^2)^sz=0 & \text{in } (-1,1)\times(0,T),\\ z=h\chi_{\mathcal{O}} & \text{in } (-1,1)^c\times(0,T),\\ z(\cdot,0)=u_0-\widehat{u}_0 & \text{in } (-1,1),\end{cases}\tag{35}$$

with $h:=g-\widehat{g}$, it is enough to prove that there exist $T>0$ and a control $h\in L^\infty((0,T);L^\infty(\mathcal{O}))$ fulfilling $h\ge-\alpha$ a.e. in $\mathcal{O}\times(0,T)$ such that $z(\cdot,T)=0$ a.e. in $(-1,1)$.

By Theorem 3.7, the null controllability of (35) with $h\in L^\infty((0,T);L^\infty(\mathcal{O}))$ is equivalent to (15). We observe that the L^1-observability inequality (15) is independent of the time interval. For that reason we can also consider the interval (t_0,T), for $t_0\in(0,T)$. Therefore, the L^1-observability inequality (15) becomes

$$\|\psi(\cdot,t_0)\|^2_{L^2(-1,1)}\le C(T-t_0)\left(\int_{t_0}^T\int_{\mathcal{O}}|\mathcal{N}_s\psi(x,t)|dxdt\right)^2.\tag{36}$$

It follows from (12) that

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)}\le\sum_{n=1}^{\infty}|\psi_{T,n}|^2e^{-2\lambda_nT}=\sum_{n=1}^{\infty}|\psi_{T,n}|^2e^{-2\lambda_n(T-t_0)}e^{-2\lambda_nt_0},\tag{37}$$

where $\psi_{T,n}:=(\psi_T,\varphi_n)_{L^2(-1,1)}$. Since $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$, it follows from (37) that

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)}\le e^{-2\lambda_1t_0}\sum_{n=1}^{\infty}|\psi_{T,n}|^2e^{-2\lambda_n(T-t_0)}=e^{-2\lambda_1t_0}\|\psi(\cdot,t_0)\|^2_{L^2(-1,1)}.\tag{38}$$

Combining (38)-(36) we get that

$$\|\psi(\cdot,0)\|^2_{L^2(-1,1)}\le e^{-2\lambda_1t_0}C(T-t_0)\left(\int_{t_0}^T\int_{\mathcal{O}}|\mathcal{N}_s\psi(x,t)|dxdt\right)^2.\tag{39}$$

By Theorem 3.4, (39) is equivalent to the existence of $h \in L^\infty((0,T);L^\infty(\mathcal{O}))$ such that

$$\|h\|^2_{L^\infty((0,T);L^\infty(\mathcal{O}))} \le e^{-2\lambda_1 t_0} C(T-t_0)\|u_0 - \widehat{u}_0\|^2_{L^2(-1,1)}. \tag{40}$$

Taking $t_0 := T/2$ and using the fact that the L^1-observability constant $C(T)$ is uniformly bounded away from $T = 0$, we can deduce from (40) that for T large enough,

$$\|h\|^2_{L^\infty(\mathcal{O}\times(0,T))} \le \alpha^2. \tag{41}$$

The estimate (41) implies that $h \ge -\alpha$ a.e. in $\mathcal{O}\times(0,T)$. We have constructed an exterior control $h \in L^\infty((0,T);L^\infty(\mathcal{O}))$ fulfilling the constraint $h \ge -\alpha$ a.e. in $\mathcal{O}\times(0,T)$, and is such that the solution z of (35) satisfies $z(\cdot,T) = 0$ a.e. in $(-1,1)$ for T large enough. If $u_0 \ge 0$, then from Theorem 2.10, we have that $u \ge 0$ a.e. in $(-1,1)\times(0,T)$. The proof is finished. □

Remark 4.1. We notice that in order for the controllability to trajectories result in Theorem 2.4 to hold, the control time T must be large enough. This is due to the positivity constraints imposed on the control function.

Before we proceed with the proof of the last main result, we need some preparations.

Lemma 4.2. *Let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c$ be an arbitrary nonempty bounded open set. Then, there are two constants $0 < C_1 \le C_2$ such that for every $x \in \mathcal{O}$, we have*

$$C_1 \le \int_{-1}^{1} \frac{dy}{|x-y|^{1+2s}} \le C_2. \tag{42}$$

Proof. Since $\overline{\mathcal{O}} \subset [-1,1]^c$, we have that there are two constants $1 < a \le b$ such that $1 < a \le |x| \le b$ for every $x \in \mathcal{O}$. Thus, we have the following two cases.

Case 1: $1 < a \le x \le b$. A simple calculation gives

$$\int_{-1}^{1} \frac{dy}{|x-y|^{1+2s}} = \frac{1}{2s}\left(\frac{1}{(x-1)^{2s}} - \frac{1}{(x+1)^{2s}}\right).$$

Define $f : [a,b] \to [0,\infty)$ by $f(x) := \frac{1}{2s}\left(\frac{1}{(x-1)^{2s}} - \frac{1}{(x+1)^{2s}}\right)$. Then, f is decreasing. Thus

$$f(b) \le f(x) \le f(a) \ \text{ for every } \ a \le x \le b. \tag{43}$$

Case 2: $-b \le x \le -a < -1$. Then

$$\int_{-1}^{1} \frac{dy}{|x-y|^{1+2s}} = \frac{1}{2s}\left(\frac{1}{(-1-x)^{2s}} - \frac{1}{(1-x)^{2s}}\right).$$

Define $\widetilde{f} : [-b,-a] \to [0,\infty)$ by $\widetilde{f}(x) := \frac{1}{2s}\left(\frac{1}{(-1-x)^{2s}} - \frac{1}{(1-x)^{2s}}\right)$. Then, $\widetilde{f}$ is increasing. Thus

$$\widetilde{f}(-b) \le f(x) \le \widetilde{f}(-a) \ \text{ for every } \ -b \le x \le -a. \tag{44}$$

Now (42) follows from (43) and (44). The proof is finished. □

Next, we recall that the non-local normal derivative of the solution ψ to the adjoint system (11) is given by

$$\mathcal{N}_s\psi(x,t) = \sum_{n=1}^{\infty} \psi_{T,n} e^{-\lambda_n(T-t)} \mathcal{N}_s\varphi_n(x). \tag{45}$$

We have the following result.

