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THE BEST CONSTANT FOR L^∞-TYPE GAGLIARDO-NIRENBERG INEQUALITIES

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This paper is dedicated to Bob Pego in respect for his many contributions to mathematical analysis and science, and for many years of friendship.

Abstract. In this paper we derive the best constant for the following L^∞-type Gagliardo-Nirenberg interpolation inequality

$$\|u\|_{L^\infty} \le C_{q,\infty,p}\|u\|_{L^{q+1}}^{1-\theta}\|\nabla u\|_{L^p}^{\theta}, \quad \theta = \frac{pd}{dp+(p-d)(q+1)},$$

where parameters q and p satisfy the conditions $p > d \ge 1$, $q \ge 0$. The best constant $C_{q,\infty,p}$ is given by

$$C_{q,\infty,p} = \theta^{-\frac{\theta}{p}}(1-\theta)^{\frac{\theta}{p}} M_c^{-\frac{\theta}{d}}, \quad M_c := \int_{\mathbb{R}^d} u_{c,\infty}^{q+1} dx,$$

where $u_{c,\infty}$ is the unique radial non-increasing solution to a generalized Lane-Emden equation. The case of equality holds when $u = Au_{c,\infty}(\lambda(x-x_0))$ for any real numbers A, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$. In fact, the generalized Lane-Emden equation in $\mathbb{R}^d$ contains a delta function as a source and it is a Thomas-Fermi type equation. For $q = 0$ or $d = 1$, $u_{c,\infty}$ have closed form solutions expressed in terms of the incomplete Beta functions. Moreover, we show that $u_{c,m} \to u_{c,\infty}$ and $C_{q,m,p} \to C_{q,\infty,p}$ as $m \to +\infty$ for $d = 1$, where $u_{c,m}$ and $C_{q,m,p}$ are the function achieving equality and the best constant of L^m-type Gagliardo-Nirenberg interpolation inequality, respectively.

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1. Introduction. Research on functional inequalities is an important topic in the Functional Analysis. In some circumstances one is interested in the exact value of the smallest admissible constant in some functional inequalities. Possible motivations for this can be described from the three respects: (i) it provides some geometrical insights (a sharp version of some functional inequality is equivalent to the Euclidean isoperimetric inequality [10]); (ii) it is helpful for the computation of the ground-state energy in a physical model; (iii) it can be used to determine sharp conditions on initial data to distinguish between global existence and finite time blow-up for some partial differential equations with competition effects from some biological or physical systems, cf. [2, 3, 5, 8, 9, 19, 33, 34].

In 1938, Sobolev [30] proved that there is a constant $C_{d,p} > 0$ such that for $d \geq 3$, $1 \leq p < d$, any function $u \in L^{\frac{pd}{d-p}}(\mathbb{R}^d)$ with $\nabla u \in L^p(\mathbb{R}^d)$, it holds that

$$\|u\|_{L^{\frac{pd}{d-p}}} \leq C_{d,p} \|\nabla u\|_{L^p}. \tag{1.1}$$

The best constant $C_{d,p}$ in (1.1) is established by Aubin and Talenti [1, 31]. Together with the interpolation inequality, it becomes the well-known Gagliardo-Nirenberg (G-N) inequality for the case $p < d$. A general G-N inequality is given by the following form (cf. [21, pp. 176, (2.3.50)] and [16, 24])

$$\|u\|_{L^{m+1}} \leq C_{q,m,p} \|u\|_{L^{q+1}}^{1-\theta} \|\nabla u\|_{L^p}^{\theta}, \quad \theta = \frac{pd(m-q)}{(m+1)[dp + (p-d)(q+1)]}, \tag{1.2}$$

where $C_{q,m,p} > 0$ is a constant, and the parameters d, q, m and p belong to the following two ranges:

(i) one range is

$$p > d, \quad q \geq 0 \text{ and } m = \infty. \tag{1.3}$$

We refer to this case as the L^∞-type G-N inequality.

(ii) the other range is

$$p \geq 1, \quad 0 \leq q < m < \sigma, \tag{1.4}$$

where σ is defined by

$$\sigma := \begin{cases} \frac{(p-1)d+p}{d-p} & \text{if } p < d, \\ \infty & \text{if } p \geq d. \end{cases} \tag{1.5}$$

This case is referred to as the L^m-type G-N inequality. For the case $m = \sigma$, this G-N inequality reduces to the Sobolev inequality (1.1) and for $m = q$, it is a trivial case.

For some special parameters d, q, m and p in (1.4) and (1.5), the best constant of the G-N inequality has been derived in terms of some closed formulas and studied widely in the literatures [6, 7, 10–13, 20, 22, 23, 33]. For $m = \infty$, the best constant $C_{q,\infty,p}$ in the inequality (1.2) was obtained in [22] only for $d = 1$. However, the best constant $C_{q,\infty,p}$ in the inequality (1.2) is not yet obtained for general parameters in (1.3) with $d \geq 2$. The goal of this paper is to derive the best constant $C_{q,\infty,p}$ of the L^∞-type G-N inequality (1.2).

For parameters in the range (1.3), the inequality (1.2) can be written as the following form

$$\|u\|_{L^\infty} \leq C_{q,\infty,p}\|\nabla u\|_{L^p}^{\theta}\|u\|_{L^{q+1}}^{1-\theta}, \quad 0<\theta=\frac{pd}{pd+(q+1)(p-d)}<1. \tag{1.6}$$

Following a standard variational method, a minimizing problem is established in the solution space

$$Y=\left\{u|\ u\in L^{q+1}(\mathbb{R}^d), \quad \nabla u\in L^p(\mathbb{R}^d)\right\}\subset L^\infty(\mathbb{R}^d),$$

and we know that there is a positive constant α such that

$$\alpha=\inf_{u\in Y}\frac{\|u\|_{L^{q+1}}^{1-\theta}\|\nabla u\|_{L^p}^{\theta}}{\|u\|_{L^\infty}}. \tag{1.7}$$

Define a functional $G: Y\to\mathbb{R}$

$$u\mapsto G(u):=\|u\|_{L^{q+1}}^{1-\theta}\|\nabla u\|_{L^p}^{\theta}. \tag{1.8}$$

The minimizing problem (1.7) is equivalent to the following minimizing problem

$$\alpha=\inf_{u\in Y,\ \|u\|_{L^\infty}=1} G(u). \tag{1.9}$$

Thanks to the rearrangement technique (see [18, Chapter 3])

$$\|h^*\|_{L^p}=\|h\|_{L^p}, \quad 1\leq p\leq\infty, \tag{1.10}$$

where h^* is the rearrangement function of h, and the Pólya-Szegő inequality [4, 27],

$$\|\nabla h^*\|_{L^p}\leq\|\nabla h\|_{L^p}, \quad 1\leq p\leq\infty, \tag{1.11}$$

we know that the minimizing problem (1.9) is equivalent (The proof of this equivalence is standard, cf. [19, Lemma 2.1]) to the following minimizing problem

$$\alpha=\inf_{u\in Y^*_{rad}} G(u), \tag{1.12}$$

where Y^*_{rad} is a non-negative radial symmetric decreasing function space

$$Y^*_{rad}=\left\{u(r)\geq 0\middle|\ \lim_{r\to 0^+}u(r)=1,\ u'(r)\leq 0\ a.e., \int_0^\infty\left(|u|^{q+1}+|u'|^p\right)r^{d-1}\,dr<\infty\right\}.$$

For any $u\in Y^*_{rad}$, we take always $u(0)=1$. And hence $Y^*_{rad}\subset C([0,\infty))$.

