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Landau-Khalatnikov-Fradkin Gauge Transformations for the Propagator and Vertex in QED and QED_2

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Abstract. The non-perturbative Landau-Khalatnikov-Fradkin (LKF) transformations describe how Green functions in quantum field theory transform under a change in the photon field's linear covariant gauge parameter (denoted ξ). The transformations are framed most simply in coordinate space where they are multiplicative. They imply that information on gauge-dependent contributions from higher order diagrams in the perturbative series is contained in lower order contributions, which is useful in multi-loop calculations. We study the LKF transformations for the propagator and the vertex in both scalar and spinor QED, in some particular dimensions. A novelty of our work is to derive momentum-space integral representations of these transformations; our expressions are also applicable to the longitudinal and transverse parts of the vertex. Applying these transformations to the tree-level Green functions, we show that the one-loop terms obtained from the LKF transformation agree with the gauge dependent parts obtained from perturbation theory. Our results will be presented in more comprehensive form elsewhere.

1. Introduction

The Landau-Khalatnikov-Fradkin (LKF) transformations [1, 2] dictates how the propagator and vertex transform under a change of the gauge used to define the longitudinal part of the photon propagator. The photon propagator, in the class of gauges covered by the LKF transformations, receives a gauge-dependent, longitudinal modification. Here, we continue a series of papers started in [3, 4, 5], where we extended the transformation to an arbitrary 2n-point amplitude in spinor and scalar QED [6], to generalise these transformations to momentum space and obtain analogous transformations for the interaction vertex. For covariant linear gauges, parameterised by the gauge parameter, ξ, a variation $\xi \to \xi + \Delta\xi$, changes the position space matter propagator to:

$$S(x', x \,|\, \xi + \Delta\xi) = S(x', x \,|\, \xi) \mathrm{e}^{\mathrm{i}\Delta\xi\left[\Delta_D(|x'-x|) - \Delta_D(0)\right]}\,, \tag{1}$$

with the gauge-fixing function (D is the space-time dimension)

$$\Delta_D(y) = -\mathrm{i}e^2(\mu)\,\mu^{4-D}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{-\mathrm{i}k\cdot y}}{k^4} = -\frac{\mathrm{i}\,e^2(\mu)}{16\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\left(\mu^2 y^2\right)^{2-\frac{D}{2}}\,. \tag{2}$$

This non-perturbative multiplicative transformation relates the complete matter propagator in two different covariant gauges. One of the consequences of this relation is that all the gauge dependent information of the propagator factorizes into an exponential factor in position space. One can fix the original gauge by setting ξ to zero, in which case the photon propagator reduces to the Landau gauge. The gauge transformation of internal photons is what defines this LKF transformations, while the transformation of external photons is well-understood by the Ward identity (a complete description of the fully-amputated vertex is given in [1]).

Often, the LKF transformations are applied to perturbative expressions for the propagator which require fixing both the dimension and the "input" propagator, $S(x', x \,|\, \xi)$. The transformed propagator then contains extra, gauge-dependent information to all orders in perturbation theory, but is only valid for the specific dimension and input selected. There are some particular cases where the LKF transformation has been found in momentum space (for example, in QED in 3 dimensions [7]) where the transformation can be applied to an arbitrary input propagator (we call them momentum-space LKF transformations).

In Ref. [8], we present a general momentum-space LKF transformation for arbitrary dimension and input, for both the propagator and the photon-amputated vertex. From this follows the transformation to the longitudinal and transverse parts of the vertex and relations between these parts of the vertex in different gauges. One application would be to constrain the transverse part of the vertex in the context of the Schwinger-Dyson equations. Here, to ramify the investigations, we consider the alternative approach of exploring the consequences of the LKF transformations for QED in $2 = 1 + 1$ space-time dimensions.

