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Does a physical pendulum ever act like a simple pendulum?

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Abstract

We show that for a physical pendulum comprising a massive sphere swinging from a massive string, there is, in general, a length of string for which its oscillatory period equals the period calculated by the simple pendulum model with a point-like mass swinging from a massless string whose model length equals the summed length of the real string and the sphere's radius.

Keywords: pendulum, kinematics, mechanics

1. Introduction

Physics laboratory courses at the undergraduate level often include an experiment involving the measurement of the period of a physical pendulum constructed with a length of string connected to a spherical mass. As pointed out by Nelson and Olsson [1], this simple system is rich in physics, as is evident from abundant discussions in the literature (see, e.g. [1–3] and references therein). The effect on the pendulum's period of amplitude, air drag, friction at the support point, buoyancy, string stretch, and the local gravitational constant have all been analyzed carefully. Another topic of interest is the effect of mass distribution on the difference in period between the physical pendulum the students have constructed and an ideal simple pendulum comprising a point mass and a massless string with length l equal to the sum of the actual string's length L and the sphere's radius, R (we ignore the very small effects of a coupling hook or loop used to attach the string to the spherical mass) [1]. Careful use of a smartphone video, sonar apparatus, or even a manually operated stopwatch [4] can yield period measurement accuracies better than 1 ms, which is generally sufficient to distinguish the two periods. This is a useful exercise for students who are just learning about moments of inertia, angular momenta, and the nature of physical models and the approximations they involve [5].

The period for the idealized simple pendulum, T_s, is

$$T_s = 2\pi\sqrt{\frac{l}{g}}, \tag{1}$$

where g is the local gravitational acceleration. This is to be compared with the period of the physical pendulum that is measured, T_p, given by

$$T_p = 2\pi\sqrt{\frac{I}{Mgd}}, \tag{2}$$

where the moment of inertia about the pivot I, mass M, and distance d from the pivot to the center of mass are calculated for the string-plus-sphere system.

Now an interesting issue arises which has not, to our knowledge (following a thorough search), been discussed in the literature. The limiting cases for L are instructive in this regard. As the thread becomes very long, its mass dominates over that of the sphere, so that, in effect, the pendulum acts as a swinging rod, with a period a factor of $\sqrt{\frac{2}{3}}$ that of T_s. On the other hand, a pendulum formed by a sphere fixed at a point on its surface will have a period $\sqrt{\frac{7}{5}}$ larger than T_s when $l = R$. One thus concludes that there must be some thread length for which $T_s = T_p$ and the physical pendulum becomes, in effect, 'simple.'

Previous work by Oliviera [6, 7] implicitly solves for this coincidence for a specific case in a different context. He considers the measurement of g with a pendulum and shows that the systematic error in its determination becomes nil for various combinations of string and sphere masses and dimensions. However, as his work has a different emphasis, he foregoes explicit discussion of the relationship between string length and pendulum period. In this paper we consider the effect of string length on period coincidence for the simple and physical pendulum models, the combination of parameters necessary for the coincidences to occur, and the general solution for the string length when the physical pendulum's period is given precisely by equation (1).

2. Analysis

We wish to compute the string length analytically, considering the somewhat idealized physical pendulum shown in figure 1, with a string length L and linear density λ, and a mass radius R and volumetric density ρ. We solve for L after fixing the other values, since this is often the parameter for which students find it easiest to choose a precise value. The moment of inertia for the system about its point of suspension, using the parallel axis theorem, is

$$\begin{aligned} I &= \frac{1}{3}L^2 m_{\text{string}} + \left(\frac{2}{5}R^2 + (L+R)^2\right)m_{\text{sphere}} \\ &= \frac{1}{3}\lambda L^3 + \left(\frac{2}{5}R^2 + (L+R)^2\right)\frac{4}{3}\pi R^3 \rho, \end{aligned} \tag{3}$$

where m_{string} and m_{sphere} are the component masses. We can most easily solve for the coincidence of the simple and physical pendulum periods by exploiting the fact that

