

Research



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Solutions to the constant Yang–Baxter equation: additive charge conservation in three dimensions

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We find all solutions to the constant Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in three dimensions, subject to an additive charge-conservation (ACC) ansatz. This ansatz is a generalization of (strict) charge-conservation, for which a complete classification in all dimensions was recently obtained. ACC introduces additional sector-coupling parameters—in three dimensions, there are four such parameters. In the generic dimension 3 case, in which all of the four parameters are non-zero, we find there is a single three parameter family of solutions. We give a complete analysis of this solution, giving the structure of the centralizer (symmetry) algebra in all orders. We also solve the remaining cases with three, two or one non-zero sector-coupling parameter(s).

1. Introduction

The Yang–Baxter equation (YBE) reads (in shorthand form)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1.1)$$

It is a fundamental equation for many applications—see for example [1–8] and references therein.

To make (1.1) explicit, one first fixes a dimension N for a vector space $V = \mathbb{C}^N$. We can also pick bases for V and $V \otimes V$. Then, we have an underlying matrix R acting on $V \otimes V$. Each matrix R_{ij} acts on $V \otimes V \otimes V$, acting on the i th and j th factors as R , and on the other factor as the identity. Thus in explicit form, the YBE reads

$$\sum_{\alpha_1, \alpha_2, \alpha_3} \mathcal{R}_{\alpha_1 \alpha_2}^{i_1 i_2} \mathcal{R}_{j_1 \alpha_3}^{\alpha_1 i_3} \mathcal{R}_{j_2 j_3}^{\alpha_2 \alpha_3} = \sum_{\beta_1, \beta_2, \beta_3} \mathcal{R}_{\beta_2 \beta_3}^{i_2 i_3} \mathcal{R}_{\beta_1 j_3}^{i_1 \beta_3} \mathcal{R}_{j_1 j_2}^{\beta_1 \beta_2}, \quad (1.2)$$

where the indices range over $0, 1, \dots, N-1$ and $\mathcal{R}_{\alpha_1 \alpha_2}^{i_1 i_2}$ is the appropriate matrix entry of R . (See also §2a.)

With various applications in mind, we impose

$$\det(R) \neq 0. \quad (1.3)$$

For some applications, the R matrices depend on spectral parameters that can be different for each R_{ij} [3,7], but in this paper, we will consider the constant YBE. By construction, any $\check{R}$ gives a representation of the braid group B_n for each n .

Observe that R will have $N^2 \times N^2$ entries and there will be, in principle, $N^3 \times N^3$ equations. It is clear that such an overdetermined set of nonlinear equations is difficult to solve, even in this constant form. Indeed, while many individual solutions are known, a complete solution is known only for dimension two [9] and for higher dimensions knowledge is far from complete. The three-dimensional upper triangular case was solved in [10], but for further progress, it is important to make a meaningful ansatz.

Recently, Martin & Rowell proposed [11] charge-conservation of the form

$$\mathcal{R}_{ij}^{kl} = 0, \quad \text{if } \{i, j\} \neq \{k, l\} \text{ as a set,} \quad (1.4)$$

as an effective constraint and with it they were able to find all solutions for all dimensions. The above constraint may be called ‘strict charge conservation’ (SCC). In this paper, we will explore the results obtained by relaxing the SCC rule to ‘additive charge conservation’ (ACC) defined by

$$\mathcal{R}_{ij}^{kl} = 0, \quad \text{if } i + j \neq k + l. \quad (1.5)$$

Observe that ACC differs from SCC first in dimension 3. In practice, this change increases the complexity of the underlying computational problem by introducing four further ‘mixing’ parameters (SCC itself having 15 parameters in dimension 3).

The paper is organized as follows. In §2, we discuss notational matters and symmetries of the problem. In §3, we present the solutions. The set of solutions is organized according to the non-vanishing conditions on the four mixing parameters (together with their symmetries). Thus, the first family of solutions is the generic case, with all parameters non-zero—it is solved in detail in §3b. The various possibilities are then addressed in turn, the last case being the set of solutions where all but one mixing parameter vanishes—§3f.

It turns out that several solutions have the ‘Hecke’ property (i.e. having precisely two distinct eigenvalues). In §4a, we use this to analyse the representations, giving a complete analysis for the generic case.

One natural realization of the constant Yang–Baxter problem is as a problem in categorical representation theory, and this is the perspective largely taken in [11] (see also [12], for example). However, here we will keep to a simple analytical setting. Direct transliteration of results between the settings is a routine exercise.

2. The set-up

For the braid group point of view, we first define

$$\check{R} = P R, \quad \text{where } \mathcal{P}_{ij}^{kl} = \delta_i^l \delta_j^k, \quad \text{i.e. } \check{\mathcal{R}}_{ij}^{kl} = \mathcal{R}_{ji}^{kl}, \quad (2.1)$$

and furthermore

$$(PR)_{12} = \check{R}_1 := \check{R} \otimes 1 \quad \text{and} \quad (PR)_{23} = \check{R}_2 := 1 \otimes \check{R},$$

acting on $V \otimes V \otimes V$. Then, the YBE in (1.1) becomes

$$\check{R}_1 \check{R}_2 \check{R}_1 = \check{R}_2 \check{R}_1 \check{R}_2, \quad (2.2)$$

i.e. the braid group version of the YBE.

(a) Presenting matrices

Set $V = \mathbb{C}^3$ with basis labelled by $\{0, 1, 2\}$. We will order this basis as the symbols suggest. Using the standard ket notation, i.e. $i \otimes j =: |ij\rangle$, we may order the basis of $V \otimes V$, for example, using lexicographical order

$$|00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle, |20\rangle, |21\rangle, |22\rangle,$$

or reverse lexicographical order (rlex)

$$|00\rangle, |10\rangle, |20\rangle, |01\rangle, |11\rangle, |21\rangle, |02\rangle, |12\rangle, |22\rangle.$$

Still another possibility is to use a ‘graded’ reverse lexicographical ordering (grlex)

$$|00\rangle, |10\rangle, |01\rangle, |20\rangle, |11\rangle, |02\rangle, |21\rangle, |12\rangle, |22\rangle.$$

The name is borrowed from monomial orderings, in which setting the symbols *are* numbers, rather than being arbitrarily associated with numbers as in our case.

The matrix entries are defined as

$$\mathcal{R}_{ij}^{kl} := \langle ij | R | kl \rangle.$$

In the present case with ACC (1.5) and the rlex ordering, we get the matrix

$$R_{rlex} = \begin{pmatrix} \mathcal{R}_{0,0}^{0,0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{R}_{1,0}^{1,0} & \cdot & \mathcal{R}_{1,0}^{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathcal{R}_{2,0}^{2,0} & \cdot & \mathcal{R}_{2,0}^{1,1} & \cdot & \mathcal{R}_{2,0}^{0,2} & \cdot & \cdot \\ \cdot & \mathcal{R}_{0,1}^{1,0} & \cdot & \mathcal{R}_{0,1}^{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathcal{R}_{1,1}^{2,0} & \cdot & \mathcal{R}_{1,1}^{1,1} & \cdot & \mathcal{R}_{1,1}^{0,2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{2,1}^{2,1} & \cdot & \mathcal{R}_{2,1}^{1,2} & \cdot \\ \cdot & \cdot & \mathcal{R}_{0,2}^{2,0} & \cdot & \mathcal{R}_{0,2}^{1,1} & \cdot & \mathcal{R}_{0,2}^{0,2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{1,2}^{2,1} & \cdot & \mathcal{R}_{1,2}^{1,2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{2,2}^{2,2} \end{pmatrix}. \quad (2.3)$$

Indeed, the ‘shape’—the non-vanishing pattern—is the same for R , R_{rlex} and $\check{R}$. The grlex matrix is obtained from this with

$$R_{grlex} = P_G R_{rlex} P_G,$$

where P_G implements the transpositions $|01\rangle \leftrightarrow |20\rangle$ and $|21\rangle \leftrightarrow |02\rangle$. Then, an ACC matrix takes the block form exemplified by

$$R_{grlex} = \begin{pmatrix} \mathcal{R}_{0,0}^{0,0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{R}_{1,0}^{1,0} & \mathcal{R}_{1,0}^{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathcal{R}_{0,1}^{1,0} & \mathcal{R}_{0,1}^{0,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}_{2,0}^{2,0} & \mathcal{R}_{2,0}^{1,1} & \mathcal{R}_{2,0}^{0,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}_{1,1}^{2,0} & \mathcal{R}_{1,1}^{1,1} & \mathcal{R}_{1,1}^{0,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathcal{R}_{0,2}^{2,0} & \mathcal{R}_{0,2}^{1,1} & \mathcal{R}_{0,2}^{0,2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{2,1}^{2,1} & \mathcal{R}_{2,1}^{1,2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{1,2}^{2,1} & \mathcal{R}_{1,2}^{1,2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathcal{R}_{2,2}^{2,2} \end{pmatrix}. \quad (2.4)$$

In order to save space, we will in the following just give the blocks as

$$R_{grlex} = \begin{bmatrix} \mathcal{R}_{0,0}^{0,0} \\ \mathcal{R}_{0,1}^{1,0} \end{bmatrix} \begin{bmatrix} \mathcal{R}_{1,0}^{0,1} & \mathcal{R}_{1,0}^{0,1} \\ \mathcal{R}_{0,1}^{1,0} & \mathcal{R}_{0,1}^{0,1} \end{bmatrix} \begin{bmatrix} \mathcal{R}_{2,0}^{2,0} & \mathcal{R}_{2,0}^{1,1} & \mathcal{R}_{2,0}^{0,2} \\ \mathcal{R}_{1,1}^{2,0} & \mathcal{R}_{1,1}^{1,1} & \mathcal{R}_{1,1}^{0,2} \\ \mathcal{R}_{0,2}^{2,0} & \mathcal{R}_{0,2}^{1,1} & \mathcal{R}_{0,2}^{0,2} \end{bmatrix} \begin{bmatrix} \mathcal{R}_{2,1}^{2,1} & \mathcal{R}_{2,1}^{1,2} \\ \mathcal{R}_{1,2}^{2,1} & \mathcal{R}_{1,2}^{1,2} \end{bmatrix} \begin{bmatrix} \mathcal{R}_{2,2}^{2,2} \end{bmatrix}. \quad (2.5)$$

Recall that $\check{R}$ is obtained from R by exchanging lower indices, which corresponds to up-down reflection within the block. In order to match with [11] (using shifted basis labels $\{0, 1, 2\} \rightsquigarrow \{1, 2, 3\}$), highlight the new parameters, and to save from writing many double indices we introduce shorthand notation for $\check{R}$:

$$\check{R} = PR = \begin{pmatrix} a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{12} & \cdot & b_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{13} & \cdot & x_1 & \cdot & b_{13} & \cdot & \cdot \\ \cdot & c_{12} & \cdot & d_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & x_2 & \cdot & a_2 & \cdot & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{23} & \cdot & b_{23} & \cdot \\ \cdot & \cdot & c_{13} & \cdot & x_4 & \cdot & d_{13} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_{23} & \cdot & d_{23} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_3 \end{pmatrix}. \quad (2.6)$$

Then, the block form is

$$\begin{bmatrix} a_1 \\ a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix} \begin{bmatrix} a_{13} & x_1 & b_{13} \\ x_2 & a_2 & x_3 \\ c_{13} & x_4 & d_{13} \end{bmatrix} \begin{bmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{bmatrix} \begin{bmatrix} a_3 \end{bmatrix}. \quad (2.7)$$

(b) Symmetries

Naturally, it is useful to consider ACC solutions to (2.2) up to transformations that preserve (2.2) and the ACC condition.

