Journal of Functional Analysis 286 (2024) 110271

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Rellich identities for the Hilbert transform ☆

María J. Carro [a,b], Virginia Naibo [c,*], María Soria-Carro [d]

[a] *Department of Analysis and Applied Mathematics, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain*
[b] *Instituto de Ciencias Matemáticas ICMAT, Madrid, Spain*
[c] *Department of Mathematics, Kansas State University, 138 Cardwell Hall, 1228 N. Martin Luther King Jr. Dr., KS 66506, USA*
[d] *Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854, USA*

A R T I C L E I N F O

Article history:
Received 9 May 2023
Accepted 24 November 2023
Available online 1 December 2023
Communicated by Stefanie Petermichl

MSC:
42B20

Keywords:
Hilbert transform
Rellich identity
A_2-conjecture
Helson–Szegö theorem

A B S T R A C T

We prove Hilbert transform identities involving conformal maps via the use of Rellich identity and the solution of the Neumann problem in a graph Lipschitz domain in the plane. We obtain as consequences new L^2-weighted estimates for the Hilbert transform, including a sharp bound for its norm as a bounded operator in weighted L^2 in terms of a weight constant associated to the Helson-Szegö theorem.

☆ First author partially supported by grants PID2020-113048GB-I00 funded by MCIN/AEI/10.13039/501100011033, CEX2019-000904-S funded by MCIN/AEI/10.13039/501100011033 and Grupo UCM-970966 (Spain). Second author partially supported by the NSF under grant DMS 2154113 and the Simons Foundation under grant 705953 (USA).
* Corresponding author.
E-mail addresses: mjcarro@ucm.es (M.J. Carro), vnaibo@ksu.edu (V. Naibo), maria.soriacarro@rutgers.edu (M. Soria-Carro).

https://doi.org/10.1016/j.jfa.2023.110271

1. Introduction and main results

Let H be the Hilbert transform, which is defined as

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy;$$

Hunt–Muckenhoupt–Wheeden [9] proved that if $1 < p < \infty$, then H is bounded from $L^p(w)$ to $L^p(w)$ if and only if $w \in A_p$. Here $L^p(w)$ is the Lebesgue space of measurable functions defined on $\mathbb{R}$ that are p-integrable with respect to the measure $w(x)dx$ and A_p denotes the Muckenhoupt class of weights on $\mathbb{R}$.

Applications in Partial Differential Equations (see Fefferman–Kenig–Pipher [5]) led to a profound study of sharp bounds for the operator norm of the Hilbert transform in terms of the A_p constant of the weight, denoted $[w]_{A_p}$ (see Section 2.1 for its definition). In particular, the A_2-conjecture for the Hilbert transform consisted in proving that the Hilbert transform satisfies analogous estimates to those by the Hardy-Littlewood maximal operator, that is,

$$||Hf||_{L^2(w)} \lesssim [w]_{A_2} ||f||_{L^2(w)} \quad \forall f \in L^2(w). \tag{1.1}$$

This conjecture was solved by Petermichl [16] in 2007 and numerous other related results followed concerning its extension to singular integral operators and other classical operators, including Petermichl [17], Hytönen [10], Hytönen et al. [12,11], Lerner [15] and Cruz-Uribe–Martell–Pérez [3].

General necessary and sufficient conditions on the weights for the boundedness of the Hilbert transform on weighted L^2 spaces were first obtained by Helson-Szegö [8] in 1960 using complex variable techniques. More precisely, they proved that

$$H : L^2(w) \longrightarrow L^2(w) \quad \iff \quad w = e^{f_1 + Kf_2},\ f_1, f_2 \in L^\infty(\mathbb{R}),\ ||f_2||_{L^\infty} < \pi/2,$$

where K is a version of the Hilbert transform for L^∞-functions (see Section 2.1). As a consequence, we have

$$w \in A_2 \quad \iff \quad w = e^{f_1 + Kf_2}, \quad f_1, f_2 \in L^\infty(\mathbb{R}),\ ||f_2||_{L^\infty} < \pi/2. \tag{1.2}$$

It is worth mentioning that the implication to the left is easy to prove, but no direct proof of the implication to the right is presently known; the reader is directed to the work García-Cuerva [6] for an interesting survey on the topic. In particular, it is proved in [6] that, if $||f_2||_{L^\infty} < \pi/2$, then

$$[e^{Kf_2}]_{A_2} \leq \sec^2 ||f_2||_{L^\infty}. \tag{1.3}$$

Given $w \in A_2$, define the *Helson-Szegö constant* of w, $[w]_{A_2(\mathrm{HS})}$, as

$$[w]^2_{A_2(\mathrm{HS})} := \inf_{\mathcal{D}_w} e^{\operatorname{osc} f_1} \sec^2 \|f_2\|_{L^\infty},$$

where $\operatorname{osc} f_1 = \sup f_1 - \inf f_1$ and

$$\mathcal{D}_w := \left\{(f_1, f_2) : w = e^{f_1 + K f_2} \text{ with } f_1, f_2 \in L^\infty(\mathbb{R}) \text{ and } \|f_2\|_{L^\infty} < \pi/2\right\}.$$

The inequality $[w]_{A_2} \le [w]^2_{A_2(\mathrm{HS})}$ follows from (1.3) and, along with (1.1), leads to

$$||Hf||_{L^2(w)} \lesssim [w]^2_{A_2(\mathrm{HS})} ||f||_{L^2(w)}.$$

One of the results in this article improves the dependence on the Helson-Szegö constant of the weight in the above inequality; indeed, we show that the dependence is linear in $[w]_{A_2(\mathrm{HS})}$. More precisely, we have the following estimate:

Theorem 1.1. *For every $w \in A_2$ and $f \in L^2(w)$ real-valued, it holds that*

$$||Hf||_{L^2(w)} \lesssim [w]_{A_2(\mathrm{HS})} ||f||_{L^2(w)}. \tag{1.4}$$

The dependence of $[w]_{A_2(\mathrm{HS})}$ in (1.4) is sharp.

The estimate (1.4) is sharp in the sense that the norm of the operator H as a bounded operator on $L^2(w)$ is comparable to $[w]_{A_2(\mathrm{HS})}$ for some $w \in A_2$. We hope that (1.4) may lead to a new proof of (1.1) by showing that $[w]_{A_2(\mathrm{HS})} \lesssim [w]_{A_2}$, which at the moment is an open question.

The estimate (1.4) along with other new L^2-weighted estimates for the Hilbert transform proved in this article are consequences of our main result on Rellich-type identities for the Hilbert transform. Such identities involve a conformal map Φ such that $\Omega = \Phi(\mathbb{R}^2_+)$ is a graph Lipschitz domain, that is, the upper part of the graph of a real-valued Lipschitz function; the map Φ extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$ and $\Phi'(x)$ exists and is non-zero for almost every $x \in \mathbb{R}$ (see details in Section 2.2). Our main result is the following theorem.

Theorem 1.2. *Let Φ be a conformal map as described in Section 2.2 and $f \in L^2(|\Phi'|^{-1})$ be real-valued. Then*

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx - 2\int_{\mathbb{R}} f\, Hf \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx, \tag{1.5}$$

and

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx = \int_{\mathbb{R}} f^2 \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx + 2\int_{\mathbb{R}} f\, Hf \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx. \tag{1.6}$$

We remark that there is a strong connection between the Helson-Szegö result (1.2) and conformal maps Φ as in the statement of Theorem 1.2. As proved in Kenig [14, Theorem 1.10 and Lemma 1.11], it holds that

$$|\Phi'| \in A_2$$

and a converse result is true in the sense that if $w \in A_2$ then there exists Φ such that $|\Phi'| \sim w$. Indeed, the latter is a consequence of (1.2): Let $w = e^{f_1+Kf_2}$ with $(f_1, f_2) \in D_w$, then it is proved in [14] that there exists a conformal map Φ from $\mathbb{R}^2_+$ onto a graph Lipschitz domain so that

$$\Phi'(x) = e^{Kf_2(x)}e^{-if_2(x)} \quad \text{a.e. } x \in \mathbb{R}; \tag{1.7}$$

therefore, $|\Phi'| = e^{-f_1}w$ and it follows that $|\Phi'| \sim w$.

