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Stability analysis for electromagnetic waveguides. Part 2: non-homogeneous waveguides

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Abstract
This paper is a continuation of Melenk et al., "Stability analysis for electromagnetic waveguides. Part 1: acoustic and homogeneous electromagnetic waveguides" (2023) [5], extending the stability results for homogeneous electromagnetic (EM) waveguides to the non-homogeneous case. The analysis is done using perturbation techniques for self-adjoint operators eigenproblems. We show that the non-homogeneous EM waveguide problem is well-posed with the stability constant scaling linearly with waveguide length L. The results provide a basis for proving convergence of a Discontinuous Petrov-Galerkin (DPG) discretization based on a full envelope ansatz, and the ultraweak variational formulation for the resulting modified system of Maxwell equations, see Part 1.

Keywords Electromagnetic waveguides · Well-posedness analysis · Perturbation of self-adjoint eigenvalue problems

Mathematics Subject Classification (2010) 78A50 · 35Q61

1 Introduction

This is the second part of our work devoted to the stability and well-posedness analysis of electromagnetic (EM) waveguides; see [5] for an introduction and the motivation for our work. In Part 1 of this work, we considered the homogeneous waveguide only, and we showed that the operator corresponding to the first-order system of Maxwell

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equations is bounded below with a constant scaled inversely with the length L of the waveguide (L is proportional to the number of wavelengths),

$$\left(\|E\|^2 + \|H\|^2\right)^{\frac{1}{2}} \le CL\left(\|\nabla \times E - i\omega H\|^2 + \|\nabla \times H + i\omega E\|^2\right)^{\frac{1}{2}},$$

where $i = \sqrt{-1}$, ω denotes the angular frequency of the light, and C is a positive constant. We use the formalism of closed operators; the electric/magnetic field (E, H) pair comes from the domain of the operator. A simple perturbation argument, given at the end of [5], shows that for a sufficiently small perturbation[1] of the dielectric constant (or relative permittivity) $\epsilon = 1 + \delta\epsilon$, the operator remains bounded below but the linear dependence of the stability constant upon L is lost. In fact, the smallness of perturbation $\delta\epsilon$ is expressed in terms of constant CL; hence, the larger the length L, the smaller $\delta\epsilon$ must be.

Step-index fibers In this paper, we extend our stability result to non-homogeneous EM waveguides. This case has importance in modeling a large number of EM waveguide applications, such as optical amplifiers which are used to achieve high-power laser outputs very efficiently [6, 8]. A typical optical fiber model is the double-clad step-index fiber—a cylindrical EM waveguide where the cross-section (or transversal domain) consists of a silica-glass fiber core surrounded by a silica-glass inner cladding and an outer polymer cladding (see Fig. 1a). The material refractive index n is slightly higher in the core than in the inner cladding which enables the propagation of core-guided transverse modes. Consequently, the permittivity $\epsilon = \epsilon(x, y, z)$, which depends on the material refractive index $n = n(x, y, z)$, is discontinuous at the core-cladding interface $\partial\Omega_{\text{core}} := \{(x, y, z) : x^2 + y^2 = r_{\text{core}}^2\}$ of a step-index fiber, as illustrated in Fig. 1b. Analogously, the material contrast at the inner-outer cladding interface $\partial\Omega_{\text{clad}} := \{(x, y, z) : x^2 + y^2 = r_{\text{clad}}^2\}$ enables propagation of cladding-guided modes by total internal reflection at the glass-polymer interface.

We note that it is a common assumption in engineering literature to consider ϵ differentiable. Indeed, it is often the case that simplified models of EM waveguide applications (e.g., some beam propagation models) entirely neglect the fact that ϵ is not differentiable. More recently, partly thanks to the increased computing capabilities, it has become possible to numerically solve EM waveguide models of realistic length based directly on the Maxwell equations [2–4] thereby avoiding such simplifying assumptions. We emphasize that the analysis in this paper considers discontinuous material parameters and is therefore directly applicable to step-index fibers.

Contributions Extending the stability analysis to the non-homogeneous waveguide problem turns out to be rather non-trivial. We begin by rewriting the Maxwell system in terms of four unknowns: transversal – E_t, H_t, and longitudinal – E_3, H_3 components of electric and magnetic fields. Assuming the exponential ansatz $e^{i\beta z}$ in the (longitudinal) z-direction, we obtain a non-standard eigenvalue problem for propagation constant β. Upon eliminating E_3, H_3, we obtain a more standard system of

[1] We use a non-dimensional version of the equations.

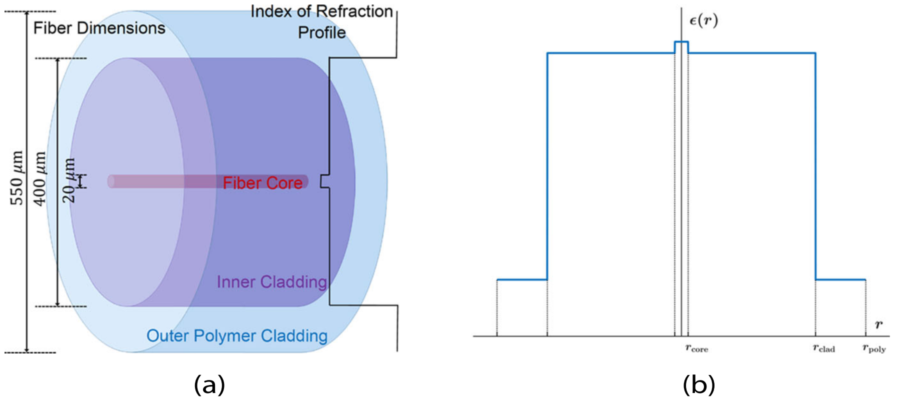

Fig. 1 **a** Schematic of a small section of a double-clad step-index fiber, taken from [1]; **b** transversal profile of the relative permittivity $\epsilon = \epsilon(r)$ in a double-clad step-index fiber

second-order equations (in x, y) with a non-self-adjoint operator, even for the homogeneous case. Only in the last step, after elimination of H_t (or E_t), we obtain a more standard E-eigenvalue problem for E_t, and the corresponding H-eigenvalue problem for H_t. The operators in the E- and H-eigenvalue problems for the homogeneous case turn out to be self-adjoint. This leads to the determination of an orthonormal eigenbasis and corresponding spectral decomposition which, upon the substitution into the original first-order system, decouples the original system into systems of first-order ordinary differential equations (ODEs). Stability analysis for the ODEs and the spectral decomposition argument led to the final result in [5].

In the non-homogeneous case, the operators in the E- and H-eigenvalue problems are not self-adjoint but they represent perturbations of self-adjoint operators. This invites the application of the classical[2] perturbation analysis for self-adjoint operators [7] that we pursue in this paper. The arguments are far from trivial, as we lose the convenient orthonormal basis argument and have to resort to a series of non-orthonormal (perturbed) eigenvectors. The decoupling argument then involves adjoint operators which need to be analyzed as well. As always with the perturbation argument, *the analysis is formal*; we proceed under the assumption that the non-orthogonal series converge as needed. In the computation of the mass matrix for the perturbed eigenvectors, we neglect the quadratic (in $\delta\epsilon$) terms and consider the *linearized mass matrix* only.

Outline The structure of the paper is as follows. We begin in Section 2 with the derivation of the various eigenvalue problems and relations between them. In Section 3, we develop the classical perturbation argument to compute the perturbed E and H eigenvectors and their counterparts for the adjoint problems. In Section 4, we arrive at our first main result; we reduce the problem to a system of decoupled systems of small subsystems of two ODEs for the coefficients in the spectral representations of E_t and H_t. Upon further reduction to a single second-order ODE, we arrive at

[2] Dating back to Lord Rayleigh.

essentially the same ODE problem as in the analysis of the homogeneous waveguide. This leads to the final estimates of E_t, H_t (and their curls) in terms of the right-hand side and to our final result presented in Section 5. We finish with short conclusions in Section 6. Finally, Appendix 1 provides additional algebraic results from the perturbation analysis. In the main body of the paper, we proceed under the customary, simplifying assumption that all perturbed eigenvalues are distinct. In Appendix 1 though, we provide additional details for the case of multiple eigenvalues as it is in the case of the step-index fiber.

In the end, our main stability result is identical with the one for the homogeneous waveguide: we show the scaling of the stability constant with length L. The (formal) perturbation analysis necessitates the assumption of a small perturbation but only in the L^∞-norm. Nowhere in our analysis do we require the dielectric constant to be differentiable, a common assumption in the engineering literature. The presented analysis thus applies to step-index fibers.

2 Eigensystems

Let E_t, E_3 denote the transversal (a 2D vector), and longitudinal (a scalar) components of a 3D vector field E. The 3D curl operator applied to vectors $(E_t, 0)$ and $(0, E_3)$ generates a 2D scalar-valued operator curl, and a 2D vector-valued operator $\nabla\times$,

$$\operatorname{curl} E_t = \operatorname{curl}(E_1, E_2) := \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}, \quad \nabla \times E_3 := (\frac{\partial E_3}{\partial y}, -\frac{\partial E_3}{\partial x}) .$$

We will be using the following 2D identities derived easily from their corresponding 3D counterparts:

$$\begin{aligned} e_z \times (e_z \times E_t) &= -E_t, & & \\ e_z \times (\nabla \times E_3) &= \nabla E_3 , & e_z \times \nabla E_3 &= -\nabla \times E_3, \\ \operatorname{curl}(e_z \times E_t) &= \operatorname{div} E_t , & \operatorname{div}(e_z \times E_t) &= -\operatorname{curl} E_t . \end{aligned} \tag{2.1}$$

For instance,

$$e_z \times E_t = e_z \times (E_x, E_y) := (-E_y, E_x) \quad (= (0, 0, 1) \times (E_x, E_y, 0) = (-E_y, E_x, 0)) .$$

The 3D Maxwell equations,

$$\nabla \times E - i\omega H = f , \qquad \nabla \times H + i\omega\epsilon E = g ,$$

are accompanied with perfect electrical conductor (PEC) boundary conditions (BCs): $n \times E = 0$ on the lateral boundary and at $z = 0$, with a non-local Dirichlet-to-Neumann (DtN) BC applied at $z = L$, see Part I of this work. The original system of equations translates into:

$$\begin{cases} \nabla \times E_3 + e_z \times \frac{\partial}{\partial z} E_t - i\omega H_t = f_t, \\ \operatorname{curl} E_t - i\omega H_3 = f_3, \\ \nabla \times H_3 + e_z \times \frac{\partial}{\partial z} H_t + i\omega\epsilon E_t = g_t, \\ \operatorname{curl} H_t + i\omega\epsilon E_3 = g_3 . \end{cases} \tag{2.2}$$

Multiplying the first and third equations by $i\omega\, e_z \times$, we obtain:

$$\begin{cases} \nabla i\omega E_3 - \frac{\partial}{\partial z} i\omega E_t + \omega^2 e_z \times H_t = i\omega\, e_z \times f_t, \\ \operatorname{curl} E_t - i\omega H_3 = f_3, \\ \nabla i\omega H_3 - \frac{\partial}{\partial z} i\omega H_t - \omega^2 e_z \times \epsilon E_t = i\omega\, e_z \times g_t, \\ \operatorname{curl} H_t + i\omega\epsilon E_3 = g_3 . \end{cases} \tag{2.3}$$

Let D denote the 2D transversal domain of the waveguide. The eigensystem corresponding to the first-order system operator, and $e^{i\beta z}$ ansatz in z, looks as follows:

$$\begin{cases} E_t \in H_0(\operatorname{curl}, D),\ E_3 \in H_0^1(D), \\ H_t \in H(\operatorname{curl}, D),\ H_3 \in H^1(D), \\ i\omega\nabla E_3 + \omega^2 e_z \times H_t = -\omega\beta E_t, \\ \operatorname{curl} E_t - i\omega H_3 = 0, \\ i\omega\nabla H_3 - \omega^2 e_z \times \epsilon E_t = -\omega\beta H_t, \\ \operatorname{curl} H_t + i\omega\epsilon E_3 = 0 . \end{cases} \tag{2.4}$$