In Section 2, Propositions 1, 3 and 4 give the Euler-Lagrange equations for critical points of the functional $G(u)$ in Y^*_{rad}. For the case $q<p-1$, by constructing an auxiliary functional and connecting it with the contact angle by a Pohozaev type identity, we show that the Euler-Lagrange equation is a free boundary problem, i.e. for some finite $R>0$,

$$(|u'|^{p-2}u')'+\frac{d-1}{r}|u'|^{p-2}u'=u^q \quad \text{for } 0<r<R, \tag{1.13}$$

$$u(0)=1,\ u(R)=u'(R)=0. \tag{1.14}$$

For the case $q\geq p-1$, the Euler-Lagrange equation is given by the following form

$$(|u'|^{p-2}u')'+\frac{d-1}{r}|u'|^{p-2}u'=u^q \quad \text{for } 0<r<\infty, \tag{1.15}$$

$$u(0)=1,\ \lim_{r\to\infty}u(r)=0. \tag{1.16}$$

We provide the decay rate of positive solutions to the problem (1.15)–(1.16) at the far field in Proposition 5. Moreover, in Proposition 6, we show that solutions to Euler Lagrange equations are also critical points of $G(u)$ in Y^*_{rad}.

Since the L^∞-space is not reflexive space, the direct method of calculus variation cannot be used for the $L^\infty(\mathbb{R}^d)$ minimizing problem (1.12). Instead, in Section 3, we will prove existence and uniqueness of solutions to the above corresponding Euler-Lagrange equations, then show that the unique solution is a minimizer of $G(u)$ in Y^*_{rad}. Hence we have the following result

THEOREM 1.1. Assume that exponents $p > d \geq 1$ and $q \geq 0$, then there is a unique solution $u(r) \in C^1((0,\infty)) \cap Y^*_{rad}$ for the problems (1.13)–(1.14) and (1.15)–(1.16), respectively. Moreover, $u'(r) < 0$ in $\{r|u(r) > 0\}$.

Moreover, we show that the Euler-Lagrange equation is the following Thomas-Fermi type equation, which contains a delta function as a source (see Proposition 10). For $q < p-1$, the solution to (1.13)–(1.14) is equivalent to the non-negative radial solution of the Thomas-Fermi type equation with a free boundary, i.e. for some $R > 0$

$$\Delta_p u + a\delta(x) = u^q, \quad \text{in } \mathcal{D}'(B(0,R)), \tag{1.17}$$

$$a := \|\nabla u\|^p_{L^p} + \|u\|^{q+1}_{L^{q+1}}, \tag{1.18}$$

$$u(0) = 1, \; u(x) = \frac{\partial u}{\partial \vec{n}}(x) = 0, \; \text{for } |x| = R, \tag{1.19}$$

where $\delta(x)$ is a delta function and $\vec{n}$ is the unit outward normal vector to $\partial B(0,R)$. When $q \geq p-1$, the solution to the problem (1.15)–(1.16) is equivalent to the positive radial solution to the Thomas-Fermi type equation

$$\Delta_p u + a\delta(x) = u^q, \quad \text{in } \mathcal{D}'(\mathbb{R}^d), \tag{1.20}$$

$$u(0) = 1, \; \lim_{|x|\to\infty} u(x) = 0. \tag{1.21}$$

This delta function in the above Thomas-Fermi type equations gives rise to a singularity $\lim_{r\to 0^+} u'(r) = \infty$ (see (3.20)) in (1.13)–(1.14) and (1.15)–(1.16). To overcome this singularity, we construct an approximation sequence by solutions of exterior problems in the domain (r_i,∞), $r_i \to 0$, and provide some delicate estimates and new techniques to finish the proof of Theorem 1.1.

In Section 4, we will derive the best constant of L^∞-type G-N inequality and show closed form solutions for some special parameters d, p and q. The main result is given by Theorem 1.2.

THEOREM 1.2. Suppose $p > d \geq 1$, $q \geq 0$, $u \in L^{q+1}(\mathbb{R}^d)$ and $\nabla u \in L^p(\mathbb{R}^d)$. Then $u \in L^\infty(\mathbb{R}^d)$ and it satisfies the following inequality

$$\|u\|_{L^\infty} \leq C_{q,\infty,p}\|u\|^{1-\theta}_{L^{q+1}}\|\nabla u\|^\theta_{L^p}, \quad \theta = \frac{pd}{dp + (p-d)(q+1)}, \tag{1.22}$$

where the best constant

$$C_{q,\infty,p} = \theta^{-\frac{\theta}{p}}(1-\theta)^{\frac{\theta}{p}} M_c^{-\frac{\theta}{d}}, \; M_c = \int_{\mathbb{R}^n} |u_{c,\infty}|^{q+1} dx. \tag{1.23}$$

Here $u_{c,\infty}$ is the unique radial solution as described by the following two cases:

- if $q < p-1$, $u_{c,\infty}$ is the unique non-increasing solution of the free boundary problem (1.13)–(1.14) and $u(r) = 0$ for $r \geq R$.
- if $q \geq p-1$, $u_{c,\infty}$ is the unique positive solution to the problem (1.15)–(1.16).

Moreover, the case of equality holds if $u = Au_{c,\infty}(\lambda|x-x_0|)$ for any $\lambda > 0$, $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$.

For the case $p > d \geq 1$ and $q = 0$, a closed form solution $u_{c,\infty}$ can be expressed in terms of an incomplete Beta function, which is defined as

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt. \tag{1.24}$$

The best constant $C_{0,\infty,p}$ is given by

$$C_{0,\infty,p} = \left(\frac{p-d}{pd}\right)^{\frac{d}{pd+p-d}} M_c^{-\frac{p}{pd+p-d}}, \; M_c = \int_{\mathbb{R}^d} u_{c,\infty}dx.$$

see Proposition 11.

For $p > d = 1$ and $q \geq 0$, the free boundary problem (1.13)–(1.14) and the problem (1.15)–(1.16) have closed form solutions respectively (see Proposition 12). And using them we deduce the best constant

$$C_{q,\infty,p} = \left(\frac{2p}{p+(p-1)(q+1)}\right)^{-\frac{p}{p+(p-1)(q+1)}}. \tag{1.25}$$

Recall the L^m-type G-N inequality

$$\|u\|_{L^{m+1}} \leq C_{q,m,p}\|u\|_{L^{q+1}}^{1-\theta}\|\nabla u\|_{L^p}^{\theta}, \quad \theta = \frac{pd(m-q)}{(m+1)[d(p-q-1)+p(q+1)]} \tag{1.26}$$

with the best constant

$$C_{q,m,p} = \theta^{-\frac{\theta}{p}}(1-\theta)^{\frac{\theta}{p}-\frac{1}{m+1}}M_c^{-\frac{\theta}{d}}, \quad M_c = \int_{\mathbb{R}^d} u_{c,m}^{q+1}dx, \tag{1.27}$$

which can be found in [20]. Here parameters p, q, m satisfy (1.4) with some restrictive conditions $p > \max\{1, \frac{2d}{d+2}\}$ and $q < \sigma - 1$, $u_{c,m}$ is described by the following two cases:

(i) if $q < p-1$, there is a finite $R_m > 0$ such that $u_{c,m}$ is the unique decreasing solution of the following free boundary problem

$$(|u'|^{p-2}u')' + \frac{d-1}{r}|u'|^{p-2}u' + u^m = u^q \quad \text{for } 0 < r < R_m, \tag{1.28}$$

$$u'(0) = 0, \; u(R_m) = u'(R_m) = 0, \tag{1.29}$$

and $u(r) = 0$ for $r \geq R_m$.

(ii) if $q \geq p-1$, $u_{c,m}$ is the unique positive decreasing solution to the following problem

$$(|u'|^{p-2}u')' + \frac{d-1}{r}|u'|^{p-2}u' + u^m = u^q \quad \text{for } 0 < r < \infty, \tag{1.30}$$

$$u'(0) = 0, \; \lim_{r\to\infty} u(r) = 0. \tag{1.31}$$

Contrasting to L^∞-type, there is no singularity at the origin in the Euler-Lagrange equations (1.28)–(1.29) and (1.30)–(1.31). In Section 5, we will show the singular limit behavior in the best constant and the solution to the Euler-Lagrange equations as $m \to \infty$ for $d = 1$, which indicates the connection between the L^m-type G-N inequality (1.26) and L^∞-type G-N inequality (1.22). The result is given by Theorem 1.3.