The main ingredients in the proof of Theorem 1.2 are the following tools: (a) the theory of solutions of the Neumann problem in a graph Lipschitz domain with data in L^2 through the use of conformal maps as developed in Carro–Naibo–Ortiz [2] and, (b) Rellich's identity, which gives that if Φ is as in the statement of Theorem 1.2, then for every harmonic function u in $\Omega = \Phi(\mathbb{R}^2_+)$ so that $M(\nabla u) \in L^2(\partial\Omega)$ and for every constant vector $e \in \mathbb{R}^2$,

$$\int_{\partial\Omega} |\nabla u|^2 \,(e\cdot\nu)\, d\sigma = 2\int_{\partial\Omega} (\partial_\nu u)\,(e\cdot\nabla u)\, d\sigma, \tag{1.8}$$

where M is the non-tangential maximal operator and $d\sigma$ denotes integration with respect to arc-length. The integral identity (1.8) is due to Rellich [18] (see also Escauriaza–Mitrea [4, (2.35)]); this identity and related versions play fundamental roles in questions on elliptic partial differential equations, inverse problems, acoustic scattering, and the multiplier method; see Agrawal–Alazard [1] and references therein.

We note that the proof of [9, Lemma 10] shows that for an infinitely differentiable function f with compact support in $\mathbb{R}$, it holds that

$$\int_{-\infty}^{\infty} (f + iHf)^2\Phi'\, dx = 0, \tag{1.9}$$

where Φ is a conformal map from $\mathbb{R}^2_+$ onto a graph Lipschitz domain such that $\Phi' = e^{Kg}e^{-ig}$ for some g satisfying $\|g\|_{L^\infty} < \frac{\pi}{2}$. Instances of (1.5) and (1.6) with Φ' instead of $1/\Phi'$ can then be deduced by taking the real and imaginary parts of (1.9). However, (1.5) and (1.6) are not explicitly shown in [9] and the proof of (1.9) is based on complex variable techniques that are different from the novel approach we use in the proof of Theorem 1.2, which holds for more general conformal maps Φ.

The paper is organized as follows. In Section 2, we present preliminaries regarding weights, the Hilbert transform, conformal mappings and the solution of the Neumann problem in a graph Lipschitz domain in the plane with data in L^2. In Section 3, we prove the Rellich's identity versions for the Hilbert transform presented in Theorem 1.2 as well as L^2-weighted estimates for H that follow from them, including (1.4). Other applications of Theorem 1.2 concerning Hilbert transform identities with power weights are discussed in Section 4. Finally, we present in Section 5 versions of Rellich identities for the Hilbert transform in weighted L^p spaces.

2. Preliminaries

In this section we present preliminaries regarding weights, the Hilbert transform, conformal mappings and the solution of the Neumann problem in a graph Lipschitz domain in the plane with data in L^2.

2.1. Muckenhoupt weights and the Hilbert transform for L^∞ functions

Let w be a weight on $\mathbb{R}$, i.e. a non-negative locally integrable function defined in $\mathbb{R}$, and $1 \le p \le \infty$. We will denote by $L^p(w)$ the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}} |f(x)|^p w(x)\, dx \right)^{\frac{1}{p}} < \infty,$$

with the corresponding changes for $p = \infty$. If $w \equiv 1$, we will use the notation $L^p(\mathbb{R})$ instead of $L^p(w)$.

For $1 < p < \infty$, a weight w defined on $\mathbb{R}$ belongs to the Muckenhoupt class A_p if

$$[w]_{A_p} = \sup_{I \subset \mathbb{R}} \left(\frac{1}{|I|} \int_I w(x) dx \right) \left(\frac{1}{|I|} \int_I w(x)^{1-p'} dx \right)^{p-1} < \infty, \tag{2.1}$$

where the supremum is taken over all intervals contained in $\mathbb{R}$. We recall that $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, where p' is the conjugate exponent of p (i.e. $1/p + 1/p' = 1$); also, $A_p \subset A_q$ if $p < q$ and for every $w \in A_p$ there exist $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.

We define the operator K as

$$Kf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} f(y) \left(\frac{1}{x-y} + \frac{\chi_{|y|>1}(y)}{y} \right) dy,$$

which allows to extend the definition of the Hilbert transform to $L^\infty(\mathbb{R})$.

For brevity we will use the notation $A \lesssim B$ to mean that $A \leq cB$, where c is a constant that may only depend on some of the parameters but not on the functions or weights involved.

2.2. Conformal maps and the Neumann problem in a graph Lipchitz domain in the plane

In this section we define the conformal maps that will be considered throughout this work and recall results from [2] on the solution of the Neumann problem in a graph Lipschitz domain with data in L^2, which is obtained via the use of conformal maps and the solution of the Neumann problem in the upper-half plane.

Let Λ be a curve in the complex plane given parametrically by $\xi(x) = x + i\gamma(x)$ for $x \in \mathbb{R}$, where γ is a real-valued Lipschitz function with constant k, and consider *the graph Lipschitz domain*

$$\Omega = \{z_1 + iz_2 \in \mathbb{C} : z_2 > \gamma(z_1)\}. \tag{2.2}$$

We have $\Lambda = \partial\Omega$ and, since Ω is simply connected, then there exists a conformal map $\Phi : \mathbb{R}^2_+ \longrightarrow \Omega$ such that $\Phi(\infty) = \infty$ and $\Phi(i) = iy_0$ for some $y_0 > \gamma(0)$. The map Φ extends as a homeomorphism from $\overline{\mathbb{R}^2_+}$ onto $\overline{\Omega}$ and $\Phi(x)$, $x \in \mathbb{R}$, is absolutely continuous when restricted to any finite interval; this implies that $\Phi'(x)$ exists for almost every $x \in \mathbb{R}$ and is locally integrable. It also holds that $\Phi'(x) \neq 0$ for almost every $x \in \mathbb{R}$, $\lim_{z\to x} \Phi'(z) = \Phi'(x)$ in the non-tangential sense for almost every $x \in \mathbb{R}$ and $|\Phi'| \in A_2$. If $\Phi'(x)$ exists and is not zero, then it is a vector tangent to $\partial\Omega$ at $\Phi(x)$. See Kenig [14, Theorems 1.1 and 1.10] for the proof of those properties. The inverse of Φ will be denoted Ψ.

Given a measurable function g defined in $\partial\Omega$, let T_Φ be given by

$$T_\Phi g(x) = |\Phi'(x)| g(\Phi(x)), \quad x \in \mathbb{R}.$$

Denote by $L^2(\partial\Omega)$ the Lebesgue space of measurable functions defined on $\partial\Omega$ that are square-integrable with respect to arc-length. We note that T_Φ is a bijection from $L^2(\partial\Omega)$ onto $L^2(|\Phi'|^{-1})$ and we have $\|g\|_{L^2(\partial\Omega)} = \|T_\Phi g\|_{L^2(|\Phi'|^{-1})}$.