And the system corresponding to the adjoint and $e^{i\gamma z}$ ansatz in z is as follows:

$$\begin{cases} F_t \in H(\operatorname{div}, D),\ F_3 \in H^1(D), \\ G_t \in H_0(\operatorname{div}, D),\ G_3 \in H_0^1(D), \\ \nabla \times F_3 + \omega^2 e_z \times \epsilon G_t = -\omega\gamma F_t, \\ i\omega(\operatorname{div} F_t - \epsilon G_3) = 0, \\ \nabla \times G_3 - \omega^2 e_z \times F_t = -\omega\gamma G_t, \\ i\omega(\operatorname{div} G_t + F_3) = 0 . \end{cases} \tag{2.5}$$

Eliminating E_3 and H_3 from system (2.4), we obtain a simplified but second-order system for E_t, H_t only:

$$\begin{cases} E_t \in H_0(\text{curl}, D), \ \text{curl}\, E_t \in H^1(D), \\ H_t \in H(\text{curl}, D), \ \frac{1}{\epsilon}\,\text{curl}\, H_t \in H_0^1(D), \\ \qquad -\nabla(\frac{1}{\epsilon}\,\text{curl}\, H_t) + \omega^2 e_z \times H_t = -\omega\beta E_t, \\ \qquad \nabla(\text{curl}\, E_t) - \omega^2 e_z \times \epsilon E_t = -\omega\beta H_t\,. \end{cases} \tag{2.6}$$

Similarly, eliminating F_3 and G_3 from system (2.5), we obtain a simplified but second-order system for F_t, G_t only:

$$\begin{cases} F_t \in H(\text{div}, D), \ \frac{1}{\epsilon}\,\text{div}\, F_t \in H_0^1(D), \\ G_t \in H_0(\text{div}, D), \ \text{div}\, G_t \in H^1(D), \\ \qquad -\nabla \times \text{div}\, G_t + \omega^2 e_z \times \epsilon G_t = -\omega\gamma F_t, \\ \qquad \nabla \times (\frac{1}{\epsilon}\,\text{div}\, F_t) - \omega^2 e_z \times F_t = -\omega\gamma G_t\,. \end{cases} \tag{2.7}$$

One can check that the operator in (2.7) corresponds to the adjoint of the operator in (2.6). Notice how the boundary conditions (BCs) on E_3, G_3 have been inherited by curl H_t and div F_t.

Reduction to single variable eigensystems Assume $\beta \neq 0$. Solving $(2.6)_2$ for H_t, we get

$$\begin{aligned} H_t &= -\frac{1}{\omega\beta}[\nabla\,\text{curl}\, E_t - \omega^2 e_z \times \epsilon E_t], \\ \text{curl}\, H_t &= \frac{\omega}{\beta}\,\text{curl}(e_z \times \epsilon E_t) = \frac{\omega}{\beta}\,\text{div}\,\epsilon E_t. \end{aligned} \tag{2.8}$$

Substituting this into $(2.6)_1$, we obtain an eigenvalue problem for E_t alone:

$$\begin{cases} E_t \in H_0(\text{curl}, D), \ \text{curl}\, E_t \in H^1(D), \ \frac{1}{\epsilon}\,\text{div}\,\epsilon E_t \in H_0^1(D), \\ \nabla \times \text{curl}\, E_t - \omega^2 \epsilon E_t - \nabla(\frac{1}{\epsilon}\,\text{div}\,\epsilon E_t) = -\beta^2 E_t\,. \end{cases} \tag{2.9}$$

Similarly, solving $(2.6)_1$ for E_t gives

$$\begin{aligned} E_t &= -\frac{1}{\omega\beta}[-\nabla(\frac{1}{\epsilon}\,\text{curl}\, H_t) + \omega^2 e_z \times H_t], \\ \text{curl}\, E_t &= -\frac{\omega}{\beta}\,\text{curl}(e_z \times H_t) = -\frac{\omega}{\beta}\,\text{div}\, H_t. \end{aligned} \tag{2.10}$$

Substituting this into $(2.6)_2$, we obtain an eigenvalue problem for H_t alone:

$$\begin{cases} H_t \in H(\text{curl}, D) \cap H_0(\text{div}, D), \ \frac{1}{\epsilon} \text{curl}\, H_t \in H_0^1(D), \ \text{div}\, H_t \in H^1(D), \\ \epsilon \nabla \times (\frac{1}{\epsilon} \text{curl}\, H_t) - \omega^2 \epsilon H_t - \nabla(\text{div}\, H_t) = -\beta^2 H_t \, . \end{cases} \tag{2.11}$$

Note that the BC $n \times E_t = 0$ implies the BC $n \cdot H_t = 0$. We proceed in the same way with the adjoint. Solving $(2.7)_2$ for G_t leads to

$$\begin{aligned} G_t &= -\frac{1}{\omega\gamma} [\nabla \times (\frac{1}{\epsilon} \text{div}\, F_t) - \omega^2 e_z \times F_t], \\ \text{div}\, G_t &= \frac{\omega}{\gamma} \text{div}(e_z \times F_t) = -\frac{\omega}{\beta} \text{curl}\, F_t, \end{aligned} \tag{2.12}$$

and substituting this into $(2.7)_1$, we obtain an eigenvalue problem for F_t alone:

$$\begin{cases} F_t \in H_0(\text{curl}, D) \cap H(\text{div}, D), \ \frac{1}{\epsilon} \text{div}\, F_t \in H_0^1(D), \ \text{curl}\, F_t \in H^1(D), \\ \nabla \times \text{curl}\, F_t - \omega^2 \epsilon F_t - \epsilon \nabla(\frac{1}{\epsilon} \text{div}\, F_t) = -\gamma^2 F_t \, . \end{cases} \tag{2.13}$$

Note that the BC $n \cdot G_t = 0$ implies the BC $n \times F_t = 0$. Similarly, solving $(2.7)_1$ for F_t, yields

$$\begin{aligned} F_t &= -\frac{1}{\omega\beta} [-\nabla \times \text{div}\, G_t) + \omega^2 e_z \times \epsilon G_t], \\ \text{div}\, F_t &= -\frac{\omega}{\beta} \text{div}(e_z \times \epsilon G_t) = -\frac{\omega}{\beta} \text{curl}\, \epsilon G_t, \end{aligned} \tag{2.14}$$

and substituting this into $(2.7)_2$, we obtain an eigenvalue problem for G_t alone:

$$\begin{cases} G_t \in H_0(\text{div}, D), \ \text{div}\, G_t \in H^1(D), \ \frac{1}{\epsilon} \text{curl}\, \epsilon G_t \in H_0^1(D), \\ \nabla \times (\frac{1}{\epsilon} \text{curl}\, \epsilon G_t) - \omega^2 \epsilon G_t - \nabla(\text{div}\, G_t) = -\gamma^2 G_t \, . \end{cases} \tag{2.15}$$

Lemma 1 *The following holds:*

(a) Let $(-\omega\beta, (E_t, H_t))$ be an eigenpair for system (2.6). Then, $(-\beta^2, E_t)$ solves (2.9), and $(-\beta^2, H_t)$ solves (2.11).

(b) Conversely, if $(-\beta^2, E_t)$ is an eigenpair for (2.9), and we define H_t by

$$H_t = \frac{1}{\omega(\pm\beta)} \left(-\nabla \,\text{curl}\, E_t + \omega^2 e_z \times \epsilon E_t \right),$$

then $(-\omega(\pm\beta), (E_t, H_t))$ is an eigenpair for system (2.6). Each eigenpair for (2.9) generates two eigenpairs for (2.6).

(c) Similarly, if $(-\beta^2, H_t)$ is an eigenpair for (2.11), and we define E_t by

$$E_t = \frac{1}{\omega(\pm\beta)} \left(\nabla(\frac{1}{\epsilon} \text{curl}\, H_t) - \omega^2 e_z \times H_t \right),$$

then $(-\omega(\pm\beta), (E_t, H_t))$ *is an eigenpair for system* (2.6)*. Each eigenpair for (2.11) generates two eigenpairs for* (2.6)*.*

Proof We have already proved (a). To prove (b), one checks that the formula for H_t and (2.9) imply (just algebra) equation $(2.6)_1$. Same procedure to prove (c). □

In particular, Lemma 1 implies that eigenproblems (2.9) and (2.11) have the same eigenvalues β^2.

Lemma 2 *The following holds:*

(a) Let $(\omega\gamma, (F_t, G_t))$ *be an eigenpair for system* (2.7)*. Then,* $(-\gamma^2, G_t)$ *solves* (2.15) *and* $(-\gamma^2, F_t)$ *solves* (2.13)*.*
(b) Conversely, if $(-\gamma^2, F_t)$ *is an eigenpair for* (2.13)*, and we define* G_t *by*

$$G_t = \frac{1}{\omega(\pm\gamma)}\left(-\nabla \times (\frac{1}{\epsilon}\operatorname{div} F_t) + \omega^2 e_z \times F_t\right),$$

then $(\omega(\pm\gamma), (F_t, G_t))$ *is an eigenpair for system* (2.7)*. Each eigenpair for* (2.13) *generates two eigenpairs for* (2.7)*.*
(c) Similarly, if $(-\gamma^2, G_t)$ *is an eigenpair for* (2.15)*, and we define* F_t *by*

$$F_t = \frac{1}{\omega(\pm\gamma)}\left(\nabla \times \operatorname{div} G_t - \omega^2 e_z \times \epsilon G_t\right),$$

then $(\omega(\pm\gamma), (F_t, G_t))$ *is an eigenpair for system* (2.7)*. Each eigenpair for* (2.15) *generates two eigenpairs for* (2.7)*.*

In particular, Lemma 2 implies that the eigenproblems (2.13) and (2.15) have the same eigenvalues γ^2.