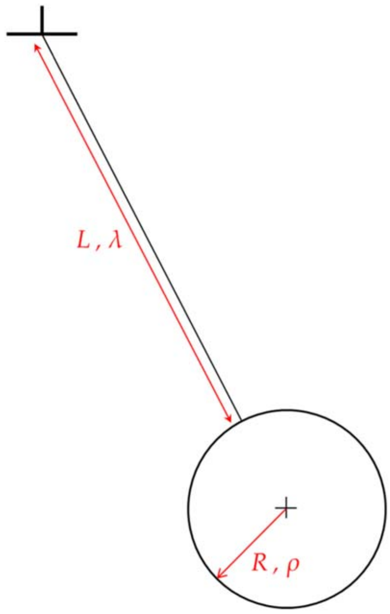

Figure 1. Relevant dimensions and densities of the physical pendulum; $l = L + R$.

$$T_s = T_p \Rightarrow T_s^2 = T_p^2, \tag{4}$$

so

$$\frac{4\pi^2}{g}(L + R) = \frac{4\pi^2}{g}\left(\frac{\frac{1}{3}\lambda L^3 + \left(\frac{2}{5}R^2 + (L + R)^2\right)\frac{4}{3}\pi R^3 \rho}{\frac{1}{2}\lambda L^2 + \frac{4}{3}\pi R^3 \rho (L + R)}\right), \tag{5}$$

and

$$L^3 + 3RL^2 - \frac{16\pi}{5}\frac{R^5 \rho}{\lambda} = 0. \tag{6}$$

This is a cubic equation for L in standard form. Since the zeroth-order term is negative and real, a factorization of equation (6) must give rise to at least one real positive root. More generally, it will produce three different real roots if the solution parameter

$$\gamma = \frac{4\pi}{5}\frac{R^2 \rho}{\lambda} \tag{7}$$

is less than unity, three real roots of which two will be equal if $\gamma = 1$, and one real root if $\gamma > 1$.

What might a reasonable range of γ values be? The difficulty of equation (7) is that it compares a volumetric and linear density. If we assume an approximately cylindrical thread, $\lambda = \pi \rho_{\text{string}} r_{\text{string}}^2$, where ρ_{string} and r_{string} are the string's volume density and radius, we can express the condition for a single real solution in terms of the ratio of radii:

Figure 2. The difference between the physical and simple model periods for a pendulum having a spherical mass of stainless steel with radius 1 cm, supported by a cotton thread of radius 0.5 mm. The vertical dashed line indicates the 'magic' length of 0.18 m where $T_s = T_p$.

$$\frac{4}{5}\left(\frac{R}{r_{\text{string}}}\right)^2 > \frac{\rho_{\text{string}}}{\rho}. \tag{8}$$

We may reasonably assume that our pendulum is constructed using a string of cotton thread, fishing line, or, possibly, a plastic like Kevlar. (The latter choice is good because of its tensile strength and relative inextensibility.) All of these materials have densities $< 1.8\ \text{g cm}^{-3}$ [8]. If we choose a practicable metal for the spherical mass and exclude magnesium, the density ratio in the above inequality will be less than unity. The squared ratio of sphere radius to string radius will generally be > 10. Unless we work with, e.g., a pendulum made of a small wood sphere suspended by a thick braided copper cord, we can be quite certain that there will be a single L for which $T_s = T_p$.

The general solution for the positive root of equation (6), using Cardano's Method [9] with $\gamma > 1$, can be written as

$$L = -\frac{R}{\alpha^{1/3}}(\alpha^{2/3} + \alpha^{1/3} + 1), \tag{9}$$

$$\text{where}\alpha \equiv 1 - \frac{8\pi}{5}\frac{R^2\rho}{\lambda} - 4\sqrt{\frac{\pi}{5}\frac{R^2\rho}{\lambda}}\sqrt{\frac{4\pi}{5}\frac{R^2\rho}{\lambda} - 1}\,. \tag{10}$$

The exponentiated values of α in equation (9) correspond to its unique real cube root or the square thereof.