Lemma 4.3. *Let $\mathcal{O} \subset \overline{\mathcal{O}} \subset [-1,1]^c$ be an arbitrary nonempty bounded open set. Let ψ be the unique weak solution of the dual system (11). If $\psi_T \in L^2(-1,1)$, then $\mathcal{N}_s\psi \in L^\infty(\mathcal{O}\times(0,T))$.*

Proof. Firstly, notice that by definition (see (7)) we have that

$$\mathcal{N}_s\psi(x,t) = C_s\int_{-1}^1 \frac{\psi(x,t)-\psi(y,t)}{|x-y|^{1+2s}}\,dy, \;\; x\in[-1,1]^c, \; t\in(0,T).$$

Let $(x,t) \in \mathcal{O}\times(0,T) \subset [-1,1]^c\times(0,T)$. Since $\psi = 0$ in $(-1,1)^c\times(0,T)$, it follows from the previous identity that

$$\mathcal{N}_s\psi(x,t) = -C_s\int_{-1}^1 \frac{\psi(y,t)}{|x-y|^{1+2s}}\,dy.$$

Using the Cauchy-Schwarz inequality we get that

$$|\mathcal{N}_s\psi(t,x)| \le C_s\left(\int_{-1}^1 \frac{dy}{|x-y|^{2+2s}}\right)^{\frac{1}{2}} \|\psi(\cdot,t)\|_{L^2(-1,1)}. \tag{46}$$

Proceeding as the proof of (42) we obtain that there is a constant $C>0$ such that

$$\int_{-1}^1 \frac{dy}{|x-y|^{2+2s}} \le C. \tag{47}$$

Secondly, since $\psi \in C([0,T];L^2(-1,1))$, we can deduce from (46)-(47) that there is a constant $C>0$ such that for a.e. $x\in\mathcal{O}$ and every $t\in[0,T]$, we have the following estimate:

$$|\mathcal{N}_s\psi(t,x)| \le C_s\left(\int_{-1}^1 \frac{dy}{|x-y|^{2+2s}}\right)^{\frac{1}{2}} \|\psi(\cdot,t)\|_{L^2(-1,1)} \le C\|\psi\|_{C([0,T];L^2(-1,1))}.$$

We have shown that $\mathcal{N}_s\psi \in L^\infty(\mathcal{O}\times(0,T))$. In addition, we have that there is a constant $C>0$ such that $\|\mathcal{N}_s\psi\|_{L^\infty(\mathcal{O}\times(0,T))} \le C\|\psi\|_{C([0,T];L^2(-1,1))}$. The proof is finished. $\square$

We recall that $\mathcal{M}(\mathcal{O}\times(0,T))$ is the space of Radon measures endowed with the norm

$$\|\mu\|_{\mathcal{M}(\mathcal{O}\times(0,T))} := \sup\left\{\int_{\mathcal{O}\times(0,T)} \xi(x,t)d\mu(x,t) \,:\, \xi\in C_c(\overline{\mathcal{O}}\times[0,T],\mathbb{R}),\; \max_{\overline{\mathcal{O}}\times[0,T]}|\xi| = 1\right\}.$$

Next, we introduce our notion of solutions to the system (1) with an exterior measure datum, which is defined by transposition.

Definition 4.4. Let $u_0 \in L^2(-1,1)$, $T>0$ and $g\in\mathcal{M}(\mathcal{O}\times(0,T))$. We shall say that a function $u\in L^1((-1,1)\times(0,T))$ is a very weak solution of (1) defined by transposition, if it satisfies the identity

$$\int_{\mathcal{O}\times(0,T)} \mathcal{N}_s\psi(x,t)dg(x,t) = \int_{-1}^1 u_0(x)\psi(x,0)dx - \langle u(\cdot,T),\psi_T\rangle_{L^1(-1,1),L^\infty(-1,1)}, \tag{48}$$

where for every $\psi_T \in C[-1,1]$, the function $\psi \in C([-1,1]\times[0,T])$ is the unique weak (strong) solution of

$$\begin{cases} -\partial_t \psi + (-\partial_x^2)^s \psi = 0 & \text{in } (-1,1)\times(0,T), \\ \psi = 0 & \text{in } (-1,1)^c \times (0,T), \\ \psi(\cdot,T) = \psi_T & \text{in } (-1,1). \end{cases} \tag{49}$$

Now we are ready to give the proof of the last main result.

Proof of Theorem 2.5. By definition of the minimal controllability time $T_{\min}$, for every $T > T_{\min}$, there exists a control function $g \in L^\infty((0,T);L^\infty(\mathcal{O}))$ such that the system (1) is null controllable at time T. Therefore, we have that for each

$$T_k := T_{\min} + \frac{1}{k}, \quad k \geq 1,$$

there exists a sequence of non–negative controls

$$\{g^{T_k}\}_{k\geq 1} \subset L^\infty((0,T_k);L^\infty(\mathcal{O}))$$

such that the associated solutions $(u^k)_{k\geq 1}$ of (1) with initial data $u^k(\cdot,0) = u_0$ a.e. in $(-1,1)$, satisfy $u^k(x,T_k) = \widehat{u}(x,T_k)$ for a.e. $x \in (-1,1)$. We extend these controls by $\widehat{g}$ in $(T_k, T_{\min}+1)$ to get a new sequence of controls $\{g^{T_k}\}_{k\geq 1} \subset L^\infty(\mathcal{O}\times(0,T_{\min}+1))$.