For $g \in L^2(\partial\Omega)$, consider the classical Neumann problem in Ω:

$$\Delta v = 0 \text{ on } \Omega, \qquad \partial_\nu v = g \text{ on } \partial\Omega \quad \text{and} \quad \mathcal{M}_\alpha(\nabla v) \in L^2(\partial\Omega). \tag{2.3}$$

Here Δ is the Laplace operator, ν denotes the outward unit normal vector to $\partial\Omega$, $\partial_\nu v = \nabla v \cdot \nu$ and the equality $\partial_\nu v = g$ is meant in the non-tangential convergence sense. For $0 < \alpha < \arctan(1/k)$, $\mathcal{M}_\alpha$ denotes the non-tangential maximal operator given by

$$\mathcal{M}_\alpha(F)(\xi) = \sup_{z \in \Gamma_\alpha(\xi)} |F(z)|, \quad \xi \in \partial\Omega,$$

for a complex-valued function F defined in Ω and

$$\Gamma_\alpha(\xi) = \{z_1 + iz_2 \in \mathbb{C} : z_2 > \mathrm{Im}(\xi) \text{ and } |\mathrm{Re}(\xi) - z_1| < \tan(\alpha)|z_2 - \mathrm{Im}(\xi)|\}.$$

Inspired by tools and techniques from Kenig [13,14], it was proved in [2, Theorem 1.4] that for every $g \in L^2(\partial\Omega)$, $v = u_{T_\Phi g} \circ \Psi$ is a solution of the Neumann problem (2.3) and

$$\|\mathcal{M}_\alpha(\nabla v)\|_{L^2(\partial\Omega)} \lesssim \|g\|_{L^2(\partial\Omega)},$$

where, for $f : \mathbb{R} \to \mathbb{C}$, u_f is defined by

$$u_f(x,y) := -\frac{1}{\pi} \int_{\mathbb{R}} \log\left(\frac{\sqrt{(x-t)^2+y^2}}{1+|t|}\right) f(t)\, dt, \quad (x,y) \in \mathbb{R}^2_+. \tag{2.4}$$

We note that the integral on the right-hand side of (2.4) is absolutely convergent for all f satisfying $\int_{\mathbb{R}} \frac{|f(t)|}{1+|t|}\, dt < \infty$; in particular, it is well defined for any $f \in L^2(w)$ with $w \in A_2$. As shown in [2], u_f is a solution of the Neumann problem in the upper half plane: More precisely, if $w \in A_2$ and $f \in L^2(w)$, then u_f is harmonic in $\mathbb{R}^2_+$, $\nabla u \cdot (0,-1) = f$ on $\mathbb{R}$ in the sense of non-tangential convergence and

$$\|\mathcal{M}_\alpha(\nabla u_f)\|_{L^2(w)} \lesssim \|f\|_{L^2(w)}.$$

It follows that

$$\partial_x u_f(x,y) = -(Q_y * f)(x) \quad \text{and} \quad \partial_y u_f(x,y) = -(P_y * f)(x),$$

where, for $y > 0$, P_y is the Poisson kernel and Q_y is the conjugate of the Poisson kernel. As a consequence,

$$\partial_x u_f(x,0) = -Hf(x) \quad \text{and} \quad \partial_y u_f(x,0) = -f(x), \tag{2.5}$$

for almost every $x \in \mathbb{R}$ in the non-tangential convergence sense.

For easier referencing, we state as a theorem the following result mentioned in Section 1.

Theorem A *(see proof of Lemma 1.11 in [14]). If $w \in A_2$ and $w = e^{f_1+Kf_2}$ with $(f_1, f_2) \in D_w$, then there exists a conformal map Φ_w from $\mathbb{R}^2_+$ onto a graph Lipschitz domain so that*

$$\Phi_w'(x) = e^{Kf_2(x)}e^{-if_2(x)} \quad a.e.\ x \in \mathbb{R}. \tag{2.6}$$

In particular $|\Phi_w'| = e^{-f_1}w = e^{Kf_2}$, $\mathrm{Arg}\,\Phi_w' = -f_2$ and $|\Phi_w'| \sim w$.

3. Rellich's identities and new L^2-weighted estimates for H

In this section we prove the Rellich's identity versions for the Hilbert transform stated in Theorem 1.2 and new L^2-weighted estimates for H that follow from them. We start with some preliminaries in Section 3.1, give the proof of Theorem 1.2 in Section 3.2 and present the L^2-weighted estimates for H in Section 3.3.

3.1. Preliminaries

Let Ω, Φ and Ψ be as in Section 2.2. We can parametrize the curve $\partial\Omega$ using the conformal map Φ, that is,

$$\partial\Omega = \{z \in \mathbb{C} : z = \Phi(x) \text{ for some } x \in \mathbb{R}\}.$$

We write $\Phi = \Phi_1 + i\Phi_2$, where $\Phi_1 = \mathrm{Re}(\Phi)$ and $\Phi_2 = \mathrm{Im}(\Phi)$, and analogously, $\Psi = \Psi_1 + i\Psi_2$. For convenience, we will sometimes denote a complex number $z = z_1 + iz_2$ using vector notation, i.e., $z = (z_1, z_2)$. Let $\nu(z)$ be the outward unit normal vector to $\partial\Omega$; then,

$$\nu(z) = \frac{\big(\Phi_2'(\Psi(z)), -\Phi_1'(\Psi(z))\big)}{|\Phi'(\Psi(z))|}. \tag{3.1}$$

Since Ψ is also a conformal map, it satisfies the Cauchy–Riemann equations:

$$\partial_1\Psi_1 = \partial_2\Psi_2 \quad \text{and} \quad \partial_2\Psi_1 = -\partial_1\Psi_2, \tag{3.2}$$

where ∂_1 and ∂_2 denote the partial derivatives with respect to the first and second variable, respectively. The following lemma will be useful in the proof of Theorem 1.2.

Lemma 3.1. *For almost every $x \in \mathbb{R}$, it holds that*

$$(\partial_1\Psi_1)(\Phi(x)) = \frac{\Phi_1'(x)}{|\Phi'(x)|^2} = \mathrm{Re}\left(\frac{1}{\Phi'(x)}\right),$$
$$(\partial_2\Psi_1)(\Phi(x)) = \frac{\Phi_2'(x)}{|\Phi'(x)|^2} = -\,\mathrm{Im}\left(\frac{1}{\Phi'(x)}\right).$$

Proof. Since $\Psi(\Phi(x,y)) = (x,y)$, we have

$$\Psi_1(\Phi(x,y)) = x, \quad \text{for all } (x,y) \in \overline{\mathbb{R}^2_+}.$$

Differentiating with respect to x, we get

$$(\partial_1\Psi_1)(\Phi(x,y))\partial_x\Phi_1(x,y) + (\partial_2\Psi_1)(\Phi(x,y))\partial_x\Phi_2(x,y) = 1,$$

and similarly, differentiating with respect to y, we have

$$(\partial_1\Psi_1)(\Phi(x,y))\partial_y\Phi_1(x,y) + (\partial_2\Psi_1)(\Phi(x,y))\partial_y\Phi_2(x,y) = 0.$$

Furthermore, since Φ satisfies the Cauchy–Riemann equations, $\partial_x\Phi_1 = \partial_y\Phi_2$ and $\partial_y\Phi_1 = -\partial_x\Phi_2$. Set $y = 0$, and denote $\Phi_1'(x) = \partial_x\Phi_1(x,0)$, $\Phi_2'(x) = \partial_x\Phi_2(x,0)$, and $\Phi' = \Phi_1' + i\Phi_2'$. Recall that $\Phi'(x)$ exists and is non-zero for almost every $x \in \mathbb{R}$. Then

$$(\partial_1\Psi_1)(\Phi(x))\Phi_1'(x) + (\partial_2\Psi_1)(\Phi(x))\Phi_2'(x) = 1 \tag{3.3}$$

$$-(\partial_1\Psi_1)(\Phi(x))\Phi_2'(x) + (\partial_2\Psi_1)(\Phi(x))\Phi_1'(x) = 0. \tag{3.4}$$

Multiplying (3.3) by $\Phi_2'(x)$ and (3.4) by $\Phi_1'(x)$, and adding both equations, we see that

$$(\partial_2\Psi_1)(\Phi(x)) = \frac{\Phi_2'(x)}{|\Phi'(x)|^2}, \quad \text{for a.e. } x \in \mathbb{R}.$$

Finally, substituting this expression into (3.4) when $\Phi_2'(x) \neq 0$, we get

$$(\partial_1\Psi_1)(\Phi(x)) = \frac{\Phi_1'(x)}{|\Phi'(x)|^2}, \quad \text{for a.e. } x \in \mathbb{R}.$$

If $\Phi_2'(x) = 0$, then (3.3) gives $(\partial_1\Psi_1)(\Phi(x)) = \frac{1}{\Phi_1'(x)} = \frac{\Phi_1'(x)}{|\Phi'(x)|^2}$. □

3.2. Proof of Theorem 1.2

Let $f \in L^2(|\Phi'|^{-1})$ be real-valued; since $T_\Phi : L^2(\partial\Omega) \to L^2(|\Phi'|^{-1})$ is invertible, there exists a unique $g \in L^2(\partial\Omega)$ such that $f = T_\Phi g$. As explained in Section 2.2, a solution v of the Neumann problem (2.3) with datum g can be represented by $v = u \circ \Psi$, where $u = u_f$ as given in (2.4) and (2.5) holds. Then the solution v satisfies Rellich's identity (1.8):

$$\int_{\partial\Omega} |\nabla v|^2 \,(e\cdot\nu)\, d\sigma = 2\int_{\partial\Omega} g\,\,(e\cdot\nabla v)\, d\sigma,$$

for any constant vector $e = (e_1, e_2)$.