2. Properties of Momentum–Space LKF–Transformations

As discussed in detail in [8], we can find a momentum-space LKF transformation in terms of momentum integrals. To do so we start from an arbitrary momentum space propagator[1] in the gauge ξ, denoted $\mathcal{S}(p', p \,|\, \xi)$. Fourier-transforming to position space, we apply the LKF transformation (1) and then an inverse Fourier transformation to return to momentum space. This allows the LKF transformation to be written completely in momentum space as

$$\mathcal{S}(p', p \,|\, \xi + \Delta\xi) = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \mathcal{S}(p' - q, p + q \,|\, \xi), \tag{3}$$

which is nothing but the convolution of the Fourier transformed LKF factor, given by

$$\Pi_D(q, \Delta\xi) := \int \mathrm{d}^D x \, \mathrm{e}^{\mathrm{i}\Delta\xi \left[\Delta_D(x) - \Delta_D(0)\right] + \mathrm{i}\, q\cdot x}, \tag{4}$$

with the input propagator. This transformation is given for an arbitrary dimension and input.

In a similar way, we consider the "photon-amputated" vertex, $\Lambda(x', x, z \,|\, \xi)$, which retains its external matter propagators. As noted by Burden and Roberts [10], its position space LKF transformation is the same as for the propagator. As elaborated upon in section 4, it follows that its momentum-space LKF transformation goes through in the same way, simply by convolution with Π_D. In [8] we present the LKF transformations in the context of QED in $D = 3$ and $D = 4$ space-time dimensions. Here, we instead discuss the application of the LKF transformation to the propagator in $D = 2$ space-time dimensions and the application of this momentum-space LKF transformation to extract some general information in the case of the photon-amputated vertex.

[1] Our conventions are set in Minkowski space-time with metric signature $(-,+,+,\dots)$ and Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$, following [9] with the exceptions of the opposite sign on electric charge and choosing all momenta to be incoming. The propagator is the inverse of the Dirac operator $\big(-i\not{D}(x) + m\big)$.

3. LKF Transformation for the Spinor Propagator in $D = 2$

In the two dimensional (i.e., $1 + 1$ dimensions) case, a subtlety arises due to poles in the dimensional regularization parameter, ϵ, appearing in the LKF exponent. We regulate these by writing $D = 2 - 2\epsilon$ in (2), finding

$$\Delta_{2-2\epsilon}(y) = -\frac{\mathrm{i}e^2}{16\pi}(\mu y)^2\pi^\epsilon\,\Gamma(-1-\epsilon)\,(\mu y)^{2\epsilon} = -\frac{\mathrm{i}e^2\mu^2y^2}{16\pi}\left[\frac{1}{\epsilon} + \log\left(\pi y^2\mu^2\right) + \gamma_E - 1\right] + \mathcal{O}(\epsilon)\,. \quad (5)$$

As $\Delta_{2-2\epsilon}(0) = 0$ for $\epsilon > 0$ in view of $y^2\ln(y^2) \to 0$ for $y^2 \to 0$, the poles are *not* cancelled when $\Delta_{2-2\epsilon}(0)$ is subtracted (as it happens in the 4 dimensional case [3]). This implies an essential singularity for the LKF transformation, in the limit $\epsilon \to 0$. The poles, of arbitrary order, must thus be taken into account at all orders in the transformation. This, along with the physically interesting aspects of two dimensional QED, make the proceeding calculations of both theoretical and practical interest.