Let φ_1 be the first non-negative eigenfunction of $(-\partial_x^2)_D^s$ (see (6)) and consider the problem

$$\begin{cases} -\partial_t \psi + (-\partial_x^2)^s \psi = 0 & \text{in } (-1,1)\times(0,T_{\min}+1), \\ \psi = 0 & \text{in } (-1,1)^c \times (0,T_{\min}+1), \\ \psi(\cdot,T_{\min}+1) = \varphi_1 & \text{in } (-1,1). \end{cases} \tag{50}$$

Firstly, since $\varphi_1 \in D((-\partial_x^2)_D^s)$, it follows from Remark 3.3 that the solution ψ of (50) has the following regularity:

$$\psi \in C^1([0,T_{\min}+1], L^2(-1,1)) \cap C([0,T_{\min}+1]; \\ D((-\partial_x^2)_D^s)) \hookrightarrow C([-1,1]\times[0,T_{\min}+1]).$$

Secondly, due to Theorem 2.10 we have that there is a constant $\alpha > 0$ such that

$$\psi(x,t) \geq \alpha > 0 \quad \forall (x,t) \in (-1,1)\times(0,T_{\min}+1). \tag{51}$$

Besides, using (42) and (51) we get that for a.e. $(x,t) \in \mathcal{O}\times(0,T)$,

$$\mathcal{N}_s\psi(x,t) = C_s \int_{-1}^1 \frac{-\psi(y,t)}{|x-y|^{1+2s}}dy \leq -C_s\alpha \int_{-1}^1 \frac{1}{|x-y|^{1+2s}}dy \leq -C_sC_1\alpha.$$

Therefore, taking $\beta := C_sC_1\alpha > 0$, we get that

$$\mathcal{N}_s\psi(x,t) \leq -\beta, \quad \text{for a.e. } (x,t) \in \mathcal{O}\times(0,T_{\min}+1).$$

Using the positivity of g^{T_k} and (48), we can deduce that there is a constant $M > 0$ such that

$$\begin{aligned} \beta\|g^{T_k}\|_{L^1(\mathcal{O}\times(0,T_{\min}+1))} &= \beta \int_0^{T_{\min}+1} \int_{\mathcal{O}} g^{T_k}(x,t)dxdt \\ &\leq \int_0^{T_{\min}+1} \int_{\mathcal{O}} -\mathcal{N}_s\psi(x,t)\; g^{T_k}(x,t)dxdt \end{aligned}$$

$$= \langle u(\cdot, T_{\min}+1), \varphi_1\rangle_{L^1(-1,1),L^\infty(-1,1)} - \int_{-1}^{1} u_0(x)\psi(x,0)dx$$
$$\leq M,$$

where the last estimate follows from the continuous dependence of solutions on the initial data. We have shown that the sequence $\{g^{T_k}\}_{k\geq 1}$ is bounded in $L^1(\mathcal{O}\times(0,T_{\min}+1))$, and hence, it is bounded in $\mathcal{M}(\mathcal{O}\times(0,T_{\min}+1))$. Thus, there exists $\widetilde{g}\in\mathcal{M}(\mathcal{O}\times(0,T_{\min}+1))$ such that, up to a subsequence if necessary,

$$g^{T_k} \rightharpoonup \widetilde{g} \quad \text{weakly}-\star \;\text{ in } \mathcal{M}(\mathcal{O}\times(0,T_{\min}+1)), \qquad \text{as } k\to\infty.$$

It is also clear that $\widetilde{g}$ satisfies the non-negativity constraint.

Next, for each k large enough and $T_{\min} < T_0 < T_{\min}+1$, using (48) and the fact that g^{T_k} is a trajectory control, we get that for every $\psi_{T_0}\in L^\infty(-1,1)$,

$$\int_0^{T_0}\int_{\mathcal{O}} \mathcal{N}_s\psi(x,t)dg^{T_k}(x,t) = \int_{-1}^{1} u_0(x)\psi(x,0)dx - \langle \widehat{u}(\cdot,T_0),\psi_{T_0}\rangle_{L^1(-1,1),L^\infty(-1,1)}. \tag{52}$$

In particular, we get that $\mathcal{N}_s\psi\in C(\overline{\mathcal{O}}\times[0,T])$. Thus, by the weak-$\star$ convergence, taking the limit of (52) as $k\to\infty$, we get that

$$\int_0^{T_0}\int_{\mathcal{O}} \mathcal{N}_s\psi(x,t)d\widetilde{g}(x,t) = \int_{-1}^{1} u_0(x)\psi(x,0)dx - \langle \widehat{u}(\cdot,T_0),\psi_{T_0}\rangle_{L^1(-1,1),L^\infty(-1,1)}. \tag{53}$$

The identity (53) together with (48) imply that $u(x,T_0)=\widehat{u}(x,T_0)$ for a.e. $x\in(-1,1)$. Finally, taking the limit as $T_0\to T_{\min}$ and using the fact that

$$|\widetilde{g}|(\mathcal{O}\times(T_{\min},T_0)) = |\widehat{g}|(\mathcal{O}\times(T_{\min},T_0)) = 0, \qquad \text{as } T_0\to T_{\min},$$

we can deduce that $u(x,T_{\min})=\widehat{u}(x,T_{\min})$ for a.e. $x\in(-1,1)$. The proof is complete. □

5. **Numerical simulations.** Our main Theorems 2.4 and 2.5 state that the non-local heat equation (1) is controllable from every initial datum $u_0\in L^2(-1,1)$ to any positive trajectory $\widehat{u}$, by using a non-negative control $g\in L^\infty(\mathcal{O}\times(0,T))$, whenever $1/2<s<1$, $\mathcal{O}\subset\overline{\mathcal{O}}\subset(-1,1)^c$ (for Theorem 2.4), $\mathcal{O}\subset\overline{\mathcal{O}}\subset[-1,1]^c$ (for Theorem 2.5), is an arbitrary bounded open set, and the controllability time is large enough. Moreover, in the minimal controllability time $T_{\min}$, this same result is achieved with controls in the space of Radon measures.