- (i) **Scaling** symmetry: equation (2.2) and the ACC condition is invariant under rescaling $\check{R}$ by a non-zero complex number.
- (ii) **Transpose** symmetry: the ACC is preserved under transpose: $\check{R} \mapsto \check{R}^T$; and of course (2.2) is satisfied by $\check{R}^T$ if it is satisfied by $\check{R}$ quite generally. Indeed, note that from the form of (1.2), it is easy to see that if $\check{R}_{ij}^{k,l}$ solves the equation, then so does $\check{R}_{k,l}^{i,j}$. The effect on the variable choices in (2.6) are $b_{ij} \leftrightarrow c_{ij}$ and $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$.
- (iii) **Left-right (LR)** symmetry: changing the ordering of the basis from lex to rlex the resulting matrix will also be a solution. This can be seen in matrix entries because if $\check{R}_{ij}^{k,l}$ solves equation (2.2), then so does $\check{R}_{ji}^{l,k}$. This corresponds to reflecting each of the blocks in (2.7) across both the diagonal and the skew-diagonal, i.e. $a_{ij} \leftrightarrow d_{ij}$, $b_{ij} \leftrightarrow c_{ij}$ as well as $x_1 \leftrightarrow x_4$ and $x_2 \leftrightarrow x_3$.
- (iv) **02- or |0⟩ ↔ |2⟩** symmetry: while (2.2) is clearly invariant under local basis changes, the ACC condition is not. However, the local basis change (permutation) $|j\rangle \leftrightarrow |2-j\rangle$ with indices $\{0, 1, 2\}$ taken modulo 3 does preserve the form of an ACC matrix: the span of the $|ij\rangle$ with $i+j=2$ is preserved, while the $|ij\rangle$ with $i+j=1$ and $i+j=3$ are interchanged as are the vectors $|00\rangle$ and $|22\rangle$. The effect on the block form (2.7) is to interchange the pairs of 1×1 and 2×2 blocks followed by a reflection across both the diagonal and skew-diagonal of each block.

Of course, these symmetries can be composed with one another and, discounting the rescaling, one finds that the group of such symmetries is the dihedral group of order 8. This can be seen by tracking the orbit of the 2×2 matrix $\begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix}$, since there are no symmetries that fix it. Indeed,

we see that there are four forms it can take, generated by the reflections across the diagonal and, independently across the skew-diagonal, and two positions in (2.7), it can occupy.

3. The solutions

For constant Yang–Baxter solutions, a necessary and sufficient set of constraint equations on the indeterminate matrix entries arises as follows. First compute, say,

$$A_R := \check{R}_1 \check{R}_2 \check{R}_1 - \check{R}_2 \check{R}_1 \check{R}_2, \quad (3.1)$$

which we call the *braid anomaly* so that the constraints are obtained from $A_R = 0$.

The SCC case in which all x_i vanish was solved in [11]. Note that in ACC, some x_i can be non-zero and there will be mixing between more states $|ij\rangle$, but always with $i + j$ constant, so this is a computationally relatively modest generalization. However, the full symmetry of indices that exists for SCC is now broken. This ansatz-relaxing obviously increases the complexity of the system of cubic equations, but they can still be solved, as given below.

We organize the solutions according to which x_i s are vanishing. In principle, there are $2^4 - 1 = 15$ cases (excluding the SCC case), but we can use the above symmetries in order to omit some x configurations. This leads to the following classification into six cases:

- (i) All x_i are non-zero. See §3a,b.
- (ii) Precisely one x vanishes, by symmetry it can be assumed to be x_4 . See §3c.
- (iii) $x_3x_4 \neq 0$ and $x_1 = x_2 = 0$, related by the LR symmetry to $x_1x_2 \neq 0$ and $x_3 = x_4 = 0$. See §3d.
- (iv) $x_1x_3 \neq 0$ and $x_2 = x_4 = 0$, related to $x_2x_4 \neq 0$ and $x_1 = x_3 = 0$ by transposition. See §3c.
- (v) $x_1x_4 \neq 0$, $x_2 = x_3 = 0$, related to $x_2x_3 \neq 0$ and $x_1 = x_4 = 0$ by transposition, §3e.
- (vi) Only one x is non-zero, by symmetry it can be assumed to be x_4 , §3f.

As noted, the solution of constant Yang–Baxter is equivalent to solving $A_R = 0$. We write out A_R explicitly in appendix Aa. We solve for the various cases as above in the following §3a–f. In the first of these, we treat case 1 relatively gently. After that, we will proceed more rapidly through all cases.

(a) The $x_1x_2x_3x_4 \neq 0$ solutions

Recall the ACC ansatz for $\check{R}$, which is as in (2.6). Consider now the refinement of this ansatz indicated by the block structure

$$[1] \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} a & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1x_3+b}{b} & x_3 \\ \frac{x_1^2x_3^2}{b^3} & -\frac{x_1(ab+x_1x_3)}{ab^2} & -\frac{x_1x_3}{ab} \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} [1],$$

that is

$$\check{R}_j = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a & \cdot & x_1 & \cdot & b & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{x_3(a-1)}{b} & \cdot & \frac{x_1x_3+b}{b} & \cdot & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{x_3^2x_1^2}{b^3} & \cdot & -\frac{x_1(ab+x_1x_3)}{ab^2} & \cdot & -\frac{x_3x_1}{ab} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}. \quad (3.2)$$

Here, the parameters a_{13}, b_{13}, x_1, x_3 are indeterminate (we write $a = a_{13}, b = b_{13}$) but the remaining parameters are replaced with functions of these four as shown.

Proposition 3.1. (I) Consider the ansatz for $\check{R}$ in (2.6). If we leave parameters a_{13}, b_{13}, x_1, x_3 indeterminate (here we write $a = a_{13}, b = b_{13}$) but replace the remaining parameters with functions of these four as shown in (3.2) then the braid anomaly A_R has an overall factor

$$x_3^2x_1^2a + a^2b^2 + x_1bx_3a - ab^2 - x_1bx_3 = b^2a(a-1) + bx_3x_1(a-1) + ax_3^2x_1^2.$$

That is, we have a family of solutions obeying $\check{R}_1\check{R}_2\check{R}_1 = \check{R}_2\check{R}_1\check{R}_2$ with free non-zero parameters (say) a, x_1, x_3 , and parameter b determined by

$$\frac{b}{x_1x_3} = \frac{-(1/a) \pm \sqrt{(1/a^2) - (4/(a-1))}}{2} \quad \text{or} \quad \frac{x_1x_3}{b} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4a^2(a-1)}}{2a}, \quad (3.3)$$

and the remaining entries determined as in (3.2) above.

(II) If $x_1x_2x_3x_4 \neq 0$ then the above (with $a \neq 1$) gives the complete set of solutions up to overall rescaling.

Proof. (I) is simply a brutal but straightforward calculation, plugging in to A_R as given for example in appendix Aa. For (II), we proceed as follows. The matrix A_R is rather large to write out (again see appendix Aa), but a subset of its entries is

$$5_R = \{-a_{12}b_{12}c_{12} - a_{12}a_{12}(a_{12} - a_1), \quad -b_{23}c_{23}d_{23} - a_3d_{23}(d_{23} - a_3), \quad (3.4)$$

$$-b_{23}x_1x_3, \quad -b_{12}x_1x_3, \quad (a_{12} - d_{12})x_1x_3, \quad ((-d_{12} + a_1)b_{13} - b_{12}^2)x_1, \quad (3.5)$$

$$a_{13}(b_{12}c_{12} - b_{23}c_{23}) + a_{12}a_{23}(a_{12} - a_{23}), \quad (d_{12} - d_{23})x_1x_2, \quad (d_{23} - a_{23})x_1x_3, \quad (3.6)$$

$$-a_{12}x_1x_3 - a_{13}b_{13}d_{13}, \quad -a_{12}x_2x_4 - a_{13}c_{13}d_{13}, \quad a_{12}c_{12}d_{12}, \quad a_{23}c_{23}d_{23}, \quad (3.7)$$

$$-a_{13}b_{13}x_4 + (a_1a_{12} - a_{12}a_2 - a_1a_{13})x_1, \quad (3.8)$$

$$(a_1d_{13} + a_2d_{12} - a_1d_{12})x_3 + b_{13}d_{13}x_2, \quad (a_{13}d_{13} + a_2a_{23} - a_{13}a_{23})x_3 + (a_3b_{13} - b_{23}^2)x_2, \quad (3.9)$$

$$a_{13}c_{13}x_3 + (a_{13}a_3 - a_{23}a_3 + a_2a_{23})x_2, \quad a_{12}c_{13}x_3 + (-a_2d_{23} + a_{13}d_{23} - b_{23}c_{12} + a_2^2)x_2, \quad (3.10)$$

$$(a_1c_{13} - c_{12}^2)x_3 + (a_{13}d_{13} - d_{12}d_{13} + a_2d_{12})x_2, \quad (3.11)$$

$$-a_2x_1x_2 + a_{13}d_{12}^2 - a_{13}^2d_{12} - a_{23}b_{13}c_{13} + a_{23}b_{12}c_{12}, \quad (3.12)$$

$$a_1x_3x_4 + d_{13}x_1x_2 - a_2d_{12}^2 - b_{12}c_{12}d_{12} + a_2^2d_{12}, \quad (3.13)$$

$$a_{13}x_3x_4 + a_3x_1x_2 - a_{23}b_{23}c_{23} - a_2a_{23}^2 + a_2^2a_{23}, \dots \}. \quad (3.14)$$

Imposing $A_R = 0$ and $x_1x_3 \neq 0$, we thus get $b_{12} = b_{23} = 0$ and $a_{12} = d_{12}$ from (3.5). Note that $\check{R}$ is invertible, so $a_1, a_3 \neq 0$ and

$$a_{12}d_{12} - b_{12}c_{12} \neq 0 \quad \text{and} \quad a_{23}d_{23} - b_{23}c_{23} \neq 0. \quad (3.14)$$

Thus, $a_{12}, d_{12}, a_{23}, d_{23} \neq 0$. Thus, $d_{12} = a_{23} = a_{12} = a_1 = d_{23} = a_3$ and $c_{12} = c_{23} = 0$ from (3.6). Note that if $\check{R}$ is a solution then so is any non-zero scalar multiple, so we first scale $\check{R}$ by an overall factor, so that $a_1 = 1$. This confirms the form for $\check{R}$ above outside the 3×3 block.

Observe now from (3.6) that $a_{13}b_{13}d_{13} = -x_1x_3 \neq 0$, and that we may either replace $a_{13} = -(x_1x_3/b_{13}d_{13})$, or $d_{13} = -(x_1x_3/a_{13}b_{13})$. The latter gives the form of d_{13} in the Proposition.

Before proceeding, we will need to show that $d = 1$ cannot occur here. (Recall $a = a_{13}$, and write also $d = d_{13}$.) Comparing (3.12) and (3.13), we find

$$(a - 1)x_3x_4 = (d - 1)x_1x_2.$$

So $a = 1$ if and only if $d = 1$. So if $a = 1$ then $b_{13} = -x_1x_3$ and $c_{13} = -x_2x_4$ (consider (3.6i/ii)).