Lemma 3 $(-\beta^2, E_t)$ *is an eigenpair for problem (2.9) if and only if* $(-\beta^2, e_z \times E_t)$ *is an eigenpair for (2.15). Similarly,* $(-\beta^2, H_t)$ *is an eigenpair for problem (2.11) if and only if* $(-\beta^2, e_z \times H_t)$ *is an eigenpair for (2.13). In particular, this implies that all four individual eigenproblems share the same eigenvalues.*

Proof Use identities (2.1). □

3 A perturbation analysis

In this section, we will use the classical perturbation theory for self-adjoint operators to analyze two eigenvalue problems:

- the electric eigenvalue problem (2.9)

$$\nabla \times \operatorname{curl} E_t - \omega^2 \epsilon E_t - \nabla(\frac{1}{\epsilon}\operatorname{div} \epsilon E_t) = -\beta^2 E_t \ (E \text{ problem}) \tag{3.1}$$

- and the magnetic eigenvalue problem (2.11)

$$\epsilon \nabla \times (\frac{1}{\epsilon} \operatorname{curl} H_t) - \omega^2 \epsilon H_t - \nabla(\operatorname{div} H_t) = -\gamma^2 H_t \ (H \text{ problem}) . \tag{3.2}$$

We have already argued that the problems share the same eigenvalues. Problem (3.1) is a perturbation of a self-adjoint eigenvalue problem for the electric field representing the homogeneous waveguide,

$$\underbrace{\nabla \times \operatorname{curl} E - \omega^2 E - \nabla(\operatorname{div} E)}_{=:AE} = -\beta^2 E, \tag{3.3}$$

where $E = E_t$. We have learned in [5] that the problem admits two families of eigenvectors:

$$\begin{aligned} E_i &= \nabla \times \psi_i, \ \beta_i^2 = \omega^2 - \mu_i, \\ E_j &= \nabla \phi_j, \quad\ \ \beta_j^2 = \omega^2 - \lambda_j, \end{aligned} \tag{3.4}$$

where (μ_i, ψ_i) and (λ_j, ϕ_j) are Neumann and Dirichlet eigenpairs for the Laplace operator. We will consistently use indices i and j to denote the two families. Problem (3.2) is a perturbation of a self-adjoint eigenvalue problem for the magnetic field representing the homogeneous waveguide,

$$\underbrace{\nabla \times \operatorname{curl} H - \omega^2 H - \nabla(\operatorname{div} H)}_{=:BE} = -\gamma^2 H, \tag{3.5}$$

where $H = H_t = G_t$. The problem admits two families of eigenvectors:

$$\begin{aligned} H_i &= \nabla \psi_i, \quad\ \ \beta_i^2 = \omega^2 - \mu_i, \\ H_j &= \nabla \times \phi_j, \ \beta_j^2 = \omega^2 - \lambda_j, \end{aligned} \tag{3.6}$$

where (μ_i, ψ_i) and (λ_j, ϕ_j) denote again the Neumann and Dirichlet eigenpairs for the Laplace operator. Whenever we distinguish between the two families of eigenvectors, we will consistently use indices i and j or k and l to denote them. The two unperturbed problems look the same but they differ in the boundary conditions. The corresponding perturbed eigenpairs are

$$(-\beta^2 - \delta\beta^2, E + \delta E), \quad (-\gamma^2 - \delta\gamma^2, H + \delta H) .$$

3.1 Perturbation analysis for the E eigenvalue problem

We will present now in detail the analysis for the first perturbed problem. The operator A is self-adjoint in $L^2(D)$, so its eigenvalues are real and the eigenvectors form an L^2-orthonormal basis. Consider now a perturbation,

$$\epsilon := 1 + \delta\epsilon, \qquad E := E + \delta E, \quad \beta^2 := \beta^2 + \delta\beta^2,$$

where $E = E_t$. Inserting these perturbations into (3.1) and linearizing, we obtain the corresponding linearized problem:

$$A(\delta E) + \beta^2 \delta E = \omega^2 \delta\epsilon E - \nabla(\delta\epsilon \operatorname{div} E) + \nabla \operatorname{div}(\delta\epsilon E) - \delta\beta^2 E \ (\delta E \text{ problem}). \tag{3.7}$$

Consider now problem (3.3) and (3.7) for a specific eigenpair [3] $(-\beta_i^2, E_i)$. Representing the perturbation in eigenbasis E_j, we have

$$\begin{aligned}
\delta E_i &= \sum_j (\delta E_i, E_j) E_j\,, \\
A(\delta E_i) &= \sum_j (\delta E_i, E_j)(-\beta_j^2) E_j\,, \\
(A(\delta E_i), E_k) &= \sum_j (-\beta_j^2)(\delta E_i, E_j)\underbrace{(E_j, E_k)}_{=\delta_{jk}} = (-\beta_k^2)(\delta E_i, E_k)\,.
\end{aligned}$$

Taking the L^2-product of (3.7) with E_k, we obtain

$$(-\beta_k^2)(\delta E_i, E_k) = -\beta_i^2(\delta E_i, E_k) - \delta\beta_i^2 \underbrace{(E_i, E_k)}_{=\delta_{ik}} + \omega^2(\delta\epsilon E_i, E_k) - (\nabla(\delta\epsilon \operatorname{div} E_i), E_k) + (\nabla \operatorname{div}(\delta\epsilon E_i), E_k)\,,$$

or,

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta\beta_i^2 \delta_{ik} = \omega^2(\delta\epsilon E_i, E_k) - (\nabla(\delta\epsilon \operatorname{div} E_i), E_k) + (\nabla \operatorname{div}(\delta\epsilon E_i), E_k)\,. \tag{3.8}$$

Assumption A We assume now that the eigenvalues are *distinct* (simple). This is a customary assumption in the perturbation argument to simplify the presentation. The case of multiple eigenvalues is more complicated and is discussed in Appendix 1.

Under the assumption of distinct (simple) eigenvalues, for $k = i$, we get a formula for the perturbation $\delta\beta_i^2$:

$$\delta\beta_i^2 = \omega^2(\delta\epsilon E_i, E_i) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_i) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i)\,. \tag{3.9}$$

For $k \neq i$, formula (3.8) allows us to compute perturbation δE_i; the i-th component of δE_i comes from a normalization argument.

Assumption B We scale the perturbed eigenvectors by the condition

$$(\delta E_i, E_i) = 0\,. \tag{3.10}$$

Note that this scaling assumption is reasonable. Normalization $\|E_i + \delta E_i\|^2 = 1$ plus linearization imply that $\Re(\delta E_i, E_i) = 0$. Scaling *complex-valued* vectors allows also

[3] We do not distinguish now between the two families of eigenvectors.

Table 1 Mass term $(\delta E, E)$ for different families of eigenvectors

	$E_k = \nabla \times \psi_k$	$E_l = \nabla \phi_l$
$\delta E_i = \delta(\nabla \times \psi_i)$	$\dfrac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_k - \mu_i}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_i, E_l)}{\lambda_l - \mu_i}$
$\delta E_j = \delta(\nabla \phi_j)$	$\dfrac{\omega^2(\delta\epsilon E_j, E_k)}{\mu_k - \lambda_j}$	$\dfrac{(\omega^2 - \lambda_l)(\delta\epsilon E_j, E_l) + \lambda_j\lambda_l(\delta\epsilon\phi_j, \phi_l)}{\lambda_l - \lambda_j}$

for fixing a phase, i.e., multiplying the vector with an arbitrary $e^{i\theta}$ factor. Selecting a proper θ we can always make the imaginary part of $(\delta E_i, E_i)$ vanish as well. See also the discussion in Section 4.

We have:

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) = \omega^2(\delta\epsilon E_i, E_k) + (\delta\epsilon \operatorname{div} E_i, \operatorname{div} E_k) - (\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_k)\,.$$

Linearized mass matrices We shall now compute linearized mass matrices for the E-eigenproblem, and the two families of eigenvectors. Table 1 presents the results for the $(\delta E_i, E_j)$ term.

We can now compute the linearized mass matrix:

$$(\delta E_i, E_j) + (E_i, \delta E_j) = (\delta E_i, E_j) + \overline{(\delta E_j, E_i)}\,.$$

The second term is obtained by swapping indices in Table 1 and changing the order of the arguments in the L^2-inner products to account for conjugation. For instance, for the first term,

$$\frac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_k - \mu_i} \quad\to\quad \frac{\omega^2(\delta\epsilon E_k, E_i)}{\mu_i - \mu_k} \quad\to\quad \frac{\omega^2(\delta\epsilon E_i, E_k)}{\mu_i - \mu_k}\,.$$

Table 2 presents selected (those that we will need) elements of the linearized mass matrix.

Curl-curl coupling Let $E_i + \delta E_i$ be the perturbed eigenvectors for the electric eigenproblem.

$$(\operatorname{curl} \delta E_i, \operatorname{curl} E_j) + (\operatorname{curl} E_i, \operatorname{curl} \delta E_j)\,.$$

Table 2 Linearized mass matrix $(\delta E_i, E_k) + (E_i, \delta E_k)$ for different families of eigenvectors

	$E_k = \nabla \times \psi_k$	$E_l = \nabla \phi_l$
$E_i = \nabla \times \psi_i$	0	not needed
$E_j = \nabla \phi_j$	not needed	$-(\delta\epsilon E_j, E_l)$

We have:

$$\begin{aligned}\delta E_i &= \sum_k (\delta E_i, E_k) E_k \qquad \text{(summation over both curls and grads)},\\ \operatorname{curl} \delta E_i &= \sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k \ \text{(summation over curls only)}.\end{aligned}$$

Hence,

$$\begin{aligned}(\operatorname{curl} \delta E_i, \operatorname{curl} E_j) &= \Big(\sum_k (\delta E_i, \nabla \times \psi_k) \mu_k \psi_k, \operatorname{curl} E_j \Big)\\ &= \begin{cases} \sum_k (\delta E_i, \nabla \times \psi_k)(\mu_k \psi_k, \mu_j \psi_j) = (\delta E_i, \nabla \times \psi_j) \mu_j & \text{if } E_j = \nabla \times \psi_j \\ 0 & \text{if } E_j \text{ is a gradient.} \end{cases}\end{aligned}$$

As for the linearized mass matrix, we need to evaluate only the interaction for E_i and E_j being curls only or grads only. Consequently, the linearized curl-curl mass matrix is non-zero only if both eigenvectors are curls, and it is equal to

$$(\delta E_i, E_j)\mu_j + (E_i, \delta E_j)\mu_i = \mu_j \frac{\omega^2 (\delta\epsilon E_i, E_j)}{\mu_j - \mu_i} + \mu_i \frac{\omega^2 (\delta\epsilon E_i, E_j)}{\mu_i - \mu_j} = \omega^2 (\delta\epsilon E_i, E_j) \tag{3.11}$$

for $E_i = \nabla \times \psi_i$, $E_j = \nabla \times \psi_j$.

3.2 Perturbation analysis for the H eigenvalue problem

The linearized problem is:

$$B\delta H + \gamma^2 \delta H = -\delta\epsilon \nabla \times \operatorname{curl} H - \nabla \times (\delta\epsilon \operatorname{curl} H) + \omega^2 \delta\epsilon H - \delta\gamma^2 H,$$

where the operator B is formally the same as operator A for the E problem (but the BCs are different). Performing the same analysis as for the E problem, we get

$$(\gamma_i^2 - \gamma_k^2)(\delta H_i, H_k) + \delta\gamma_i^2 \delta_{ik} = \omega^2 (\delta\epsilon H_i, H_k) - (\delta\epsilon \nabla \times \operatorname{curl} H_i, H_k) + (\nabla \times (\delta\epsilon \operatorname{curl} H_i), H_k)\,. \tag{3.12}$$

Under **Assumption A** of distinct (simple) eigenvalues, for $k = i$, we get a formula for the perturbation $\delta\gamma_i^2$:

$$\delta\gamma_i^2 = \omega^2 (\delta\epsilon H_i, H_i) - (\delta\epsilon \nabla \times \operatorname{curl} H_i, H_i) + (\nabla \times (\delta\epsilon \operatorname{curl} H_i), H_i)\,. \tag{3.13}$$

For $k \neq i$, formula (3.12) allows us to compute the perturbation δH_i; the i-th component of δH_i comes from a normalization assumption.

Table 3 Linearized mass matrix $(\delta H_i, H_k) + (H_i, \delta H_k)$ for different families of eigenvectors

	$H_k = \nabla \psi_k$	$H_l = \nabla \times \phi_l$
$H_i = \nabla \psi_i$	0	not needed
$H_j = \nabla \times \phi_j$	not needed	$(\delta\epsilon H_j, H_l)$

Assumption C We assume

$$(\delta H_i, H_i) = 0 \,. \tag{3.14}$$

Assumption C implies that the perturbed eigenvector $H_i + \delta H_i$ is (approximately) of length one, see the discussion in Section 4.

After integrating the last term in (3.12) by parts, we obtain

$$(\gamma_i^2 - \gamma_k^2)(\delta H_i, H_k) = \omega^2(\delta\epsilon\ H_i, H_k) - (\delta\epsilon\nabla \times \mathrm{curl}\ H_i, H_k) + (\delta\epsilon\ \mathrm{curl}\ H_i, \mathrm{curl}\ H_k)\,.$$

The computations of the (required) elements of the linearized stiffness matrices are fully analogous to those for the electric eigenvalue problem. Table 3 presents selected (those that we need) elements of the linearized mass matrix.