The aim of this final section is to present some numerical examples confirming these theoretical conclusions. To this end, we shall first discuss how to approximate the following exterior problem:

$$\begin{cases} \partial_t u + (-\partial_x^2)^s u = 0 & \text{in } (-1,1)\times(0,T),\\ u = g & \text{in } (-1,1)^c\times(0,T),\\ u(\cdot,0)=0 & \text{in } (-1,1).\end{cases} \tag{54}$$

In what follows, we will employ a FE approach, which is based on the variational formulation associated to (54). Notice that (54) is not the classical one-dimensional boundary problem, in which the non-homogeneous datum g is supported on the boundary $\{-1\}\times(0,T)$ or $\{1\}\times(0,T)$. The fact that g is supported in the exterior of the domain $(-1,1)$ introduces some difficulties in the approximation process which requires a more careful analysis.

We impose the exterior condition in (54) by using the approach from [6] (see also [4] for the stationary problem). We first approximate the Dirichlet problem (54) by the fractional Robin problem

$$\begin{cases}\partial_t u^n + (-\partial_x^2)^s u^n = 0 & \text{in } (-1,1)\times(0,T),\\ \mathcal{N}_s u^n + n\kappa u^n = n\kappa g & \text{in } (-1,1)^c\times(0,T),\\ u^n(\cdot,0)=u_0 & \text{in } (-1,1),\end{cases} \tag{55}$$

where $n\in\mathbb{N}$ is a fixed, $\kappa\in L^1(-1,1)^c\cap L^\infty(-1,1)^c$ is a given non-negative function. Indeed, it has been shown in the aforementioned reference that the weak solution u^n to (55) converges to a very weak solution u to (54), at a rate of $\mathcal{O}(n^{-1})$. More precisely, if we let the solution space of u^n to be

$$H^s_\kappa(-1,1) := \Big\{u:\mathbb{R}\to\mathbb{R} \text{ measurable and } \|u\|_{H^s_\kappa(-1,1)}<\infty\Big\},$$

where

$$\|u\|^2_{H^s_\kappa(-1,1)} := \int_{-1}^1 |u|^2\,dx + \int_{(-1,1)^c}|u|^2\kappa\,dx + \int_{\mathbb{R}^2\setminus((-1,1)^c)^2}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dxdy,$$

then the following result holds (cf. [6, Theorem 5.3]).

Theorem 5.1. *Let $g\in H^1((0,T);H^s(-1,1)^c)$ and*

$$u^n\in L^2((0,T);H^s_\kappa(-1,1)\cap L^2((-1,1)^c)\cap H^1((0,T);(H^s_\kappa(-1,1)\cap L^2(-1,1)^c)^*)$$

be the weak solution of (55). Let $u\in L^2((0,T);H^s(\mathbb{R}))\cap H^1((0,T);\widetilde{H}^{-s}(-1,1))$ be the weak solution of (54). Then, there is a constant $C>0$, independent of n, such that

$$\|u-u^n\|_{L^2((0,T);L^2(\mathbb{R}))}\le \frac{C}{n}\|u\|_{L^2((0,T);H^s(\mathbb{R}))}. \tag{56}$$

In particular, u^n converges strongly to u in $L^2((0,T);L^2(-1,1)) = L^2((-1,1)\times(0,T))$ as $n\to+\infty$.

Thus for a sufficiently large n, (55) approximates (54) well. In view of that, for the remainder of this section, instead of (54) we will consider (55) with $n=10^9$, giving an approximation of the order $\mathcal{O}(10^{-9})$.

Concerning now the control problem, we discretize (55) in the interval $(-2,2)$ by assuming that the control function g is supported in a subset $\mathcal{O}$ of $((-2,2)\setminus(-1,1))$. In that case, we can take $\kappa=1$ and the control function g to be supported in $\mathcal{O}\times(0,T)$ by multiplying it with the characteristic function $\chi_{\mathcal{O}\times(0,T)}$. In other words, we will consider the following control problem:

$$\begin{cases}\partial_t u^n + (-\partial_x^2)^s u^n = 0 & \text{in } (-1,1)\times(0,T),\\ \mathcal{N}_s u^n + n u^n = n g\chi_{\mathcal{O}\times(0,T)} & \text{in } ((-2,2)\setminus(-1,1))\times(0,T),\\ u^n(\cdot,0)=u_0 & \text{in } (-1,1).\end{cases} \tag{57}$$

For the target trajectory, we consider

$$\widehat{u}(x,T) := \frac{\Gamma\left(\frac12\right)2^{-2s}e^T}{\Gamma(1+s)\Gamma\left(\frac12+s\right)}\left(1-|x|^2\right)^s_+, \tag{58}$$

which is known (see for instance [6]) to be the exact solution to the Dirichet problem evaluated at the final time T, i.e., $\widehat{u}$ satisfies

$$\begin{cases} \partial_t \widehat{u} + (-\partial_x^2)^s \widehat{u} = z_{exact} + e^t & \text{in } (-1,1)\times(0,1), \\ \widehat{u} = z_{exact} & \text{in } ((-2,2)\setminus(-1,1))\times(0,1), \\ \widehat{u}(\cdot,0) = z_{exact}(\cdot,0) & \text{in } (-1,1), \end{cases} \tag{59}$$

where

$$z_{exact}(x,t) := \frac{\Gamma\left(\frac{1}{2}\right)2^{-2s}e^t}{\Gamma(1+s)\Gamma\left(\frac{1}{2}+s\right)}\left(1-|x|^2\right)_+^s.$$