Evaluating (3.9ii)–(3.10) here, we find

$$a_2^2 - a_2 + a - a_2 + d - ad = (a_2 - 1)^2 - (a - 1)(d - 1) = 0.$$

So if $a = 1$ then $a_2 = 1$. Further, if $a = 1$ then (3.11) becomes $-x_1x_2 - b_{13}c_{13} = -x_1x_2 - x_1x_2x_3x_4 = 0$ so $x_3x_4 = -1$. But if $a = 1$ then (18) gives $x_4 = (-x_1)/(-x_1x_3) = 1/x_3$ – a contradiction. We conclude here that $(a - 1) \neq 0$; and hence $(d - 1) \neq 0$.

From (3.8), we have two formulae for $b_{13}x_2$. Equating we have

$$\frac{(a_1d_{13} + a_2d_{12} - a_1d_{12})}{d_{13}} = (a_{13}d_{13} + a_2a_{23} - a_{13}a_{23}),$$

that is

$$\frac{(d_{13} + a_2 - 1)}{d_{13}} = (a_{13}d_{13} + a_2 - a_{13}), \quad \text{thus} \quad \frac{(a_2 - 1)(1 - d_{13})}{d_{13}} = -a_{13}(1 - d_{13}).$$

Since $d_{13} - 1 \neq 0$ we have $a_2 - 1 = -a_{13}d_{13} = x_1x_3/b_{13}$ giving a_2 as in the Proposition. Plugging back in we find $(d_{13} + a_2 - 1)x_3 = -b_{13}d_{13}x_2$, $((a_{13} - 1)x_3/b_{13}) = x_2$ as in the Proposition.

From (3.7), we have

$$x_4 = x_1 \frac{1 - a_2 - a_{13}}{a_{13}b_{13}} = x_1 \frac{-x_1x_3 - a_{13}b_{13}}{a_{13}b_{13}^2},$$

as in the Proposition. Finally, from (3.9), we now have

$$c_{13} = -\frac{(a_{13} + a_2 - 1)x_2}{a_{13}x_3} = -\frac{(a_{13}b_{13} + x_1x_3)(a_{13} - 1)}{a_{13}b_{13}^2}$$

and

$$\begin{aligned} c_{13}x_3 &= -(a_{13} + a_2(a_2 - 1))x_2 = -(a + \frac{(x_1x_3 + b)}{b} \frac{x_1x_3}{b}) \frac{(a_{13} - 1)x_3}{b} \\ &= \frac{-x_3((ab^2 + x_1x_3(x_1x_3 + b_{13}))(a_{13} - 1))}{b_{13}^3}. \end{aligned}$$

Equating the two formulae for c_{13} , and noting that $a_{13} - 1 \neq 0$, we have

$$a_{13}(x_1x_3)^2 + (a_{13} - 1)b_{13}x_1x_3 + (a_{13} - 1)a_{13}b_{13}^2 = 0.$$

Plugging back in to (3.9), we obtain c_{13} as in the Proposition, so we are done. ■

(b) Case 1: $x_1x_2x_3x_4 \neq 0$ revisited

In this section, we solve the case in which $x_1x_2x_3x_4 \neq 0$ again, but leaning directly on the appendix (as we shall below for the remaining cases). Since all x_i are non-zero, we conclude from equations (A 5)–(A 8) that $b_{12} = c_{12} = b_{23} = c_{23} = 0$, and from (A 11)–(A 13) $a_{12} = d_{12} = a_{23} = d_{23}$. Then since $a_{12} \neq 0$ from (A 29), we get $a_{12} = 1$ and since $a_3 \neq 0$ from (A 30) $a_3 = 1$.

Now from (A 18), we find $a_{13}b_{13}d_{13} \neq 0$ and we can solve $d_{13} = -(x_1x_3/a_{13}b_{13})$, and from (A 23) $c_{13} = b_{13}x_2x_4/x_1x_3$. Then from (A 70), we find $x_4 = x_1(1 - a_2 - a_{13})/a_{13}b_{13}$. Now it turns out that some equations factorize, for example (A 94) can be written as $x_1(a_{13} - 1)[(a_2 - 1)b_{13} - x_1x_3] = 0$. If we were to choose $a_{13} = 1$, we reach a contradiction: from (A 58), we get $a_2 = 1$ and then (A 50)

and (A 66) are contradictory since $x_i \neq 0$. Thus, we can solve $a_2 = (b_{13} + x_1 x_3)/b_{13}$, and then from (A 50) $x_2 = x_3(a_{13} - 1)/b_{13}$.

After this, all remaining non-zero equations simplify to

$$b_{13}^2 a_{13}(a_{13} - 1) + b_{13} x_1 x_3(a_{13} - 1) + a_{13} x_1^2 x_3^2 = 0.$$

This biquadratic equation can be resolved using Weierstrass elliptic function $\wp$:

$$a = -\wp + \frac{5}{12}, \quad \beta = 6 \frac{12\wp + 7 + 12\wp'}{(12\wp - 5)(12\wp + 7)} \quad \text{and} \quad (\wp')^2 = 4\wp^3 - \frac{1}{12}\wp + \frac{7 \cdot 23}{2^3 3^3},$$

where $a_{13} = a$ and $b_{13} = x_1 x_3 \beta$. The solution in block form now reads

$$[1] \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} a & x_1 & \beta x_1 x_3 \\ \frac{a-1}{\beta x_1} & \frac{\beta+1}{\beta} & x_3 \\ 1 & \frac{-(a\beta+1)}{a\beta^2 x_3} & \frac{-1}{a\beta} \end{bmatrix} \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} [1], \quad (3.15a)$$

with constraint

$$\beta^2 a(a-1) + \beta(a-1) + a = 0. \quad (3.15b)$$

(c) Cases 2 and 4: $x_1 x_3 \neq 0$ and $x_4 x_2 = 0$

From (A 7), (A 8), we get $b_{12} = b_{23} = 0$ and then since the matrix is non-singular, we must have $a_{12} d_{12} a_{23} d_{23} \neq 0$. Then, from (A 1), (A 2), we get $c_{12} = c_{23} = 0$, from (A 29), (A 31) $a_{12} = d_{12} = 1$ (recall that we have scaled $a_1 = 1$) and from (A 84), (A 85) $a_{23} = d_{23} = 1$. Next from (A 30) $a_3 = 1$. Since $x_1 x_3 \neq 0$, we have from (A 18) that $a_{13} b_{13} d_{13} \neq 0$ and then from (A 24), we find $c_{13} = 0$.

To continue, we consider first the case $x_2 = 0$, x_4 free. Then, from (A 52), we get $a_{13} = 1$ and from (A 76) $d_{13} = 1 - a_2$. Then since $d_{13} \neq 0$ we cannot have $a_2 = 1$ but (A 106) is $x_3(a_2 - 1)^2 = 0$, a contradiction.

Next assume $x_2 \neq 0$, $x_4 = 0$. Then, from (A 66), we get $d_{13} = 1$ and from (A 70) $a_{13} = 1 - a_2$ but (A 98) yields $a_2 = 1$, which is in contradiction with $a_{13} \neq 0$.

Thus, there are no solutions in this case.

(d) Case 3: $x_1 = x_2 = 0$, $x_3 x_4 \neq 0$

From (A 16), we get $a_{23} = a_{12}$ and from (A 94) and (A 104) $b_{13} = b_{12}^2$ and $c_{13} = c_{12}^2$. On the basis of (A 42) and (A 46), we can divide the problem into two branches: Case 3.1: $a_{12} = 0$, $b_{12} c_{12} \neq 0$, and Case 3.2: $a_{12} d_{12} \neq 0$, $b_{12} = c_{12} = 0$.

(i) Case 3.1: $a_{12} = 0$, $b_{12} c_{12} \neq 0$

From (A 46), we get $a_{13} = 0$ and then from (A 41) $d_{12} = 1 - b_{12} c_{12}$ and from (A 76) $d_{13} = (1 - a_2)(1 - b_{12} c_{12})$. Then, from (A 100) and (A 101), $b_{23} = a_2^2/c_{12}$, $c_{23} = a_2^2/b_{12}$ and from (A 95) $a_3 = a_2^4/(b_{12} c_{12})^2$. Since $a_3 \neq 0$, we have $a_2 \neq 0$ and can solve d_{23} from (A 43): $d_{23} = (a_2^4 - (b_{12} c_{12})^3)/(b_{12} c_{12})^2$.

Now (A 82) factorizes as $(a_2^2 - b_{12} c_{12})(a_2^2 + a_2 b_{12} c_{12} + (b_{12} c_{12})^2) = 0$.

Case 3.1.1: If we choose the first factor and set $b_{12} = a_2^2/c_{12}$ the remaining equations simplify to $x_3 = a_2(a_2^2 - 1)^2/x_4$, yielding the first solution ($a_2 \rightarrow a$, $c_{12} \rightarrow c$);

$$[1] \begin{bmatrix} \cdot & \frac{a^2}{c} \\ c & 1 - a^2 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{a^4}{c^2} \\ \cdot & a & \frac{a(a^2 - 1)^2}{x_4} \\ c^2 & x_4 & (a+1)(a-1)^2 \end{bmatrix} \begin{bmatrix} \cdot & \frac{a^2}{c} \\ c & 1 - a^2 \end{bmatrix} [1]. \quad (3.16)$$

The eigenvalues of this solution are $1, -a^2$ and a^3 with multiplicities 5, 3 and 1, respectively.

Case 3.1.2: For the second solution, we solve (A 82) by $b_{12} = a\omega/c_{12}$, where ω is a cubic root of unity $\omega \neq 1$. Then, the remaining equation is solved by $x_3 = a_2(a_2 - 1)(1 - \omega a_2)/x_4$ and we have

$$[1] \begin{bmatrix} \cdot & \frac{\omega a}{c} \\ c & 1 - \omega a \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{\omega^2 a^2}{c^2} \\ \cdot & a & \frac{(\omega^2 - a)(a - 1)a}{x_4} \\ c^2 & x_4 & (1 - \omega a)(1 - a) \end{bmatrix} \begin{bmatrix} \cdot & \frac{a^2}{c} \\ \omega^2 a c & \omega a(a - 1) \end{bmatrix} [\omega a^2]. \quad (3.17)$$

The eigenvalues are $\{1, -\omega a, \omega a^2\}$ each with multiplicity 3.

For both solutions $a \neq 0, 1$ and for the first $a \neq -1$. Note that for the second case if $a = -1$, there are only two eigenvalues: 1 and ω .

(ii) Case 3.2: $a_{12}d_{12} \neq 0, b_{12} = c_{12} = 0$

From equations (A 9), (A 29) we get $a_{12} = d_{12} = 1$ and from (A 44), (A 47) $b_{23} = c_{23} = 0$. Due to non-singularity, we may now assume $a_{23}d_{23} \neq 0$ and then from (A 10), (A 16), we get $a_{23} = d_{23} = 1$. Since $a_3 \neq 0$ (A 30) yields $a_3 = 1$. Now from (A 69) and (A 100), we get $a_2 = 1, d_{13} = 0$, after which we get a contradiction in (A 50).

(e) Case 5: $x_2x_3 \neq 0, x_1 = x_4 = 0$

This case contains many solutions and therefore it is necessary to do some basic classification first. We do this on the basis of the 2×2 blocks.