Finally, the curl-curl linearized mass matrix for the grad eigenvectors vanishes, and for the curl eigenvectors $H_i = \nabla \times \phi_i$, $H_j = \nabla \times \phi_j$ the entries look as follows:

$$\begin{aligned}
(\mathrm{curl}\,\delta H_i, \mathrm{curl}\ H_j) + &\ (\mathrm{curl}\ H_i, \mathrm{curl}\,\delta H_j) = (\delta H_i, H_j)\lambda_j + (H_i, \delta H_j)\lambda_j \\
&= \frac{\lambda_j}{\lambda_j - \lambda_i}\left[\omega^2(\delta\epsilon H_i, H_j) - \underbrace{(\delta\epsilon\nabla \times \mathrm{curl}\ H_i, H_j)}_{=\lambda_i(\delta\epsilon H_i, H_j)} + \underbrace{(\delta\epsilon\ \mathrm{curl}\ H_i, \mathrm{curl;}\ H_j)}_{=\lambda_i\lambda_j(\delta\epsilon\phi_i, \phi_j)}\right] \\
&+ \frac{\lambda_i}{\lambda_i - \lambda_j}\overline{\left[\omega^2(\delta\epsilon H_j, H_i) - \lambda_j(\delta\epsilon H_j, H_i) + \lambda_j\lambda_i(\delta\epsilon\phi_j, \phi_i)\right]} \\
&= \omega^2(\delta\epsilon H_i, H_j) + \lambda_i\lambda_j(\delta\epsilon\phi_i, \phi_j)\,.
\end{aligned} \tag{3.15}$$

4 Stability analysis

We return to system (2.3). We test the first equation with F_t and the third equation with $G_t, n \cdot G_t = 0$ on ∂D, to obtain

$$\left\{\begin{aligned}
-(i\omega E_3, \mathrm{div}\ F_t) + \omega^2(e_z \times H_t, F_t) - \tfrac{\partial}{\partial z} i\omega(E_t, F_t) &= i\omega\,(e_z \times f_t, F_t), \\
\mathrm{curl}\ E_t - i\omega H_3 &= f_3, \\
-(i\omega H_3, \mathrm{div}\ G_t) - \omega^2(e_z \times \epsilon E_t, G_t) - \tfrac{\partial}{\partial z} i\omega(H_t, G_t) &= i\omega\,(e_z \times g_t, G_t), \\
\mathrm{curl}\ H_t + i\omega\epsilon E_3 &= g_3\,.
\end{aligned}\right.$$

Note that, when integrating by parts the first terms, we have used the fact that $E_3 = 0$ and $n \cdot G_t = 0$ on ∂D. Solving the second and fourth equations in (2.2) for E_3 and

H_3, we get

$$E_3 = \frac{1}{i\omega\epsilon} g_3 - \frac{1}{i\omega\epsilon} \operatorname{curl} H_t \,, \qquad H_3 = -\frac{1}{i\omega} f_3 + \frac{1}{i\omega} \operatorname{curl} E_t \,.$$

Substituting this into the first and the third equations, we obtain a system of two variational equations for E_t, H_t :

$$\begin{cases} (\frac{1}{\epsilon} \operatorname{curl} H_t, \operatorname{div} F_t) + \omega^2 (e_z \times H_t, F_t) - \frac{\partial}{\partial z} i\omega (E_t, F_t) = i\omega\, (e_z \times f_t, F_t) + (\frac{1}{\epsilon} g_3, \operatorname{div} F_t), \\ -(\operatorname{curl} E_t, \operatorname{div} G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) - \frac{\partial}{\partial z} i\omega (H_t, G_t) = i\omega\, (e_z \times g_t, G_t) - (f_3, \operatorname{div} G_t)\,. \end{cases} \tag{4.1}$$

The variational eigenvalue problem

$$\begin{cases} E_t \in H_0(\operatorname{curl}, D),\, H_t \in H(\operatorname{curl}, D), \\ (\frac{1}{\epsilon} \operatorname{curl} H_t, \operatorname{div} F_t) + \omega^2 (e_z \times H_t, F_t) = -\omega\beta (E_t, F_t), \\ -(\operatorname{curl} E_t, \operatorname{div} G_t) - \omega^2 (e_z \times \epsilon E_t, G_t) = -\omega\beta (H_t, G_t), \\ \qquad\qquad\qquad\qquad \forall F_t \in H(\operatorname{div}, D),\, \forall G_t \in H_0(\operatorname{div}, D)\,, \end{cases}$$

is equivalent to the eigenproblem (2.6). Similarly, switching the roles of (E_t, H_t) and (F_t, G_t) above, we obtain the adjoint variational eigenvalue problem, which is equivalent to (2.7).

Consider now system (4.1). We expand the unknowns into a series of the perturbed eigenvectors:

$$\begin{aligned} E_t &= \sum_i \alpha_i E_{t1,i} + \sum_j \tau_j E_{t2,j}, \\ H_t &= \sum_i \delta_i H_{t1,i} + \sum_j \eta_j H_{t2,j}, \end{aligned}$$

where $\alpha_i, \tau_j, \delta_i, \eta_j$ are functions of z, and

$$\begin{aligned} E_{t1,i} &= \nabla \times \psi_i + \delta E_{t1,i}, \qquad & E_{t2,j} &= \nabla \phi_j + \delta E_{t2,j}, \\ H_{t1,i} &= \nabla \psi_i + \delta H_{t1,i}, \qquad & H_{t2,j} &= \nabla \times \phi_j + \delta H_{t2,j} \end{aligned}$$

are the two E and H families of (perturbed) eigenvectors. Let

$$\begin{aligned} F_{t1,i} &= \nabla \times \psi_i + \delta F_{t1,i}, \qquad & F_{t2,j} &= \nabla \phi_j + \delta F_{t2,j}, \\ G_{t1,i} &= \nabla \psi_i + \delta G_{t1,i}, \qquad & G_{t2,j} &= \nabla \times \phi_j + \delta G_{t2,j} \end{aligned}$$

be the corresponding families of perturbed adjoint eigenvectors.

Scaling of the eigenvectors The unperturbed E eigenvectors are scaled to provide an orthonormal basis, i.e., $\|\nabla \times \psi_i\| = \|\nabla \phi_j\| = 1$. This implies that the unperturbed H eigenvectors are also unit vectors as $\|\nabla \psi_i\| = \|\nabla \times \psi_i\|$, etc. The unperturbed F and G eigenvectors coincide with the E and H eigenvectors. We learned in Section 3 that the perturbations $\delta E_{t1,i}$ are scaled by the condition $(\delta E_{t1,i}, \nabla \times \psi_i) = 0$. This implies that the perturbed eigenvector is, up to the linearization, a unit vector as well,

$$(\nabla \times \psi_i + \delta E_{t1,i}, \nabla \times \psi_i + \delta E_{t1,i}) \approx (\nabla \times \psi_i, \nabla \times \psi_i) + (\nabla \times \psi_i, \delta E_{t1,i}) + (\delta E_{t1,i}, \nabla \times \psi_i) = 1\,.$$

The same comment applies to all remaining perturbed eigenvectors. Note additionally that the bi-orthogonality condition $(E_i, F_i) = 1$ is also (approximately) satisfied,

$$(\nabla \times \psi_i + \delta E_{t1,i}, \nabla \times \psi_i + \delta F_{t1,i}) \approx \underbrace{(\nabla \times \psi_i, \nabla \times \psi_i)}_{=1} + \underbrace{(\nabla \times \psi_i, \delta F_{t1,i})}_{=0} + \underbrace{(\delta E_{t1,i}, \nabla \times \psi_i)}_{=0} = 1\,.$$

Decoupling the equations Let $-\beta^2$ be an eigenvalue for eigenproblems (2.9) and (2.11) with the corresponding eigenvectors E_t, H_t scaled as discussed above. In order to invoke Lemma 1(b), we have to replace H_t with cH_t where the constant c is computed by comparing the eigenvector cH_t with H_t given by relation (2.8),

$$cH_t = \frac{1}{\omega\beta}[-\nabla \operatorname{curl} E_t + \omega^2 e_z \times \epsilon E_t]\,.$$

The pair (E_t, cH_t) constitutes then an eigenvector for system (2.6) corresponding to root β of β^2 selected in such a way that $e^{i\beta z}$ represents an outgoing wave.[4] We proceed similarly with the adjoint eigenvectors. Let $-\gamma^2$ be an eigenvalue for problems (2.13) and (2.15) with the corresponding eigenvectors F_t, H_t. After scaling the second component, the pair (F_t, dG_t) constitutes an eigenvector for system (2.7) corresponding to a root γ of γ^2. The constant d is obtained[5] by comparing dG_t with G_t given by (2.12), cf. Lemma 2,

$$dG_t = \frac{1}{\omega\beta}[-\nabla \times (\frac{1}{\epsilon} \operatorname{div} F_t) + \omega^2 e_z \times F_t]\,.$$

Case $\beta^2 \neq \gamma^2$ This implies $\beta \neq \gamma$. Multiplying system (2.6) with the pair (F_t, dG_t), we obtain the bi-orthogonality condition,

$$c(BH_t, F_t) + d(CE_t, G_t) = 0,$$

where B and C denote the operators on the left-hand side of (2.6). But testing with the adjoint eigenpair $(F_t, -dH_t)$ (corresponding to the eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0.$$

Consequently, we have,

$$(BH_t, F_t) = 0 \qquad \text{and} \qquad (CE_t, G_t) = 0\,.$$

[4] The choice depends upon the ansatz in time.

[5] We learn in Appendix 1 that d is real.

Case $\beta^2 = \gamma^2$ and $\beta = \gamma$ Multiplying system (2.6) with the pair (F_t, G_t), we obtain

$$c(BH_t, F_t) + d(CE_t, G_t) = -\omega\beta[1 + cd].$$

Testing with the adjoint eigenpair $(F_t, -dG_t)$ (corresponding to the eigenvalue $-\gamma \neq \beta$), we obtain also

$$c(BH_t, F_t) - d(CE_t, G_t) = 0.$$

Consequently, we have

$$\theta := (BH_t, F_t) = -\frac{\omega\beta}{2c}[1 + cd] \quad \text{and} \quad \nu := (CE_t, G_t) = -\frac{\omega\beta}{2d}[1 + cd]\,.$$

Theorem 1 *Testing in (4.1) with $(F_{t1,i}, dG_{t1,i})$ and with $(F_{t2,i}, dG_{t2,i})$, we obtain a decoupled system of ODEs for the coefficients α_i, δ_i:*

$$\begin{cases} \theta_1\delta_i - i\omega\alpha_i' = r_1(z) := (i\omega\, e_z \times f_t, F_{t1,i}) + (\frac{1}{\epsilon}g_3, \operatorname{div} F_{t1,i}), \\ \nu_1\alpha_i - i\omega\delta_i' = r_2(z) := (i\omega\, e_z \times g_t, G_{t1,i}) - (f_3, \operatorname{div} G_{t1,i}), \end{cases} \tag{4.2}$$

and for the coefficients τ_i, η_i:

$$\begin{cases} \theta_2\eta_i - i\omega\tau_i' = s_1(z) := (i\omega\, e_z \times f_t, F_{t2,i}) + (\frac{1}{\epsilon}g_3, \operatorname{div} F_{t2,i}), \\ \nu_2\tau_i - i\omega\eta_i' = s_2(z) := (i\omega\, e_z \times g_t G_{t2,i}) - (f_3, \operatorname{div} G_{t2,i}), \end{cases} \tag{4.3}$$

where θ_k, ν_k, $k = 1, 2$, are the values of the coefficients θ, ν for the two families of eigenvectors.

We refer to Appendix 1, for the computation of the constants c, d, θ, ν using the perturbation analysis, and the final values of θ, ν listed in Table 4. The constants take different values for the two families of E and F eigenvectors. For the homogeneous waveguide, the systems reduce to the ones in [5].