In position space, the singular part of the multiplicative LKF factor is

$$\left.\mathrm{e}^{\mathrm{i}\Delta\xi\left[\Delta_{2-2\epsilon}(x)-\Delta_{2-2\epsilon}(0)\right]}\right|_{\epsilon^{-1}} = \exp\left(-\frac{\Delta\xi\alpha\mu^2}{4\epsilon}(x-x')^2\right). \quad (6)$$

We will also require the finite contribution for this case:

$$\left.\mathrm{e}^{\mathrm{i}\Delta\xi\left[\Delta_{2-2\epsilon}(x)-\Delta_{2-2\epsilon}(0)\right]}\right|_{\epsilon^{0}} = \exp\left(-\frac{\Delta\xi\alpha\mu^2}{4}(x-x')^2\left\{\log[\pi\mu^2(x-x')^2] + \gamma_E - 1\right\}\right). \quad (7)$$

3.1. Perturbative Verification: Spinor QED

To deal with the complicated pole structure, a perturbative calculation must be carried out about the physical dimension, maintaining $D = 2 - 2\epsilon$ throughout. We have the modest aim of verifying the LKF transformation to ($\mathcal{L} = 1$)-loop order, beginning from the bare, tree-level propagator, to verify that the known results for the self-energy can be recovered with this technique. As such, the tree-level propagator in momentum space defines our reference gauge (in two dimensions we will implement the Clifford algebra of the Dirac equation using the Pauli matrices $\gamma^1 = \sigma_1$ and $\gamma^2 = \mathrm{i}\sigma_2$ without continuation when using dimensional regularisation),

$$S_0^D(p,p'\,|\,\xi) = \frac{(2\pi)^D\delta^{(D)}(p+p')}{-\not{p}'+m} = (2\pi)^D\delta^{(D)}(p+p')\,\frac{\not{p}'+m}{p'^2+m^2}, \quad (8)$$

which maintains our convention that all momenta are incoming (the metric is $g^{\mu\nu} = \mathrm{diag}(-1,1)$, consistent with our 4-dimensional conventions). Rather than applying (3), which would contain poles to all orders, we carry out the transformation between position and momentum space. At one-loop order, corrections linear in ϵ that arise in these transformations are sufficient – it is precisely these linear corrections that combine with the $\frac{1}{\epsilon}$ pole in (5).

We begin by transforming the tree-level propagator to position space using the integrals collected in Appendix A, finding (μ is an arbitrary scale introduced on dimensional grounds and $K_\alpha(z)$ is the modified Bessel function of the second kind)

$$S_0^{2-2\epsilon}(x,x'\,|\,\xi) = \frac{\mu^{2\epsilon}}{(2\pi)^{1-\epsilon}}\left[m + \mathrm{i}\not{\partial}_x\right]\left(\frac{x}{m}\right)^\epsilon K_{-\epsilon}(mx)\,. \quad (9)$$

We multiply by the LKF factor as per (1) and compute the Fourier transform in $D = 2 - 2\epsilon$ dimensions (multiplying now also by $\mu^{-2\epsilon}$). Note that integrating the term with the derivative by parts we arrive at (we suppress the global momentum conserving δ-function for brevity):

$$S_{0^+}^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi) = \left[m+\not{p}'\right]\left(\frac{p}{m}\right)^{\epsilon}\int_0^\infty \mathrm{d}x\, x J_{-\epsilon}(px)K_{-\epsilon}(mx)\Big\{1+ \frac{1}{4}\alpha x^2\mu^2\Delta\xi\Big[\frac{1}{\epsilon}+\log\left(\pi x^2\mu^2\right)+\gamma_E-1\Big]+\mathcal{O}(\alpha^2\Delta\xi^2)\Big\} + \frac{\Delta\xi}{(2\pi)^{1-\epsilon}}\int \mathrm{d}^3x\,\left(\frac{x}{m}\right)^{\epsilon}K_{-\epsilon}(mx)\left(\not{\partial}_x\Delta_{2-2\epsilon}(x)\right)\mathrm{e}^{i\Delta\xi\Delta_{2-2\epsilon}(x)+\mathrm{i}p'\cdot x}\,. \tag{10}$$