THEOREM 1.3. Let $u_{c,m}$ and $u_{c,\infty}$ be respectively the unique non-increasing radial solutions of the problem (1.28)–(1.29) and the problem (1.13)–(1.14)(or the problem (1.30)–(1.31) and the problem (1.15)–(1.16)) in the one-dimensional case. Then the following facts hold

$$u_{c,m}(r) \to u_{c,\infty}(r) \quad \text{for any } r > 0, \quad C_{q,m,p} \to C_{q,\infty,p}, \quad \text{as } m \to \infty.$$

Moreover, for $q < p - 1$, let R_m and R_∞ be the free boundaries for $u_{c,m}$ and $u_{c,\infty}$ respectively. Then we have $R_m \to R_\infty$ as $m \to \infty$, where R_∞ and R_m defined in [20, formula (3.2)] are respectively given by

$$R_\infty = \left(\frac{p}{(p-1)(q+1)}\right)^{-1/p} \frac{p}{p-q-1}, \text{ and}$$
$$R_m = \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{(m+1)^{\frac{p-q-1}{p(m-q)}}(q+1)^{\frac{1}{p}-\frac{p-q-1}{p(m-q)}}}{m-q} \mathcal{B}\left(1-\frac{1}{p}, \frac{p-(q+1)}{p(m-q)}\right). \tag{1.32}$$

For simplicity, we will use the same function $u = u(x)$ and $u = u(r)$ to represent a radial solution with $u(x) = u(|x|)$ in this paper. It should be clear according to the content of the text.

2. Euler-Lagrange equations for L^∞-type G-N inequalities. In the beginning of this section, we derive the Euler-Lagrange equations for critical points of the functional $G(u)$ in Y^*_{rad}.

PROPOSITION 1. Assume that $\bar{u}(r) \in Y^*_{rad}$ is a critical point of $G(u)$, then there is $\lambda_0 > 0$ such that the re-scaling function $u(r) = \bar{u}(\lambda_0 r)$ satisfies the following equation in the classical sense

$$(|u'|^{p-2}u')' + \frac{d-1}{r}|u'|^{p-2}u' = u^q, \ 0 < r < R, \tag{2.1}$$

and the boundary conditions

$$\lim_{r\to 0^+} u(r) = 1, \quad \lim_{r \to R^-} u(r) = 0, \tag{2.2}$$

for some $0 < R \le +\infty$.

Proof. **Step 1** (Re-scaling and admissible variation). Let $u_1(r) := \bar{u}(\lambda_1 r)$, $\lambda_1 > 0$ be a re-scaling parameter to be determined by (2.4). Noticing the scaling invariant of $G(u)$ for $u_1(r) = \bar{u}(\lambda_1 r)$, we have

$$G(u_1) = G(\bar{u}). \tag{2.3}$$

Hence if $\bar{u}(r) \in Y^*_{rad}$ is the critical point of $G(u)$, then u_1 is also a critical point ($\frac{\delta G(u_1)}{\delta u} = 0$), and by choosing λ_1, it holds that

$$\|u_1\|_{L^{q+1}} = 1, \quad \|\nabla u_1\|^p_{L^p} =: a_1. \tag{2.4}$$

Since $u_1 \in Y^*_{rad}$, we have that $u_1(r)$ is continuous in $[0, \infty)$.[1] Denote

$$R_1 := \inf\{r > 0 | u_1(r) = 0\} \in \mathbb{R}^+ \cup \{+\infty\}.$$

For any admissible variation $\phi \in C_0^\infty(0, R_1)$ at u_1, i.e., there is an $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, one has $u_1 + \varepsilon\phi \in Y^*_{rad}$. Then from a direction computation and using (2.4), we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} G(u_1 + \varepsilon\phi) = S_d \int_0^{R_1} \theta a_1^{\frac{\theta}{p}-1} \left(r^{d-1}|u_1'|^{p-2}u_1'\right)\phi'(r) + (1-\theta) a_1^{\frac{\theta}{p}} u_1^q r^{d-1}\phi(r)\, dr = 0.$$

This implies that u_1 satisfies the following generalized Lane-Emden equation in the distribution sense

$$-\theta\left(r^{d-1}|u_1'|^{p-2}u_1'\right)' + (1-\theta)a_1 r^{d-1}u_1^q = 0, \quad \text{in } (0, R_1), \tag{2.5}$$

$$\lim_{r\to 0^+} u_1(r) = 1, \qquad \lim_{r\to R_1^-} u_1(r) = 0, \tag{2.6}$$

where $0 < R_1 \leq +\infty$.

Step 2 (Normalization). We re-scale the function u_1 as $u(r) = u_1(\lambda r)$, where λ will be given in (2.7). From (2.3), we know that u is also a critical point of $G(u)$ in Y^*_{rad}. From (2.5) we deduce that u satisfies the following equation

$$-\theta\lambda^{-p}\left(r^{d-1}|u'|^{p-2}u'\right)' + (1-\theta)a_1 r^{d-1}u^q = 0, \quad 0 < r < \frac{R_1}{\lambda} =: R.$$

Taking

$$\lambda = \left(\frac{\theta}{(1-\theta)a_1}\right)^{1/p}, \quad \text{i.e. } \theta\lambda^{-p} = (1-\theta)a_1, \tag{2.7}$$

we have that u satisfies (2.1)–(2.2) in the distribution sense.

Step 3 (u satisfies (2.1)–(2.2) in the classical sense). The purpose of this step is to prove that solutions of (2.1)–(2.2) in the distribution sense are also classical solutions in any closed interval of $(0, R)$. We need only to show that equation (2.1) is uniformly elliptic, i.e., to prove $|u'(r)| \geq C$ for some $C > 0$ in any closed interval of $(0, R)$. That will be a consequence of the following claim.

Claim. If there is $0 < r_* \leq R$ such that $u'(r_*) = 0$, then $u(r_*) = 0$.

Proof of Claim. If not, then $u(r_*) > 0$. Noticing that $\lim_{r\to 0^+} u(r) = 1$ and $u'(r) \leq 0$, then for any fixed $r > r_*$, we have $\int_{r_*}^r s^{d-1}u^q(s)\, ds < \infty$. Moreover, by the continuity of $u(r)$ in $(0, \infty)$, we know that there is $r^* : r_* < r^* < \infty$ such that if $r \in [r_*, r^*)$, $u(r) > 0$. Hence integrating (2.1) from r_* to r^*, we deduce

$$(r^*)^{d-1}|u'(r^*)|^{p-2}u'(r^*) = \int_{r_*}^{r^*} s^{d-1}u^q(s) ds. \tag{2.8}$$

Notice that $(r^*)^{d-1}|u'(r^*)|^{p-2}u'(r^*) \leq 0$ due to $u'(r^*) \leq 0$ and the right side of the above equation is positive. That is a contradiction. This completes the proof of this claim.

Since $R := \inf\{r > 0 | u(r) = 0\} \in \mathbb{R}^+ \cup \{+\infty\}$, then we have $r_* = R$ by the claim. So, we have that $|u'(r)| > 0$ in $(0, R)$. Therefore, from regularity of solutions to the elliptic

[1] If $u_1 \in Y^*_{rad}$, we know that for any $0 < a < b < +\infty$, $u_1 \in W^{1,p}_{rad}([a, b])$. Hence $u_1(r)$ is continuous in $(0, \infty)$. The continuity at $x = 0$ is given by $\lim_{r\to 0^+} u_1(r) = 1$.

equation, we know that any weak solution of equation (2.5) in Y^*_{rad} is also a classical solution in any closed interval of $(0, R)$. Hence equation (2.1) holds in the classical sense. □

In the following, we show the three important results: (i) For the case $q < p-1$, all critical points of $G(u)$ in Y^*_{rad} satisfy the free boundary problem (1.13)–(1.14). Here we need to derive a Pohozaev type identity and use it to prove that the contact angle is zero. (ii) For the case $q \geq p-1$, we derive the Euler-Lagrange equations (1.15)–(1.16) for the critical points of $G(u)$ in Y^*_{rad}. Solutions to the Euler-Lagrange equations (1.15)–(1.16) are positive and have decay properties at the infinity. (iii) We show that the solution to the Euler-Lagrange equations in Y^*_{rad} is also a critical point of $G(u)$ up to a re-scaling.