Our goal is to derive a Rellich's type identity for the function u. To this end, we need to compute the partial derivatives of v in terms of u. Let $z \in \partial\Omega$, with $z = \Phi(x)$, for some $x \in \mathbb{R}$. Using Lemma 3.1, (2.5) and (3.2), we obtain

$$(\partial_1 v)(\Phi(x)) = -H(T_\Phi g)(x)\,\mathrm{Re}\left(\frac{1}{\Phi'(x)}\right) - T_\Phi g(x)\,\mathrm{Im}\left(\frac{1}{\Phi'(x)}\right), \tag{3.5}$$

$$(\partial_2 v)(\Phi(x)) = H(T_\Phi g)(x)\,\mathrm{Im}\left(\frac{1}{\Phi'(x)}\right) - T_\Phi g(x)\,\mathrm{Re}\left(\frac{1}{\Phi'(x)}\right). \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$|\nabla v(\Phi(x))|^2 = \big((H(T_\Phi g)(x))^2 + (T_\Phi g(x))^2\big)\frac{1}{|\Phi'(x)|^2}. \tag{3.7}$$

Making the change of variables $z = \Phi(x)$, and using (3.7) and (3.1), we see that

$$\begin{aligned}\int_{\partial\Omega} |\nabla v|^2\ (e\cdot\nu)\,d\sigma &= \int_{\mathbb{R}} \big((H(T_\Phi g))^2 + (T_\Phi g)^2\big)\left((e_1,e_2)\cdot\frac{(\Phi_2', -\Phi_1')}{|\Phi'|^2}\right) dx\\ &= -e_1\int_{\mathbb{R}} \big((H(T_\Phi g))^2 + (T_\Phi g)^2\big)\operatorname{Im}\left(\frac{1}{\Phi'}\right) dx\\ &\quad - e_2\int_{\mathbb{R}} \big((H(T_\Phi g))^2 + (T_\Phi g)^2\big)\operatorname{Re}\left(\frac{1}{\Phi'}\right) dx.\end{aligned}$$

Similarly, we get

$$\begin{aligned}2\int_{\partial\Omega} g\ (e\cdot\nabla v)\,d\sigma &= -2e_1\int_{\mathbb{R}} T_\Phi g\, H(T_\Phi g)\operatorname{Re}\left(\frac{1}{\Phi'}\right) dx - 2e_1\int_{\mathbb{R}} (T_\Phi g)^2\operatorname{Im}\left(\frac{1}{\Phi'}\right) dx\\ &\quad + 2e_2\int_{\mathbb{R}} T_\Phi g\, H(T_\Phi g)\operatorname{Im}\left(\frac{1}{\Phi'}\right) dx - 2e_2\int_{R} (T_\Phi g)^2\operatorname{Re}\left(\frac{1}{\Phi'}\right) dx.\end{aligned}$$

Therefore, Rellich's identity yields

$$\begin{aligned}&2e_1\int_{\mathbb{R}} T_\Phi g\, H(T_\Phi g)\operatorname{Re}\left(\frac{1}{\Phi'}\right) dx - e_2\int_{\mathbb{R}} \big((H(T_\Phi g))^2 - (T_\Phi g)^2\big)\operatorname{Re}\left(\frac{1}{\Phi'}\right) dx\\ &= e_1\int_{\mathbb{R}} \big((H(T_\Phi g))^2 - (T_\Phi g)^2\big)\operatorname{Im}\left(\frac{1}{\Phi'}\right) dx + 2e_2\int_{\mathbb{R}} T_\Phi g\, H(T_\Phi g)\operatorname{Im}\left(\frac{1}{\Phi'}\right) dx.\end{aligned}$$

Recalling that $T_\Phi g = f$, (1.5) and (1.6) immediately follow from the above identity, taking $e = (0,1)$ and $e = (1,0)$, respectively. □

Remark 3.2. Observe that for $\Omega = \mathbb{R}^2_+$, we have $\Phi(x,y) = (x,y)$. If $y = 0$, then $\Phi'(x) = 1$, and noting that $\int_{\mathbb{R}} fHf\,dx = 0$, (1.5) recovers the well-known identity $\|Hf\|_{L^2} = \|f\|_{L^2}$ for the Hilbert transform in $L^2(\mathbb{R})$.

3.3. L^2-weighted estimates for the Hilbert transform

In this section we prove new L^2-weighted estimates for the Hilbert transform that are consequences of Theorem 1.2. In what follows Φ is a conformal map as described in Section 2.2.

We first state and prove Theorem 3.3, which gives L^2-weighted estimates for H with constants in terms of $\varphi(\tau^2)$, where $\tau = \tan \| \operatorname{Arg} \Phi' \|_{L^\infty}$ and

$$\varphi(s) := \inf_{0<\varepsilon<1} \frac{\varepsilon + s}{(1-\varepsilon)\varepsilon} = 1 + 2s + 2\sqrt{s^2 + s}, \quad s \geq 0.$$

We then prove Theorem 1.1, which follows from Theorem 3.3 and gives sharp L^2-weighted estimates for H in terms of the Helson-Szegö constant of the weight as discussed in Section 1.

We end the section with Corollary 3.4, which gives an identity for the norm of H as a bounded operator on $L^2(\operatorname{Re}(\frac{1}{\Phi'}))$, and Corollary 3.5, which gives a uniform bound for the norm of H as a bounded operator on $L^2(|\Phi'|^{-1})$ when $\Phi(\mathbb{R}^2_+)$ is a monotone Lipschitz domain.

Theorem 3.3. *Let Φ be a conformal map as described in Section 2.2 and $\tau = \tan \| \operatorname{Arg} \Phi' \|_{L^\infty}$. Then for all $f \in L^2(|\Phi'|^{-1})$ real-valued, we have*

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx \leq \varphi(\tau^2) \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx, \tag{3.8}$$

$$\int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1} dx \leq (2+\sqrt{3})\varphi(\tau^2) \int_{\mathbb{R}} f^2 \, |\Phi'|^{-1} dx. \tag{3.9}$$

Proof. Proof of (3.8): We first estimate the last term on the right-hand side of (1.5). Since $\operatorname{Im}(\frac{1}{\Phi'}) = \operatorname{Re}(\frac{1}{\Phi'}) \tan(\operatorname{Arg} \frac{1}{\Phi'}) = -\operatorname{Re}(\frac{1}{\Phi'}) \tan(\operatorname{Arg} \Phi')$ and $\operatorname{Re}(\frac{1}{\Phi'}) = \frac{\operatorname{Re}(\Phi')}{|\Phi'|^2} \geq 0$, it follows that

$$\int_{\mathbb{R}} f Hf \left| \operatorname{Im}\left(\frac{1}{\Phi'}\right) \right| dx \leq \tau \int_{\mathbb{R}} f Hf \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx.$$

Therefore, using the Cauchy-Schwarz inequality and (1.5), we get

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx \leq \left(1 + \frac{\tau^2}{\varepsilon}\right) \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx + \varepsilon \int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx.$$

Equivalently,

$$\int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx \leq \frac{\varepsilon + \tau^2}{(1-\varepsilon)\varepsilon} \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx,$$

and the result follows by taking the infimum in $0 < \varepsilon < 1$.