Since $\Delta_D(x)$ is a function only of the magnitude of x, the partial derivative can be written as $\not{\partial}\Delta_D(x) = \frac{\not{x}}{x}\Delta'_D(x)$, following which we generate the $\not{x}$ via a derivative $i\not{\partial}_{p'}$ acting on the Fourier exponent. In this way the second integral in the equation above can be cast in the simpler form $-i\not{\partial}_{p'}\left[\left(\frac{p'}{m}\right)^{\epsilon}\int_0^\infty \mathrm{d}x\, J_{-\epsilon}(p'x)K_{-\epsilon}(mx)\Delta'_{2-2\epsilon}(x)\mathrm{e}^{\mathrm{i}\Delta\xi\Delta_{2-2\epsilon}(x)}\right]$. So far we have been working with this contribution to all orders in $\alpha\Delta\xi$. For the one-loop corrections, we reuse (5), keeping contributions up to linear order in $\alpha\Delta\xi$. This leads to the following expansion in ϵ for the LKF-transformed propagator,

$$S_{0^+}^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi) = \left[m+\not{p}'\right]D_0^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi) - \frac{\not{p}'}{2p'}\alpha\,\Delta\xi\,\mu^2\,\partial_{p'}\left\{\frac{1}{p'^2+m^2}\frac{1}{\epsilon}-\frac{2\log\left(1+\frac{p'^2}{m^2}\right)+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_E+1}{p'^2+m^2}+\mathcal{O}(\epsilon,\alpha\Delta\xi)\right\}, \tag{11}$$

where $D_0^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi)$ is just the LKF-transformed scalar propagator, found here as part of the spinor calculation to be (up to corrections of $\mathcal{O}(\epsilon,\alpha^2\Delta\xi^2)$),

$$D_0^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi) = \frac{1}{p^2+m^2}-\frac{\alpha\mu^2\Delta\xi}{\epsilon}\frac{p^2-m^2}{(p^2+m^2)^3} + \alpha\mu^2\,\Delta\xi\,\frac{p^2-m^2}{(p^2+m^2)^3}\left[2\log\left(1+\frac{p^2}{m^2}\right)+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_E-\frac{4p^2}{p^2-m^2}\right]+\mathcal{O}(\epsilon,(\alpha\Delta\xi)^2)\,. \tag{12}$$

Computing the derivative we can give the result decomposed in the basis $\{\mathbb{1},\gamma^1,\gamma^2\}$ as

$$S_{0^+}^{2-2\epsilon}(p,p'\,|\,\xi+\Delta\xi) = S_0^2(p,p'\,|\,\xi) - \alpha\mu^2\Delta\xi\Big\{\frac{p^2-m^2}{(p^2+m^2)^3}\frac{1}{\epsilon}-\frac{p^2-m^2}{(p^2+m^2)^3}\left[2\log\left(1+\frac{p^2}{m^2}\right)+\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_E-\frac{4p^2}{p^2-m^2}\right] +\not{p}'\Big[-\frac{2m^2}{(p^2+m^2)^3}\frac{1}{\epsilon}+\frac{2m^2\log\left(1+\frac{p'^2}{m^2}\right)+m^2\log\left(\frac{m^2}{4\pi\mu^2}\right)+m^2(\gamma_E-1)+p^2}{(p^2+m^2)^3}\Big]\Big\}+\mathcal{O}((\alpha\Delta\xi)^2)\,. \tag{13}$$

After amputating the external spinors, this corresponds to the gauge-dependent part of the well-known one-loop electron self-energy calculated in dimensional regularisation given in, for example, [11, 12]. This asserts that the dimensional regularisation of the LKF factor is consistent as long as one carries out the appropriate Fourier transformations with the same dimensional deformation in order to account for the poles structure in (5).