For the first 2×2 block (the '12' block), consider equations (A 1), (A 2), (A 9), (A 29) and (A 31). The solutions to these equations can be divided into the following:

- α : $a_{12}d_{12} \neq 0$. Then, one finds $b_{12} = c_{12} = 0$ and $a_{12} = d_{12} = a_1$.
- β : $a_{12} \neq 0, d_{12} = 0$ and $b_{12}c_{12} \neq 0$, then $a_{12} = a_1 - b_{12}c_{12}/a_1$.
- γ : $d_{12} \neq 0, a_{12} = 0$ and $b_{12}c_{12} \neq 0$, then $d_{12} = a_1 - b_{12}c_{12}/a_1$.
- δ : $a_{12} = d_{12} = 0$.

The results for the other 2×2 block (the '23' block) are obtained by index changes, including $a_1 \rightarrow a_3$, we denote them as α' , etc.

In principle, there would be $4 \times 4 = 16$ cases, but we can omit several using the known symmetries. First of all for the '12' block, we can omit γ because it is related to β by LR symmetry. The list of cases is as follows:

- (i) (α, α') : $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 & \cdot \\ \cdot & a_3 \end{bmatrix} [a_3]$.
- (ii) (α, β') : $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 - b_{23}c_{23}/a_3 & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.
- (iii) (α, δ') : $[a_1] \begin{bmatrix} a_1 & \cdot \\ \cdot & a_1 \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.
- (iv) (β, β') : $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} a_3 - b_{23}c_{23}/a_3 & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.
- (v) (β, γ') : $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & a_3 - b_{23}c_{23}/a_3 \end{bmatrix} [a_3]$.
- (vi) (β, δ') : $[a_1] \begin{bmatrix} a_1 - b_{12}c_{12}/a_1 & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3]$.

$$(vii) (\delta, \delta'): [a_1] \begin{bmatrix} \cdot & b_{12} \\ c_{12} & \cdot \end{bmatrix} [3 \times 3] \begin{bmatrix} \cdot & b_{23} \\ c_{23} & \cdot \end{bmatrix} [a_3].$$

Here, we have omitted (α, γ') , (β, α') , (δ, α') , (δ, β') and (δ, γ') , because they are related to entries in the above list of seven by some symmetry. Specifically, note that the vanishing of x_1 and x_4 and the non-vanishing of x_2x_3 is preserved under LR-symmetry and the $|0\rangle \leftrightarrow |2\rangle$ symmetry, but *not* the transpose symmetry. Moreover, the composition of the LR and 02-symmetries has the effect of simply interchanging the pairs of 2×2 and 1×1 blocks.

(i) Case 5.1 (α, α')

We scale to $a_1 = 1$ and from (A 30) get $a_3 = 1$. According to (A 86) $a_2^2 - a_2 = 0$ and then from (A 106) and (A 108), we get $d_{13} = -b_{13}x_2/x_3$ and $c_{13} = -c_{13}x_2/x_3$ but then the 3×3 block matrix becomes singular. Therefore, no solutions for this subcase.

(ii) Case 5.2 (α, β')

From (A 39) and (A 43), we get $c_{13} = c_{23}^2/a_3$ and $b_{13} = b_{23}a_3/c_{23}$. Next from (A 58) $d_{13} = 0$ and from (A 76) $a_2 = 1$ and from (A 54) $a_{23} = a_{13}$. After setting $a_3 = -x_3c_{23}^2/x_2$ from (A 104), the GCD of the remaining equations is $(x_3c_{23} - x_2b_{23})(x_2 + x_3c_{23})^2$ and we get two solutions: ($c_{23} \rightarrow c$, $b_{23} \rightarrow b$)

$$5.2.1: x_2 = -x_3c_{23}^2$$

$$[1] \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} 1-bc & \cdot & \frac{b}{c} \\ -x_3c^2 & 1 & x_3 \\ c^2 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1-bc & b \\ c & \cdot \end{bmatrix} [1] \quad (3.18)$$

The eigenvalues are $-bc$ with multiplicity 2 and 1 with multiplicity 7.

$$5.2.2: x_2 = x_3c_{23}/b_{23}$$

$$[1] \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} 1-bc & \cdot & -b^2 \\ \frac{x_3c}{b} & 1 & x_3 \\ -\frac{c}{b} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1-bc & b \\ c & \cdot \end{bmatrix} [-bc]. \quad (3.19)$$

The eigenvalues are $-bc$ with multiplicity 3 and 1 with multiplicity 6.

(iii) Case 5.3 (α, δ')

From (A 54) and (A 62), we get $a_{13} = d_{13} = 0$ and from (A 72) $a_2 = 1$. Next, (A 40) and (A 43) yield $c_{13} = b_{13} = a_3$ and (A 102) $x_3 = -x_2a_3$. The remaining equations are satisfied with $a_3 = \epsilon_1$ and $c_{23} = \epsilon_2$, where $\epsilon_j^2 = 1$. The result is

$$[1] \begin{bmatrix} 1 & \cdot \\ \cdot & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \epsilon_1 \\ x_2 & 1 & -x_2\epsilon_1 \\ \epsilon_1 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \epsilon_2 \\ \epsilon_2 & \cdot \end{bmatrix} [\epsilon_1]$$

The eigenvalues are 1 and -1 with multiplicity 7 and 2 if $\epsilon_1 = 1$ and 6 and 3 otherwise. However, when $\epsilon_1 = -1$, this is a special case of (3.19) by setting $b = c = -1$. For $\epsilon_1 = 1$, we may take $b = c = 1$ in (3.18). Thus, this case may be discarded *a posteriori* as a subcase.

(iv) Case 5.4 (β, β')

From (A 37) and (A 39), we get $c_{13} = c_{12}^2$ and $a_3 = c_{23}^2/c_{12}^2$. Next, since $a_{12} = 1 - b_{12}c_{12} \neq 0$, we get $d_{13} = 0$ from (A 61). From (A 41) $b_{13} = b_{12}/c_{12}$ and from (A 78) $a_{13} = 1 - b_{12}c_{12}$. For non-singularity, we must have $a_2 \neq 0$ and then from (A 82), we get $b_{23} = b_{12}$ and from (A 43) $c_{23} = c_{12}$. Now from

(A 74), we find $a_2 = -x_3 c_{12}^2 / x_2$ and after that the remaining equations factorize and we have two solutions:

$$5.4.a \quad x_2 = -c_{12}^2 x_3$$

$$[1] \begin{bmatrix} 1-bc & b \\ c & . \end{bmatrix} \begin{bmatrix} 1-bc & . & \frac{b}{c} \\ -x_3 c^2 & 1 & x_3 \\ c^2 & . & . \end{bmatrix} \begin{bmatrix} 1-bc & b \\ c & . \end{bmatrix} [1]. \quad (3.20)$$

Eigenvalues are 1 with multiplicity 6 and $-bc$ with multiplicity 3.

$$5.4.b \quad x_2 = c_{12} x_3 / b_{12}$$

$$[1] \begin{bmatrix} 1-bc & b \\ c & . \end{bmatrix} \begin{bmatrix} 1-bc & . & \frac{b}{c} \\ \frac{x_3 c}{b} & -bc & x_3 \\ c^2 & . & . \end{bmatrix} \begin{bmatrix} 1-bc & b \\ c & . \end{bmatrix} [1]. \quad (3.21)$$

Eigenvalues are 1 with multiplicity 5 and $-bc$ with multiplicity 4.

(v) Case 5.5 (β, γ')

Since the matrix is non-singular, we must have $a_2 \neq 0$. From (A 38) and (A 41), we get $c_{13} = c_{12}^2$ and $b_{13} = b_{12}/c_{12}$, and from (A 39) and (A 43) $b_{12} = b_{23}^2 c_{12}/a_3$, $c_{23} = b_{23} c_{12}^2/a_3$. Then, we get from several equations the condition $a_{13} d_{13} = 0$. If both $a_{13} = d_{13} = 0$, we would get from (A 78) $a_{12} = 0$, which would lead to case δ' . Therefore, we have two branches:

5.5.1 Assume $a_{13} = 0$, $d_{13} \neq 0$. From (A 76), we get $x_3 = -x_2 b_{23}^2/a_3$ and then since $a_{12} \neq 0$ equation (A 102) yields $a_2 = 1$. From (A 66), we get $d_{13} = 1 - b_{23}^2 c_{12}^2/a_3$. If we use (A 81) to eliminate second and higher powers of a_3 of equation (A 82), it factorizes as $(a_3 - 1)(1 + b_{23} c_{12}) = 0$, and we get two branches:

5.5.1.1 If we choose $a_3 = 1$ all other equations are satisfied with $b_{23} = \omega^2/c_{12}$, where $\omega^3 = 1$ but we must have $\omega \neq 1$ to stay in the (β, γ') case.

$$[1] \begin{bmatrix} 1-\omega & \frac{\omega}{c} \\ c & . \end{bmatrix} \begin{bmatrix} . & . & \frac{\omega}{c^2} \\ x_2 & 1 & \frac{-x_2 \omega}{c^2} \\ c^2 & . & 1-\omega \end{bmatrix} \begin{bmatrix} . & \frac{\omega^2}{c} \\ c\omega^2 & 1-\omega \end{bmatrix} [1] \quad (3.22)$$

The eigenvalues are 1 with multiplicity 6 and ω with multiplicity 3.

5.5.1.2 Now, we choose $b_{23} = -1/c_{12}$ and then the remaining equations are satisfied with $a_3 = \varsigma = \pm i$.

$$[1] \begin{bmatrix} \varsigma+1 & \frac{-\varsigma}{c} \\ c & . \end{bmatrix} \begin{bmatrix} . & . & \frac{-\varsigma}{c^2} \\ x_2 & 1 & \frac{x_2 \varsigma}{c^2} \\ c^2 & . & \varsigma+1 \end{bmatrix} \begin{bmatrix} . & \frac{-1}{c} \\ \varsigma c & \varsigma+1 \end{bmatrix} [\varsigma]. \quad (3.23)$$

The eigenvalues are 1 with multiplicity 5 and ς with multiplicity 4.

5.5.2 The case $d_{13} = 0$, $a_{13} \neq 0$ is obtained by 02-symmetry from 5.5.1. Indeed, we see that the form (β, γ') is invariant under the $|0\rangle \leftrightarrow |2\rangle$ symmetry, with the 3×3 block having the following pairs interchanged (a_{13}, d_{13}) , (b_{13}, c_{13}) , (x_2, x_3) and $(x_1, x_4) = (0, 0)$. Thus, any solution obtained for $d_{13} = 0$ and $a_{13} \neq 0$ may be transformed into a solution with $d_{13} \neq 0$ and $a_{13} = 0$.

(vi) Case 5.6 (β, δ')

Since in this case $a_{12} \neq 0$, we have from (A 54) and (A 63) that $a_{13} = d_{13} = 0$ but then (A 78) implies $a_{12} = 0$, a contradiction.

(vii) Case 5.7 (δ, δ')

From $\det \neq 0$, we get $a_2 \neq 0$ and then (A 81) and (A 82) imply $c_{23} = c_{12}$ and $b_{23} = b_{12}$. Next from (A 38) and (A 42), we get $c_{13} = c_{12}^2$ and $b_{13} = b_{12}^2$. Equation (A 39) then gives $a_3 = 1$ and (A 40) implies $c_{12} = 1/b_{12}$. After this (A 52) and (A 66) yield $a_{13} = d_{13} = 0$. The remaining equations are satisfied with $a_2 = \epsilon$, $\epsilon = \pm 1$.

$$[1] \begin{bmatrix} \cdot & b \\ 1 & \cdot \\ \frac{1}{b} & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & b^2 \\ x_2 & \epsilon & x_3 \\ \frac{1}{b^2} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & b \\ 1 & \cdot \\ \frac{1}{b} & \cdot \end{bmatrix} [1]. \quad (3.24)$$

The eigenvalues are 1 and -1 with multiplicities 5 and 4 if $\epsilon = -1$ and multiplicities 6 and 3 otherwise.