2.1. *Case $q < p-1$: Compact support, zero contact angle and free boundary problem.* In this subsection, we prove that all critical points of $G(u)$ satisfy the free boundary problem (1.13)–(1.14). First, we show that a solution to (2.1)–(2.2) has a compact support in $[0, +\infty)$ (see Proposition 2). Next we show that the solution has a zero-contact-angle at the boundary of the compact support (see Lemma 2.1 and Lemma 2.2). Finally we use the zero-contact-angle result to derive a complete free boundary problem (1.13)–(1.14) (see Proposition 3).

Proposition 2. Assume that $\bar{u}(r) \in Y^*_{rad}$ is a critical point of $G(u)$. Let $u(r) = \bar{u}(\lambda_0 r)$ for some $\lambda_0 > 0$ satisfy (2.1)–(2.2). If $p > 1$, $0 \leq q < p-1$, then there is $R \in (0, \infty)$ such that $u(R) = 0$.

Proof. For a radial decreasing non-negative function $u \in Y^*_{rad}$, there only exist two cases: (i) there exists a finite R such that $u(R) = 0$; (ii) $u(r) > 0$ for all $r > 0$, and hence $u(r) \to 0$, $u'(r) \to 0$ as $r \to \infty$.

Inspired by the work [26, Theorem 5.1], using a contradiction method, we show that the second case cannot happen. Indeed, if (ii) holds, then $u > 0$ is a solution to the following problem

$$(|u'|^{p-2}u')' + \frac{d-1}{r}|u'|^{p-2}u' = u^q, \quad 0 < r < \infty, \tag{2.9}$$

$$\lim_{r \to 0^+} u(r) = 1, \quad \lim_{r \to \infty} u(r) = 0. \tag{2.10}$$

Multiplying u' to both sides of (2.9), we get

$$\frac{d}{dr}\left(\frac{p-1}{p}|u'(r)|^p - \frac{u^{q+1}(r)}{q+1}\right) + \frac{d-1}{r}|u'(r)|^p = 0. \tag{2.11}$$

Integrating (2.11) from r to $+\infty$ and utilizing the fact $u(r) \to 0$ and $u'(r) \to 0$ as $r \to \infty$, we have

$$\frac{p-1}{p}|u'(r)|^p - \frac{u^{q+1}(r)}{q+1} = \int_r^{+\infty} \frac{d-1}{s}|u'(s)|^p ds. \tag{2.12}$$

Hence from (2.12), it holds that

$$-u'(r) \geq \left(\frac{p}{(p-1)(q+1)}\right)^{1/p} u^{\frac{q+1}{p}}(r). \tag{2.13}$$

Using the method of separation of variable for (2.13) and integrating the resulting inequality from 0 to r, $r \in (0, +\infty)$, we obtain

$$\int_{u(r)}^{1} u^{-\frac{q+1}{p}} du \geq \left(\frac{p}{(p-1)(q+1)}\right)^{1/p} r \quad \text{for all } r > 0,$$

which gives

$$r \leq \left(\frac{p}{(p-1)(q+1)}\right)^{-1/p} \frac{p}{p-q-1}\left(1 - u^{\frac{p-q-1}{p}}(r)\right). \tag{2.14}$$

Noticing that $p > 1$, $q < p-1$, by (2.14) we have

$$r \leq \left(\frac{p}{(p-1)(q+1)}\right)^{-1/p} \frac{p}{p-q-1}. \tag{2.15}$$

Taking $r \to +\infty$, we obtain a contradiction from (2.15). Hence the second case cannot happen, i.e., there exists a finite R such that $u(R) = 0$.

□

Now we show that solutions to (2.1)–(2.2) have a zero contact angle at the boundary of the compact support by constructing an auxiliary energy functional

$$\mathcal{G}(u) := \frac{p-d}{p} \int_{\mathbb{R}^d} |\nabla u|^p dx - \frac{d}{q+1} \int_{\mathbb{R}^d} u^{q+1} dx. \tag{2.16}$$

Lemma 2.1. Let $\bar{u}(r) \in Y^*_{rad}$ be a critical point of $G(u)$. Then there is $\lambda_0 > 0$ such that the re-scaling function $u(r) = \bar{u}(\lambda_0 r)$ is a zero point of the energy functional $\mathcal{G}(u)$ defined in (2.16), i.e.,

$$\mathcal{G}(u) = 0. \tag{2.17}$$

Proof. From (2.4), the re-scaling function $u_1(r) = \bar{u}(\lambda_1 r)$, $\lambda_1 > 0$ satisfies

$$1 = \int_{\mathbb{R}^d} u_1^{q+1} dx, \quad a_1 = \int_{\mathbb{R}^d} |\nabla u_1|^p dx. \tag{2.18}$$

Let $u(r) = u_1(\lambda r)$, where λ is given by (2.7). Thus from (2.18) we deduce

$$\int_{\mathbb{R}^d} u^{q+1} dy = \frac{1}{\lambda^d}, \quad \int_{\mathbb{R}^d} |\nabla u|^p dy = a_1 \lambda^{p-d}.$$

Hence

$$\begin{aligned}\mathcal{G}(u) &= \frac{p-d}{p} \int_{\mathbb{R}^d} |\nabla u|^p dx - \frac{d}{q+1} \int_{\mathbb{R}^d} u^{q+1} dx \\ &= \frac{p-d}{p} a_1 \lambda^{p-d} - \frac{d}{q+1} \lambda^{-d}.\end{aligned}$$

Using (2.7) and the definition (1.8) of θ, we have

$$\mathcal{G}(u) = \frac{p-d}{p}\left(\frac{\theta}{1-\theta}\right) \lambda^{-p} \lambda^{p-d} - \frac{d}{q+1} \lambda^{-d} = 0,$$

i.e., (2.17) holds.

□

LEMMA 2.2. Let $u(r)$ be a solution to the problem (2.1)–(2.2) in Y^*_{rad}. Assume that $u(r)$ has a touchdown point R (i.e. $u(R) = 0$). Then the following relation between the energy functional defined by (2.16) and the contact angle holds

$$\mathcal{G}(u) = \frac{(p-1)S_d}{p} \lim_{r\to R^-} r^d |u'(r)|^p, \tag{2.19}$$

where S_d is the surface area of d-dimensional unit ball.

Proof. Now we prove (2.19) by using a similar idea to the proof of the Pohozaev identity. Introduce the energy function

$$H(r) := \frac{p-1}{p}|u'(r)|^p - \frac{u^{q+1}(r)}{q+1}. \tag{2.20}$$

Using (2.11), we have the following energy-dissipation relation

$$\frac{dH(r)}{dr} + \frac{d-1}{r}|u'(r)|^p = 0. \tag{2.21}$$

Multiplying r^d to (2.21) and integrating the resulting equation from r to R_0, for any fixed $0 < R_0 < R$, we obtain that

$$R_0^d H(R_0) - r^d H(r) - d\int_r^{R_0} s^{d-1}H(s)ds + (d-1)\int_r^{R_0} |u'(s)|^p s^{d-1} ds = 0.$$

By (2.20), the above equation can be written as the following form

$$\begin{aligned} r^d\left(\frac{p-1}{p}|u'(r)|^p - \frac{u^{q+1}(r)}{q+1}\right) =& R_0^d H(R_0) - \frac{p-d}{p}\int_r^{R_0} s^{d-1}|u'(s)|^p ds \\ &+ \frac{d}{q+1}\int_r^{R_0} s^{d-1}u^{q+1}(s)ds. \end{aligned} \tag{2.22}$$

Since $u(r) \in Y^*_{rad}$, we have that the limit of the right side of (2.22) exists as $r \to 0^+$. Hence taking the limit for both sides of (2.22), we have

$$\begin{aligned} \frac{p-1}{p}\lim_{r\to 0^+} r^d|u'(r)|^p =& R_0^d H(R_0) - \frac{p-d}{p}\int_0^{R_0} s^{d-1}|u'(s)|^p ds \\ &+ \frac{d}{q+1}\int_0^{R_0} s^{d-1}u^{q+1}(s)ds. \end{aligned} \tag{2.23}$$