We focus on the following two specific situations:

- **Case 1**: Set the initial datum to be
$$u_0(x) := \frac{1}{2}\cos\left(\frac{\pi}{2}x\right).$$
In this case, we have that $u_0 < \widehat{u}(\cdot,T)$ in $(-1,1)$ for every $T>0$, where $\widehat{u}$ is as in (58).
- **Case 2**: Set the initial datum to be
$$u_0(x) := 1.8\cos\left(\frac{\pi}{2}x\right).$$
In this case, we have that $u_0 > \widehat{u}(\cdot,T)$ in $(-1,1)$ for every $T>0$, where $\widehat{u}$ is as in (58).

In both cases, we first estimate numerically $T_{\min}$ by formulating the minimal-time control problem as an optimization problem. We show that in this computed minimal time, the fractional heat equation (1) is controllable from $u_0 \in L^2(-1,1)$ to the given trajectory $\widehat{u}(\cdot,T)$ (cf. (58)) by means of a non-negative control g. Secondly, we will show that, for $T < T_{\min}$ this controllability result is not achieved.

In all cases, we choose the sub-interval $\mathcal{O} = (1.7, 1.9) \subset ((-2,2)\setminus[-1,1])$ as the control region. Moreover, we focus on the case $\frac{1}{2} < s < 1$, where we know that (1) is controllable. In particular, we will always take $s = 0.8$.

5.1. **Case 1:** $u_0 < \widehat{u}(\cdot,T)$**.** We first consider the case where the initial datum u_0 is below the final target $\widehat{u}(\cdot,T)$. We begin by estimating the minimal controllability time $T_{\min}$ by solving an optimization problem. Next we address the numerical constrained controllability of (1) in a time horizon $T \geq T_{\min}$. Finally, we consider the case where $T < T_{\min}$.

5.1.1. *Calculation of the minimal controllability time* $T_{\min}$. To obtain $T_{\min}$, we consider the following constrained optimization problem:

$$\text{minimize} \quad T \tag{60}$$

subject to

$$\begin{cases} T > 0, \\ \partial_t u^n + (-\partial_x^2)^s u^n = 0 & \text{in } (-1,1)\times(0,T), \\ \mathcal{N}_s u^n + n u^n = n g \chi_{\mathcal{O}\times(0,T)} & \text{in } ((-2,2)\setminus(-1,1))\times(0,T), \\ u^n(\cdot,0) = u_0 \geq 0 & \text{in } (-1,1), \\ g \geq 0 & \text{in } \mathcal{O}\times(0,T), \\ u^n(\cdot,T) = \widehat{u}(\cdot,T), \end{cases} \tag{61}$$

which we solve using CasADi open-source tool for nonlinear optimization and algorithmic differentiation [2]. We stress that, in the above optimization problem, both T and g will be considered as variables which need to be computed.

The PDE in (61) is discretized over a uniform partition of the space interval $(-2,2)$ as follows:

$$-2 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 2,$$

where $x_i = x_{i-1} + h$, for all $i \in \{0,1,\ldots,N\}$, with h denoting the distance between two consecutive points. We use $\mathfrak{M}$ to denote a mesh with points $\{x_i : i = 0,1,\ldots,N\}$. In all our examples we have set $N = 210$.

We use globally continuous piece-wise linear finite element method on the aforementioned mesh to discretize in space. We denote the resulting finite element space by $\mathbb{V}_h$. We apply Backward-Euler, on a grid $t_k = \frac{Tk}{M}$, $k = 0,\ldots,M$, to discretize in time. In all our experiments, we have set $N = 210$ and $M = 300$. Then, given $u_h^0 = u_0$, for $k = 1,\ldots,M$, we need to solve for $u_h^k \in \mathbb{V}_h$ via

$$\int_{-1}^{1} \frac{u_h^k - u_h^{k-1}}{\delta t} v\ dx + n\mathcal{F}(u_h^k, v) + \int_{(-1,1)^c} n u_h^k v\ dx = \int_{\mathcal{O}} n g^k v\ dx, \quad \forall v \in \mathbb{V}_h, \tag{62}$$

where the closed bilinear form $\mathcal{F}$ is given by

$$\mathcal{F}(u,v) := \frac{C_s}{2} \int\int_{\mathbb{R}^2\setminus(\mathbb{R}\setminus(-1,1)^c)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\ dxdy.$$

The approximation of $\mathcal{F}(u_h^k, v)$ is carried out by using the approach of [8].

By solving (60)-(61) we obtain that $T_{\min} = 0.4739$. Next, we solve the state equation with $T = T_{\min}$, the results are given in Figure 2. We clearly notice that in this time horizon, we are able to steer the initial datum u_0 to the desired target $\widehat{u}$ while maintaining the positivity of the solution.