Remark 1 While we use the perturbation analysis to evaluate the constants c, d, θ, ν, the decoupling result in Theorem 1 is general and valid for arbitrary ϵ.

5 Estimation of E_t, H_t and curl E_t, curl H_t

Recall that $i\omega H_3 = \operatorname{curl} E_t - f_3$. Estimating curl E_t is thus equivalent to estimating H_3. Similarly, $i\omega\epsilon E_3 = -\operatorname{curl} H_t - g_3$, so that estimating curl H_t is equivalent to estimating E_3.

Table 4 Coefficients δ and ν for the two families of eigenvectors

	First family	Second family
$\theta + \delta\theta$	$-\omega^2$	$-\beta^2 - \omega^2(\delta\epsilon\, \nabla\psi, \nabla\psi)$
$\nu + \delta\nu$	$\beta^2 + \lambda^2(\delta\epsilon\, \phi, \phi)$	$\omega^2(1 + (\delta\epsilon\, \nabla\phi, \nabla\phi))$

5.1 Estimation of E_t and H_t with their spectral components

If the L^2 mass matrix corresponding to the perturbed eigenvectors represents a bounded operator in L^2, then we can bound the L^2-norm of E_t with the sum of its spectral components. More precisely,

$$
\begin{aligned}
\|E_t\|^2 &\le 2\left[\|\sum_{i=1}^{\infty}\alpha_i E_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\tau_j E_{t1,j}\|^2\right] \\
&= 2\lim_{N\to\infty}\left[(\sum_{i=1}^{N}\alpha_i E_{t1,i}, \sum_{k=1}^{N}\alpha_k E_{t1,k}) + (\sum_{j=1}^{N}\tau_j E_{t2,j}, \sum_{l=1}^{N}\tau_l E_{t2,l})\right] \\
&= 2\lim_{N\to\infty}\left[\sum_{i,k=1}^{N}\alpha_i\overline{\alpha_k}(E_{t1,i}, E_{t1,k}) + \sum_{j,l=1}^{N}\tau_j\overline{\tau_l}(E_{t2,j}, E_{t2,l})\right] \\
&\le \lim_{N\to\infty} 2C\left[\sum_{i=1}^{N}|\alpha_i|^2 + \sum_{j=1}^{N}|\tau_j|^2\right] \\
&= 2C\left[\sum_{i=1}^{\infty}|\alpha_i|^2 + \sum_{j=1}^{\infty}|\tau_j|^2\right]
\end{aligned}
$$

where C is assumed to be independent of N. Note that we do not need any information about the off-diagonal terms $(E_{t1,i}, E_{t2,j})$. According to the formulas from Table 2, we have $C = 1 + \|\delta\epsilon\|_{L^\infty(D)}$.

Similarly,

$$
\begin{aligned}
\|H_t\|^2 &\le 2\left[\|\sum_{i=1}^{\infty}\delta_i H_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\eta_j H_{t2,j}\|^2\right] \\
&\le 2C\left[\sum_{i=1}^{\infty}|\delta_i|^2 + \sum_{j=1}^{\infty}|\eta_j|^2\right]
\end{aligned}
$$

where, by the results from Table 3, $C = 1 + \|\delta\epsilon\|_{L^\infty(D)}$ as well.

After integrating in z, we get

$$
\begin{aligned}
\int_0^L \|E_t\|^2\,dz &\le 2C\left[\sum_{i=1}^{\infty}\int_0^L|\alpha_i|^2\,dz + \sum_{j=1}^{\infty}\int_0^L|\tau_j|^2\,dz\right], \\
\int_0^L \|H_t\|^2\,dz &\le 2C\left[\sum_{i=1}^{\infty}\int_0^L|\delta_i|^2\,dz + \sum_{j=1}^{\infty}\int_0^L|\eta_j|^2\,dz\right].
\end{aligned}
\tag{5.1}
$$

5.2 Estimation of curl E_t, curl H_t with spectral components of E_t, H_t

By the same token, using (3.11), we obtain,

$$\begin{aligned}
\|\operatorname{curl} E_t\|^2 &\le 2\left[\|\sum_{i=1}^{\infty}\alpha_i \operatorname{curl} E_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\tau_j \operatorname{curl} E_{t2,j}\|^2\right] \\
&= 2\lim_{N\to\infty}\left[(\sum_{i=1}^{N}\alpha_i \operatorname{curl} E_{t1,i}, \sum_{k=1}^{N}\alpha_k \operatorname{curl} E_{t1,k}) + (\sum_{j=1}^{N}\tau_j \operatorname{curl} E_{t2,j}, \sum_{l=1}^{N}\tau_l \operatorname{curl} E_{t2,l})\right] \\
&= 2\lim_{N\to\infty}\left[\sum_{i,k=1}^{N}\alpha_i\overline{\alpha_k}(\operatorname{curl} E_{t1,i}, \operatorname{curl} E_{t1,k}) + \sum_{j,l=1}^{N}\tau_j\overline{\tau_l}(\operatorname{curl} E_{t2,j}, \operatorname{curl} E_{t2,l})\right] \\
&\approx 2\sum_{i=1}^{\infty}(\mu_i + \omega^2\|\delta\epsilon\|_{L^\infty(D)})|\alpha_i|^2 \,.
\end{aligned}$$

Note that, like for the homogeneous case, the perturbed gradients do not contribute (the linearized perturbed curl mass matrix is zero).

Similarly, using formula (3.15),

$$\begin{aligned}
\|\operatorname{curl} H_t\|^2 &\le 2\left[\|\sum_{i=1}^{\infty}\alpha_i \operatorname{curl} H_{t1,i}\|^2 + \|\sum_{j=1}^{\infty}\tau_j \operatorname{curl} H_{t2,j}\|^2\right] \\
&\lesssim 2\sum_{i=1}^{\infty}(\lambda_i\|\epsilon\|_{L^\infty(D)} + \omega^2\|\delta\epsilon\|_{L^\infty(D)})|\eta_i|^2 \,.
\end{aligned}$$

Note again that the perturbed gradients do not contribute.

5.3 Estimation of spectral components α_i, δ_i

We focus now on the ODE boundary-value problem (4.2) for the coefficients α and δ,

$$\begin{cases}
\alpha(0) = 0,\ \delta(L) = \sqrt{\frac{\nu}{\theta}}\alpha(L), \\
\theta\delta - i\omega\alpha' = r_1, \\
\nu\alpha - i\omega\delta' = r_2,
\end{cases} \tag{5.2}$$

where $\theta = \theta_1$ and $\nu = \nu_1$ are the coefficient values for the first family of eigenvectors. Testing the second equation with $\delta\alpha$, $\delta\alpha(0) = 0$, integrating the derivative term by parts, and utilizing the impedance BC, we obtain

$$i\omega(\delta, \delta\alpha') = -\nu(\alpha, \delta\alpha) + i\omega\alpha(L)\delta\alpha(L) + (r_2, \delta\alpha)\,.$$

Testing the first equation with $\delta\alpha'$ and using the formula above, we obtain the final variational problem for the coefficient α:

$$\begin{cases} \alpha(0) = 0 \\ (\alpha', \delta\alpha') + \kappa^2(\alpha, \delta\alpha) + \kappa\alpha(L)\delta\alpha(L) = \frac{i}{\omega}(r_1, \delta\alpha') - \frac{\theta}{\omega^2}(r_2, \delta\alpha) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \delta\alpha \;:\; \delta\alpha(0) = 0 \end{cases} \tag{5.3}$$

where $\kappa = i\frac{\sqrt{\theta v}}{\omega}$. For the homogeneous waveguide, $\kappa = i\beta$ and the equation coincides with that derived in [5]. For the non-homogeneous waveguide,

$$\kappa = i\sqrt{\beta^2 + \omega^2(\delta\epsilon\nabla\psi, \nabla\psi)}\,.$$

The perturbed κ is still of order β. As $\theta = \theta_1 = -\omega^2$, the right-hand side reduces to

$$\frac{i}{\omega}(r_1, \delta\alpha') + (r_2, \delta\alpha)\,. \tag{5.4}$$

The following lemma was proved in [5].

Lemma 4 *Let $I = (0, L)$. Consider two problems: Find $q_1, q_2 \in H^1_{(0}(I) := \{v \in H^1(I) \,:\, v(0) = 0\}$ such that*

$$\begin{aligned} (q_1', v') + \lambda^2(q_1, v) + \lambda q_1(L)v(L) &= (f, v) \quad v \in H^1_{(0}(I), \\ (q_2', v') + \lambda^2(q_2, v) + \lambda q_2(L)v(L) &= (f, v') \;\; v \in H^1_{(0}(I), \end{aligned}$$

where $f \in L^2(I)$. Then, denoting $\|q\|^2_{1,\beta} := \|q'\|^2 + \beta^2\|q\|^2$, we have:

- *Case (i): $\lambda = i\beta, \beta > 0$. There exists a constant $C > 0$, depending only on a lower bound for $L\beta$ such that*

$$\begin{aligned} \|q_1\|^2_{1,\beta} &\le CL^2\|f\|^2, \\ \|q_2\|^2_{1,\beta} &\le CL^2\beta^2\|f\|^2. \end{aligned}$$

- *Case (ii): $\lambda = \beta, \beta > 0$. There exists a constant $C > 0$, depending only on a lower bound for $L\beta$, such that*

$$\begin{aligned} \|q_1\|^2_{1,\beta} &\le C\beta^{-2}\|f\|^2 \;\; \text{and hence } \|q_1\|^2 \le C\beta^{-4}\|f\|^2, \\ \|q_2\|^2_{1,\beta} &\le C\|f\|^2 \qquad\;\; \text{and hence } \|q_2\|^2 \le C\beta^{-2}\|f\|^2\,. \end{aligned}$$

We will use Lemma 4 to estimate the L^2-norms of coefficients α_j, δ_j by the L^2-norms of the right-hand sides $r_{1,j}, r_{2,j}$ and, in turn, the $L^2(0, L)$-norms of $r_{1,j}, r_{2,j}$ by the L^2-norms of f_t, f_3, g_t, g_3. While the stability of propagating modes (Case (i) in Lemma 4) implies the linear dependence of the stability constant upon L, the stability of evanescent modes (Case (ii) in Lemma 4) provides the desired asymptotic scaling properties in terms of the eigenvalues $|\beta_i| \approx \mu_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}}$. Note that, since the number of propagating modes is finite, their stability does not affect the asymptotic scaling properties with $|\beta_i|$. We will skip the dependence of the stability constants C upon L but keep track of the dependence upon the eigenvalues $|\beta_i| \approx \mu_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}}$.

Estimation of α_j We will consider the four terms contributing to the right-hand side (5.4) and estimate the corresponding solutions α_j, one at a time. By linearity, this will imply the estimate for the ultimate coefficients α_j.

Term 1 $i\omega(e_z \times f_t, F_{t1,j})$ contributing to r_1. Skipping the factor $i\omega$, we have:

$$
\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\,dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2 && \text{(Lemma 4(ii)}_2\text{)}\\
&\lesssim 2\sum_j \int_0^L \beta_j^{-2} [|(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2] && \text{(Young's inequality)}\\
&\lesssim 2\sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2 && \text{(linearization, } \beta_j^{-2} \lesssim 1\text{)}\\
&\leq 2\int_0^L \|e_z \times f_t\|^2\,dz\\
&= 2\int_0^L \|f_t\|^2\,dz\,.
\end{aligned}
$$

Remark 2 Note that the application of Young's inequality and neglection of the second-order terms reduces the estimation of coefficients α_j to the case of the homogeneous waveguide. The ODE systems (4.2) and (4.3) are identical with those for the homogeneous waveguide except for the values of θ_i, ν_i which are different but of the same order as for the homogeneous system. Hence the estimation of coefficients $\alpha_i, \delta_i, \tau_j, \eta_j$ in the perturbed case is identical with the estimation for the homogeneous waveguide. For the reader's convenience, we estimate explicitly each term, repeating arguments from [5].