Proof of (3.9): Using (3.8) and again that $\operatorname{Im}(\frac{1}{\Phi'}) = -\operatorname{Re}(\frac{1}{\Phi'}) \tan(\operatorname{Arg} \Phi')$ and $\operatorname{Re}(\frac{1}{\Phi'}) \geq 0$, we have

$$\int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \le \varphi(\tau^2) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx + \tau \int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx.$$

We next use (1.5) and the Cauchy-Schwarz inequality to control the second term on the right-hand side:

$$\begin{aligned}\tau \int_{\mathbb{R}} (Hf)^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx &\le \tau \int_{\mathbb{R}} f^2 \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx + \frac{\tau}{\varepsilon} \int_{\mathbb{R}} f^2 \left|\operatorname{Im}\left(\frac{1}{\Phi'}\right)\right| dx \\ &\quad + \varepsilon\tau \int_{\mathbb{R}} (Hf)^2 \left|\operatorname{Im}\left(\frac{1}{\Phi'}\right)\right| dx \\ &\le \left(\tau + \frac{\tau}{\varepsilon}\right) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx + \varepsilon\tau \int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \\ &\le \left(\frac{1+\tau}{\varepsilon}\right) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx + \varepsilon\tau \int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx,\end{aligned}$$

where $\varepsilon > 0$ is such that $0 < \varepsilon\tau < 1$. Combining both estimates, we see that

$$(1-\varepsilon\tau) \int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \le \left(\varphi(\tau^2) + \frac{1+\tau}{\varepsilon}\right) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx.$$

Setting $\delta = \varepsilon\tau$, we obtain

$$\int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \le \varphi(\tau^2) \frac{\delta + (1+\tau)\tau/\varphi(\tau^2)}{(1-\delta)\delta} \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx.$$

Taking the infimum over all $0 < \delta < 1$ leads to

$$\int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \le \varphi(\tau^2)\varphi((1+\tau)\tau/\varphi(\tau^2)) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx.$$

Since $(1+\tau)\tau/\varphi(\tau^2) \le 1/2$, we have $\varphi((1+\tau)\tau/\varphi(\tau^2)) \le \varphi(1/2) = 2 + \sqrt{3}$; therefore, it follows that

$$\int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \le (2+\sqrt{3})\varphi(\tau^2) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx,$$

as desired. □

We next present the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $w \in A_2$, we have $w^{-1} = e^{-f_1 - Kf_2}$ with $(f_1, f_2) \in \mathcal{D}_w$. Let $\Phi = \Phi_{w^{-1}}$ be as given in Theorem A; then $|\Phi'| = e^{-Kf_2}$ and $\operatorname{Arg}\Phi' = f_2$.

By (3.9) and using that $\varphi(s) \sim 1 + s$ for $s \geq 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} (Hf)^2 w\, dx &= \int_{\mathbb{R}} (Hf)^2 e^{f_1 + Kf_2}\, dx \\ &\leq e^{\sup f_1} \int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1}\, dx \\ &\lesssim e^{\sup f_1} \varphi(\tan^2 \|f_2\|_{L^\infty}) \int_{\mathbb{R}} f^2 |\Phi'|^{-1}\, dx \\ &\lesssim e^{\operatorname{osc} f_1} (1 + \tan^2 \|f_2\|_{L^\infty}) \int_{\mathbb{R}} f^2 w\, dx.\end{aligned}$$

Taking infimum over all pairs $(f_1, f_2) \in \mathcal{D}_w$, we obtain (1.4).

The fact that the dependence of $[w]_{A_2(\mathrm{HS})}$ in (1.4) is sharp follows from Remark 4.4. □

The next corollary is a direct consequence of (1.5).

Corollary 3.4. *If Φ is a conformal map as described in Section 2.2, then*

$$||H||^2_{L^2\left(\mathrm{Re}\left(\frac{1}{\Phi'}\right)\right) \to L^2\left(\mathrm{Re}\left(\frac{1}{\Phi'}\right)\right)} = \sup_{\substack{||f||_{L^2\left(\mathrm{Re}\left(\frac{1}{\Phi'}\right)\right)} = 1 \\ f \text{ real-valued}}} \left| 1 - 2 \int_{\mathbb{R}} f Hf \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx \right| \tag{3.10}$$

It is worth noting that for the integrals on the right hand side of (3.10), only the values $Hf(x)$ for x in the support of f are needed.

A *monotone graph Lipschitz domain* is a graph Lipschitz domain Ω as described in Section 2.2 for which the function γ is monotone. Note that if Φ is a conformal map associated to Ω as given in Section 2.2, then Ω is a monotone graph Lipschitz domain if and only if $\operatorname{Im}(\Phi') \geq 0$ almost everywhere or $\operatorname{Im}(\Phi') \leq 0$ almost everywhere. We have the following result for conformal maps associated to monotone graph Lipschitz domains:

Corollary 3.5. *Let Φ be a conformal map as described in Section 2.2 such that $\Phi(\mathbb{R}^2_+)$ is a monotone graph Lipschitz domain. If $f \in L^2(|\Phi'|^{-1})$ is real-valued, then*

$$||Hf||_{L^2(|\Phi'|^{-1})} \leq \sqrt{\sqrt{2}\left(1 + 2\sqrt{2} + 2\sqrt{2 + \sqrt{2}}\right)}\, ||f||_{L^2(|\Phi'|^{-1})}. \tag{3.11}$$

Proof. Since $\Phi(\mathbb{R}^2_+)$ is a monotone graph Lipschitz domain, then $\operatorname{Im}\left(\frac{1}{\Phi'}\right) \geq 0$ almost everywhere or $\operatorname{Im}\left(\frac{1}{\Phi'}\right) \leq 0$ almost everywhere.

Assume first that $\operatorname{Im}\left(\frac{1}{\Phi'}\right) \geq 0$ almost everywhere. Adding (1.5) and (1.6), we have

$$\begin{aligned}
\int_{\mathbb{R}} (Hf)^2 |\Phi|^{-1} dx &\leq \int_{\mathbb{R}} (Hf)^2 \left(\operatorname{Re}\left(\frac{1}{\Phi'}\right) + \operatorname{Im}\left(\frac{1}{\Phi'}\right)\right) dx \\
&= \int_{\mathbb{R}} f^2 \left(\operatorname{Re}\left(\frac{1}{\Phi'}\right) + \operatorname{Im}\left(\frac{1}{\Phi'}\right)\right) dx + 2\int_{\mathbb{R}} f\, Hf \left(\operatorname{Re}\left(\frac{1}{\Phi'}\right) - \operatorname{Im}\left(\frac{1}{\Phi'}\right)\right) dx \\
&\leq \sqrt{2}\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx + 2\sqrt{2}\int_{\mathbb{R}} |f\, Hf||\Phi'|^{-1} dx.
\end{aligned}$$

Letting $0 < \varepsilon < 1/\sqrt{2}$ and applying the Cauchy-Schwarz inequality, the second term in the last line is controlled by

$$2\sqrt{2}\int_{\mathbb{R}} |f\, Hf||\Phi'|^{-1} dx \leq \frac{\sqrt{2}}{\varepsilon}\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx + \sqrt{2}\,\varepsilon \int_{\mathbb{R}} (Hf)^2 |\Phi'|^{-1} dx.$$

Setting $\delta = \sqrt{2}\,\varepsilon$, we then obtain

$$\int_{\mathbb{R}} (Hf)^2 |\Phi|^{-1} dx \leq \sqrt{2}\frac{1+\frac{1}{\varepsilon}}{1-\sqrt{2}\varepsilon}\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx = \sqrt{2}\frac{\delta+\sqrt{2}}{\delta(1-\delta)}\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx.$$

Taking infimum over $0 < \delta < 1$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}} (Hf)^2 |\Phi|^{-1} dx &\leq \sqrt{2}\,\varphi(\sqrt{2})\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx \\
&= \sqrt{2}\left(1 + 2\sqrt{2} + 2\sqrt{2+\sqrt{2}}\right)\int_{\mathbb{R}} f^2 |\Phi'|^{-1} dx.
\end{aligned}$$

The case $\operatorname{Im}\left(\frac{1}{\Phi'}\right) \leq 0$ follows analogously by subtracting (1.5) and (1.6). □

4. Hilbert transform identities with power weights

In this section we investigate further the identities (1.5) and (1.6) when $\Phi(\mathbb{R}^2_+)$ is a cone, and obtain Hilbert transform identities with power weights. We will consider two types of cones: Symmetric cones about the imaginary axis and monotone cones (i.e. cones that are monotone graph Lipschitz domains).