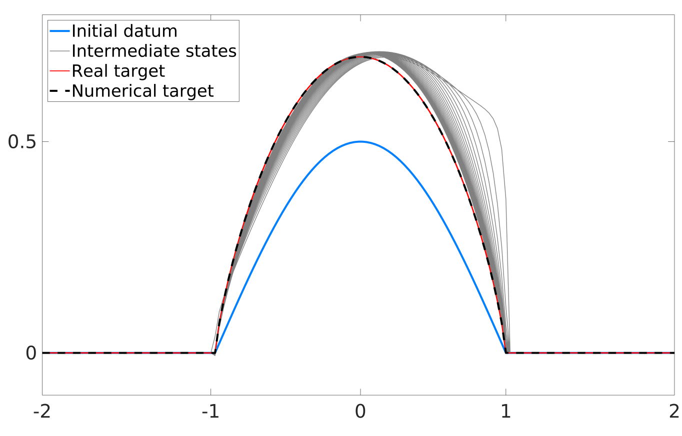

FIGURE 2. Evolution of the solution to (55) in the time interval $(0, T_{\min})$ with $s = 0.8$. The blue line is the initial configuration u_0. The red line is the target $\widehat{u}(\cdot, T)$ ($T = T_{\min}$) configuration. The black dashed line is the numerical solution at $T = T_{\min}$

.

The Figures 3 and 4 show the behavior of the control from $t = 0$ to $T = T_{\min}$. Since the amplitude of control impulses is comparatively large, therefore, we have used logarithmic scale to plot Figure 4. We notice that at first, the control produces an initial shock and as a result it raises the value of the solution close to the final target. After an intermediate period, it shows an impulsive behavior to adjust to the trajectory of the desired state. Notice that the controllability at $T = T_{\min}$ and the impulsive behavior are both according to our theoretical results.

Intuitively the behavior of the control in Figures 3 and 4 is natural. Our goal is to reach a target which is above the initial datum u_0. This means that the control needs to countervail the dissipation of the solution of (57), by acting on it from the very beginning with a positive force.

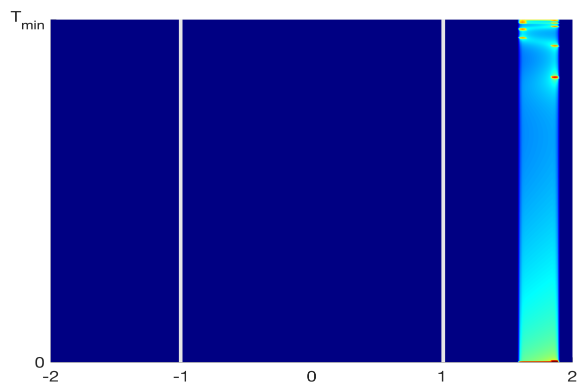

FIGURE 3. Minimal-time control: space-time distribution of the control. The white lines delimit the dynamics region $(-1, 1)$.

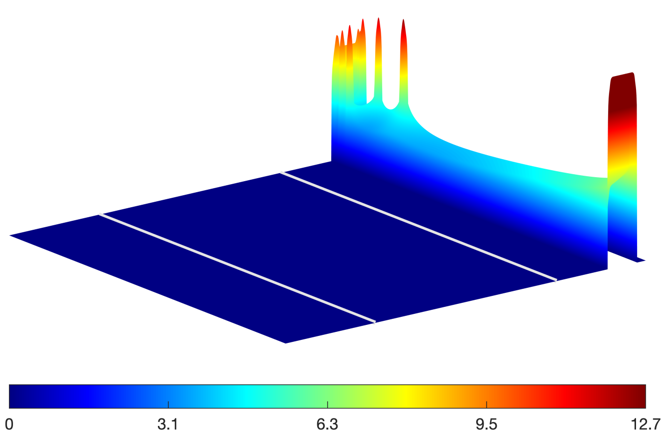

FIGURE 4. Minimal-time control: intensity of the impulses in logarithmic scale. In the (x, t) plane in blue the time t varies from $t = 0$ (bottom) to $t = T_{\min}$ (top).

5.1.2. *Lack of controllability when $T < T_{\min}$.* In this section, we conclude our discussion on Case 1 by showing the lack of controllability of (1) when the time horizon $T < T_{\min}$.

To this end, we employ a classical gradient method implemented in the DyCon Computational Toolbox ([1]) to solve the following optimization problem:

$$\min \; \|u(\cdot,T) - \widehat{u}(\cdot,T)\|^2_{L^2(-1,1)} \tag{63}$$

subject to the constraints (61).

We choose a time horizon $T = 0.2 < T_{\min}$ and solve the constrained optimization problem (63)-(61).

In Figure 5 we notice that we cannot control the solution to (1) any longer. The positive control displayed in Figure 6 is trying to push the initial datum u_0 to the desired target but since $T < T_{\min}$, we are unable to steer u_0 to $\widehat{u}(\cdot,T)$.

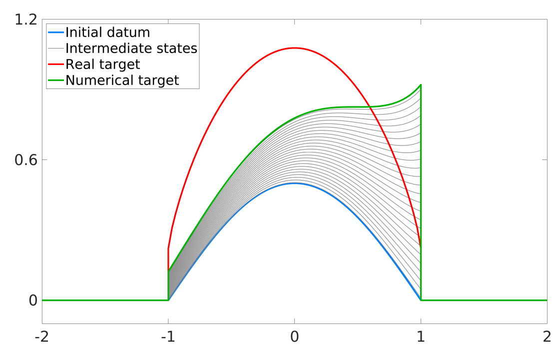

FIGURE 5. Evolution in the time interval $(0, 0.2)$ of the solution to (57) with $s = 0.8$ and $n = 10^9$. The equation is not controllable to the desired trajectory.