Notice that $r^d|u'(r)|^p \geq 0$. Hence there is a constant $C \geq 0$ such that

$$\lim_{r\to 0^+} r^d|u'(r)|^p = C.$$

Now we claim $C = 0$. If $C > 0$, then there is $\delta > 0$ such that

$$r^d|u'(r)|^p \geq \frac{C}{2} \quad \text{for } 0 < r \leq \delta,$$

which means

$$r^{d-1}|u'(r)|^p \geq \frac{C}{2}r^{-1} \quad \text{for } 0 < r \leq \delta.$$

Integrating above inequality from 0 to δ, we deduce

$$\infty > \int_0^\delta s^{d-1}|u'(s)|^p ds \geq \frac{C}{2}\int_0^\delta r^{-1}dr = +\infty.$$

This is a contradiction. Hence it holds that

$$\lim_{r\to 0^+} r^d|u'(r)|^p = 0. \tag{2.24}$$

Therefore, using (2.20), (2.24) and taking the limit for (2.23) as $R_0 \to R^-$, we have

$$\begin{aligned}\frac{p-1}{p}\lim_{R_0\to R^-} R_0^d|u'(R_0)|^p &= \lim_{R_0\to R^-} R_0^d H(R_0)\\ &= \frac{p-d}{p}\int_0^R r^{d-1}|u'(r)|^p dr - \frac{d}{q+1}\int_0^R r^{d-1}u^{q+1}(r)dr\\ &= \frac{p-d}{p}\frac{1}{S_d}\int_{B_R(0)}|\nabla u|^p dx - \frac{d}{q+1}\frac{1}{S_d}\int_{B_R(0)} u^{q+1}dx = \frac{1}{S_d}\mathcal{G}(u).\end{aligned}$$

Hence (2.19) holds. □

Finally, we show that all critical points of $G(u)$ satisfy the free boundary problem (1.13)–(1.14) up to a re-scaling.

Proposition 3. Assume $p > 1$, $0 \le q < p-1$. Let $\bar u(r) \in Y^*_{rad}$ be a critical point of $G(u)$. Then there is $\lambda_0 > 0$ such that the re-scaling function $u(r) = \bar u(\lambda_0 r)$ satisfies the free boundary problem (1.13)–(1.14).

Proof. As a direct consequence of (2.19) and $\mathcal{G}(u) = 0$, one knows that $u'(R) = 0$. In the other words, the contact angle is zero. This case is the so-called complete wetting regime in Young's law [17]. □

2.2. *Case $q \ge p-1$: Positivity and decay property.* In this subsection, we show that solutions to (2.1)–(2.2) are positive (see Proposition 4). And decay properties of solutions to the problem (1.15)–(1.16) are proved in Proposition 5.

Proposition 4. Assume $p > 1$, $q \ge p-1$. Let $u(r)$ be a solution of (2.1)–(2.2). Then $u(r) > 0$ for any $0 < r < \infty$.

Proof. Now we only need to prove that $R = \infty$ for $p > 1$, $q \ge p-1$. If not, $R < \infty$. By Proposition 3, we have $u'(R) = 0$. Multiplying r^{d-1} to equation (2.1) and using $u'(r) \le 0$, we have

$$(r^{d-1}|u'(r)|^{p-1})' + r^{d-1}u^q(r) = 0,\ 0 < r < R.$$

We extend the function u to $u = 0$ for $r \ge R$. Let $\Omega_\varepsilon := \mathbb{R}^d \setminus \overline{B_\varepsilon(0)}, \forall \varepsilon > 0$ be a domain without the origin. For any $\phi(x) = \phi(|x|) \ge 0$ and $\phi \in C_c^\infty(\Omega_\varepsilon)$, it holds that

$$\int_\varepsilon^\infty (-\phi'(r)r^{d-1}|u'(r)|^{p-1} + \phi(r)r^{d-1}u^q(r))dr = 0.$$

And noticing $\nabla u \in L^p(\mathbb{R}^d)$, we have

$$\int_{\Omega_\varepsilon} (\nabla\phi\cdot\nabla u|\nabla u|^{p-2} + \phi u^q)dx = 0.$$

Hence we have for any $\bar R > R > 0$

$$\Delta_p u = u^q \quad \text{ in } \mathcal{D}(B_{\bar R}(0)\setminus\overline{B_\varepsilon(0)}).$$

Moreover, by Step 3 in the proof of Proposition 1 and $u'(R) = 0$, we know $u \in C^1(B_{\bar R}(0)\setminus \overline{B_\varepsilon(0)})$. Positivity of u in $B_{\bar R}(0)\setminus\overline{B_\varepsilon(0)}$ is a direct consequence of the Strong Maximum

Principle given by Pucci and Serrin [29, Theorem 1.1.1]. To prove this positivity, we only need to verify the necessary and sufficient condition for the Strong Maximum Principle: $f(s) > 0$ for $s \in (0, \delta)$ and $\int_{0^+} \frac{ds}{H^{-1}(F(s))} = \infty$ (in the same notations as that in [29], $f(s) = s^q$, $F(s) = \frac{s^{q+1}}{q+1}$, $H(s) = \frac{p-1}{p} s^p$). While the condition $\int_{0^+} \frac{ds}{H^{-1}(F(s))} = \infty$ holds if and only if $q \geq p - 1$.

Therefore, positivity in $B_{\bar{R}}(0) \setminus \overline{B_\varepsilon(0)}$ is a contradiction with $u \equiv 0$ in $B_{\bar{R}}(0) \setminus \overline{B_R(0)}$. □

PROPOSITION 5. Let $u(r)$ be a solution of the problem (1.15)–(1.16). Then $u(r)$ satisfies the following decay estimate

$$\lim_{r \to \infty} r^{d-1} |u'(r)|^{p-1} = 0. \tag{2.25}$$

Moreover, $u(r)$ satisfies the following decay rates

(i) for $q > p - 1$, it holds that

$$u(r) + r|u'(r)| \leq C_{p,q} r^{-\frac{p}{q+1-p}} \quad \text{for } r > 0; \tag{2.26}$$

(ii) for $q = p - 1$, it holds that

$$u(r) + |u'(r)| \leq C_p e^{-(p-1)^{-\frac{1}{p}} r} \quad \text{for } r > 0. \tag{2.27}$$

Proof. **Step 1.** We prove the decay estimate (2.25). Since the function $u(r)$ satisfies equation (1.15), hence we have

$$(r^{d-1} |u'(r)|^{p-1})' = -u^q r^{d-1} < 0 \quad \text{for any } r > 0.$$

So, $r^{d-1}|u'(r)|^{p-1}$ is decreasing in r. Notice that $r^{d-1}|u'(r)|^{p-1} \geq 0$. Hence there is a constant $C \geq 0$ such that

$$\lim_{r \to \infty} r^{d-1} |u'(r)|^{p-1} = C.$$

Now we claim $C = 0$. If $C > 0$, we have

$$r^{d-1} |u'(r)|^{p-1} \geq C \quad \text{for any } r > 0,$$

which means

$$-u'(r) \geq C r^{-\frac{d-1}{p-1}} \quad \text{for any } r > 0.$$

Integrating above inequality from r to ∞ for any $r > 0$ and using the fact $\lim_{r \to \infty} u(r) = 0$, we obtain

$$u(r) \geq C \frac{p-1}{p-d} r^{1-\frac{d-1}{p-1}} \Big|_r^\infty = +\infty. \tag{2.28}$$

This is a contradiction, i.e., (2.25) holds.

