HOW DO MATHEMATICS TEACHERS LEARN TO CREATE A MATHEMATICAL STORYLINE IN PROBLEM-BASED LESSONS?

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*Building on student work (SW) in mathematics classroom discussion requires complex decision-making from mathematics teachers. Previous literature on problem-based lessons recommends selecting and sequencing pieces of SW in a way that creates a mathematical storyline, but there is rarely any empirical evidence on how mathematics teachers can master such practices. We use the case of Story*Circles*, a lesson-based professional development program, to show how iterative processes in which teachers were engaging with SW assisted them in developing heuristics for a careful selection and sequencing of SW. The results show that these processes involved 1) the teachers' emerging awareness of features of SW; and 2) an evolving capacity to relate these features to the lesson goal. We discuss design features that fostered these changes.*

BACKGROUND AND RATIONALE

Building on student thinking in mathematics lessons is a core aspect of responsive teaching. Instructional practices purported to support teachers' attempts to foreground student ideas include posing mathematically rich problems, monitoring students as they work on such problems, and then selecting several students to share work in a sequence that can be leveraged to support a productive whole-class discussion. The latter two practices are known as *selecting* and *sequencing* (Smith & Stein, 2011). The literature recommends that teachers select and sequence in ways that create "mathematically coherent storylines" (ibid, p. 44). Like any other good story, a crucial component of a mathematical storyline is its culmination, which, in the case of mathematics lessons, means that the discussion of a problem leads to the mathematical goal of the lesson (Kazemi & Hintz, 2014). Despite the importance of the lesson goal, empirical studies show that teachers tend to overlook it when justifying how they select and sequence (Ayalon & Rubel, 2022; Dunning, 2022). This suggests that attending to the lesson goal when selecting SW is part of teachers' tacit knowledge (Herbst & Chazan, 2011). For these reasons, we suspect that teachers' learning of such practices may be enhanced through participation in "infrastructures that support the interplay of knowledge and knowing" (Cook & Brown, 1999, p. 381). Below, we describe how a collaborative practice-based professional development (PD), Story*Circles* (details below), supported teachers in sharing and expanding their ways of knowing related to a careful selection of SW. Compared to the abundance of literature about classroom discussions, the practices of selecting and sequencing are under-researched, despite their importance. Our contribution to this burgeoning body of research is in unpacking the notion of "mathematical storylines" by exploring how geometry teachers became explicit and deliberate about their decisions related to SW. We ask:

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*What justifications for selecting and sequencing SW do teachers make available when participating in Story*Circles*? How do teachers' justifications evolve over time?*

THEORETICAL FRAMING

To explore how teachers make decisions about SW we draw on some of the key ideas of the *practical rationality* framework (Herbst & Chazan, 2011). First, we build on the idea that teachers' actions are always related to the norms of the instructional situation in which they operate. Instructional situations are events in classrooms that teachers recognize as familiar and mentally categorize as a situation of a particular type (Herbst & Chazan, 2011). In the US high-school context, common instructional situations in geometry include *construction* and *proof* (Herbst et al., 2018). It follows that the norms which are related to particular instructional situations are subject-specific, that is, allude to the mathematical content that is being taught. Second, we apply this idea to describe categories of teacher perception (Herbst et al., 2021), namely, aspects to which mathematics teachers attend when they examine and make decisions upon SW: the category of normativity alludes to the SW alignment with the teacher expectations, based on how the teacher framed the instructional situation. For example, if a problem is framed as one expecting students to do a construction, teachers may consider a sketched figure as less normative than a SW where construction tools were used. A second category is the serviceability of the SW, which attends to the alignment with the lesson goal. That is, a sketch that suggested a connection to the lesson goal might be considered more serviceable (even though it is less normative) than a construction that provides no leverage for progress. This emerging framework enables us to discern subject-specific aspects of teachers' decisions.