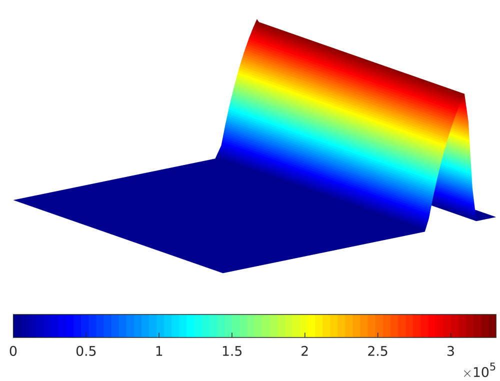

FIGURE 6. Evolution in the time interval $(0, 0.2)$ of the control function computed through the minimization process (63)-(61).

5.2. **Case 2:** $u_0 > \widehat{u}(\cdot, T)$**.** Let us now consider the case of an initial datum u_0 which is greater than the final target $\widehat{u}(\cdot, T)$. As in the previous case, we first solve the optimization problem (60)-(61) using `CasADi` to determine $T_{\min}$. We obtain $T_{\min} = 0.5713$. Figure 7 shows that in this time horizon the fractional heat equation (1) is controllable and we can reach $\widehat{u}(\cdot, T)$ from u_0. We again observe that the minimal-time control has an impulse nature, see Figures 8 and 9.

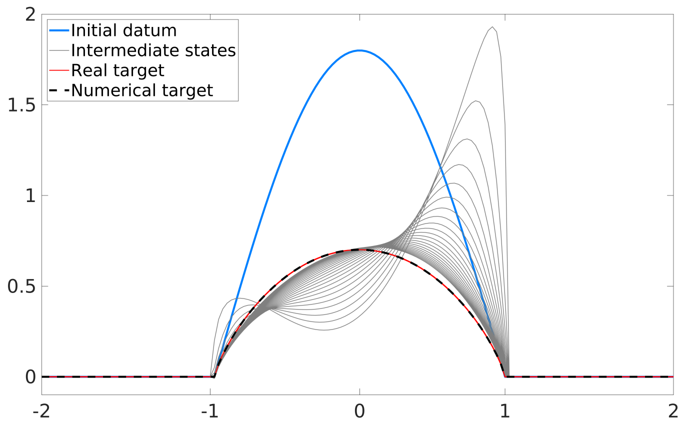

FIGURE 7. Evolution of the solution to (55) in the time interval $(0, T_{\min})$ with $s = 0.8$. The blue line is the initial configuration u_0. The red line is the target $\widehat{u}(\cdot, T)$ we aim to reach. The black dashed line is the target we computed numerically.

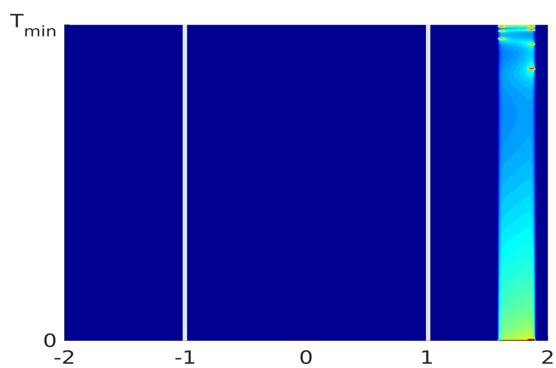

FIGURE 8. Minimal-time control: space-time distribution of the control. The white lines delimit the dynamics region $(-1, 1)$.

Notice that, this time, we want to reach a target which is below the initial datum u_0. To achieve that, the control acts by countervailing the natural dissipation of the fractional heat process, by acting on the solution to (1) with a positive force. In the end, increases its intensity to reach the desired trajectory.

Since g is not allowed to push itself down (due to the constraints), intuitively we expected to see g to be inactive, at least initially, to let the equation dissipate under the action of the heat semigroup. The control becames active only when the

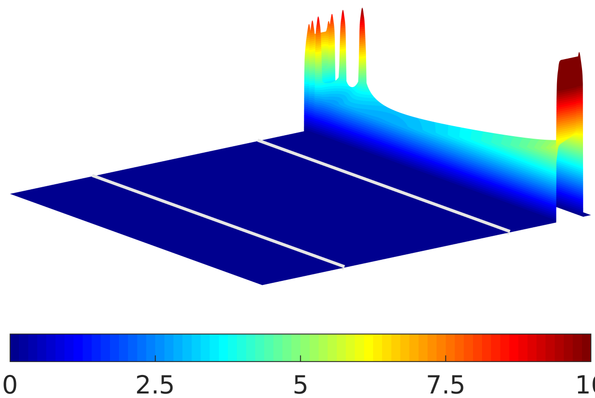

FIGURE 9. Minimal-time control: intensity of the impulses in logarithmic scale. In the (x,t) plane in blue the time t varies from $t=0$ (bottom) to $t=T_{\min}$ (top).

solution is close to the final target to do final adjustments. This is what has been observed in [10] when the control is in the interior of the domain $(-1,1)$. However, our numerical experiments show that this intuition is no longer valid in the case of the exterior control. This is another example of the fact that the action of the exterior control is very different than the existing notion of interior or boundary controls.

Finally, when considering a time horizon $T<T_{\min}$ we again notice that we cannot reach the desired trajectory $\widehat{u}(\cdot,T)$. In fact, since we want to reach a final target which is below the initial datum u_0, the natural approach is to push down the state with a "negative" action. However, since the control is not allowed to do this because of the non-negativity constraint, its best option is to remain inactive for the entire time interval and to let the solution diffuses under the action of the fractional heat semigroup (see Figures 10 and 11). But this is not sufficient to reach the target in the time horizon provided.

6. **Concluding remarks.** In this paper, we have studied the exterior controllability of trajectories for a one-dimensional fractional heat equation under nonnegativity state and control constraints. This extends our previous analysis presented in [10] for the case of interior controls.