Step 2 (The decay rate of u). From (2.20) and (2.21), we have

$$\frac{d}{dr} |u'(r)|^p - \frac{p}{(p-1)(q+1)} \frac{d}{dr} u^{q+1}(r) + \frac{p(d-1)}{p-1} \frac{1}{r} |u'(r)|^p = 0. \tag{2.29}$$

Integrating (2.29) from r to ∞ and using $u(r)$, $u'(r) \to 0$ as $r \to \infty$, we obtain

$$|u'(r)|^p - \frac{p}{(p-1)(q+1)} u^{q+1}(r) = \frac{p(d-1)}{p-1} \int_r^\infty \frac{1}{s} |u'(s)|^p ds.$$

Thus

$$|u'(r)|^p - \frac{p}{p-1}\frac{u^{q+1}(r)}{q+1} \geq 0 \quad \text{for } r > 0,$$

i.e.,

$$-u'(r) \geq \left(\frac{p}{(p-1)(q+1)}\right)^{\frac{1}{p}} u^{\frac{q+1}{p}}(r).$$

Using the method of separation of variable for above formula and integrating the resulting inequality from 0 to r, $r \in (0, \infty)$, we deduce

$$u(r) \leq C_{p,q} r^{-\frac{p}{q+1-p}} \quad \text{for } r > 0,\ q > p-1; \tag{2.30}$$

$$u(r) \leq e^{-(p-1)^{-\frac{1}{p}} r} \quad \text{for } r > 0,\ q = p-1. \tag{2.31}$$

Step 3 (The refined decay rate of $u'(r)$). Again multiplying r^k, $k = \frac{p(d-1)}{p-1}$ on both sides of (2.29), it holds that

$$\frac{d}{dr}(r^k|u'(r)|^p) - \frac{p}{(p-1)(q+1)} r^k \frac{d}{dr} u^{q+1}(r) = 0.$$

The decay rate in (2.25) implies $\lim_{r\to\infty} r^k |u'(r)|^p = 0$. Hence integrating the above equality from r to ∞ gives

$$r^k|u'(r)|^p + \frac{p}{(p-1)(q+1)} \int_r^\infty s^k \frac{d}{ds} u^{q+1}(s) ds = 0.$$

Using (2.30), we can directly check that $\lim_{r\to\infty} r^k u^{q+1}(r) = 0$ due to $k - \frac{p(q+1)}{q+1-p} < 0$ for $p > d$. Hence using the integration by parts, we have

$$r^k|u'(r)|^p = \frac{p}{(p-1)(q+1)} r^k u^{q+1}(r) + \frac{p}{(p-1)(q+1)} k \int_r^\infty s^{k-1} u^{q+1}(s) ds.$$

Thus we get

$$|u'(r)|^p = \frac{p}{(p-1)(q+1)} u^{q+1}(r) + \frac{p}{(p-1)(q+1)} k r^{-k} \int_r^\infty s^{k-1} u^{q+1}(s) ds. \tag{2.32}$$

Using (2.30), (2.31) and (2.32), a direct computation gives that for any $r > 0$

$$r|u'(r)| \leq C_{p,q} r^{-\frac{p}{q+1-p}} \quad \text{for } q > p-1; \tag{2.33}$$

$$|u'(r)| \leq C_p e^{-(p-1)^{-\frac{1}{p}} r} \quad \text{for } q = p-1. \tag{2.34}$$

Hence (2.30) and (2.33) give (2.26). Formulas (2.31) and (2.34) imply (2.27). □

2.3. *Solutions to Euler Lagrange equations are critical points of $G(u)$.* Since the zero contact angle in the free boundary condition (1.14) provides a C^1 zero extension for the case $q < p-1$, we can recast the free boundary problem (1.13)–(1.14) into the problem (1.15)–(1.16) as in Lemma 2.3.

LEMMA 2.3. Let $u(r)$ be a solution to the free boundary problem (1.13)–(1.14), and $u(r)=0$ for $r\geq R$. Then the zero extension solution $u(r)\in C^1(0,\infty)$ is a non-negative solution to the following problem in the distribution sense

$$(|u'|^{p-2}u')'+\frac{d-1}{r}|u'|^{p-2}u'=u^q,\ 0<r<+\infty, \tag{2.35}$$

$$\lim_{r\to 0^+}u(r)=1,\ \lim_{r\to\infty}u(r)=0. \tag{2.36}$$

Proof. Since u is a solution to the free boundary problem (1.13)–(1.14), by Step 3 in the proof of Proposition 1, we know that u is also a classical solution in $(0,R)$. Notice that $u'(R)=0$, which allows us to make a C^1-zero extension, i.e., extend it to $u(r)=0$ for $r\geq R$. Thus we have that the solution u is a C^1-non-negative solution to (2.35)–(2.36) in $(0,\infty)$. □

PROPOSITION 6. Let $u(r)$ be a solution to (2.35)–(2.36) in Y^*_{rad}. Then for any $\lambda>0$, the re-scaling function $u_\lambda(r)=u(\frac{r}{\lambda})$ is a critical point of $G(u)$ in Y^*_{rad}.

Proof. **Step 1.** In this step, we show that $\mathcal{G}(u)=0$, $\mathcal{G}(u)$ is defined by (2.16).

Since u satisfies equations (2.35)–(2.36) and the decay estimates (2.26)–(2.27), hence by (2.19), we have $\mathcal{G}(u)=0$, i.e.,

$$\int_{\mathbb{R}^d}|\nabla u|^p dx=\frac{pd}{(q+1)(p-d)}\int_{\mathbb{R}^d}u^{q+1}dx. \tag{2.37}$$

Step 2. We prove that for any $\lambda>0$, the re-scaling function $u_\lambda(r)=u(\frac{x}{\lambda})$ is a critical point of $G(u)$ in Y^*_{rad}.

In fact, it is directly verified that for any admissible variation $\phi\in C^1_c(0,\infty)$ at u_λ (i.e., there is an $\varepsilon_0>0$ such that for any $|\varepsilon|<\varepsilon_0$ one has $u_\lambda+\varepsilon\phi_\lambda\in Y^*_{rad}$), we have

$$\begin{aligned}&\frac{1}{G(u_\lambda)}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}G(u_\lambda+\varepsilon\phi_\lambda)\\ &=-\theta\|\nabla u_\lambda\|_{L^p}^{-p}\int_0^\infty \phi_\lambda'|u_\lambda'|^{p-1}s^{d-1}\,ds+(1-\theta)\|u_\lambda\|_{L^{q+1}}^{-q-1}\int_0^\infty \phi_\lambda u_\lambda^q s^{d-1}\,ds\\ &=-\theta\|\nabla u\|_{L^p}^{-p}\int_0^\infty \phi'|u'|^{p-1}r^{d-1}\,dr+(1-\theta)\|u\|_{L^{q+1}}^{-q-1}\int_0^\infty \phi(r)u^q(r)r^{d-1}\,dr.\end{aligned} \tag{2.38}$$

Together with (2.37), we deduce

$$\begin{aligned}&\frac{1}{G(u_\lambda)}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}G(u_\lambda+\varepsilon\phi_\lambda)\\ &\quad=-\theta\|\nabla u\|_{L^p}^{-p}\left(\int_0^\infty \phi'|u'|^{p-1}r^{d-1}\,dr-\int_0^\infty \phi(r)u^q(r)r^{d-1}\,dr\right).\end{aligned}$$

Noticing that u is a distribution solution to (2.35)–(2.36) in Y^*_{rad}, then it holds that

$$\frac{1}{G(u_\lambda)}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}G(u_\lambda+\varepsilon\phi_\lambda)=0.$$

Hence any re-scaling function of u is a critical point of $G(u)$ in Y^*_{rad}.

□

3. Existence and uniqueness for Euler-Lagrange equations in L^∞ case. In this section, we prove existence and uniqueness of solutions to the Euler-Lagrange equations (2.35)–(2.36) of L^∞-type G-N inequalities. We also show that the Euler-Lagrange equations are equivalent to some Thomas-Fermi type equations.

3.1. *Existence.* In this subsection, we prove existence of solutions $u(r)$ to the problem (2.35)–(2.36). First, we show that there is a singularity of $u'(r)$ at $r=0$: $u'(r)\sim Cr^{-\frac{d-1}{p-1}}$ at $r\to 0$ (Proposition 7). We then prove existence through a limit of a sequence of solutions in the exterior domain (r_i,∞), $r_i\to 0$. The main ingredients of the convergence proof are: (i) Comparison principle (Lemma 3.1); (ii) Uniform low bound nearby $r=0$ (Lemma 3.3); (iii) Application of the Dini theorem.