Term 2 $(\frac{1}{\epsilon}g_3, \operatorname{div} F_{t1,j})$ contributing to r_1.

$$
\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\,dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j} + \delta F_{t1,j}))|^2 && \text{(Lemma 4(ii)}_2\text{)}\\
&\leq 2\sum_j \int_0^L \beta_j^{-2} [|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j}))|^2 + |(\frac{1}{\epsilon}g_3, \operatorname{div}(\delta F_{t1,j}))|^2] && \text{(Young's lemma)}\\
&\lesssim 2\sum_j \int_0^L |(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t1,j}))|^2 && \text{(linearization, } \beta_j^{-2} \lesssim 1\text{)}\\
&\lesssim 0 && \text{(div } F_{t1,j} = 0\text{)}
\end{aligned}
$$

Term 3 $i\omega(e_z \times g_t, G_{t1,j})$ contributing to r_2. We follow exactly the same reasoning as for Term 1. Note that Lemma 4(ii)$_1$ gives us even a better factor β^{-4}.

Term 4 $(f_3, \operatorname{div} G_{t1,j})$ contributing to r_2.

$$\begin{aligned}
\sum_j \int_0^L |\alpha_j|^2\,dz &\lesssim \sum_j \int_0^L \beta_j^{-4} |(f_3, \operatorname{div}(G_{t1,j} + \delta G_{t1,j}))|^2 && \text{(Lemma 4(ii)}_1\text{)}\\
&\le 2 \sum_j \int_0^L \beta_j^{-4} [|(f_3, \operatorname{div}(G_{t1,j}))|^2 + |(f_3, \operatorname{div}(\delta G_{t1,j}))|^2] && \text{(Young's lemma)}\\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-4} |(f_3, \operatorname{div}(G_{t1,j}))|^2 && \text{(linearization)}\\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-4} \mu_j |(f_3, \mu_j^{1/2}\psi_j)|^2 \\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2} |(f_3, \mu_j^{1/2}\psi_j)|^2 && (\beta_j^{-2}\mu_j \approx O(1))\\
&\lesssim 2 \sum_j \int_0^L |(f_3, \mu_j^{1/2}\psi_j)|^2 && (\beta_j^{-2} \lesssim 1)\\
&\lesssim 2 \sum_j |(f_3, \mu_j^{1/2}\psi_j)|^2 = 2\|f_3\|^2 .
\end{aligned}$$

Estimation of curl $\boldsymbol{E_t}$ In the estimation of curl E_t, we need to estimate

$$\sum_i \int_0^L \underbrace{(\mu_i + \|\delta\epsilon\|_{L^\infty(D)})}_{\sim\beta_i^2} |\alpha_i|^2\,dz .$$

We follow exactly the same strategy as above. In all cases, we can accommodate the extra $\mu_i \approx \beta_i^2$ factor.

Estimation of $\boldsymbol{\delta_j}$ The first equation of system (5.2) implies

$$\omega^2 \|\delta\|_{L^2(I)} \le \omega \|\alpha'\|_{L^2(I))} + \|r_1\|_{L^2(I)} .$$

Estimation of the derivatives $\sum_i \|\alpha_i'\|^2_{L^2(I))}\|$ is done in exactly the same way as for α_i's, except that Lemma 4(ii) delivers now less by a factor of β_j^{-2}. However, we can spare it in all of the four discussed cases. It remains to estimate r_1. We proceed in the same way as before.

Term 1 $(e_z \times f_t, F_{t1,j})$ contributing to r_1. We have:

$$\begin{aligned}
&\sum_j \int_0^L |(e_z \times f_t, F_{t1,j} + \delta F_{t1,j})|^2 \\
&\quad \lesssim 2 \sum_j \int_0^L [|(e_z \times f_t, F_{t1,j})|^2 + |(e_z \times f_t, \delta F_{t1,j})|^2] \text{ (Young's inequality)} \\
&\quad \lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t1,j})|^2 \qquad \text{(linearization)} \\
&\quad \leq 2 \int_0^L \|e_z \times f_t\|^2 \, dz = 2 \int_0^L \|f_t\|^2 \, dz \, .
\end{aligned}$$

Term 2 $(\frac{1}{\epsilon} g_3, \operatorname{div} G_{t1,j})$ contributing to r_1.

$$\begin{aligned}
&\sum_j \int_0^L |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j} + \delta F_{t1,j}))|^2 \\
&\quad \leq 2 \sum_j \int_0^L [|(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 + |(\frac{1}{\epsilon} g_3, \operatorname{div}(\delta F_{t1,j}))|^2] \text{ (Young's lemma)} \\
&\quad \lesssim 2 \sum_j \int_0^L |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t1,j}))|^2 \qquad \text{(linearization)} \\
&\quad \lesssim 0 \qquad (\operatorname{div} F_{t1,j} = 0)
\end{aligned}$$

5.4 Estimation of spectral components τ_i, η_i

We use exactly the same techniques to estimate the spectral components corresponding to the second families of E and H eigenvectors. We will point out only to the differences between the two cases. The ODE boundary-value problem (4.3) for the coefficients τ and η takes the same form as system (5.2),

$$\begin{cases}
\beta(0) = 0, \ \eta(L) = \sqrt{\frac{\theta_2}{\theta_1}} \beta(L), \\
\theta\eta - i\omega\beta' = s_1, \\
\nu\beta - i\omega\eta' = s_2,
\end{cases} \tag{5.5}$$

where

$$\begin{aligned}
s_1 &= (i\omega \, e_z \times f_t, F_{t2,j}) + (\frac{1}{\epsilon} g_3, \operatorname{div} F_{t2,j}), \\
s_2 &= (i\omega \, e_z \times g_t G_{t2,j}) - (f_3, \operatorname{div} G_{t2,j}) \, .
\end{aligned}$$

The coefficients θ, ν are now different. By the results from Appendix 1 in the “Computation of scaling coefficients $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{\nu}$” section,

$$\theta = \theta_2 = \beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi), \qquad \nu = \nu_2 = \omega^2 + \omega^2(\delta\epsilon\, \nabla\phi, \nabla\phi)\,.$$

This gives for the homogeneous waveguide $\kappa = i\beta$, and for the non-homogeneous case,

$$\kappa = i\,\frac{\sqrt{(\beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi))(\omega^2 + \omega^2(\delta\epsilon\, \nabla\phi, \nabla\phi))}}{\omega}$$

which is still $i\beta$ to leading order.

The first essential difference is the scaling in the right-hand side of the second-order problem (5.3),

$$\frac{i}{\omega}(s_1, \delta\alpha') - \frac{\theta}{\omega^2}(s_2, \delta\alpha) = \frac{i}{\omega}(s_1, \delta\alpha') - \frac{(\beta^2 + \lambda(\delta\epsilon\, \lambda^{\frac{1}{2}}\phi, \lambda^{\frac{1}{2}}\phi))}{\omega^2}(s_2, \delta\alpha)\,.$$

The coefficient in front of s_1 is the same as before, but the coefficient θ in front of s_2 is now of order β^2.

Estimation of $\boldsymbol{\tau}_j$ We now discuss the four terms contributing to the right-hand side above.

Term 1 $i\omega(e_z \times f_t, F_{t2,j})$ contributing to s_1. The estimate is identical with that for α_i. Skipping constant terms, we have:

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2} |(e_z \times f_t, F_{t2,j} + \delta F_{t2,j})|^2 && \text{(Lemma 4(ii)}_2\text{)}\\
&\lesssim 2 \sum_j \int_0^L \beta_j^{-2}[|(e_z \times f_t, F_{t2,j})|^2 + |(e_z \times f_t, \delta F_{t2,j})|^2] && \text{(Young's inequality)}\\
&\lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t2,j})|^2 && \text{(linearization, } \beta_j^{-2} \lesssim 1\text{)}\\
&\leq 2 \int_0^L \|e_z \times f_t\|^2\, dz = 2 \int_0^L \|f_t\|^2\, dz\,.
\end{aligned}$$

Term 2 $(\frac{1}{\epsilon}g_3, \operatorname{div} F_{t2,j})$ contributing to s_1. The situation is now different as $\operatorname{div} F_{t2,j} = \operatorname{div}\operatorname{grad}\phi_j = -\lambda_j\phi_j$.

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2}|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j} + \delta F_{t2,j}))|^2 && \text{(Lemma 4(ii)}_2\text{)}\\
&\le 2\sum_j \int_0^L \beta_j^{-2}[|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j}))|^2 + |(\frac{1}{\epsilon}g_3, \operatorname{div}(\delta F_{t2,j}))|^2] && \text{(Young's lemma)}\\
&\lesssim 2\sum_j \int_0^L \beta_j^{-2}|(\frac{1}{\epsilon}g_3, \operatorname{div}(F_{t2,j}))|^2 && \text{(linearization)}\\
&\lesssim 2\sum_j \int_0^L |(\frac{1}{\epsilon}g_3, \lambda_j^{\frac{1}{2}}\phi_j))|^2 && (\beta_j^{-2}\lambda_j \lesssim 1)\\
&\lesssim \|\frac{1}{\epsilon}g_3\|^2 \lesssim \|g_3\|^2 .
\end{aligned}$$

Term 3 $i\omega(e_z \times g_t, G_{t2,j})$ contributing to s_2. We follow exactly the same reasoning as for Term 1. Lemma 4(ii)$_1$ gives us a better factor β^{-4} but there is a factor of order β^2 in front of s_2.

Term 4 $(f_3, \operatorname{div} G_{t2,j})$ contributing to s_2. Compared with the estimate for α_j, we lose again a factor β_j^{-2} due to the term in front of s_2.

$$\begin{aligned}
\sum_j \int_0^L |\tau_j|^2\, dz &\lesssim \sum_j \int_0^L \beta_j^{-2}|(f_3, \operatorname{div}(G_{t2,j} + \delta G_{t2,j}))|^2 && \text{(Lemma 4(ii)}_1\text{)}\\
&\le 2\sum_j \int_0^L \beta_j^{-2}[|(f_3, \operatorname{div}(G_{t2,j}))|^2 + |(f_3, \operatorname{div}(\delta G_{t2,j}))|^2] && \text{(Young's lemma)}\\
&\lesssim 2\sum_j \int_0^L \beta_j^{-2}|(f_3, \operatorname{div}(G_{t2,j}))|^2 && \text{(linearization)}\\
&\lesssim 2\sum_j \int_0^L \beta_j^{-2}\lambda_j|(f_3, \lambda_j^{1/2}\phi_j)|^2 && (\operatorname{div} G_{t2,j} = -\lambda\phi_j)\\
&\lesssim 2\sum_j \int_0^L |(f_3, \lambda_j^{1/2}\phi_j)|^2 && (\beta_j^{-2}\lambda_j \approx O(1))\\
&= 2\|f_3\|^2 .
\end{aligned}$$

Estimation of $\boldsymbol{\eta}_j$ From the first equation in system (5.5), we get

$$\eta = \frac{1}{\theta}(i\omega\beta' + s_1) \quad \Rightarrow \quad \|\eta\|^2_{L^2(I)} \lesssim \frac{1}{\lambda^2}\|\tau'\|^2_{L^2(I)} + \frac{1}{\lambda^2}\|s_1\|^2_{L^2(I)}$$

as $|s_1| \approx \lambda$. In order to estimate curl H_t, we need a more demanding estimate for

$$\sum_j \lambda_j\|\eta_j\|^2_{L^2(I)} \lesssim \sum_j \lambda_j^{-1}\|\tau_j'\|^2_{L^2(I)} + \lambda_j^{-1}\|s_{1,j}\|^2_{L^2(I)} .$$

In the estimation of the derivatives τ_j', we lose a factor of β_j^2 but we gain it back with the factor λ_j^{-1} above. We also need the additional factor when estimating the second term. The details are as follows.