4.1. Symmetric cones

Let Ω be a cone with aperture $\alpha\pi$, with $\alpha \in (0, 2)$, which is symmetric about the imaginary axis (see Fig. 1). Consider the conformal map $\Phi : \mathbb{R}^2_+ \to \Omega$ such that

$$\Phi(z) = e^{i\frac{(1-\alpha)}{2}\pi} z^{\alpha} = ie^{-i\frac{\alpha}{2}\pi} e^{\alpha(\log|z| + i\,\mathrm{Arg}(z))}, \tag{4.1}$$

where we chose the branch cut $\{iy : y \le 0\}$, so that Φ is analytic on $\mathbb{R}^2_+$.

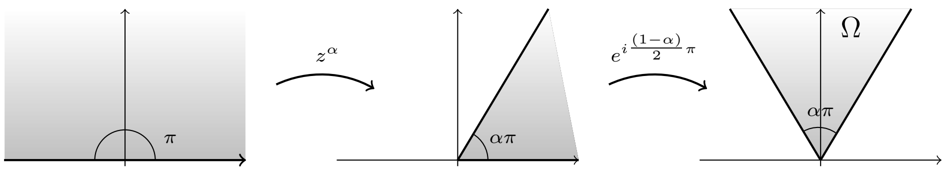

Fig. 1. Symmetric cone with aperture $\alpha\pi$.

We have the following result.

Theorem 4.1. *Fix $\beta \in (-1, 1)$. For any $f \in L^2(|x|^\beta)$ real-valued, it holds that*

$$\int_{\mathbb{R}} (Hf)^2 |x|^\beta\, dx = \int_{\mathbb{R}} f^2 |x|^\beta\, dx + 2\cot\left(\frac{(1-\beta)\pi}{2}\right) \int_{\mathbb{R}} fHf\,\mathrm{sgn}(x)|x|^\beta\, dx.$$

Remark 4.2. Note that $\cot\left(\frac{(1-\beta)\pi}{2}\right)$ blows up when $\beta = -1$ and $\beta = 1$. This is consistent with the well-known fact that $|x|^\beta \in A_2$ if and only if $\beta \in (-1, 1)$.

Proof. Fix $\beta \in (-1, 1)$. Let $\alpha = 1 - \beta \in (0, 2)$, and consider the conformal map Φ given in (4.1). We need to compute $\mathrm{Re}\left(\frac{1}{\Phi'}\right)$ and $\mathrm{Im}\left(\frac{1}{\Phi'}\right)$ on $\mathbb{R}$. If $x > 0$, then $\Phi(x) = ie^{-i\frac{\alpha}{2}\pi} x^\alpha$. Differentiating with respect to x, we get

$$\Phi'(x) = \alpha \sin\left(\frac{\alpha\pi}{2}\right) x^{\alpha-1} + i\alpha\cos\left(\frac{\alpha\pi}{2}\right) x^{\alpha-1}.$$

Similarly, if $x < 0$, then $\Phi(x) = ie^{i\frac{\alpha}{2}\pi}(-x)^\alpha$, and thus,

$$\Phi'(x) = \alpha \sin\left(\frac{\alpha\pi}{2}\right) (-x)^{\alpha-1} - i\alpha\cos\left(\frac{\alpha\pi}{2}\right) (-x)^{\alpha-1}.$$

Note that $|\Phi'(x)|^2 = \alpha^2 |x|^{2(\alpha-1)}$. Therefore,

$$\mathrm{Re}\left(\frac{1}{\Phi'}\right) = \alpha^{-1}\sin\left(\frac{\alpha\pi}{2}\right)|x|^{1-\alpha} \quad \text{and} \quad \mathrm{Im}\left(\frac{1}{\Phi'}\right) = -\alpha^{-1}\cos\left(\frac{\alpha\pi}{2}\right)\mathrm{sgn}(x)|x|^{1-\alpha}.$$

Substituting these expressions into (1.5), and using that $\beta = 1 - \alpha$, we obtain the result. □

Corollary 4.3. *If $\beta \in (-1, 1)$, it holds that*

$$\|H\|^2_{L^2(|x|^\beta)\to L^2(|x|^\beta)} \ge 1 - \frac{2(1+\beta)}{\pi}\cot\left(\frac{(1-\beta)\pi}{2}\right) \int_1^\infty |x|^{-2-\beta}\log|1-x|\, dx.$$

Proof. By Theorem 4.1, with $f_r = \chi_{(0,r)}$ and $H(f_r) = \frac{1}{\pi}\log\frac{|x|}{|x-r|}$, we get that

$$\begin{aligned}\int_{\mathbb{R}} (H(f_r))^2 |x|^\beta\,dx &= \frac{r^{1+\beta}}{1+\beta} + \frac{2}{\pi}\cot\left(\frac{(1-\beta)\pi}{2}\right)\int_0^r |x|^\beta \log\frac{|x|}{|x-r|}\,dx \\ &= \frac{r^{1+\beta}}{1+\beta} - \frac{2}{\pi}\cot\left(\frac{(1-\beta)\pi}{2}\right) r^{1+\beta}\int_1^\infty |y|^{-2-\beta}\log|1-y|\,dy \\ &= \|f_r\|^2_{L^2(|x|^\beta)}\left(1 - \frac{2(1+\beta)}{\pi}\cot\left(\frac{(1-\beta)\pi}{2}\right)\int_1^\infty |y|^{-2-\beta}\log|1-y|\,dy\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|H\|^2_{L^2(|x|^\beta)\to L^2(|x|^\beta)} &\geq \frac{\|H(f_r)\|^2_{L^2(|x|^\beta)}}{\|f_r\|^2_{L^2(|x|^\beta)}} \\ &= 1 - \frac{2(1+\beta)}{\pi}\cot\left(\frac{(1-\beta)\pi}{2}\right)\int_1^\infty |x|^{-2-\beta}\log|1-x|\,dx. \quad \square\end{aligned}$$

Remark 4.4. We observe that the right-hand side behaves as $(1+\beta)^{-2}$ when $\beta \to -1^+$. Furthermore, $|x|^\beta = e^{\beta\log|x|} = e^{\beta\frac{\pi}{2}K(\operatorname{sgn} x)} = e^{f_1 + Kf_2}$, with $f_1 \equiv 0$ and $f_2 = \frac{\beta\pi}{2}\operatorname{sgn} x$. Hence,

$$[|x|^\beta]^2_{A_2(\mathrm{HS})} \leq \sec^2(\tfrac{\beta\pi}{2}) \sim (1+\beta)^{-2} \quad \text{as } \beta \to -1^+.$$

Therefore, the dependence of $[w]_{A_2(\mathrm{HS})}$ in Theorem 1.1 is sharp.

4.2. Monotone cones

Let Ω be a monotone cone of aperture $\alpha\pi$ (see Fig. 2); then we must have $\alpha \in (1/2, 3/2)$. Here the definition of Φ changes according to $\alpha \in (1/2, 1)$ or $\alpha \in [1, 3/2)$. We will only study the first case, since results from the second case can be deduced analogously.

Let $\alpha \in (1/2, 1)$; for each $\theta \in [1-\alpha, 1/2]$, we define $\Phi : \mathbb{R}^2_+ \to \Omega$ such that

$$\Phi(z) = e^{i\theta\pi} z^\alpha = e^{i\theta\pi} e^{\alpha(\log|z| + i\,\mathrm{Arg}(z))}, \tag{4.2}$$

where we chose the branch cut $\{iy : y \leq 0\}$.

We next present several identities that follow from (1.5) and (1.6). Although it is possible that some of these identities may be known, we were unable to find them in the literature.