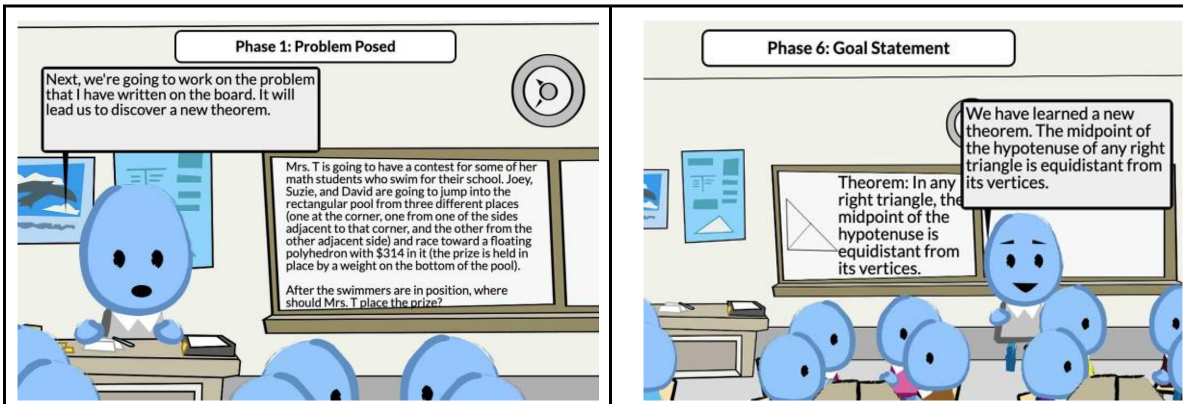
METHODS

The design of the PD environment

Story*Circles* is a collaborative PD where secondary mathematics teachers anticipate a lesson through iterative phases of scripting, visualizing, and arguing about it (Herbst & Milewski, 2018), in online synchronous and asynchronous activities. Inspired by Japanese lesson study, each Story*Circle* focuses on participants' attempts to improve one lesson, initially sketched in a storyboard, as they see fit. Importantly, the goal of Story*Circles* is to foster teachers' peer argumentation about practice (and not, for example, to direct them to teach a specific lesson or to include specific moves).

At the beginning of a Story*Circle*, participants view one version of the storyboarded lesson. Over six weeks they are engaged in various activities where they discuss key decision points in the lesson and script more scenes accordingly. The constants of the lesson-to-be-revised are the posed problem (the first frame of the storyboarded lesson, see Figure 1a) and the culminating institutionalization of the instructional goal (last frame, Figure 1b). In the particular Story*Circle* considered here, the pool problem lesson (in geometry) was represented with nine frames, that showed the arc of the lesson as a sequence of phases: Problem Posed (Phase 1); Getting Your Feet Wet (Phase 2); Whole Class Check-In (Phase 3); Redirecting the Work (Phase 4); Whole

Class Discussion (Phase 5); Goal Statement (Phase 6). The lesson starts with a problem that US teachers tend to frame as a construction (Figure 1a) and culminates with arriving at the theorem stating that "the midpoint of the hypotenuse of any right triangle is equidistant from its vertices" (See Figure 1b). A main resource for the participants' work on the lesson is a collective examination of samples of SW, as will be further detailed. The facilitator was an experienced geometry teacher who previously participated in Story*Circles*. A diverse group (in terms of gender, ethnicity, experience, institutions, and more) of seven teachers participated in the entire PD cycle.



1a 1b

Figure 1. Scenes at the beginning and the end of the pool problem lesson (© 2021, The Regents of the University of Michigan, used with permission)

Data sources and analysis

Focusing on participants' decision-making when interacting with SW, the data sources used for this analysis are videos and transcripts from two synchronous meetings in which SW was the focus: the second meeting (M2) and the fifth (M5). To identify how participants' justifications were made available to peers through discussions, we performed a content analysis guided by the categories of normativity and serviceability (Herbst et al., 2021). The code "normativity" was used when participants referred to the norms of the instructional situation of construction (e.g., when mentioning tools or precision) to justify a decision to select or sequence SW in particular ways. The code "serviceability" was used when they referred to the lesson goal (mediated by mentioning mathematical ideas or objects) to justify a decision to select or sequence SW. The process included: (a) segmenting the transcript into sets of utterances associated with each SW sample and then into idea units that include justifications; (b) coding the segments using the top-down categories together with bottom-up themes; (c) comparing the analyses of M2 and M5, focusing on the place of the instructional goal – namely, attention to serviceability – in participants' justifications; (d) identifying key moments in the talk that illustrate the evolution of justifications.