4. Momentum-space LKF Transformation for the Vertex

In this section we return to a general space-time dimension to discuss the LKF transformation of the QED 3-point interaction vertex. As mentioned in the introduction it is convenient to work with the photon-amputated vertex, $\Lambda^\mu(p', p\,|\,\xi)$, related to the fully amputated vertex, $\Gamma^\mu(p', p\,|\,\xi)$, through the relation

$$\Lambda^\mu(p', p, k\,|\,\xi) = \mathcal{S}(p\,|\,\xi)\Gamma^\mu(p', p, k\,|\,\xi)\mathcal{S}(p'\,|\,\xi)\,. \tag{14}$$

Although physical interest is in the fully amputated vertex, the LKF transformation of the photon-amputated vertex is far simpler: by virtue of having two external matter propagators, in position space it transforms multiplicatively, analogously to (1). Decomposing the vertex to extract the momentum conserving δ-function,

$$\Lambda^\mu(p', p, k\,|\,\xi) = (2\pi)^D\delta^D(p' + p + k)\Lambda^\mu(p', p\,|\,\xi)\,. \tag{15}$$

the reduced vertex satisfies the momentum-space LKF transformation

$$\Lambda^\mu(p', p\,|\,\xi + \Delta\xi) = \int \frac{\mathrm{d}^Dq}{(2\pi)^D}\Pi_D(q, \Delta\xi)\,\Lambda^\mu(p' - q, p + q\,|\,\xi). \tag{16}$$

In [8] we use this to obtain the fully non-perturbative form of the LKF-transformed vertex and also verify that it correctly reproduces the known results for scalar and spinor QED in $D = 3$ and $D = 4$ space-time dimensions.

The Ward-Fradkin-Green-Takahashi identity provides a natural decomposition of the vertex [13] into a sum of a longitudinal part,

$$\Lambda^\mu_{\rm L}(p', p\,|\,\xi) = \sum_i \widetilde{\lambda}_i(p', p\,|\,\xi)L^\mu_i\,, \tag{17}$$

which completely satisfies the identity alone, and a transverse part,

$$\Lambda^\mu_{\rm T}(p', p\,|\,\xi) = \sum_j \widetilde{\tau}_j(p', p\,|\,\xi)T^\mu_j\,, \qquad k_\mu\Lambda^\mu_{\rm T}(p', p\,|\,\xi) = 0\,, \quad \Lambda^\mu_{\rm T}(p, p\,|\,\xi) = 0 \tag{18}$$

(we reserve the commonly used notation λ and τ for the fully amputated vertex). The specific vector structure of the vertices is theory dependent, as is shown below. We calculate the longitudinal and transverse parts of the LKF-transformed photon-amputated vertex, obtainng some relations between the parts of the vertex (i.e., the $\widetilde{\lambda}$'s and $\widetilde{\tau}$'s) in two different gauges. This could be useful in the context of the Schwinger-Dyson equations, where one way to deal with the infinite tower of coupled differential equations is to truncate them by giving an Ansatz for the full non-perturbative vertex. The Ansatz is constructed based on some basic principles, including that it should transform under a change of the covariant gauge parameter as the LKF transformation dictates.

4.1. Scalar Case

For the scalar case, the vertex can be decomposed into two structures as

$$\Lambda^\mu(p', p\,|\,\xi) = \Lambda^\mu_{\rm L}(p', p\,|\,\xi) + \Lambda^\mu_{\rm T}(p', p\,|\,\xi) = \widetilde{\lambda}(p', p\,|\,\xi)L^\mu + \widetilde{\tau}(p', p\,|\,\xi)T^\mu, \tag{19}$$

with

$$L^\mu = (p - p')^\mu\,, \qquad T^\mu = p\cdot k\,p'^\mu - p'\cdot k\,p^\mu = \frac{(p^2 - p'^2)k^\mu + k^2(p - p')^\mu}{2}\,. \tag{20}$$

The idea is to start with an arbitrary vertex and apply the momentum-space LKF transformation; we then find the longitudinal and transverse parts of the LKF-transformed vertex, which, as we will see, mixes the longitudinal and transverse parts of the input vertex.