Now consider a normal physical pendulum, in which we use cotton sewing thread ($\rho_{\text{string}} = 1.54\text{gcm}^{-3}$) of radius 0.5 mm, supporting a stainless-steel sphere ($\rho = 7.96\text{gcm}^{-3}$) with a 1 cm radius. This results in $\gamma = 1654 > 1$. Suppose also that a student assumes (falsely) that the approximation is approached asymptotically, and so selects a long thread—say, two meters. The measured period of this pendulum, T_p, would be 2.829 s. Compared with $T_s = 2.846$ s, there is a 17 ms difference, which is easy to measure with ~5% accuracy, assuming oscillatory amplitudes of less than 5° so that deviations of the period from that of a simple harmonic oscillator due to second-order amplitude corrections do not exceed 1 ms

[3, 4]. The difference (in ms) of the physical and simple periods for varying L is plotted in figure 2. Solving equation (6), we find that the 'magic' length for which $T_s = T_p$ is 0.18 m. At very small values of L, the period difference for the two cases both converge to the required 37 ms for a 1 cm radius uniform solid sphere (this is not explicitly shown in the graph, where the upper ordinate is truncated at 20 ms). Following the discussion in the introduction, the curve must converge in the large-L limit to the asymptotic functional form

$$T_s - T_p = \left(1 - \sqrt{\frac{2}{3}}\right)T_s = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right)\sqrt{\frac{L}{g}}. \quad (11)$$

3. Discussion

In addition to asking students to simply compare their measured values of T_p with T_s , asking them to determine whether or not there are any lengths of string they could use to render $T_p = T_s$ will provide them with a rigorous exercise in both analytical analysis and the application of concepts they have learned in their study of moments of inertia and rotational motion. One could prompt students by asking them to calculate the difference between the periods for the physical pendulum and the simple pendulum in the $L = 0$ and $L \to \infty$ limits. We recommend that they also be asked to confirm experimentally the accuracy of their prediction of a 'magic' string length.

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Data availability statement

The data cannot be made publicly available upon publication because they are owned by a third party and the terms of use prevent public distribution. The data that support the findings of this study are available upon reasonable request from the authors.

Appendix : Solvability of the cubic equation (6)

We begin with equation (6):

$$0 = L^3 + 3RL^2 - \frac{16\pi}{5}\frac{R^5\rho}{\lambda}. \quad (6a)$$

We wish to solve equation (6) for the variable L, representing the length of the pendulum thread. This means that we are solving a cubic equation where

$$0 = x^3 + ax^2 + bx + c; a = 3R,\ b = 0,\ c = -\frac{16\pi}{5}\frac{R^5\rho}{\lambda}. \quad (A.1)$$

Following the notation of Press et al [9], we write

$$Q \equiv \frac{a^2 - 3b}{9} = R^2,\ Z \equiv \frac{2a^3 - 9ab + 27c}{54} = R^3 - \frac{8\pi}{5}\frac{R^5\rho}{\lambda}. \tag{A.2}$$

There are three real roots when

$$Z^2 < Q^3,\ \frac{4\pi}{5}\frac{R^2\rho}{\lambda} < 1. \tag{A.3}$$

To solve equation (6) analytically when the converse of relationship (A.3) holds, giving only one real root except in the case of precise equality, we define

$$A \equiv -(Z \pm \sqrt{Z^2 - Q^3})^{1/3}. \tag{A.4}$$

The ambiguous sign in equation (A.4) is selected to satisfy

$$\pm \mathrm{Re}\,[Z^* \sqrt{Z^2 - Q^3}] \geqslant 0, \tag{A.5}$$

but since Z and Q are real, this procedure chooses whatever sign Z has. From (A.2) one can show that a non-negative Z is given by the condition,

$$\frac{8\pi}{5}\frac{R^2\rho}{\lambda} > 1, \tag{A.6}$$

which is contradicted by (A.3), meaning that, for the case with one real root, Z is always negative and (A.4) always uses the minus sign. One must also introduce

$$B \equiv \begin{cases} \frac{Q}{A} & \text{if} \quad A \neq 0 \\ 0 & \text{if} \quad A = 0 \end{cases}, \tag{A.7}$$

but the zero case is irrelevant for our analysis of the physical pendulum model. One can check by Equations (A.2, A.4) that this case only occurs for a point-mass with zero radius. It is then sufficient to take the values of A and B to follow the steps given in [9] to retrieve equations (9) and (10).

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