We introduce the following exterior Dirichlet problem on (r_0,∞), which was studied in [29]

$$(|u'|^{p-2}u')'+\frac{d-1}{r}|u'|^{p-2}u'=u^q,\quad r_0<r<\infty, \tag{3.1}$$

$$u(r_0)=1,\quad \lim_{r\to\infty}u(r)=0. \tag{3.2}$$

From [29, Theorem 4.3.1] and [29, Theorem 4.3.2], we know that the problem (3.1)–(3.2) has a unique solution $u(r)\in C^1[r_0,\infty)$ satisfying $u'(r)<0$ when $u(r)>0$. Furthermore, this solution $u(r)$ is non-increasing in $[r_0,+\infty)$, although this statement is not directly stated in [29, Theorem 4.3.1], the non-increase of u is a consequence in their proof [29, p. 94, line 1-4]. See also the proof of Proposition 7. We refer to $u(r)$ as a C^1 non-increasing solution.

Proposition 7 is to give a characterization of singularity of $u'(r)$ at $r=0$.

Proposition 7. For $p>d\geq 1$, $q>0$, and any $r_0>0$, the non-increasing solution $u(r)$ to the exterior problem (3.1)–(3.2) on (r_0,∞) satisfies for any $r>r_0$

$$r^{d-1}|u'(r)|^{p-1}=\int_r^\infty s^{d-1}u^q(s)ds<\infty; \tag{3.3}$$

$$\int_r^\infty s^{d-1}|u'(s)|^p ds+\int_r^\infty s^{d-1}u^{q+1}(s)ds=r^{d-1}u(r)|u'(r)|^{p-1}. \tag{3.4}$$

Proof. From the proof of [29, Theorem 4.3.1], we know that the non-increasing solution $u(r)$ to the problem (3.1)–(3.2) is the limit of a non-increasing function sequence $\{u_j(r)\}_1^\infty$, which is a solution to the following truncated exterior problem

$$(|u'|^{p-2}u')'+\frac{d-1}{r}|u'|^{p-2}u'=u^q,\ r_0<r<r_0+j, \tag{3.5}$$

$$u(r_0)=1,\quad u(r_0+j)=0, \tag{3.6}$$

$$u'(r)\leq 0,\ \text{in } [r_0,r_0+j], \tag{3.7}$$

and satisfies $u_j(r)<u_{j+1}(r)\leq 1$ for $r>r_0$. This implies that

(i) $|u_j'(r)|\leq|u_{j-1}'(r)|\leq\cdots\leq|u_1'(r)|$;
(ii) there is $u(r)$ such that $\lim_{j\to\infty}u_j(r)=u(r)$.

Multiplying r^{d-1} to (3.5) and integrating the resulting equation from r to r_0+j, $r_0 < r < r_0+j$, we obtain

$$(r_0+j)^{d-1}|u_j'(r_0+j)|^{p-1} - r^{d-1}|u_j'(r)|^{p-1} + \int_r^{r_0+j} s^{d-1}u_j^q(s)ds = 0, \tag{3.8}$$

which means

$$I_j := \int_r^{r_0+j} s^{d-1}u_j^q(s)ds \le r^{d-1}|u_j'(r)|^{p-1} \le r^{d-1}|u_1'(r)|^{p-1} \quad \text{for any } j \in \mathbb{N}^+. \tag{3.9}$$

Extending $u_j(r) = 0$ for $r \ge r_0+j$, we have $\int_r^\infty s^{d-1}u_j^q(s)ds < r^{d-1}|u_1'(r)|^{p-1}$. Hence by the Monotone Convergence Theorem, we have

$$\int_r^\infty s^{d-1}u^q(s)ds = \lim_{j\to\infty}\int_r^\infty s^{d-1}u_j^q(s)ds \le r^{d-1}|u_1'(r)|^{p-1} < \infty.$$

Similar to the process to obtain (2.25), we can get that the solution of the problem (3.1)–(3.2) also has the decay property

$$\lim_{r\to\infty} r^{d-1}|u'(r)|^{p-1} = 0. \tag{3.10}$$

Hence taking the limit for (3.8), we have (3.3).

Multiplying $u(r)$ on both sides of equation (3.1), integrating it from r to ∞, and using the facts $u(r) \to 0$, $r^{d-1}|u'|^{p-1} \to 0$ as $r \to \infty$, we have

$$\int_r^\infty s^{d-1}|u'(s)|^p ds + \int_r^\infty s^{d-1}u^{q+1}(s)ds = r^{d-1}u(r)|u'(r)|^{p-1}, \ r > r_0.$$

□

In order to show existence of solutions to the problem (2.35)–(2.36) (see Proposition 8), first we prove Lemmas 3.1–3.3.

LEMMA 3.1 (Comparison principle). Let u_1 and u_2 be C^1 non-increasing solutions to the exterior problem (3.1)–(3.2) on (r_1,∞) and (r_2,∞), respectively. Then if $r_1 < r_2$, we have that for $r \in \{r|u_2(r) > 0\}$,

$$u_1(r) < u_2(r), \qquad u_1'(r) > u_2'(r). \tag{3.11}$$

Proof. Since $u_1(r_1) = 1$ and $u_1'(r) < 0$ when $u_1(r) > 0$, we have $u_1(r) < 1$ in $(r_1, r_2]$. Hence $u_2(r_2) > u_1(r_2)$ due to $u_2(r_2) = 1$. Using a contradiction method, we assume that there is $r_* :\ r_2 < r_* < \infty$ such that $u_2(r_*) = u_1(r_*) =: m^* > 0$. Considering the following problem

$$(|u'|^{p-2}u')' + \frac{d-1}{r}|u'|^{p-2}u' = u^q, \ r_* < r < \infty, \tag{3.12}$$

$$u(r_*) = m^*, \quad \lim_{r\to\infty} u(r) = 0, \tag{3.13}$$

we know that solutions $u_1(r)$ and $u_2(r)$ defined in $[r_*,\infty)$ are C^1 non-increasing solutions to (3.12)–(3.13). The uniqueness of the C^1 non-increasing solution to the problem (3.12)–(3.13) implies that $u_1(r) = u_2(r)$ for $r \in [r_*,\infty)$. By ODE theory, we know that $u_1(r) = u_2(r)$ for any $r \ge r_2$, which is a contradiction with $u_2(r_2) > u_1(r_2)$. Hence for an $r \in \{r|u_2(r) > 0\}$, $u_2(r) > u_1(r)$.

From (3.3) and the comparison principle, we deduce that for any $r \in \{r|u_2(r) > 0\}$

$$r^{d-1}|u_2'(r)|^{p-1} = \int_r^\infty s^{d-1}u_2^q(s)ds > \int_r^\infty s^{d-1}u_1^q(s)ds = r^{d-1}|u_1'(r)|^{p-1},$$

which implies $u_2'(r) < u_1'(r)$. This is the proof of Lemma 3.1. □

LEMMA 3.2 (Uniform estimate in r_j). Let u_1 and u_2 be C^1 non-increasing solutions to the exterior problem (3.1)–(3.2) on (r_1, ∞) and (r_2, ∞), respectively. Then if $r_1 < r_2$, we have $\mu_{r_1} \le \mu_{r_2} + \frac{r_2^d - r_1^d}{d}$.