Term 1 $(e_z \times f_t, F_{t2,j})$ contributing to s_1. This term in painless. We do not need the additional factor $\lambda_j^{-1} \approx \beta_j^{-2}$.

$$\begin{aligned}
&\sum_j \int_0^L |(e_z \times f_t, F_{t2,j} + \delta F_{t2,j})|^2 \\
&\quad \lesssim 2 \sum_j \int_0^L [|(e_z \times f_t, F_{t2,j})|^2 + |(e_z \times f_t, \delta F_{t2,j})|^2] \text{ (Young's inequality)} \\
&\quad \lesssim 2 \sum_j \int_0^L |(e_z \times f_t, F_{t2,j})|^2 && \text{(linearization)} \\
&\quad \leq 2 \int_0^L \|e_z \times f_t\|^2 \, dz = 2 \int_0^L \|f_t\|^2 \, dz \,.
\end{aligned}$$

Term 2 $(\frac{1}{\epsilon} g_3, \operatorname{div} G_{t2,j})$ contributing to s_1. The presence of the additional factor is now essential.

$$\begin{aligned}
&\sum_j \beta_j^{-2} \int_0^L |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j} + \delta F_{t2,j}))|^2 \\
&\quad \leq 2 \sum_j \int_0^L \beta_j^{-2} [|(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j}))|^2 + |(\frac{1}{\epsilon} g_3, \operatorname{div}(\delta F_{t2,j}))|^2] \text{ (Young's lemma)} \\
&\quad \lesssim 2 \sum_j \beta_j^{-2} \int_0^L |(\frac{1}{\epsilon} g_3, \operatorname{div}(F_{t2,j}))|^2 && \text{(linearization)} \\
&\quad \lesssim 2 \sum_j \beta_j^{-2} \lambda_j \int_0^L |(\frac{1}{\epsilon} g_3, \lambda_j^{\frac{1}{2}} \phi_j))|^2 && (\operatorname{div} F_{t2,j} = -\lambda_j \phi_j) \\
&\quad \lesssim 2 \sum_j \int_0^L |(\frac{1}{\epsilon} g_3, \lambda_j^{\frac{1}{2}} \phi_j))|^2 && (\beta_j^{-2} \lambda_j \lesssim 1) \\
&\quad \leq 2 \|\frac{1}{\epsilon} g_3\|^2 \lesssim \|g_3\|^2 \,.
\end{aligned}$$

5.4.1 Final result

We arrive at our final result.

Theorem 2 *Assume that $\epsilon = 1 + \delta\epsilon$ where the perturbation $\delta\epsilon$ is sufficiently*[6] *small in L^∞-norm, and that it vanishes near the boundary.*[7] *Under the technical assumption that the eigenvalues of the homogeneous waveguide are simple, there exists then a constant $C > 0$ such that under the assumption of validity of the performed linearizations,*

$$\|E\|^2 + \|H\|^2 \leq C L^2 \left(\|\nabla \times E - i\omega H\|^2 + \|\nabla \times H + i\omega\epsilon E\|^2 \right)$$

for all (E, H) from the domain of the operator.

Remark 3 We have proved the theorem under the simplifying assumption of distinct (simple) eigenvalues β_i^2 of the homogeneous waveguide problem, see Assumption A in Section 3.1. Extending the proof to the case of multiple eigenvalues requires techniques discussed in Appendix 1. We believe (conjecture) that this is possible using the presented techniques although a presentation of it would

[6] To justify the perturbation analysis.

[7] See Assumption D in Appendix 1

add additional complexity to the already technical material. For a start, we mention that, in the case of equal Dirichlet and Neumann Laplace eigenvalues, we cannot exclude the possibility of the perturbed Maxwell eigenvalues being complex.

6 Conclusions

We have extended the stability analysis for homogeneous electromagnetic waveguides from [5] to the case of a non-homogeneous waveguide with a perturbed dielectric constant $\epsilon = 1 + \delta\epsilon$. The analysis was done using the classical (formal) perturbation theory for eigenproblems involving a self-adjoint operator under the assumption of 'smallness' of perturbation the $\delta\epsilon$ but with no assumptions on its derivatives. In particular, the results hold for discontinuous perturbations $\delta\epsilon$ and are therefore applicable to the case of step-index fibers.

Appendix 1. Perturbation analysis continued

In this section, we provide additional results obtained from the perturbation analysis. The results below require the perturbation analysis for the adjoint F and G problems that follow the same lines as for the E and H problems. We skip the details and present only the final results that were used in Section 4.

Computation of scaling coefficients c, d, θ, ν

First family of eigenvectors

We first investigate perturbations of E, F eigenvectors $\nabla \times \psi_i$ and the corresponding H, G eigenvectors $\nabla\psi_i$.

Perturbations of coefficients c, d Let $\nabla \times \psi + \delta E$ and $\nabla\psi + \delta H$ be the perturbed E and H eigenvectors corresponding to a perturbed eigenvalue $\beta + \delta\beta$. Here ψ is a Neumann eigenvector of the Laplacian, $-\Delta\psi = \mu\psi$, and $\beta^2 = \omega^2 - \mu$. The perturbed coefficient $c + \delta c$ is defined by the relation:

$$(c + \delta c)(\nabla\psi + \delta H) = \frac{\beta - \delta\beta}{\omega\beta^2}\left[-\nabla\,\mathrm{curl}(\nabla \times \psi + \delta E) + \omega^2 e_z \times (1 + \delta\epsilon)(\nabla \times \psi + \delta E)\right].$$

We first compute the value of c. Testing with $\nabla\psi$, we get,

$$c = \frac{1}{\omega\beta}([-\nabla \underbrace{(-\Delta\psi)}_{=\mu\psi} + \omega^2 \underbrace{(e_z \times (\nabla \times \psi)}_{=\nabla\psi}], \nabla\psi) = \frac{1}{\omega\beta}\underbrace{[-\mu + \omega^2]}_{=\beta^2} = \frac{\beta}{\omega}.$$

Linearizing both sides, and testing with $\nabla\psi$, we get,

$$\begin{aligned}\delta c \underbrace{(\nabla\psi, \nabla\psi)}_{=1} + c \underbrace{(\delta H, \nabla\psi)}_{=0} = &-\tfrac{\delta\beta}{\omega\beta^2} \underbrace{(-\nabla\,\mathrm{curl}\,\nabla \times \psi + \omega^2\nabla\psi, \nabla\psi)}_{=\beta^2} \\ &+ \tfrac{1}{\omega\beta} \underbrace{(-\nabla\,\mathrm{curl}\,\delta E + \omega^2 e_z \times \delta E, \nabla\psi)}_{=0} \\ &+ \tfrac{1}{\omega\beta}\omega^2 \underbrace{(\delta\epsilon\, e_z \times (\nabla \times \psi), \nabla\psi)}_{=(\delta\epsilon\nabla\psi, \nabla\psi)}.\end{aligned}$$

In the end,

$$\delta c = -\frac{\delta\beta}{\omega} + \frac{\omega}{\beta}(\delta\epsilon\nabla\psi, \nabla\psi)\,.$$

Similarly, perturbation δd is defined by the relation:

$$(d+\delta d)(\nabla\psi+\delta G) \approx \frac{\beta-\delta\beta}{\omega\beta^2}\Big[-\nabla\times((1-\delta\epsilon)\,\mathrm{div}(\nabla\times\psi+\delta F))+\omega^2(e_z\times(\nabla\times\psi+\delta F))\Big]\,.$$

This yields:

$$d = \frac{\omega}{\beta} \quad \text{and} \quad \delta d = -\delta\beta\frac{\omega}{\beta^2}\,.$$

Perturbations of coefficients entering the decoupled system of equations We are now ready to compute the perturbation of coefficient

$$\theta = -\frac{\omega\beta}{2c}[1+cd] = -\frac{\omega\beta}{2}[\frac{1}{c}+d] = -\omega^2\,.$$

We have:

$$\begin{aligned}\delta\theta &= -\frac{\omega}{2}[\frac{1}{c}+d]\,\delta\beta + \frac{\omega\beta}{2}\frac{1}{c^2}\,\delta c - \frac{\omega\beta}{2}\,\delta d\\ &= -\frac{\omega^2}{\beta}\delta\beta + \frac{\omega^3}{2\beta}[-\frac{\delta\beta}{\omega}+\frac{\omega}{\beta}(\delta\epsilon\nabla\psi,\nabla\psi)] + \frac{\omega^2}{2\beta}\delta\beta\\ &= -\frac{\omega^2}{\beta}\delta\beta + \frac{\omega^4}{2\beta^2}(\delta\epsilon\nabla\psi,\nabla\psi)\,.\end{aligned}$$

Finally, using formula (3.9) for $\delta\beta^2$ and utilizing $\delta\beta^2 = 2\beta\,\delta\beta$, we obtain,

$$\delta\theta = -\frac{\omega^4}{2\beta^2}(\delta\epsilon\nabla\times\psi, \nabla\times\psi) + \frac{\omega^4}{2\beta^2}(\delta\epsilon\nabla\psi,\nabla\psi) = 0\,.$$

We proceed with the coefficient

$$\nu = -\frac{\omega\beta}{2d}[1+cd] = -\frac{\omega\beta}{2}[\frac{1}{d}+c] = -\beta^2\,.$$

We have:

$$\begin{aligned}\delta\nu &= -\frac{\omega}{2}[\frac{1}{d}+c]\,\delta\beta + \frac{\omega\beta}{2}\frac{1}{d^2}\,\delta d - \frac{\omega\beta}{2}\,\delta c\\ &= -\beta\,\delta\beta - \frac{\beta}{2}\delta\beta - \frac{\omega\beta}{2}[-\frac{\delta\beta}{\omega}+\frac{\omega}{\beta}(\delta\epsilon\nabla\psi,\nabla\psi)]\\ &= -\beta\,\delta\beta - \frac{\omega^2}{2}(\delta\epsilon\nabla\psi,\nabla\psi)\,.\end{aligned}$$

Using again formula (3.9) for $\delta\beta^2$, we obtain,

$$\delta\nu = -\frac{\omega^2}{2}(\delta\epsilon\nabla\times\psi,\nabla\times\psi) - \frac{\omega^2}{2}(\delta\epsilon\nabla\psi,\nabla\psi) = -\omega^2(\delta\epsilon\nabla\psi,\nabla\psi)\,.$$

Second family of eigenvectors

Next, we record the results for similar computations for perturbations of the second family of E, F eigenvectors: $\nabla \phi_j$ and the corresponding H, G eigenvectors $\nabla \times \phi_j$.

$$\begin{aligned} c &= -\frac{\omega}{\beta} \\ d &= -\frac{\beta}{\omega} \\ \theta = -\frac{\omega\beta}{2c}[1+cd] &= \beta^2 \\ \nu = -\frac{\omega\beta}{2d}[1+cd] &= \omega^2 . \end{aligned}$$

The perturbations are as follows:

$$\begin{aligned} \delta c &= \frac{\omega}{\beta^2}\delta\beta - \frac{\omega}{\beta}(\delta\epsilon\, \nabla\phi, \nabla\phi) \\ \delta d &= \frac{\delta\beta}{\omega} - \frac{\lambda^2}{\omega\beta}(\delta\epsilon\, \phi, \phi) \end{aligned}$$

and,

$$\begin{aligned} \delta\theta &= \beta\,\delta\beta + \frac{\beta^3}{2\omega}\delta c - \frac{\omega\beta}{2}\delta d \\ &= \beta\,\delta\beta - \frac{\beta^2}{2}(\delta\epsilon\, \nabla\times\phi, \nabla\times\phi) + \frac{\lambda^2}{2}(\delta\epsilon\, \phi, \phi) \\ &= \lambda^2(\delta\epsilon\, \phi, \phi) \\ \delta\nu &= \frac{\omega^2}{\beta}\delta\beta + \frac{\omega^3}{2\beta}\delta d - \frac{\omega\beta}{2}\delta c \\ &= \frac{\omega^2}{\beta}\delta\beta - \frac{\omega^2\lambda^2}{2\beta^2}(\delta\epsilon\, \phi, \phi) + \frac{\omega^2}{2}(\delta\epsilon\, \nabla\phi, \nabla\phi) \\ &= \omega^2(\delta\epsilon\, \nabla\phi, \nabla\phi) . \end{aligned}$$

The results for both families are summarized in Table 4.