(f) Case 6: $x_1 = x_2 = x_3 = 0, x_4 \neq 0$

From the outset, it is best to divide this into two cases depending on whether or not b_{12} vanishes.

(i) Case 6.1: $b_{12} = 0$ therefore $a_{12}d_{12} \neq 0$

Then, from (A 1) $c_{12} = 0$ and from (A 29) and (A 31) $a_{12} = d_{12} = 1$. From (A 94), we get $b_{13} = 0$, and hence $a_{13}a_2d_{13} \neq 0$ and then from (A 90), (A 86) and (A 66) $a_{13} = a_2 = d_{13} = 1$, which leads to a contradiction with (A 68).

(ii) Case 6.2: Now that $b_{12} \neq 0$, we get from (A 68) and (A 72) $a_{13} = d_{13} = 0$

From (A 72) $a_{13} = a_{12}$ and from (A 94) $b_{13} = b_{12}^2$ and then from (A 98) and (A 99) $a_{12} = a_{23} = 0$ and therefore $c_{12}b_{23}c_{23} \neq 0$. Next from (A 46) $c_{13} = c_{12}^2$ and from (A 100) and (A 101) $b_{23} = a_2^2/c_{12}$, $b_{12} = a_2^2/c_{23}$ and from (A 95) $a_3 = c_{23}^2/c_{12}^2$.

At this point, we divide the problem into two branches on whether or not d_{12} vanishes.

Case 6.2.1: $d_{12} = 0$. Then, from (A 68) $d_{13} = 0$ and from (A 95) $c_{23} = a_2^2c_{12}$ and from (A 83) $d_{23} = a_2(1 - a_2^2)$. After this, the remaining equations imply that we must have either $a_2 = \epsilon$, $\epsilon = \pm 1$ or $a_2 = \omega$ with $\omega^3 = 1$, $\omega \neq 1$. After changing $c_{12} \rightarrow c$, one solution is

$$[1] \begin{bmatrix} \cdot & \frac{1}{c} \\ \cdot & \cdot \\ c & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2} \\ \cdot & \epsilon & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{1}{c} \\ \cdot & \cdot \\ c & \cdot \end{bmatrix} [1]. \quad (3.25)$$

The eigenvalues are 1 and -1 , with multiplicities 5 and 4 if $\epsilon = -1$, otherwise multiplicities 6 and 3. Note that this solution may be obtained from (3.24) by setting $x_2 = 0$ and taking the transpose, but this violates the case 5 assumption that $x_2 \neq 0$.

The second solution is

$$[1] \begin{bmatrix} \cdot & \frac{1}{c} \\ \cdot & \cdot \\ c & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2} \\ \cdot & \omega & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{\omega^2}{c} \\ \cdot & \cdot \\ \omega^2 c & \omega - 1 \end{bmatrix} [\omega]. \quad (3.26)$$

The eigenvalues are 1, ω and -1 , each with multiplicity 3.

Case 6.2.2: $d_{12} \neq 0$. From (A 64) and (A 68), we get $d_{13} = d_{23} = d_{12}(1 - a_2)$ and then from (A 63) $a_2 = 1$. The remaining equations are solved by $d_{12} = (c_{23} - c_{12})/c_{23}$ and $c_{23} = c_{12}\omega$ with $\omega^3 = 1$, $\omega \neq 1$, yielding

$$[1] \begin{bmatrix} \cdot & \frac{1}{c\omega} \\ c & \omega + 2 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \frac{1}{c^2\omega^2} \\ \cdot & 1 & \cdot \\ c^2 & x_4 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \frac{1}{c} \\ \omega c & \cdot \end{bmatrix} [\omega^2]. \quad (3.27)$$

The eigenvalues are $1, \omega^2$ and $-\omega^2$, each with multiplicity 3.

Altogether, we have established the following:

Theorem 3.2. *For the YBE (2.2) in three dimensions, the complete list of solutions satisfying ACC but not SCC (see [11] for SCC) is given, up to noted symmetries (see §2b, in the formulae (3.15)–(3.27), and collected in table 1).*

4. Analysis of the generic representations (3.2/3.3)

Constant Yang–Baxter solutions can be of considerable intrinsic interest. But they are also often interesting because of their symmetry algebras. In the XXZ case (one of the SCC cases), for example, the symmetry algebra is the quantum group $U_q sl_2$. This holds true in all ranks (i.e. all system sizes n)—as we go up in ranks, we simply see more of the symmetry algebra—i.e. the action of the symmetry algebra on n -fold tensor space has a smaller kernel as n increases. It is not immediate that such a strong outcome would hold in general. But it is interesting to investigate.

In §4a, we analyse our new solutions. (In §4b, we recall some classical facts about the classical cases for comparison.)

(a) Analysis of the generic solution: spectrum of $\check{R}$

Now, we consider the solution in (3.2/3.3). Observe that the trace of the 3×3 block is

$$a + \frac{x_1 x_3 + b}{b} - \frac{x_1 x_3}{ab} = \frac{a^2 b + ab x_1 x_3 + ab^2 - x_1 x_3}{ab}.$$

Consider $\check{R}_j - 1$, so that all but the 3×3 block is zero. Restricting to the 3×3 block of $\check{R}$, call it $\check{r}$, we have

$$\begin{aligned} \check{r} - 1_3 &= \begin{bmatrix} a & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1 x_3 + b}{b} & x_3 \\ \frac{x_3^2 x_1^2}{b^3} & -\frac{x_1(ab + x_1 x_3)}{ab^2} & -\frac{x_3 x_1}{ab} \end{bmatrix} - 1_3 \\ &= \begin{bmatrix} a-1 & x_1 & b \\ \frac{x_3(a-1)}{b} & \frac{x_1 x_3}{b} & x_3 \\ \frac{x_3^2 x_1^2}{b^3} & -\frac{x_1(ab + x_1 x_3)}{ab^2} & -\frac{b(x_3 x_1 + ab)}{ab^2} \end{bmatrix}. \end{aligned} \quad (4.1)$$

Note that $x_3^2 x_1^2 / b^3 = -(a-1)(ab + x_1 x_3) / ab^2$ so this is clearly rank 1. Thus, only one eigenvalue of $\check{R} - 1$ is not 0, and so only one eigenvalue of $\check{R}$ is not 1. We have

$$\text{Trace}(\check{R} - 1) = a - 1 + \frac{x_1 x_3}{b} - \frac{b(x_1 x_3 + ab)}{ab^2} = \frac{a^2 b^2 + ab x_1 x_3 - b(x_1 x_3 + ab)}{ab^2} - 1.$$

Table 1. Table of all ACC solutions to the Yang–Baxter equation (2.2) in rank-3. Here, x denotes a non-zero variable possibly with further constraints described in the text; ω is a primitive third root of unity; and ς is a primitive fourth root of unity. In the continuous/discrete parameter column, entry 3/1 means a 3-free-parameter family, not counting overall scaling, with 1 discrete parameter (which always take on exactly two values). (Hyphens and omitted ‘names’ correspond to choices leading to no solution.)

soln. name	non-zero x, ς	block form	parameters cont./discrete	eigenvalues, degeneracies
1		(3.15)	3/0	$\begin{pmatrix} 1 & , & x \\ \times 8 & & \times 1 \end{pmatrix}$
2		-	-	-
3.1.1		(3.16)	3/0	$\begin{pmatrix} 1 & , & -x^2 & , & x^3 \\ \times 5 & & \times 3 & & \times 1 \end{pmatrix}$
3.1.2		(3.17)	3/1	$\begin{pmatrix} 1 & , & -\omega x & , & \omega x^2 \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$
4		-	-	-
5.2.1		(3.18)	3/0	$\begin{pmatrix} 1 & , & x \\ \times 7 & & \times 2 \end{pmatrix}$
5.2.2		(3.19)	3/0	$\begin{pmatrix} 1 & , & x \\ \times 6 & & \times 3 \end{pmatrix}$
5.4.a		(3.20)	3/0	$\begin{pmatrix} 1 & , & x \\ \times 6 & & \times 3 \end{pmatrix}$
5.4.b		(3.21)	3/0	$\begin{pmatrix} 1 & , & x \\ \times 5 & & \times 4 \end{pmatrix}$
5.5.1.1		(3.22)	2/1	$\begin{pmatrix} 1 & , & \omega \\ \times 6 & & \times 3 \end{pmatrix}$
5.5.1.2		(3.23)	2/1	$\begin{pmatrix} 1 & , & \varsigma \\ \times 5 & & \times 4 \end{pmatrix}$
5.7		(3.24)	3/1	$\begin{pmatrix} 1 & , & -1 \\ \times 5 & & \times 4 \end{pmatrix} / \begin{pmatrix} 1 & , & -1 \\ \times 6 & & \times 3 \end{pmatrix}$
6.2.1		(3.25)	2/1	$\begin{pmatrix} 1 & , & -1 \\ \times 5 & & \times 4 \end{pmatrix} / \begin{pmatrix} 1 & , & -1 \\ \times 6 & & \times 3 \end{pmatrix}$
6.2.1'		(3.26)	2/1	$\begin{pmatrix} 1 & , & \omega & , & -1 \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$
6.2.2		(3.27)	2/1	$\begin{pmatrix} 1 & , & \omega & , & -\omega \\ \times 3 & & \times 3 & & \times 3 \end{pmatrix}$

The other eigenvalue of $\tilde{R}$ is

$$\lambda_2 = \frac{a(a-1)b^2 + (a-1)bx_1x_3}{ab^2} = -\left(\frac{x_1x_3}{b}\right)^2 = -\left(\frac{-(a-1) \pm \sqrt{(a-1)^2 - 4a^2(a-1)}}{2a}\right)^2$$

—note from (3.3) that this depends only on a .

In particular, each of our braid representations (varying the parameters appropriately) is a Hecke representation.

We see that the eigenvalue λ_2 can be varied over an open interval (in each branch, it is a continuous function of a , small for a close to 1; and large for large negative a). So (by Hecke representation theory, specifically that the Hecke algebras are generically semisimple, and abstract considerations [13,14]) the representation is generically semisimple.

Returning to (4.1), we have

$$\check{r} - 1_3 = \left[1, \frac{x_3}{b}, \frac{-ab - x_1x_3}{ab^2} \right]^t [a - 1, x_1, b] = \frac{1}{ab^2} \begin{bmatrix} ab^2 \\ abx_3 \\ -ab - x_1x_3 \end{bmatrix} [a - 1, x_1, b]$$

and

$$\check{R} - 1_9 = \left[0, 0, 1, 0, \frac{x_3}{b}, 0, \frac{-ab - x_1x_3}{ab^2}, 0, 0 \right]^t [0, 0, a - 1, 0, x_1, 0, b, 0, 0].$$

Armed with this, we have a Temperley–Lieb category representation (i.e. an embedded TQFT—we assume familiarity with the standard U_qsl_2 version, which can be used for comparison—see [15, Sec.6.2] and references therein). In this form, the duality is going to be skewed (not a simple conjugation) but should be workable. In particular, the loop parameter is

$$[0, 0, a - 1, 0, x_1, 0, b, 0, 0] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \frac{x_3}{b} \\ 0 \\ \frac{(-ab - x_1x_3)}{ab^2} \\ 0 \\ 0 \end{bmatrix} = (a - 1) + \frac{x_1x_3}{b} - \frac{b(ab + x_1x_3)}{ab^2}$$

$$= \frac{a^2b^2 - 2ab^2 + (a - 1)bx_1x_3}{ab^2} = \frac{a - 1}{a} \left(a + \frac{x_1x_3}{b} \right) - 1 = \lambda_2 - 1,$$

which, note, depends only on a .