For $1/2<s<1$, when the interior and exterior controllabilities for the unconstrained fractional heat equation hold in any positive time $T>0$, we have shown that the introduction of state or control constraints creates a positive minimal time $T_{\min}$ for achieving the same result. Moreover, we have also proved that, in this minimal time, the exterior constrained controllability holds with controls in the space of Radon measures.

Our results, which are in the same spirit of the analogous ones obtained in [10, 35, 40], are supported by the numerical simulations in Section 5.

We present hereafter a non-exhaustive list of open problems and perspectives related to our work.

1. **Extension to the multi-dimensional case.** Our analysis, based on spectral techniques, applies only to a one-dimensional fractional heat equation.

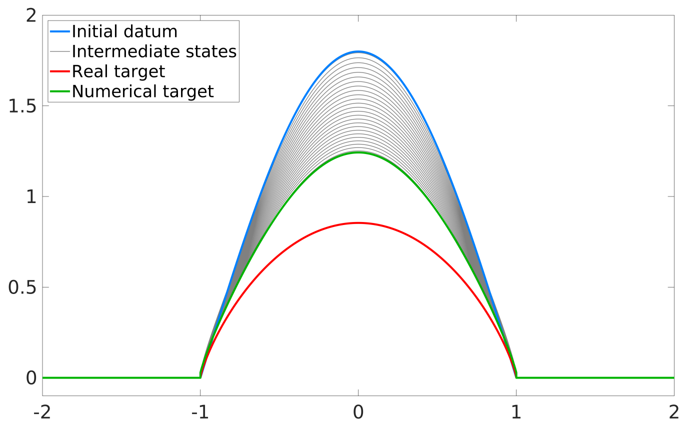

FIGURE 10. Evolution of the solution to (55) in the time interval $(0, T_{\min})$ with $s = 0.8$. The blue line is the initial configuration u_0. The red line is the target $\widehat{u}(\cdot, T)$ we aim to reach. The green line is the target we computed numerically. The equation is not controllable.

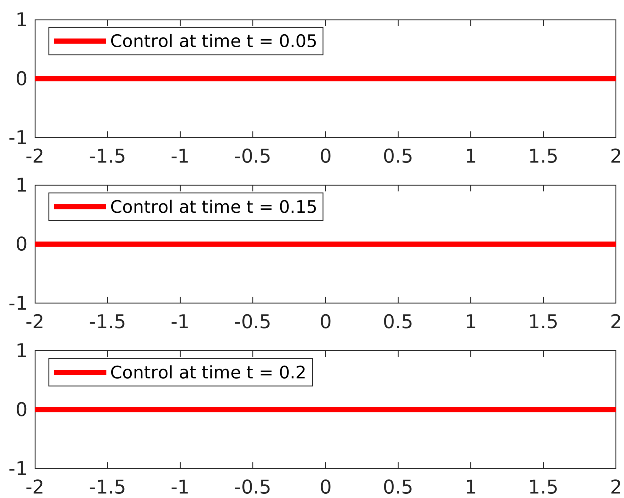

FIGURE 11. Control corresponding to the dynamics of Figure 10. The control is inactive for the entire time horizon.

The extension to multi-dimensional problems on bounded domains $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is still completely open, even in the unconstrained case. This would require different tools such as Carleman estimates. Nevertheless, obtaining Carleman estimates for the fractional Laplacian is a very difficult issue which has been considered only partially, and only for problems defined on the whole Euclidean space $\mathbb{R}^N$ (see, e.g., [45]). The case of bounded domains remains currently unaddressed and it is quite challenging. As one expects, the main difficulties come from the non-local nature of the fractional Laplacian, which makes classical PDEs techniques more delicate or even impossible to use.

2. **Strict positivity of the minimal controllability time.** In Theorems 2.4 and 2.5 we have shown the constrained controllability of trajectories for the non-local heat equation (1) in a large enough time horizon. Moreover, in the minimal controllability time $T_{\min}$, this result is achieved with controls in the space of Radon measures. This need of a large enough time horizon for constrained controllability arose when employing the dissipativity of the fractional heat semi-group in the proof of Theorem 2.4. Furthermore, as we have seen in Section 5, this is also supported by numerical evidences showing that, when the time horizon is too narrow, the equation fails to be controllable. These observations suggest that the minimal controllability time needs to be strictly positive, $T_{\min} > 0$. This is indeed the case when considering the constrained controllability problem for the local heat equation (see [35, 40]) and for the non-local one with interior control ([10]). Nevertheless, the strategies developed in the aforementioned references to prove the strict positivity of $T_{\min}$ do not seem to be applicable in the context of the present paper. This issue then remains an interesting open problem which shall need a deeper investigation.
3. **Lower bounds for the minimal constrained controllability time.** In Section 5, we gave some numerical lower bound for the minimal constrained controllability time. Nevertheless, we cannot ensure that the bounds we presented are optimal. This raises the very important issue of obtaining analytical lower bounds for the controllability time. In particular, to understand how it depends on the order s of the fractional Laplacian is evidently a fundamental point to be clarified. This question was already addressed in [35, 40] for the local heat equation but, as we discussed in [10, Section 4.4], the methodology developed in those works does not apply immediately to our case. Therefore, there is the necessity to adapt the techniques of [35, 40], or to develop new ones.
4. **Convergence results for the minimal time.** The minimal time $T_{\min}$ in the simulations of Section 5 is just an approximation computed by solving numerically the optimization problem (60)-(61). The validity of these computational results should be confirmed by showing that this minimal time of control for the discrete problem converges towards the continuous one as the mesh-sizes tend to zero. This could be done by adapting the procedure presented in [35, Section 5.3]. Nevertheless, we have to mention that, in order to corroborate this procedure, it is required the knowledge of an analytic lower bound for $T_{\min}$ which, at the present stage, is unknown (see Item 3 above).

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