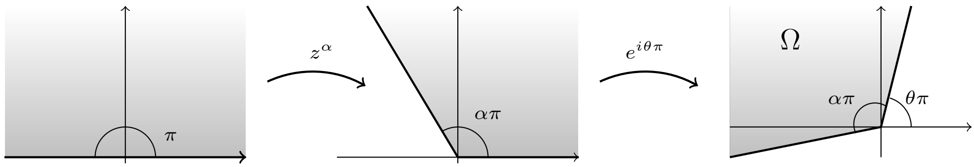

Fig. 2. Monotone cone with aperture $\alpha\pi$.

Theorem 4.5. *Fix $\beta \in (0, 1/2)$ and $\theta \in [\beta, 1/2]$. For any $f \in L^2(|x|^\beta)$ real-valued, it holds that*

$$\int_{\mathbb{R}} (Hf)^2\, a_1(\operatorname{sgn} x)|x|^\beta\, dx = \int_{\mathbb{R}} f^2\, a_1(\operatorname{sgn} x)|x|^\beta\, dx + 2\int_{\mathbb{R}} fHf\, a_2(\operatorname{sgn} x)|x|^\beta\, dx,$$
$$\int_{\mathbb{R}} (Hf)^2\, a_2(\operatorname{sgn} x)|x|^\beta\, dx = \int_{\mathbb{R}} f^2\, a_2(\operatorname{sgn} x)|x|^\beta\, dx - 2\int_{\mathbb{R}} fHf\, a_1(\operatorname{sgn} x)|x|^\beta\, dx,$$

where a_1 and a_2 are the functions given by

$$a_1(s) = \cos(\theta\pi)\frac{1+s}{2} + \cos((\theta-\beta)\pi)\frac{1-s}{2},$$
$$a_2(s) = \sin(\theta\pi)\frac{1+s}{2} + \sin((\theta-\beta)\pi)\frac{1-s}{2},$$

for any s in $\mathbb{R}$.

Remark 4.6. Note that if $\beta \in (0, 1/2)$ and $\theta \in [\beta, 1/2]$, then $0 \le \theta - \beta < 1/2$, and

$$\min\Big\{\cos(\theta\pi), \sin(\theta\pi), \cos((\theta-\beta)\pi), \sin((\theta-\beta)\pi)\Big\} \ge 0.$$

Therefore, $a_1(\operatorname{sgn} x) \ge 0$ and $a_2(\operatorname{sgn} x) \ge 0$ for all $x \in \mathbb{R}$.

Proof. Fix $\beta \in (0, 1/2)$ and $\theta \in [\beta, 1/2]$. Let $\alpha = 1 - \beta \in (1/2, 1)$, and consider the conformal map given in (4.2). We proceed as in the previous proof. If $x > 0$, then $\Phi(x) = e^{i\theta\pi}x^\alpha$. Differentiating with respect to x, we get

$$\Phi'(x) = \alpha\cos(\theta\pi)x^{\alpha-1} + i\alpha\sin(\theta\pi)x^{\alpha-1}.$$

Similarly, if $x < 0$, then $\Phi(x) = e^{i(\alpha+\theta)\pi}(-x)^\alpha$. Therefore,

$$\Phi'(x) = -\alpha\cos((\alpha+\theta)\pi)(-x)^{\alpha-1} - i\alpha\sin((\alpha+\theta)\pi)(-x)^{\alpha-1}.$$

Since $|\Phi'(x)|^2 = \alpha^2|x|^{2(\alpha-1)}$, and $\beta = 1 - \alpha$, it follows that

$$\begin{aligned}\operatorname{Re}\left(\frac{1}{\Phi'}\right) &= \alpha^{-1}\left(\cos(\theta\pi)\tfrac{1+\operatorname{sgn} x}{2} - \cos((\alpha+\theta)\pi)\tfrac{1-\operatorname{sgn} x}{2}\right)|x|^{1-\alpha} \\ &= (1-\beta)^{-1}a_1(\operatorname{sgn} x)|x|^\beta\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\left(\frac{1}{\Phi'}\right) &= -\alpha^{-1}\left(\sin(\theta\pi)\tfrac{1+\operatorname{sgn}x}{2} - \sin((\alpha+\theta)\pi)\tfrac{1-\operatorname{sgn}x}{2}\right)|x|^{1-\alpha} \\ &= -(1-\beta)^{-1}a_2(\operatorname{sgn}x)|x|^{\beta},\end{aligned}$$

where a_1 and a_2 are defined as in the statement. Substituting in (1.5) and (1.6) we conclude the desired results. □

When $\beta = \theta$, Theorem 4.5 gives the following result:

Corollary 4.7. *If $\beta \in (0, 1/2)$ and $f \in L^2(|x|^\beta)$ is real-valued, it holds that*

$$\begin{aligned}&\cos(\beta\pi)\int_0^\infty (Hf)^2\,|x|^\beta\,dx + \int_{-\infty}^0 (Hf)^2|x|^\beta\,dx \\ &\quad = \cos(\beta\pi)\int_0^\infty f^2\,|x|^\beta\,dx + \int_{-\infty}^0 f^2\,|x|^\beta\,dx + 2\sin(\beta\pi)\int_0^\infty fHf\,|x|^\beta\,dx, \quad (4.3)\end{aligned}$$

$$\begin{aligned}\sin(\beta\pi)\int_0^\infty (Hf)^2\,|x|^\beta\,dx &= \sin(\beta\pi)\int_0^\infty f^2\,|x|^\beta\,dx \\ &\quad - 2\cos(\beta\pi)\int_0^\infty fHf\,|x|^\beta\,dx - 2\int_{-\infty}^0 fHf\,|x|^\beta\,dx. \quad (4.4)\end{aligned}$$

Several interesting identities follow as particular cases of Corollary 4.7:

Identity (4.3) and $\operatorname{supp}(f) \subset (0,\infty)$: If $\beta \in (0, 1/2)$ and $f \in L^2(|x|^\beta)$ is real-valued, then

$$\begin{aligned}&\cos(\beta\pi)\int_0^\infty (Hf)^2\,|x|^\beta\,dx + \int_{-\infty}^0 (Hf)^2|x|^\beta\,dx \\ &\quad = \cos(\beta\pi)\int_0^\infty f^2\,|x|^\beta\,dx + 2\sin(\beta\pi)\int_0^\infty fHf\,|x|^\beta\,dx.\end{aligned}$$

As $\beta \to 1/2$ we obtain

$$\int_{-\infty}^0 (Hf)^2|x|^{\frac{1}{2}}\,dx = 2\int_0^\infty fHf\,|x|^{\frac{1}{2}}\,dx.$$

Identity (4.3) and supp$(f) \subset (-\infty, 0)$: If $\beta \in (0, 1/2)$ and $f \in L^2(|x|^\beta)$ is real-valued, then

$$\cos(\beta\pi) \int_0^\infty (Hf)^2 \, |x|^\beta \, dx + \int_{-\infty}^0 (Hf)^2 |x|^\beta \, dx = \int_{-\infty}^0 f^2 \, |x|^\beta \, dx.$$

As $\beta \to 1/2$, we have

$$\int_{-\infty}^0 (Hf)^2 |x|^{\frac{1}{2}} \, dx = \int_{-\infty}^0 f^2 \, |x|^{\frac{1}{2}} \, dx.$$

Identity (4.4) and supp$(f) \subset (0, \infty)$: If $\beta \in (0, 1/2)$ and $f \in L^2(|x|^\beta)$ is real-valued, then

$$\sin(\beta\pi) \int_0^\infty (Hf)^2 \, |x|^\beta \, dx = \sin(\beta\pi) \int_0^\infty f^2 \, |x|^\beta \, dx - 2\cos(\beta\pi) \int_0^\infty fHf \, |x|^\beta \, dx.$$

As $\beta \to 1/2$, we get

$$\int_0^\infty (Hf)^2 \, |x|^{\frac{1}{2}} \, dx = \int_0^\infty f^2 \, |x|^{\frac{1}{2}} \, dx.$$

Identity (4.4) and supp$(f) \subset (-\infty, 0)$: If $\beta \in (0, 1/2)$ and $f \in L^2(|x|^\beta)$ is real-valued, then

$$\sin(\beta\pi) \int_0^\infty (Hf)^2 \, |x|^\beta \, dx = -2 \int_{-\infty}^0 fHf \, |x|^\beta \, dx.$$

As $\beta \to 1/2$, it follows that

$$\int_0^\infty (Hf)^2 \, |x|^{\frac{1}{2}} \, dx = -2 \int_{-\infty}^0 fHf \, |x|^{\frac{1}{2}} \, dx.$$

5. Rellich identities for the Hilbert transform in L^p

Theorem 1.2 states Rellich's identities for the Hilbert transform for functions in weighted L^2-spaces. In this section we present versions of such identities for pairs of functions on weighted Lebesgue spaces. Our main result is the following theorem.