(b) Irreducible representation content of the generic solution ρ_n

The following analysis gives us an invariant, and thus a way to classify solutions $\check{R}$ (or equivalently R).

Thus, in principle, we can classify R -matrices according to the B_n -representation structure (the irrep content and so on) for each (and all) n . In general, such an approach is very hard (due to the limits on knowledge of the braid groups B_n and their representation theory). Certain properties can, however, make the problem more tractable.

In our case, call the representation ρ_n (or just ρ if no ambiguity arises, or to denote the monoidal functor from the braid category, as in [11]). Depending on the field we are working over, this might mean the rep with indeterminate parameters, or a generic point in parameter space (i.e. the rep variety or a point on that variety).

Since this $\check{R}$ has two eigenvalues (see §4a), we have a Hecke representation—a representation of the algebra $H_n = H_n(q)$, a quotient of the group algebra of B_n for each n , for some q . (With the same understanding about parameters.)

Since eigenvalue $\lambda_1 = 1$, this $H_n(q)$ is essentially in the ‘Lusztig’ convention—we can write t_i for the braid generators in H_n , so

$$R_i = \rho(t_i) = \rho_n(t_i);$$

then the quotient relation is

$$(t_i - 1)(t_i + q) = 0, \quad (4.2)$$

for some $q = -\lambda_2$, as in [16]. Here, it is convenient to define

$$U_i = \frac{t_i - 1}{\alpha},$$

so $\alpha U_i(\alpha U_i + 1 + q) = 0$, i.e. $\alpha U_i^2 = -(1 + q)U_i$.

In a convention/parameterization as in (4.2), the operator

$$e' = 1 - t_1 - t_2 + t_1 t_2 + t_2 t_1 - t_1 t_2 t_1,$$

is an unnormalized idempotent, and hence

$$\rho_n(e') = 1 - R_1 - R_2 + R_1 R_2 + R_2 R_1 - R_1 R_2 R_1,$$

is an unnormalized (possibly zero) idempotent, whenever $\check{R}$ gives such a Hecke representation.

In our case, in fact, e' is zero (by direct computation)

$$\rho_n(e') = 0. \quad (4.3)$$

Note that

$$\alpha^3 U_1 U_2 U_1 = (t_1 - 1)(t_2 - 1)(t_1 - 1) = t_1 t_2 t_1 - t_1 t_2 - t_2 t_1 - t_1 t_1 + 2t_1 + t_2 - 1,$$

so in our case

$$\rho_n(\alpha^3 U_1 U_2 U_1) = (R_1 - 1)(R_2 - 1)(R_1 - 1) = R_1 R_2 R_1 - R_1 R_2 - R_2 R_1 - R_1 R_1 + 2R_1 + R_2 - 1,$$

so

$$\rho_n(\alpha^3 U_1 U_2 U_1) = -R_1^2 + R_1 = -R_1(R_1 - 1) = \rho_n(q\alpha U_1) \quad \text{so } \rho_n(U_1 U_2 U_1) = \rho_n\left(\frac{q}{\alpha^2} U_1\right),$$

so if we put $\alpha = \pm\sqrt{q}$ then we have the relations of the usual generators for Temperley-Lieb [17].

We assume familiarity with the generic irreducible representations of H_n , which we write, up to isomorphism, as L_λ with $\lambda \vdash n$ an integer partition of n . The idempotent e' induces the irrep L_{1^3} . The unnormalized idempotent inducing the irrep L_3 is

$$e'_3 = 1 + \frac{1}{q}(R_1 + R_2) + \frac{1}{q^2}(R_1 R_2 + R_2 R_1) + \frac{1}{q^3} R_1 R_2 R_1. \quad (4.4)$$

This gives

$$L_3(e'_3) = \frac{1}{q^3}(1 + q)(1 + q + q^2),$$

which gives the normalization factor, so

$$e_3 = \frac{q^3}{(1 + q)(1 + q + q^2)} e'_3.$$

The generalization to irrep L_n in rank n will hopefully be clear (in fact we will not really need it except for checking).

We can write χ_λ for the irreducible character associated with irrep L_λ . That is,

$$\chi_\lambda(t_i) = \text{Trace}(L_\lambda(t_i)).$$

We can evaluate these characters in various ways, but a simple device is the restriction rule for the inclusion $H_{n-1} \otimes 1_1 \hookrightarrow H_n$; together with the easy cases

$$\chi_n(t_i) = 1 \quad \text{and} \quad \chi_{1^n}(t_i) = \lambda_2. \quad (4.5)$$

For example,

$$\chi_{2,1}(t_i) = \chi_2(t_i) + \chi_{1^2}(t_i) = 1 + \lambda_2$$

and so on.

Observe that the eigenvalues of R_i , specifically $R_1 = \check{R} \otimes 1_3$, are three copies each of the eigenvalues of $\check{R}$. Hence, there are 24 eigenvalues $\lambda_1 = 1$ and three copies of the other eigenvalue, call it λ_2

$$\chi_\rho(t_i) = 3(8 + \lambda_2) = 24 + 3\lambda_2.$$

The 1d irrep L_3 , when present, contributes 1 eigenvalue $\lambda_1 = 1$. The 2d irrep $L_{2,1}$ contributes 1 eigenvalue $\lambda_1 = 1$ and 1 of the other eigenvalue λ_2 . The 1d irrep L_{1^3} contributes just 1 of the other eigenvalue λ_2 . Since $e' = 0$ the multiplicity of this irrep in ρ is 0. Therefore, all the three eigenvalues λ_2 come from $L_{2,1}$ summands. The identity (4.3) therefore tells us that the irreducible content of our representation of H_3 (the Hecke quotient of B_3) is

$$\rho = 21 L_3 + 3 L_{2,1}. \quad (4.6)$$

(The sum is generically but not necessarily always direct.) In particular, we have re-verified

Proposition 4.1. *Representation ρ is a representation of Temperley–Lieb.*

Note that it follows from the tensor construction that this TL property holds (i.e. the image of e' continues to vanish) for all n .

Next, we address the question of faithfulness of ρ_n as a TL representation, and determine the centralizer, for all n .

Write m_λ for the multiplicity of the generic irrep L_λ in our rep ρ (the generic character is well-defined in all specializations, but the corresponding rep is not irreducible in all specializations)

$$\chi_{\rho_n} = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \quad (4.7)$$

Note that integer partitions can be considered as vectors (‘weights’ in Lie theory) and hence added. For example, if $\mu = (\mu_1, \mu_2, \mu_3, \dots, \mu_l)$ then

$$\mu + 11 = \mu + (1, 1) = \mu + (1, 1, 0, \dots, 0) = (\mu_1 + 1, \mu_2 + 1, \mu_3, \dots, \mu_l).$$

Stability Lemma. The multiplicity m_μ at level $n - 2$ is the same as $m_{\mu+11}$ at level n . *Outline Proof* The method of ‘virtual Lie theory’ works here (e.g. [14,18]). Let us define

$$U_i = \check{R}_i - 1,$$

our rank-1 operator. Thus, U_i is itself an unnormalized idempotent—indeed it is, up to scalar, the image of the cup-cap operator in the TL diagram algebra.

Write T_n for TL on n strands. Recall that $U_1 T_n$ is a left T_{n-2} right T_n bimodule. Recall the algebra isomorphism $U_1 T_n U_1 \cong T_{n-2}$; and recall that $T_n / T_n U_1 T_n \cong k$, where k is the ground field (for us it is $\mathbb{C}$). It follows that the category $T_{n-2} - \text{mod}$ embeds in $T_n - \text{mod}$, with the embedding functor given by

$$M \mapsto T_n U_1 \otimes_{T_{n-2}} M. \quad (4.8)$$

The irrep $L_\mu = L_{\mu_1, \mu_2}$ is taken to $L_{\mu+11} = L_{\mu_1+1, \mu_2+1}$. Here, L_n is the module not hit by the embedding—this is the module corresponding to $T_n / T_n U_1 T_n \cong k$, so the one that is annihilated by the localization $M \mapsto U_1 M$.

The Theorem below is a corollary of this lemma.

It might also be of interest to show how to compute the further multiplicities m_μ by direct calculation. For $n = 4$, we have $\chi_{\rho_4}(t_i) = 3\chi_{\rho_3}(t_i) = 72 + 9\lambda_2$. A direct calculation gives $\chi_{\rho_4}(e_4) = 55$ so $m_4 = 55$, and we have $\chi_{\rho_4}(t_1) = 55 + m_{3,1}(2 + \lambda_2) + m_{2,2}(1 + \lambda_2)$. We have $2m_{3,1} + m_{2,2} = 72 - 55 = 17$ and $m_{3,1} + m_{2,2} = 9$, giving $m_{3,1} = 8$ and $m_{2,2} = 1$.

Observe that this is in agreement with the Stability Lemma.

For $n = 5$, we have $\chi_{\rho_5}(t_i) = 3\chi_{\rho_4}(t_i) = 216 + 27\lambda_2$. A direct calculation gives $\chi_{\rho_5}(e_5) = 144$ and so we have $\chi_{\rho_5}(t_1) = 144 + m_{4,1}(3 + \lambda_2) + m_{3,2}(3 + 2\lambda_2)$. We have $3m_{4,1} + 3m_{3,2} = 216 - 144 = 72$ and $m_{4,1} + 2m_{3,2} = 27$, giving $m_{4,1} = 21$ and $m_{3,2} = 3$.

We observe a pattern of repeated multiplicities, in agreement with the Stability Lemma

m_λ	1	3	8	21	55	144
λ	11		2			
		21		3		
	22		31		4	
		32		41		5
	33					

Besides the Stability Lemma or a direct calculation, the last entry above may be guessed based on Perron–Frobenius applied to the Hamiltonian $H = \sum_i \tilde{R}_i$ —if some power of H is positive, then there is a unique largest magnitude eigenvalue, and hence the corresponding multiplicity is 1. We know from the XXZ chain, which has the same eigenvalues but different multiplicities, that $\lambda = mm$ gives the largest eigenvalue when $n = 2m$.

Theorem 4.2. *The multiplicity m_n in (4.7) is given by A001906 from Sloane/OEIS [19], with all other multiplicities m_μ determined by the Stability Lemma.*

The Temperley–Lieb algebras are generically semisimple; and a representation of a semisimple algebra is faithful if and only if every irrep appears as a summand. The latter is immediate from the Theorem, so generical faithfulness of our representations ρ_n is similarly immediate.

This brings us back to the original question about the stability of the centralizer as n varies—the possibility of an overarching symmetry algebra analogous to U_qsl_2 in the XXZ case. Of course, by Schur’s Lemma, the Stability Lemma exactly says that there is a limit symmetry algebra, with all finite cases simply quotients of this limit. But the combinatorial fact does not of itself imply that the symmetry algebra is something as beautiful as a quantum group (cf. appendix B).

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Appendix A. The equations

(a) The cubic constraints

Here, we write out the system of cubics corresponding to entries in A_R as in (3.1), hence the cubics that must vanish, in the ACC ansatz.