Applying the LKF transformation to the longitudinal part of the vertex, we get

$$\Lambda_L^\mu(p', p \,|\, \xi + \Delta\xi) = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\lambda}(p' - q, p + q \,|\, \xi) L^\mu + 2 \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\lambda}(p' - q, p + q \,|\, \xi) q^\mu \,, \tag{21}$$

and using the usual tools of tensor decomposition, we can express the second integral in terms of scalar integrals:

$$\widetilde{I}_\lambda^\alpha := \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\lambda}(p' - q, p + q \,|\, \xi) q^\alpha \equiv \widetilde{L}_\lambda L^\alpha + \widetilde{T}_\lambda T^\alpha \,, \tag{22}$$

with

$$\widetilde{L}_\lambda = \frac{\widetilde{I}_\lambda \cdot p + \widetilde{I}_\lambda \cdot p'}{p^2 - p'^2} \,, \qquad \widetilde{T}_\lambda = \frac{(p'^2 - p \cdot p')\widetilde{I}_\lambda \cdot p + (p^2 - p \cdot p')\widetilde{I}_\lambda \cdot p'}{(p^2 - p'^2)[(p \cdot p')^2 - p^2 p'^2]} \,. \tag{23}$$

It is simple to show that the LKF transformation of a transverse vertex remains transverse, which implies that the longitudinal part of the LKF-transformed vertex arise only from the LKF transformation of the original longitudinal part. Then, the longitudinal part of the LKF transformed vertex is given by

$$\widetilde{\lambda}(p', p \,|\, \xi + \Delta\xi) = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\lambda}(p' - q, p + q \,|\, \xi) + 2\widetilde{L}_\lambda \,. \tag{24}$$

Further comments about the WFGT identity and some general remarks are provided in [8].

Calculated in a similar way, the transverse part of the LKF-transformed vertex is given by

$$\widetilde{\tau}(p', p \,|\, \xi + \Delta\xi) = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\tau}(p' - q, p + q \,|\, \xi) + k^2 \widetilde{T}_\tau + 2\widetilde{T}_\lambda \,. \tag{25}$$

where the tensor decomposition now is performed in the basis $\{T^\mu, K^\mu\}$, with

$$\widetilde{I}_\tau^\alpha := \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Pi_D(q, \Delta\xi)\, \widetilde{\tau}(p' - q, p + q \,|\, \xi) q^\alpha = \widetilde{T}_\tau T^\alpha + \widetilde{K}_\tau k^\alpha \,, \tag{26}$$

$$\widetilde{T}_\tau = \frac{\widetilde{I}_\tau \cdot p\, p' \cdot k - \widetilde{I}_\tau \cdot p'\, p \cdot k}{k^2[(p \cdot p')^2 - p^2 p'^2]} \,, \quad \widetilde{K}_\tau = \frac{\widetilde{I}_\tau \cdot k}{k^2} \,. \tag{27}$$

Note the transverse part *does* receive a contribution (via $\widetilde{T}_\lambda$) from the longitudinal part of the input vertex. We think that this relation for the transverse part could be useful in restricting Ansätze for the non-perturbative form of the vertex in the context of Schwinger-Dyson equations.

As for the propagator, in [8], we apply these transformations to the tree-level photon-amputated vertex in this scalar QED case in arbitrary dimension, getting an all-orders in α expression, which reproduces the order α contribution of the vertex as reported in [14].