FINDINGS

The analysis identified that participants' focus on serviceability shifted across their participation in Story*Circles*, with them hardly attending to serviceability in M2 (30% of codes) while serviceability became a core aspect of their arguments in M5 (74% of codes). The following processes were identified: (1) The group's emerging awareness of serviceable aspects of SW; and (2) an evolving capacity to relate these aspects to the lesson goal, while disregarding other aspects. Below, we illustrate these processes. "F" is short for facilitator, and all of the participants' names are pseudonyms.

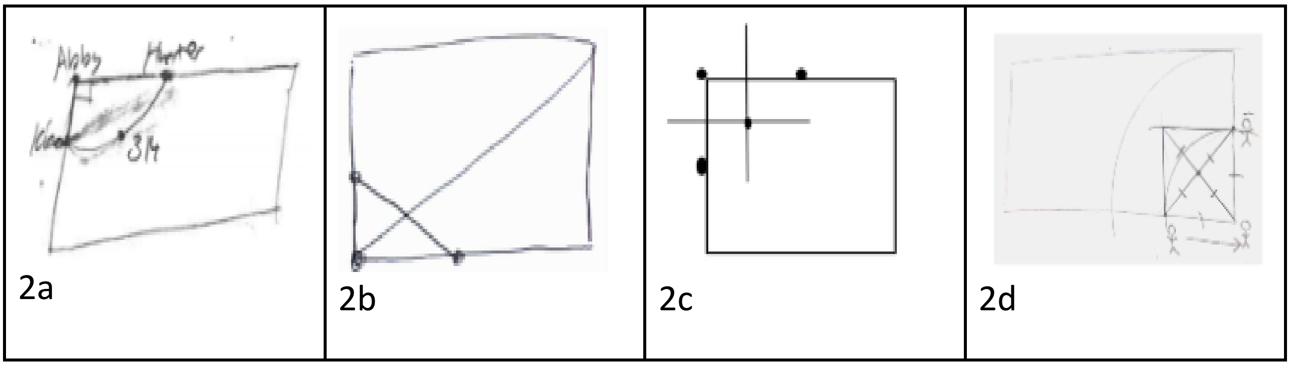


Figure 2. Samples of student work that were discussed in M2 (© 2021, The Regents of the University of Michigan, used with permission)

WEEK 2 MEETING (M2): OVERLOOKING THE LESSON GOAL

Prior to M2, the teachers participated in an introduction session and two asynchronous activities in which they perused and annotated an initial version of the pool problem lesson and nine pieces of SW, respectively (see examples for the SW in Figure 2). M2, which was aimed at focusing participants' attention on the practices of selecting and sequencing SW, began with the question: "What pieces of work catch your attention? [...] Are there ideas that you particularly want to make sure that you brought out?". The participants mainly attended to the first question, discussing SW that were unexpected (Figure 2a) or non-normative (Figure 2b), and then coalesced toward selecting Figure 2b as the first work they want to be shared on the board:

246 Ira: There'd be more than one person who just draws the rectangle, puts the swimmers there, [I'd] say okay where are your tools? What did you do?

273 Ran: I'd rather start with this [Figure 2b], because I think everyone would understand [...] I think it's better to start out with a more general one.

We hear the participants as claiming that selecting 2b would be justifiable on account of their perceiving that piece of work as accessibile, general, representative, and non-normative – the latter aspect suggests that the SW was partly selected as a non-example that is used to direct students to follow the norms of the situation (using construction tools). Notably, none of the justifications mentioned the lesson goal. The facilitator then asked which SW could be useful, a move that led to the first emergence of serviceability in a justification; one participant suggested Figure 2c, "Obviously, because that's a perpendicular bisector, that gets right to where you're trying to go." The word "obviously" suggests that justification for selecting SW that aligns with the lesson goal seems to her as taken-as-shared knowledge that might not be worth mentioning. Following the failed attempts to discuss the lesson goal based on features of the SW, the facilitator tried again:

341 F: What would be the mathematical ideas that we want to make sure got out on the table? Can you imagine bringing up SW to pull these ideas out?