Are the perturbed eigenvalues $\beta^2 + \delta\beta^2$ Real?

We recall the main results concerning the E eigenvalues for the homogeneous waveguide obtained in [5].

Lemma 5 *Let (λ_i, ϕ_i) and $(\mu_j \psi_j)$ denote the Dirichlet and Neumann eigenpairs of the Laplacian in domain D. The eigenvalues β_i^2 are classified into the following three families.*

(a) *$\beta^2 = \omega^2 - \mu_j$ with μ_j distinct from all λ_i. The corresponding eigenvectors are curls:*

$$E = \nabla \times \psi_j ,$$

with multiplicity of β^2 equal to the multiplicity of μ_j.

(a) *$\beta^2 = \omega^2 - \lambda_i$ with λ_i distinct from all μ_j. The corresponding eigenvectors are gradients:*

$$E = \nabla\phi_i ,$$

with multiplicity of β^2 equal to the multiplicity of λ_i.

(c) $\beta^2 = \omega^2 - \mu_j = \omega^2 - \lambda_i$ *for* $\mu_j = \lambda_i$*. The corresponding eigenvectors are linear combinations of curls and gradients:*

$$E = a\nabla \times \psi_j + b\nabla\phi_i\,, \quad a, b \in \mathbb{C}\,,$$

with multiplicity of β^2 equal to the sum of multiplicities of μ_j and λ_i.

Analogous results hold for the H eigenproblem (2.11).

In Section 3, under the assumption of distinct (simple) eigenvalues, we derived the following formula for the perturbation of E eigenvalues and eigenvectors:

$$(\beta_i^2 - \beta_k^2)(\delta E_i, E_k) + \delta\beta_i^2\delta_{ik} = \omega^2(\delta\epsilon E_i, E_k) - \big(\nabla(\delta\epsilon \operatorname{div} E_i), E_k\big) + \big(\nabla \operatorname{div}(\delta\epsilon E_i), E_k\big)\,. \quad \text{(A.1)}$$

For $k = i$, we obtain the formula for the perturbation $\delta\beta_i^2$:

$$\delta\beta_i^2 = \omega^2(\delta\epsilon E_i, E_i) - \big(\nabla(\delta\epsilon \operatorname{div} E_i), E_i\big) + \big(\nabla \operatorname{div}(\delta\epsilon E_i), E_i\big)\,. \quad \text{(A.2)}$$

The case of multiple eigenvalues will be discussed momentarily. We investigate now whether the perturbed eigenvalues remain real. The first term on the right-hand side of (A.2) is always real. The second term is real as well as,

$$-\big(\nabla(\delta\epsilon \operatorname{div} E_i), E_i\big) = \big(\delta\epsilon\ \operatorname{div} E_i, \operatorname{div} E_i\big)\,.$$

Assumption D: Perturbation $\delta\epsilon$ is zero near the boundary ∂D.

We use the assumption to rewrite the third term as:

$$\big(\nabla \operatorname{div}(\delta\epsilon E_i), E_i\big) = -\big(\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i\big)\,.$$

Consider now the three cases discussed in Lemma 5. In the first case, $E_i = \nabla \times \psi_i$, the term is zero. For the second case, $E_i = \nabla\phi_i$, the term is:

$$-\big(\operatorname{div}(\delta\epsilon E_i), \operatorname{div} E_i\big) = -\big(\operatorname{div}(\delta\epsilon\nabla\phi_i), -\lambda_i\phi_i\big) = \lambda_i\big(\operatorname{div}(\delta\epsilon\nabla\phi_i), \phi_i\big) = -\lambda_i(\delta\epsilon\nabla\phi_i, \nabla\phi_i)$$

which is real as well. This concludes the analysis for the case when Dirichlet and Neumann eigenvalues of the Laplace operator are distinct.

Multiple eigenvalues

The third case is the most difficult to analyze as we are dealing with a multiple eigenvalues. The standard perturbation theory does not cover the case. Formula (A.2) is invalid for the case of a multiple eigenvalue because the right-hand side may depend upon the choice of an eigenvector from the eigenspace. Additionally, formula (A.1) does not allow to compute components of δE_i corresponding to other eigenvectors from the same eigenspace. Instead, we have to restrict the original perturbed operator to the eigenspace corresponding to the multiple eigenvalues, and consider the perturbed eigenvalue problem directly. Let us assume for simplicity that both $\lambda = \mu$ are simple eigenvalues, i.e., we are dealing with a double eigenvalue $\beta^2 = \omega^2 - \lambda = \omega^2 - \mu$. Recall the linearized operator:

$$\begin{aligned} A_\epsilon E &:= \nabla \times \operatorname{curl} E - \omega^2\epsilon E - \nabla\big(\tfrac{1}{\epsilon} \operatorname{div}(\epsilon E)\big) \\ &\approx \nabla \times \operatorname{curl} E - \omega^2(1 + \delta\epsilon)E - \nabla\big((1 - \delta\epsilon) \operatorname{div}((1 + \delta\epsilon)E)\big) \\ &\approx \underbrace{\nabla \times \operatorname{curl} E - \omega^2 E - \nabla(\operatorname{div} E)}_{=AE} + \underbrace{\nabla(\delta\epsilon \operatorname{div} E) - \nabla(\operatorname{div}(\delta\epsilon)E) - \omega^2\delta\epsilon E}_{\delta A\, E}\,. \end{aligned}$$

Let ψ, ϕ be Neumann and Dirichlet Laplace eigenvectors corresponding to a common eigenvalue $\mu = \lambda$, and let $E_1 = \nabla \times \psi$ and $E_2 = \nabla\phi$ be the two eigenvectors spanning the two-dimensional eigenspace corresponding to eigenvalue $\beta^2 = \omega^2 - \lambda$ of operator A. We have:

$$\begin{aligned}
\delta A\, E_1 &= -\nabla\big(\operatorname{div}(\delta\epsilon E_1)\big) - \omega^2\delta\epsilon E_1 \\
\delta A\, E_2 &= \underbrace{(\nabla(\delta\epsilon \operatorname{div}\nabla\phi)}_{=-\lambda\nabla(\delta\epsilon\phi)} - \big(\nabla \operatorname{div}(\delta\epsilon\nabla\phi)\big) - \omega^2\delta\epsilon\nabla\phi\,.
\end{aligned}$$

The perturbed operator restricted to the eigenspace in terms of its spectral components looks as follows.

$$\begin{pmatrix} (\delta A\, E_1, E_1) & (\delta A\, E_1, E_2) \\ (\delta A\, E_2, E_1) & (\delta A\, E_2, E_2) \end{pmatrix} = \begin{pmatrix} -\omega^2 \underbrace{(\delta\epsilon\nabla\times\psi, \nabla\times\psi)}_{=:a} & (\lambda-\omega^2)\underbrace{(\delta\epsilon\nabla\times\psi, \nabla\phi)}_{=:c} \\ -\omega^2\underbrace{(\delta\epsilon\nabla\times\psi, \nabla\phi)}_{=:c} & (\lambda-\omega^2)\underbrace{(\delta\epsilon\nabla\phi, \nabla\phi)}_{=:b} - \lambda^2\underbrace{(\delta\epsilon\phi, \phi)}_{=:d} \end{pmatrix}$$

Let further simplify the matrix notation to

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The characteristic equation for perturbed eigenvalues $\delta\lambda$ reads as follows.

$$(\delta\beta^2)^2 - (\delta\beta)(A + D) + (AD - BC) = 0$$

with the discriminant

$$\Delta = (A + D)^2 - 4(AD - BC) = (A - D)^2 + 4BC\,.$$

This gives:

$$\begin{aligned}
\Delta &= (-\omega^2 a - (\lambda-\omega^2)b + \lambda^2 d)^2 - 4\omega^2(\lambda-\omega^2)c^2 \\
&= (\omega^2 a + (\lambda-\omega^2)b - \lambda^2 d)^2 - 4\omega^2(\lambda-\omega^2)c^2\,.
\end{aligned}$$

Can discriminant Δ be negative ? It is certainly positive for $\lambda \le \omega^2$. In other words, the propagating modes remain propagating or become purely evanescent. However, for $\lambda > \omega^2$, it is difficult to exclude the possibility of the discriminant becoming negative.[8]

The analysis of multiple eigenvalues can now easily be extended to the case of multiple Neumann and Dirichlet eigenvalues for the Laplace operator. For simplicity, let us consider only the case of double eigenvalues, relevant for the cylindrical waveguide and the step-index fiber. Recall again the formula for the perturbed operator,

$$(\delta A\, E, F) = -(\delta\epsilon \operatorname{div} E, \operatorname{div} F) + \big(\operatorname{div}(\delta\epsilon E), \operatorname{div} F\big) - \omega^2(\delta\epsilon E, F)\,.$$

[8] Among other things, we tried to compute the derivative $d\Delta/d\lambda$ at $\lambda = \omega^2$, but we could not show that it must be positive.

In the case of multiple Neumann eigenvalue μ, $E_1 = \nabla \times \psi_1$, $E_2 = \nabla \times \psi_2$, and we obtain

$$\begin{pmatrix} (\delta A\, E_1, E_1) & (\delta A\, E_1, E_2) \\ (\delta A\, E_2, E_1) & (\delta A\, E_2, E_2) \end{pmatrix} = -\omega^2 \begin{pmatrix} \underbrace{(\delta\epsilon \nabla \times \psi_1, \nabla \times \psi_1)}_{=:A} & \underbrace{(\delta\epsilon \nabla \times \psi_1, \nabla \times \psi_2)}_{=:B} \\ \underbrace{(\delta\epsilon \nabla \times \psi_2, \nabla \times \psi_1)}_{=:B} & \underbrace{(\delta\epsilon \nabla \times \psi_2, \nabla \times \psi_2)}_{=:D} \end{pmatrix}$$

Upon inspection, we see that the discriminant of the characteristic equation is always positive.

$$\delta = (A + D)^2 - 4(AD - B^2) = (A - D)^2 + 4B^2 \,.$$

For the step-index fiber, $B = 0$, and $A = D$. The perturbed eigenvalue remains a double eigenvalue.

In the case of multiple Dirichlet eigenvalue λ, $E_1 = \nabla\phi_1$, $E_2 = \nabla\phi_2$, and we obtain

$$(\delta A\, E_i, E_j) = (\lambda - \omega^2)(\delta\epsilon \nabla\phi_i, \nabla\phi_j) - \lambda^2(\delta\epsilon\phi_i, \phi_j) \quad i, j = 1, 2 \,.$$

For the step-index fiber, the off-diagonal terms are zero, and the diagonal terms are equal. The perturbed eigenvalue remains double. One can show in a similar way that, in the general case, perturbed eigenvalues $\delta\beta^2$ remain real.

Lemma 6 *Assume that the perturbation $\delta\epsilon$ vanishes near the boundary ∂D. In the first two cases discussed in Lemma 5 and in the third case for $\lambda \leq \omega^2$, the perturbed eigenvalues $\beta_i^2 + \delta\beta_i^2$ are always real. In the third case, for $\lambda > \omega^2$, they may be complex.*

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Declarations

Competing interests The authors declare no competing interests.

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