4.2. Spinor Case

For the spinor case, there are more vector structures available: the full longitudinal part in this spinor case is composed of 3 longitudinal structures (L_i^μ),

$$L_1^\mu = \gamma^\mu \,, \qquad L_2^\mu = (p - p')^\mu \,, \qquad L_3^\mu = (\not{p} - \not{p}')(p - p')^\mu \,, \tag{28}$$

and the full transverse part by 8 transverse structures (T_i^μ), given by [15]

$$T_1^\mu = (p\cdot k)p'^\mu - (p'\cdot k)p^\mu\,, \qquad T_2^\mu = [(p\cdot k)p'^\mu - (p'\cdot k)p^\mu](\not{p} - \not{p}')\,, \tag{29a}$$
$$T_3^\mu = k^2\gamma^\mu - \not{k}k^\mu\,, \qquad T_4^\mu = k^2[\gamma^\mu(\not{p}-\not{p}') + p^\mu - p'^\mu] - 2k^\mu p^\lambda p'^\nu\sigma_{\lambda\nu}\,, \tag{29b}$$
$$T_5^\mu = -k_\nu\sigma^{\nu\mu}\,, \qquad T_6^\mu = (p'^2 - p^2)\gamma^\mu - \not{k}(p-p')^\mu\,, \tag{29c}$$
$$T_7^\mu = \frac{1}{2}(p'^2 - p^2)[\gamma^\mu(\not{p}-\not{p}') - p'^\mu + p^\mu] - (p-p')^\mu p^\lambda p'^\nu \sigma_{\lambda\nu}\,, \tag{29d}$$
$$T_8^\mu = \gamma^\mu p^\nu p'^\lambda \sigma_{\nu\lambda} + \not{p}' p^\mu - \not{p}p'^\mu. \tag{29e}$$

The LKF transformation of a transverse vertex still remains transverse, although the transverse structures mix. In general the LKF transformation of a longitudinal part mixes longitudinal and transverse vertices.

The full details of the LKF transformations for these transverse and longitudinal components can be found in [8]. Here we show directly the results for the spinor case (the $\widetilde{\lambda}_i$ and $\widetilde{\tau}_i$ are defined in (17) and (18)):

$$\widetilde{\lambda}_1(p',p\,|\,\xi+\Delta\xi) = I_{\lambda_1} + 4L_{1\,\lambda_3} - (p^2-p'^2)^2 T_{\lambda_3}\,, \qquad \widetilde{\lambda}_2(p',p\,|\,\xi+\Delta\xi) = I_{\lambda_2} + 2L_{\lambda_2}\,, \tag{30a}$$
$$\widetilde{\lambda}_3(p',p\,|\,\xi+\Delta\xi) = I_{\lambda_3} + 4L_{\lambda_3} + k^2 T_{\lambda_3} + 4L_{3\,\lambda_3}, \qquad \widetilde{\tau}_1(p',p\,|\,\xi+\Delta\xi) = I_{\tau_1} + 2T_{\lambda_2} + 2L_{\tau_1}\,, \tag{30b}$$
$$\widetilde{\tau}_2(p',p\,|\,\xi+\Delta\xi) = I_{\tau_2} + 4L_{\lambda_2} + 2T_{\lambda_3} + 4T_{2\lambda_3} + 4L_{3\tau_2} + 2k^2 T_{2\tau_2}, \tag{30c}$$
$$\widetilde{\tau}_3(p',p\,|\,\xi+\Delta\xi) = I_{\tau_3} + 4T_{3\lambda_3} + (p'^2-p^2)K_{\tau_2} + 2L_{1\tau_2} + 2k^2 T_{3\tau_2} + 2K_{\tau_6}\,, \tag{30d}$$
$$\widetilde{\tau}_4(p',p\,|\,\xi+\Delta\xi) = I_{\tau_4} + 2L_{\tau_4} + K_{\tau_7} + \frac{1}{2}(C_{\tau_7} - A_{\tau_7})\,, \tag{30e}$$
$$\widetilde{\tau}_5(p',p\,|\,\xi+\Delta\xi) = I_{\tau_5} + 2k^2 K_{\tau_4} + (p^2-p'^2)K_{\tau_7} + T_{5\,\tau_7}\,, \tag{30f}$$
$$\widetilde{\tau}_6(p',p\,|\,\xi+\Delta\xi) = I_{\tau_6} + 4T_{6\lambda_3} - (p^2-p'^2)T_{\lambda_3} + 2k^2 T_{6\tau_2} - k^2 K_{\tau_2} + 2L_{\tau_6}\,, \tag{30g}$$
$$\widetilde{\tau}_7(p',p\,|\,\xi+\Delta\xi) = I_{\tau_7} + 4L_{\tau_7} + T_{7\tau_7}\,, \qquad \widetilde{\tau}_8(p',p\,|\,\xi+\Delta\xi) = I_{\tau_8} + 2L_{\tau_8}\,. \tag{30h}$$