In fact, the first few cubics in A_R are unchanged (ordering 000 001 002 010 011 012 020 021 022 100 101 102 ... 222) from the strict CC ansatz. Row 000 has vanishing anomaly. Row 001 gives

$$\langle 001|A_R|001\rangle = -a_{12}b_{12}c_{12} - a_1a_{12}^2 + a_1^2a_{12} \quad \text{and} \quad \langle 001|A_R|010\rangle = -a_{12}b_{12}d_{12},$$

with all other entries vanishing. The first departure from SCC is in the 002 row, which is

$$\begin{aligned} \langle 002|A_R = [0, 0, -a_{12}x_1x_2 - a_{13}b_{13}c_{13} - a_1a_{13}^2 + a_1^2a_{13}, 0, \\ -a_{13}b_{13}x_4 + (a_1a_{12} - a_{12}a_2 - a_1a_{13})x_1, 0, -a_{12}x_1x_3 - a_{13}b_{13}d_{13}, 0, 0, 0, \\ -b_{13}c_{12}x_1 - a_{12}b_{12}x_1 + a_1b_{12}x_1, 0, \\ -b_{13}d_{12}x_1 + a_1b_{13}x_1 - b_{12}^2x_1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]. \end{aligned}$$

(b) List of equations

We give the complete list of equations that are distinct up to an overall sign, organized by the number of terms (in computations, we use the scale freedom to assume $a_1 = 1$).

$$a_{12} c_{12} d_{12} = 0, \quad (\text{A } 1)$$

$$a_{12} b_{12} d_{12} = 0, \quad (\text{A } 2)$$

$$a_{23} c_{23} d_{23} = 0, \quad (\text{A } 3)$$

$$a_{23} b_{23} d_{23} = 0, \quad (\text{A } 4)$$

$$x_2 x_4 c_{12} = 0, \quad (\text{A } 5)$$

$$x_2 x_4 c_{23} = 0, \quad (\text{A } 6)$$

$$x_1 x_3 b_{12} = 0, \quad (\text{A } 7)$$

$$x_1 x_3 b_{23} = 0 \quad (\text{A } 8)$$

$$a_{12} d_{12} (a_{12} - d_{12}) = 0, \quad (\text{A } 9)$$

$$a_{23} d_{23} (a_{23} - d_{23}) = 0, \quad (\text{A } 10)$$

$$x_1 x_2 (d_{12} - d_{23}) = 0, \quad (\text{A } 11)$$

$$x_1 x_3 (a_{12} - d_{12}) = 0, \quad (\text{A } 12)$$

$$x_1 x_3 (a_{23} - d_{23}) = 0, \quad (\text{A } 13)$$

$$x_2 x_4 (a_{12} - d_{12}) = 0, \quad (\text{A } 14)$$

$$x_2 x_4 (a_{23} - d_{23}) = 0, \quad (\text{A } 15)$$

$$x_3 x_4 (a_{12} - a_{23}) = 0, \quad (\text{A } 16)$$

$$x_1 x_3 c_{12} - a_{12} b_{12} d_{12} = 0, \quad (\text{A } 17)$$

$$x_1 x_3 d_{23} + a_{13} b_{13} d_{13} = 0, \quad (\text{A } 18)$$

$$x_1 x_3 a_{12} + a_{13} b_{13} d_{13} = 0, \quad (\text{A } 19)$$

$$x_1 x_3 c_{23} - a_{23} b_{23} d_{23} = 0, \quad (\text{A } 20)$$

$$x_1 x_3 d_{12} + a_{13} b_{13} d_{13} = 0, \quad (\text{A } 21)$$

$$x_1 x_3 a_{23} + a_{13} b_{13} d_{13} = 0, \quad (\text{A } 22)$$

$$x_2 x_4 d_{12} + a_{13} c_{13} d_{13} = 0, \quad (\text{A } 23)$$

$$x_2 x_4 a_{12} + a_{13} c_{13} d_{13} = 0, \quad (\text{A } 24)$$

$$x_2 x_4 b_{12} - a_{12} c_{12} d_{12} = 0, \quad (\text{A } 25)$$

$$x_2 x_4 d_{23} + a_{13} c_{13} d_{13} = 0, \quad (\text{A } 26)$$

$$x_2 x_4 a_{23} + a_{13} c_{13} d_{13} = 0, \quad (\text{A } 27)$$

$$x_2 x_4 b_{23} - a_{23} c_{23} d_{23} = 0, \quad (\text{A } 28)$$

$$a_{12} (a_1^2 - a_1 a_{12} - c_{12} b_{12}) = 0, \quad (\text{A } 29)$$

$$a_{23} (c_{23} b_{23} - a_3^2 + a_3 a_{23}) = 0, \quad (\text{A } 30)$$

$$d_{12} (a_1^2 - a_1 d_{12} - c_{12} b_{12}) = 0, \quad (\text{A } 31)$$

$$d_{23} (c_{23} b_{23} - a_3^2 + a_3 d_{23}) = 0, \quad (\text{A } 32)$$

$$x_1 (a_1 b_{12} - c_{12} b_{13} - a_{12} b_{12}) = 0, \quad (\text{A } 33)$$

$$x_1 (a_1 b_{13} - d_{12} b_{13} - b_{12}^2) = 0, \quad (\text{A } 34)$$

$$x_1 (c_{23} b_{13} - a_3 b_{23} + a_{23} b_{23}) = 0, \quad (\text{A } 35)$$

$$x_1 (a_3 b_{13} - d_{23} b_{13} - b_{23}^2) = 0, \quad (\text{A } 36)$$

$$x_2 (a_1 c_{12} - c_{12} a_{12} - c_{13} b_{12}) = 0, \quad (\text{A } 37)$$

$$x_2 (a_1 c_{13} - c_{12}^2 - c_{13} d_{12}) = 0, \quad (\text{A } 38)$$

$$x_2 (c_{13} a_3 - c_{13} d_{23} - c_{23}^2) = 0, \quad (\text{A } 39)$$

$$x_2 (c_{13} b_{23} - c_{23} a_3 + c_{23} a_{23}) = 0, \quad (\text{A } 40)$$

$$x_3 (a_1 b_{12} - c_{12} b_{13} - d_{12} b_{12}) = 0, \quad (\text{A } 41)$$

$$x_3 (a_1 b_{13} - a_{12} b_{13} - b_{12}^2) = 0, \quad (\text{A } 42)$$

$$x_3 (c_{23} b_{13} - a_3 b_{23} + d_{23} b_{23}) = 0, \quad (\text{A } 43)$$

$$x_3 (a_3 b_{13} - a_{23} b_{13} - b_{23}^2) = 0, \quad (\text{A } 44)$$

$$x_4 (a_1 c_{12} - c_{12} d_{12} - c_{13} b_{12}) = 0, \quad (\text{A } 45)$$

$$x_4 (a_1 c_{13} - c_{12}^2 - c_{13} a_{12}) = 0, \quad (\text{A } 46)$$

$$x_4 (c_{13} a_3 - c_{13} a_{23} - c_{23}^2) = 0, \quad (\text{A } 47)$$

$$x_4 (c_{13} b_{23} - c_{23} a_3 + c_{23} d_{23}) = 0, \quad (\text{A } 48)$$

$$x_3 x_4 (a_{12} - a_{23}) + x_1 x_2 (-d_{12} + d_{23}) = 0, \quad (\text{A } 49)$$

$$x_3 x_4 a_{23} - x_2 x_1 d_{23} + d_{13} a_{13} (d_{13} - a_{13}) = 0, \quad (\text{A } 50)$$

$$x_3 x_4 a_{12} - x_2 x_1 d_{12} + d_{13} a_{13} (d_{13} - a_{13}) = 0, \quad (\text{A } 51)$$

$$x_1 x_2 a_{12} + a_{13} (-a_1^2 + a_1 a_{13} + c_{13} b_{13}) = 0, \quad (\text{A } 52)$$

$$x_1 x_2 a_{23} + a_{13} (c_{13} b_{13} - a_3^2 + a_3 a_{13}) = 0, \quad (\text{A } 53)$$

$$x_1 x_2 b_{12} + b_{23} (-d_{23} a_{13} + a_{12} a_{13} - a_{12} a_{23}) = 0, \quad (\text{A } 54)$$

$$x_1 x_2 b_{23} + b_{12} (-d_{12} a_{13} - a_{12} a_{23} + a_{13} a_{23}) = 0, \quad (\text{A } 55)$$

$$x_1 x_2 c_{12} + c_{23} (-d_{23} a_{13} + a_{12} a_{13} - a_{12} a_{23}) = 0, \quad (\text{A } 56)$$

$$x_1 x_2 c_{23} + c_{12} (-d_{12} a_{13} - a_{12} a_{23} + a_{13} a_{23}) = 0, \quad (\text{A } 57)$$

$$x_1 x_3 a_2 + b_{13} (d_{13} a_{12} - d_{23} a_{12} + d_{23} a_{13}) = 0, \quad (\text{A } 58)$$

$$x_1 x_3 a_2 + b_{13} (d_{12} a_{13} - d_{12} a_{23} + d_{13} a_{23}) = 0, \quad (\text{A } 59)$$

$$x_2 x_4 a_2 + c_{13} (d_{12} a_{13} - d_{12} a_{23} + d_{13} a_{23}) = 0, \quad (\text{A } 60)$$

$$x_2 x_4 a_2 + c_{13} (d_{13} a_{12} - d_{23} a_{12} + d_{23} a_{13}) = 0, \quad (\text{A } 61)$$

$$x_3 x_4 b_{12} + b_{23} (d_{12} d_{13} - d_{12} d_{23} - d_{13} a_{23}) = 0, \quad (\text{A } 62)$$

$$x_3 x_4 b_{23} + b_{12} (-d_{12} d_{23} + d_{13} d_{23} - d_{13} a_{12}) = 0, \quad (\text{A } 63)$$

$$x_3 x_4 c_{12} + c_{23} (d_{12} d_{13} - d_{12} d_{23} - d_{13} a_{23}) = 0, \quad (\text{A } 64)$$

$$x_3 x_4 c_{23} + c_{12} (-d_{12} d_{23} + d_{13} d_{23} - d_{13} a_{12}) = 0, \quad (\text{A } 65)$$

$$x_3 x_4 d_{12} + d_{13} (-a_1^2 + a_1 d_{13} + c_{13} b_{13}) = 0, \quad (\text{A } 66)$$

$$x_3 x_4 d_{23} + d_{13} (c_{13} b_{13} - a_3^2 + a_3 d_{13}) = 0, \quad (\text{A } 67)$$

$$x_4 (a_1 d_{12} - a_1 d_{13} - a_2 d_{12}) - x_1 c_{13} d_{13} = 0, \quad (\text{A } 68)$$

$$x_4 (a_2 d_{23} + a_3 d_{13} - a_3 d_{23}) + x_1 c_{13} d_{13} = 0, \quad (\text{A } 69)$$

$$x_4 a_{13} b_{13} + x_1 (a_2 a_{23} + a_3 a_{13} - a_3 a_{23}) = 0, \quad (\text{A } 70)$$

$$x_4 a_{13} b_{13} + x_1 (-a_1 a_{12} + a_1 a_{13} + a_2 a_{12}) = 0, \quad (\text{A } 71)$$

$$x_4 b_{12} (a_{12} - a_{13}) + x_1 c_{12} (-d_{12} + d_{13}) = 0, \quad (\text{A } 72)$$