Here, the facilitator illustrates a heuristic for a mindful selection of SW: First thinking about the mathematical ideas that are needed for the discussion, and then selecting SW that includes these ideas or that can be used to prompt discussions about them. Nevertheless, participants mostly suggested ideas that they noticed in the SW such as Mentions "equidistant"; Properties of a rectangle; Right triangle; and Circle, overlooking the hypotenuse, midpoint, and perpendicular bisectors. As the discussion evolved, participants gradually attended to serviceable aspects (such as the generality of the theorem). Yet, the facilitator noticed that they were still not prioritizing any SW, wishing to be attentive to each and every mathematical idea and SW.

454 F: Are these all equally important ideas or do we want to make sure we're prioritizing some of these over others? The goal is to be able to have the students have a discussion that ends up with them discovering this theorem.

458 Clader: It kind of already is in order [...] talking about the equidistant part first, recognizing the rectangle […]. The only thing that obviously would be a little thrown off is if you were going to do the circles piece, I think that might not naturally lead into it, but maybe too [...]

This exchange shows the emergence of the understanding that presenting all pieces of SW, and discussing all related ideas, could impede the coherence of the lesson ("a little thrown off"). A few minutes later, the group made further progress in this direction:

496 Clader: What we need to decide is, like, which direction we're going, because there are so many different entry points to this […] Now we have all these ideas, and that's kind of where I'm stuck because […] once we decided a way [...] I think it'd be easier to determine what drawing next I'd want to lift.

501 Ira: Yeah that's where I'm stuck too [...] what if someone use the tools that found the midpoint and found the right answer and have no clue how they got it, but they're putting up the correct drawing?

This exchange shows the processes in which the participants realize the need to prioritize, an insight that emerged together with the understanding that there are multiple storylines that could be developed and focusing on one of them requires making decisions. The word "stuck" suggests that the participants' conflict is not only about what decision to take, but also about identifying that this moment involves complex decision-making that takes account not only of the features of a particular piece of SW, but also of how those features serve the goal of the lesson. That leads to an evolving awareness of more features of SW (Turn 501), where the participant claims that not all of the SW which is correct and/or normative is also useful for the whole class discussion. Overall, we identified two constraints for the creation of a mathematical storyline: 1) adherence to normativity; 2) a desire for full attentiveness, including making sure that all givens are used, instead of focusing on the question.

WEEK 5 MEETING (M5): ATTENDING TO SERVICEABILITY

M5 was conducted after the group further engaged with the samples of SW in several asynchronous activities which included, for example, scripting classroom dialogues. The meeting described below focused on the last two phases of the lesson, where the second whole class discussion (Phase 5) leads to the discovery of the theorem (Phase 6). The meeting began with the facilitator asking "What would you like to have on the table going into that last class discussion?". This time, the conversation was ample with arguments mentioning the lesson goal:

267 Ran: Most of what we had [in previous phases] had to do with rectangles, and nothing to do with hypotenuse [...] , and even the radii and the arches, those are ideas to measure, maybe, but not necessarily **conducive to the conclusion that we're looking for**.

The facilitator then asked what kind of work the participants imagine can help generate this conversation. That led to a discussion about the normativity of the work. One participant, instead of imagining, attended to his reality, saying: "I think students would draw it [the pool] rather than construct it, unless you told them to construct it" [272]. The facilitator then asked if the group thought it was important that the students construct. While in M2 participants had expressed their expectation for constructions without providing an explanation why, in this meeting most of their comments implied that construction is important for the sake of achieving the lesson goal. In other words, they saw construction as a valid alternative to proof for arriving at the theorem. The following is an illustrative exchange:

303 Clader: Towards the very end I would like to see something more formal.

309 Ran: Without doing a construction, there has to be a [different] way, because otherwise we're just theorizing. How do we know that it's equidistant? [...] At least my students, [...] if I say it's equidistant they will agree, but that does not prove. It's conducive to them to conclude that on their own.