The terms C_{τ_7}, A_{τ_7}, L_{f_i}, T_{f_i}, L_{nf_i}, T_{nf_i} and K_{τ_i} , where f stands for either $\widetilde{\lambda}$ or $\widetilde{\tau}$, are combinations of scalar integrals just as in eq. (23), and the sub-index f_i show where these terms are coming from, and

$$I_{f_i} = \int \frac{\mathrm{d}^D q}{(2\pi)^D}\Pi_D(q,\Delta\xi)\, f_i(p'-q, p+q\,|\,\xi)\,. \tag{31}$$

These expressions are defined fully in [8]; however, by analyzing where the terms are coming from, we still can derive some important results without giving their explicit expressions. We can see that there are some sub-spaces that are closed under the action of the transformation, given by $\{L_1\}$, $\{T_1\}$, $\{T_3\}$, $\{T_5\}$, $\{T_6\}$, $\{T_8\}$, $\{L_2, T_1\}$, $\{T_3, T_6\}$, $\{T_4, T_5\}$, $\{T_2, T_3, T_6\}$, $\{T_4, T_5, T_7\}$, $\{L_1, L_3, T_2, T_3, T_6\}$, and invariant subspaces given by $\{T_8\}$, $\{T_4, T_5, T_7\}$, $\{L_1, L_2, L_3, T_1, T_2, T_3, T_6\}$, respectively. We believe that, again, this could be very useful in the context of the Schwinger-Dyson equations, because now we can see that we can fix the gauge dependent form of the vertex in parts, taking into account the invariant subspaces.

5. Conclusions

We have reported on a momentum-space form of the Landau-Khalatnikov-Fradkin transformation for the matter propagator and 3-point interaction vertex in QED, for an arbitrary dimension and input. We have also explored the special case of these gauge transformations in

$D = 2$ space-time dimensions. We applied them to the photon-amputated vertex to find the longitudinal and transverse parts of the LKF-transformed vertex and we found relations between the components of the vertex determined in different covariant gauges. We anticipate that these relations which may be useful in the context of the Schwinger-Dyson equations. In ongoing work these momentum space transformations are being transferred to the fully amputated vertex, which are of more direct physical significance.

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Appendix A. Appendix A – Integrals Involving Bessel functions

We require the following integrals for the LKF transformation about $D = 2$ dimensions which we record here to avoid interrupting the main text:

$$\int_0^\infty dz\, z J_\alpha(az) K_\alpha(bz) = \frac{\left(\frac{a}{b}\right)^\alpha}{a^2 + b^2}\,; \quad \alpha > -1, \tag{A.1}$$

$$\int_0^\infty dz\, z^3 J_\alpha(az) K_\alpha(bz) = \frac{4\left(\frac{a}{b}\right)^\alpha}{(a^2 + b^2)^3}\left[b^2 - a^2 + \alpha\left(b^2 + a^2\right)\right]\,; \quad \alpha > -2 \tag{A.2}$$

$$\int_0^\infty dz\, z^3 \log(z) J_0(az) K_0(bz) = \frac{4}{(a^2 + b^2)^3} \times \left[\left(a^2 - b^2\right)\left(\log\left(1 + \frac{a^2}{b^2}\right) + \log\left(\frac{b}{2}\right) + \gamma_E - 1\right) - a^2\right]. \tag{A.3}$$

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