Theorem 5.1. *Let Φ be a conformal map as described in Section 2.2 such that $|\Phi'| \in A_p \cap A_{p'}$ for some $1 < p < \infty$. If $f \in L^p(|\Phi'|^{1-p})$ and $g \in L^{p'}(|\Phi'|^{1-p'})$ are real-valued, then*

$$\int_{\mathbb{R}} \Big(HfHg - fg\Big) \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = -\int_{\mathbb{R}} \Big(fHg + gHf\Big) \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx, \tag{5.1}$$

$$\int_{\mathbb{R}} \Big(HfHg - fg\Big) \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx = \int_{\mathbb{R}} \Big(fHg + gHf\Big) \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx. \tag{5.2}$$

Proof. We first prove (5.1) for f and g continuous with compact support. Using (1.5) and the identity (see [7, (5.1.23), p. 320])

$$(Hf)^2 - f^2 = 2H(fHf), \tag{5.3}$$

we have that

$$\int_{\mathbb{R}} H(fHf) \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = -\int_{\mathbb{R}} fHf \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx.$$

Applying this equality to the functions $f+g$, f and g, it follows that

$$\int_{\mathbb{R}} H\Big(fHg + gHf\Big) \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx = -\int_{\mathbb{R}} \Big(fHg + gHf\Big) \operatorname{Im}\left(\frac{1}{\Phi'}\right) dx.$$

Using (5.3) for the function $f+g$, f and g, we obtain

$$HfHg - fg = H\Big(fHg + gHf\Big).$$

These last two equalities give (5.1).

The identity (5.2) for f and g continuous with compact support is proved similarly using (1.6).

The general case can be obtained by density once it is shown that the integrals in (5.1) and (5.2) are absolutely convergent for $f \in L^p(|\Phi'|^{1-p})$ and $g \in L^{p'}(|\Phi'|^{1-p'})$. For such functions, using Hölder's inequality, we obtain

$$\int_{\mathbb{R}} |f\, g| \operatorname{Re}\left(\frac{1}{\Phi'}\right) dx \le \left(\int_{\mathbb{R}} |f|^p |\Phi'|^{1-p} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g|^{p'} |\Phi'|^{1-p'} dx\right)^{\frac{1}{p'}} < \infty.$$

We also have

$$\int_{\mathbb{R}} |Hf\, Hg|\, \mathrm{Re}\left(\frac{1}{\Phi'}\right) dx \leq \left(\int_{\mathbb{R}} |Hf|^p |\Phi'|^{1-p} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |Hg|^{p'} |\Phi'|^{1-p'} dx\right)^{\frac{1}{p'}}$$
$$\lesssim \left(\int_{\mathbb{R}} |f|^p |\Phi'|^{1-p} dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g|^{p'} |\Phi'|^{1-p'} dx\right)^{\frac{1}{p'}} < \infty,$$

where in the last inequality we have used the boundedness of the Hilbert transform noting that $|\Phi'|^{1-p} \in A_p$ since $|\Phi'| \in A_{p'}$ and $|\Phi'|^{1-p'} \in A_{p'}$ since $|\Phi'| \in A_p$. A similar reasoning is applied for the integrals on the right hand side of (5.1) and (5.2). □

As an application, we next present examples of the identities of Theorem 5.1 associated to power weights. The next corollary follows by considering the conformal map Φ given in (4.1) and the corresponding computations done in Section 4.1.

Corollary 5.2. *Let $1 < p < \infty$ and $\beta \in (-1, 1)$. If $f \in L^p(|x|^{\beta(p-1)})$ and $g \in L^{p'}(|x|^{\beta(p'-1)})$ are real-valued then*

$$\int_{\mathbb{R}} \Big(HfHg - fg\Big)|x|^\beta dx = \cot\left(\frac{(1-\beta)\pi}{2}\right) \int_{\mathbb{R}} \Big(fHg + gHf\Big)\, \mathrm{sgn}(x)|x|^\beta dx, \tag{5.4}$$

$$\int_{\mathbb{R}} \Big(HfHg - fg\Big)\, \mathrm{sgn}(x)|x|^\beta dx = -\tan\left(\frac{(1-\beta)\pi}{2}\right) \int_{\mathbb{R}} \Big(fHg + gHf\Big)|x|^\beta dx. \tag{5.5}$$

Data availability

No data was used for the research described in the article.

Acknowledgments

We thank the referee for their careful reading of the manuscript and their suggestions.

References

[1] S. Agrawal, T. Alazard, Refined Rellich boundary inequalities for the derivatives of a harmonic function, https://doi.org/10.48550/arXiv.2205.04756, 2022.
[2] M.J. Carro, V. Naibo, C. Ortiz-Caraballo, The Neumann problem in graph Lipschitz domains in the plane, Math. Ann. 385 (1–2) (2023) 17–57.
[3] D. Cruz-Uribe, J.M. Martell, C. Pérez, Sharp weighted estimates for classical operators, Adv. Math. 229 (1) (2012) 408–441.
[4] L. Escauriaza, M. Mitrea, Transmission problems and spectral theory for singular integral operators on Lipschitz domains, J. Funct. Anal. 216 (1) (2004) 141–171.
[5] R.A. Fefferman, C.E. Kenig, J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations, Ann. Math. (2) 134 (1) (1991) 65–124.

[6] J. García-Cuerva Abengoza, Theory of weights and functions of bounded mean oscillation, Publ. Secc. Mat., Univ. Autón. Barc. 26 (1) (1982) 111–130.
[7] L. Grafakos, Classical Fourier Analysis, third edition, Graduate Texts in Mathematics, vol. 249, Springer, New York, 2014.
[8] H. Helson, G. Szegö, A problem in prediction theory, Ann. Mat. Pura Appl. (4) 51 (1960) 107–138.
[9] R. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Am. Math. Soc. 176 (1973) 227–251.
[10] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, Ann. Math. (2) 175 (3) (2012) 1473–1506.
[11] T. Hytönen, M. Lacey, H. Martikainen, T. Orponen, M.C. Reguera, E. Sawyer, I. Uriarte-Tuero, Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on A_p weighted spaces, J. Anal. Math. 118 (1) (2012) 177–220.
[12] T. Hytönen, C. Pérez, S. Treil, A. Volberg, Sharp weighted estimates for dyadic shifts and the A_2 conjecture, J. Reine Angew. Math. 687 (2014) 43–86.
[13] C. Kenig, Weighted Hardy spaces on Lipschitz domains, in: Harmonic Analysis in Euclidean Spaces, Part 1, Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978, in: Proc. Sympos. Pure Math., vol. XXXV, Amer. Math. Soc., Providence, R.I., 1979, pp. 263–274.
[14] C. Kenig, Weighted H^p spaces on Lipschitz domains, Am. J. Math. 102 (1) (1980) 129–163.
[15] A. Lerner, A simple proof of the A_2 conjecture, Int. Math. Res. Not. 14 (2013) 3159–3170.
[16] S. Petermichl, The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic, Am. J. Math. 129 (5) (2007) 1355–1375.
[17] S. Petermichl, The sharp weighted bound for the Riesz transforms, Proc. Am. Math. Soc. 136 (4) (2008) 1237–1249.
[18] F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, Math. Z. 46 (1940) 635–636.