315 Llara: I agree, I think […] that would be where you can now show them why a construction is used instead of just answering question 23 on the test. So they can actually see that the construction now will verify.

324 Ran: And I'm thinking that what we say at the class discussion, I think there has to be something in their work that is what we're going to discuss about.

338 Labrona: I would kind of hope [that] some of the work included a perpendicular bisector so that might be something we can carry into the story, **because for me this is the justification for the eventual goal about the midpoint of the hypotenuse**.[...] I think there were a couple pieces of work that either alluded to a perpendicular bisector or more explicitly had one.

The group then selected Figure 2c and discussed how it could be further used in a discussion. Then, they returned to the issue of which student solutions are considered valid for them. In this discussion, they were not adhering to normativity anymore:

494 F: I wonder if you've got feelings about the idea of a formal or informal proof, or what would make for just a valid conversation that would satisfy?

496 Ran: I like the idea of allowing them to explore different avenues, so I'm not set on formal or informal. […] if one wants to write a proof and the other one wants to use a compass, then that's okay, **as long as they do learn the concept that I want them to learn**.

501 Llara: At least for me, as long as the students can explain it, that's the most important thing, I mean it doesn't have to be formal or informal, [...] what's important [is] that it's correct in the way they explained it for us.

532 Clader: we're trying to emphasize specific vocabulary to get them to the end [...] **the pieces of vocabulary just have to be threaded throughout each example to get them to our final conclusion**. I don't think it's a problem of how they got there as long as we can emphasize those specific pieces.

This exchange illustrates awareness of more features of SW, as well as an increased ability to set aside the procedures to solve the problem and emphasize, instead, the achievement of the instructional goal. In the participants' justifications, there is an emerging recognition of the importance of emphasizing the serviceable aspects of SW.

Overall, in this session teachers could better justify why they selected or disregarded SW, by recognizing features that allude to the lesson goal. Consequently, their choices created a blueprint for a coherent mathematical storyline.

DISCUSSION

This paper examines how teachers' justifications for selecting and sequencing were made available and evolved during their participation in Story*Circles*. We showed how teachers' decision-making became more explicit and deliberate, that is, related to the lesson goal (as Kazemi & Hintz, 2014; Smith & Stein, 2011 recommend). We highlight two notable findings: 1) by noticing more features in SW teachers were able to better justify their choices; and 2) teachers adopted the idea that creating a mathematical story involves prioritizing work that attends to the lesson goal. Although such prioritization could be in tension with attentiveness to all students (Ayalon & Rubel, 2022), teachers reconciled this tension by letting the lesson goal guide their decisions.

The following elements contributed to the emerging focus on serviceability: 1) **The phase structure of the storyboarded lesson**. Based on previous iterations of Story*Circles*, the design of the current cycle provided teachers with the arc of the lesson – that is, a structure that conveyed the sense of what is happening in each phase and how particular moments are related to the larger context of the lesson. This feature enables zooming in and out and relating decisions into larger goals; 2) **Iterative engagement with pieces of SW**. The teachers had diverse opportunities to engage with the SW, with peers and alone, in iterative activities in which the lesson goal has gradually become more salient in their interpretations. We maintain that developing heuristics for purposeful decision-making in an environment of reduced complexity is essential for a later adoption of such heuristics; 3) **Responsive facilitation**. The facilitator was attentive and hardly provided input, yet she was deliberate on building a coherent storyline in her navigation of teachers' talk. Moreover, she was modeling how to lead a discussion that builds on learners' ideas and is goal-oriented.

The contribution of this work is in unpacking how teachers can be supported in creating a mathematical storyline (Smith & Stein, 2011), by using a subject-specific lens that is sensitive to the mathematics at the core of teachers' decision-making.

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