$$x_4 b_{23} (a_{13} - a_{23}) + x_1 c_{23} (-d_{13} + d_{23}) = 0, \quad (\text{A } 73)$$

$$x_2 (a_1 a_{12} - a_1 a_{13} - a_2 a_{12}) - x_3 c_{13} a_{13} = 0, \quad (\text{A } 74)$$

$$x_2 (a_2 a_{23} + a_3 a_{13} - a_3 a_{23}) + x_3 c_{13} a_{13} = 0, \quad (\text{A } 75)$$

$$x_2 d_{13} b_{13} + x_3 (-a_1 d_{12} + a_1 d_{13} + a_2 d_{12}) = 0, \quad (\text{A } 76)$$

$$x_2 d_{13} b_{13} + x_3 (a_2 d_{23} + a_3 d_{13} - a_3 d_{23}) = 0, \quad (\text{A } 77)$$

$$x_2 b_{12} (d_{12} - d_{13}) + x_3 c_{12} (-a_{12} + a_{13}) = 0, \quad (\text{A } 78)$$

$$x_2 b_{23} (d_{13} - d_{23}) + x_3 c_{23} (-a_{13} + a_{23}) = 0, \quad (\text{A } 79)$$

$$x_1 (a_2 b_{12} - a_2 b_{23} + a_{12} b_{23} - a_{23} b_{12}) = 0, \quad (\text{A } 80)$$

$$x_2 (c_{12} a_2 - c_{12} a_{23} - a_2 c_{23} + c_{23} a_{12}) = 0, \quad (\text{A } 81)$$

$$x_3 (a_2 b_{12} - a_2 b_{23} + d_{12} b_{23} - d_{23} b_{12}) = 0, \quad (\text{A } 82)$$

$$x_4 (c_{12} a_2 - c_{12} d_{23} - a_2 c_{23} + c_{23} d_{12}) = 0, \quad (\text{A } 83)$$

$$c_{12} d_{13} b_{12} - c_{23} d_{13} b_{23} + d_{12}^2 d_{23} - d_{12} d_{23}^2 = 0, \quad (\text{A } 84)$$

$$c_{12} a_{13} b_{12} - c_{23} a_{13} b_{23} + a_{12}^2 a_{23} - a_{12} a_{23}^2 = 0, \quad (\text{A } 85)$$

$$x_1 x_2 a_1 + x_3 x_4 a_{13} + a_{12} (-a_{12} a_2 - b_{12} c_{12} + a_2^2) = 0, \quad (\text{A } 86)$$

$$x_1 x_2 a_3 + x_3 x_4 a_{13} + a_{23} (-a_{23} a_2 - b_{23} c_{23} + a_2^2) = 0, \quad (\text{A } 87)$$

$$x_1 x_2 d_{13} + x_3 x_4 a_3 + d_{23} (-b_{23} c_{23} + a_2^2 - a_2 d_{23}) = 0, \quad (\text{A } 88)$$

$$x_1 x_2 d_{13} + x_3 x_4 a_1 + d_{12} (-b_{12} c_{12} + a_2^2 - a_2 d_{12}) = 0, \quad (\text{A } 89)$$

$$x_1 x_2 a_2 - c_{12} a_{23} b_{12} + c_{13} a_{23} b_{13} - d_{12}^2 a_{13} + d_{12} a_{13}^2 = 0, \quad (\text{A } 90)$$

$$x_1 x_2 a_2 + c_{13} a_{12} b_{13} - c_{23} a_{12} b_{23} - d_{23}^2 a_{13} + d_{23} a_{13}^2 = 0, \quad (\text{A } 91)$$

$$x_3 x_4 a_2 - c_{12} d_{23} b_{12} + c_{13} d_{23} b_{13} + d_{13}^2 a_{12} - d_{13} a_{12}^2 = 0, \quad (\text{A } 92)$$

$$x_3 x_4 a_2 + c_{13} d_{12} b_{13} - c_{23} d_{12} b_{23} + d_{13}^2 a_{23} - d_{13} a_{23}^2 = 0, \quad (\text{A } 93)$$

$$x_1 (a_{13} d_{13} + a_2 d_{12} - d_{12} d_{13}) + x_4 (-b_{12}^2 + b_{13} a_1) = 0, \quad (\text{A } 94)$$

$$x_1 (a_{13} d_{13} + a_2 d_{23} - d_{13} d_{23}) + x_4 (b_{13} a_3 - b_{23}^2) = 0, \quad (\text{A } 95)$$

$$x_1 (a_1 c_{13} - c_{12}^2) + x_4 (-a_{12} a_{13} + a_{12} a_2 + a_{13} d_{13}) = 0, \quad (\text{A } 96)$$

$$x_1 (c_{13} a_3 - c_{23}^2) + x_4 (-a_{13} a_{23} + a_{13} d_{13} + a_{23} a_2) = 0, \quad (\text{A } 97)$$

$$x_1 (a_{13} d_{12} - b_{23} c_{12} + a_2^2 - a_2 d_{12}) + x_4 a_{23} b_{13} = 0, \quad (\text{A } 98)$$

$$x_1 (a_{13} d_{23} - b_{12} c_{23} + a_2^2 - a_2 d_{23}) + x_4 a_{12} b_{13} = 0, \quad (\text{A } 99)$$

$$x_1 c_{13} d_{12} + x_4 (-a_{23} a_2 + a_{23} d_{13} - b_{23} c_{12} + a_2^2) = 0, \quad (\text{A } 100)$$

$$x_1 c_{13} d_{23} + x_4 (-a_{12} a_2 + a_{12} d_{13} - b_{12} c_{23} + a_2^2) = 0, \quad (\text{A } 101)$$

$$x_2 (a_1 b_{13} - b_{12}^2) + x_3 (a_2 a_{12} + d_{13} a_{13} - a_{12} a_{13}) = 0, \quad (\text{A } 102)$$

$$x_2 (a_3 b_{13} - b_{23}^2) + x_3 (a_2 a_{23} + d_{13} a_{13} - a_{13} a_{23}) = 0, \quad (\text{A } 103)$$

$$x_2 (a_2 d_{12} - d_{12} d_{13} + d_{13} a_{13}) + x_3 (a_1 c_{13} - c_{12}^2) = 0, \quad (\text{A } 104)$$

$$x_2 (a_2 d_{23} - d_{13} d_{23} + d_{13} a_{13}) + x_3 (c_{13} a_3 - c_{23}^2) = 0, \quad (\text{A } 105)$$

$$x_2 d_{12} b_{13} + x_3 (a_2^2 - a_2 a_{23} - c_{23} b_{12} + d_{13} a_{23}) = 0, \quad (\text{A } 106)$$

$$x_2 d_{23} b_{13} + x_3 (-c_{12} b_{23} + a_2^2 - a_2 a_{12} + d_{13} a_{12}) = 0, \quad (\text{A } 107)$$

$$x_2 (c_{12} b_{23} - a_2^2 + a_2 d_{23} - d_{23} a_{13}) - x_3 c_{13} a_{12} = 0 \quad (\text{A } 108)$$

$$\text{and} \quad x_2 (a_2^2 - a_2 d_{12} - c_{23} b_{12} + d_{12} a_{13}) + x_3 c_{13} a_{23} = 0, \quad (\text{A } 109)$$

Appendix B. Aside on further analysing solutions

A step even further than the all-ranks representation theory analysis in §4 above would be to give an *intrinsic* characterization of the centralizer algebra. We do not do this, but we can briefly set the scene.

For an example $\check{R}=P$ as in (2.1) is itself a solution—this specific case, and also the corresponding P for each N . This solution is relatively simple, and completely understood in all cases, but still highly non-trivial. Of course, it factors through the symmetric group. (It is the Schur–Weyl dual to the natural general linear group action on tensor space.) Its kernel as a symmetric group representation depends on N as well as n . Assuming we work over the complex field, then the kernel is generated exactly by the rank $N+1$ antisymmetrizer. Thus, in particular, for $N=2$, we have a faithful representation of ‘classical’ Temperley–Lieb. While for $N=3$ the rank-3 antisymmetrizer does not vanish (so faithful on the corresponding algebras—e.g. [20]).

More explicitly, we have the charge-conserving decomposition

$$\begin{aligned} \rho &= (\rho_{111} \oplus \rho_{222} \oplus \rho_{333}) \oplus (\rho_{112} \oplus \rho_{122} \oplus \rho_{113} \oplus \rho_{133} \oplus \rho_{223} \oplus \rho_{233}) \oplus (\rho_{123}) \\ &\cong 3\rho_{111} \oplus 6\rho_{112} \oplus \rho_{123} \cong 10L_3 \oplus 8L_{21} \oplus L_{1^3}, \end{aligned} \quad (\text{B } 1)$$

where the bracketed sums are of isomorphic reps, and ρ_{111} is trivial; $\rho_{112} = L_3 \oplus L_{21}$; $\rho_{123} = L_3 \oplus 2L_{21} \oplus L_{1^3}$ (i.e. the regular rep). Observe that the multiplicities 10, 8, 1 are the dimensions of the corresponding GL_3 irreps (recall these may be indexed by integer partitions of at most two rows, or equivalently of at most three rows where we delete all length-3 columns) as dictated by the duality. Note that this structure will be preserved by any generic deformation.

We can characterize this in the classical way, starting with the spectrum of $\check{R}$ itself

$$\square \otimes \square = \square \oplus \square \quad (\text{B } 2)$$

$$3 \times 3 = 6 + \bar{3}, \quad (\text{B } 3)$$

$$\square \otimes \square \otimes \square = \left(\square \oplus \square \right) \otimes \square = \square \square \square \oplus 2 \cdot \square \square \oplus \emptyset \quad (\text{B } 4)$$

$$\text{and} \quad 3 \times 3 \times 3 = (6 + \bar{3}) \times 3 = 10 + 2.8 + 1 \quad (\text{B } 5)$$

cf. (B 1). Recall that this continues

$$\square \otimes \square \otimes \square \otimes \square = \square \square \square \square \oplus 3 \cdot \square \square \square \oplus 2 \cdot \square \square \oplus 3 \cdot \square$$

and

$$3 \times 3 \times 3 \times 3 = 15 + 3.15 + 2.\bar{6} + 3.3.$$

(Side note for future reference: here in each third rank up, the reps from three ranks down reappear (along with some more). This ‘three’ is one sign that we are with gl_3 or sl_3 in this case.)

Observe that the solution for $\check{R}$ in (3.2) (in §3a) certainly does not have the multiplicities in (B 3). Indeed, it agrees formally initially with

$$\square \otimes \square = \square \oplus \emptyset$$

and

$$3 \times \bar{3} = 8 + 1$$

(e.g. [21])—formally, in the sense that the symmetry needed for the symmetric group (/Hecke/braid) action is broken here. In this formal picture, it is not clear how the labels would correspond

with the Hecke algebra/symmetric group labels—we are in rank-2 (but at least there are two summands). And it is not clear how to continue. We have

$$\square \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \square \quad (\text{B } 6)$$

and

$$3 \times 8 = 15 + \bar{6} + 3, \quad (\text{B } 7)$$

for example (so at least the centralized algebra of $\square \otimes \square \otimes \square$ is—miraculously—isomorphic to the Hecke quotient of B_3). But this is nowhere close to what we have. This suggests that it is at least time to pass to the Lie supergroups again, such as $GL(2|1)$ (cf. e.g. [21–23]). (Alternatively, it could be that the construction is not dual to a quantum group action.)

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