Proof. Since $u_1'(r) \le 0$ in $(r_1, +\infty)$, we have $u_1(r) \le 1$ in $[r_1, r_2]$. Hence for any $r_1, r_2 > 0$ satisfying $r_1 < r_2$, a direct computation gives

$$\mu_{r_1} = \left(\int_{r_1}^{r_2} + \int_{r_2}^{+\infty}\right) r^{d-1}u_1^q(r)dr \le \int_{r_1}^{r_2} r^{d-1}dr + \int_{r_2}^{+\infty} r^{d-1}u_1^q(r)dr. \tag{3.14}$$

Using the comparison principle in Lemma 3.1, we have

$$\int_{r_2}^{+\infty} r^{d-1}u_1^q(r)dr \le \int_{r_2}^{+\infty} r^{d-1}u_2^q(r)dr. \tag{3.15}$$

Hence (3.14) and (3.15) imply that $\mu_{r_1} \le \mu_{r_2} + \frac{r_2^d - r_1^d}{d}$. □

LEMMA 3.3 (Uniform low bound). Let u be the C^1 non-increasing solution to the exterior problem (3.1)–(3.2) on (r_0, ∞). Then for any $r > r_0$, there is $C > 0$ independent of r_0 and r such that

$$u(r) \ge 1 - C\left(r^{\frac{p-d}{p-1}} - r_0^{\frac{p-d}{p-1}}\right). \tag{3.16}$$

Proof. Multiplying r^{d-1} to (3.1) and integrating the resulting equation from r_0 to ∞, we deduce

$$r_0^{d-1}|u'(r_0)|^{p-1} = \int_{r_0}^\infty r^{d-1}u^q(r)dr = \mu_{r_0}. \tag{3.17}$$

Again multiplying r^{d-1} to (3.1) and integrating it from r_0 to r, and using (3.17), we obtain

$$r^{d-1}|u'(r)|^{p-1} - \mu_{r_0} = -\int_{r_0}^r r^{d-1}u^q(r)dr.$$

From above equation with $r > r_0$ and $u \ge 0$, we have

$$-u'(r) \le r^{-\frac{d-1}{p-1}}\mu_{r_0}^{\frac{1}{p-1}}. \tag{3.18}$$

Integrating (3.18) from r_0 to r, we deduce

$$u(r) \ge 1 - \frac{p-1}{p-d}\mu_{r_0}^{\frac{1}{p-1}}\left(r^{\frac{p-d}{p-1}} - r_0^{\frac{p-d}{p-1}}\right). \tag{3.19}$$

By Lemma 3.2, we know that for a fixed $r_* > r_0$, $\mu_{r_0} \le \mu_{r_*} + \frac{r_*^d - r_0^d}{d} \le \mu_{r_*} + \frac{r_*^d}{d}$. Denote $C := \frac{p-1}{p-d}\left(\mu_{r_*} + \frac{r_*^d}{d}\right)^{\frac{1}{p-1}}$. Hence (3.19) implies that (3.16) holds true.

□

Proposition 8 (Existence). Assume that exponents $p > d \geq 1$ and $q > 0$, then there is a C^1 non-increasing solution $u(r)$ to the problem (2.35)–(2.36), and satisfies the following properties

(i) $u \in L^{q+1}(\mathbb{R}^d)$, $\nabla u \in L^p(\mathbb{R}^d)$, and

$$\int_0^\infty r^{d-1}|u'|^p dr + \int_0^\infty r^{d-1}u^{q+1}dr = \lim_{r\to 0^+} r^{d-1}|u'|^{p-1} = \int_0^\infty r^{d-1}u^q dr < \infty; \quad (3.20)$$

(ii) $u'(r) < 0$ for $u(r) > 0$.

Proof. Let $u_i(r)$ be the C^1 non-increasing solution to the exterior Dirichlet problem (3.1)–(3.2) on the domain $(r_i, +\infty)$ with the boundary condition $u_i(r_i) = 1$. We take a sequence $\{r_i\}_{i=1}^\infty$ satisfying $r_i > r_{i+1} > 0$ for any $i \in \mathbb{N}^+$, and $r_i \to 0^+$ as $i \to +\infty$. We will show the limit function of u_i is the solution of the problem (2.35)–(2.36).

Step 1. We prove that there is a continuous, non-negative and non-increasing function $u(r)$ such that as $i \to \infty$,

$$u_i(r) \to u(r), \quad u_i'(r) \to u'(r), \quad \text{for all } r > 0, \quad (3.21)$$

and they converge uniformly in any interval $[a, b]$ for $0 < a < b < \infty$.

Notice that $\{u_i(r)\}_{i_0}^\infty$ is continuous, non-negative and non-increasing sequence and bounded below in $[a, \infty)$, $a > 0$ $(0 < r_{i_0} < a)$. Hence by the Dini theorem the sequence $\{u_i(r)\}_{i_0}^\infty$ converges uniformly on every compact interval $[a, b]$ of $(0, \infty)$ to a non-negative, non-increasing, continuous function $u(r)$, i.e.,

$$u_i(r) \to u(r) \quad \text{for all } r > 0, \qquad \text{as } i \to +\infty, \quad (3.22)$$

and they converge uniformly in any interval $[a, b]$ for $0 < a < b < \infty$. Since $u_i(r)$ is a non-negative and non-increasing function in r, hence $u(r)$ is also a non-negative and non-increasing function. Furthermore, by the comparison principle we know that for any $i \in \mathbb{N}$, it holds that

$$u(r) \leq u_i(r) \quad \text{for all } r \in (r_i, \infty). \quad (3.23)$$

Moreover, let $u_j(r)$ be a solution to equation (3.1) with $u(r_j) = 1$. Hence by (3.11) and the Dini theorem we have

$$u_j'(r) \to u'(r) \quad \text{for all } r > 0, \quad (3.24)$$

and they converge uniformly in any interval $[a, b]$ for $0 < a < b < \infty$. Moreover, by the comparison principle we know that for any $i \in \mathbb{N}$, it holds that

$$u'(r) \geq u_i'(r) \quad \text{for all } r \in (r_i, \infty). \quad (3.25)$$

Step 2. We prove $\lim_{r\to\infty} u(r) = 0$ and $\lim_{r\to 0^+} u(r) = 1$.

Since $\lim_{r\to\infty} u_i(r) = 0$ by (3.2), we have $\lim_{r\to\infty} u(r) = 0$. On the other hand, by Lemma 3.3 we have

$$\lim_{r\to 0^+} \lim_{i\to\infty} u_i(r) \geq 1.$$

Together with $\lim_{r\to 0^+} \lim_{i\to\infty} u_i(r) \leq 1$ gives that $\lim_{r\to 0^+} u(r) = 1$. Thus the limit function $u(r)$ satisfies the boundary condition (2.36).

Step 3. We show the L^q-integrability of the limit function $u(r)$, i.e., for fixed $i_0 \in \mathbb{N}$, it holds that

$$\int_0^{+\infty} r^{d-1}u^q(r)dr < \mu_{r_{i_0}} + \frac{r_{i_0}^d}{d}. \tag{3.26}$$

Indeed, for any $\varepsilon > 0$, there exists $i_* \in \mathbb{N}$ such that $0 < r_{i_*} < \varepsilon < r_{i_0}$. And hence by (3.23), we have

$$\int_\varepsilon^\infty r^{d-1}u^q(r)dr \le \int_\varepsilon^\infty r^{d-1}u_{i_*}^q(r)dr \le \int_{r_*}^\infty r^{d-1}u_{i_*}^q(r)dr.$$

Hence by Lemma 3.2, we know that for $r_{i_*} < r_{i_0}$

$$\int_\varepsilon^\infty r^{d-1}u^q(r)dr \le \int_{i_0}^\infty r^{d-1}u_{i_0}^q(r)dr + \frac{r_{i_0}^d - r_{i_*}^d}{d}. \tag{3.27}$$

Taking $\varepsilon \to 0$, we get (3.26).

Step 4. We give $u \in Y_{rad}^*$. Mainly $u \in L^{q+1}$ and $\nabla u \in L^p$.

For any $\varepsilon > 0$, there exists $i_* \in \mathbb{N}$ such that $0 < r_{i_*} < \varepsilon < r_{i_0}$. For any $j \ge i_*$, let $u_j(r)$ be a solution to equation (3.1) with $u(r_j) = 1$. Then we have

$$\int_\varepsilon^\infty s^{d-1}|u_j'(s)|^p ds + \int_\varepsilon^\infty s^{d-1}u_j^{q+1}(s)ds = \varepsilon^{d-1}u(\varepsilon)|u_j'(\varepsilon)|^{p-1} \tag{3.28}$$

$$\le \varepsilon^{d-1}|u_j'(\varepsilon)|^{p-1} = \int_\varepsilon^\infty s^{d-1}u_j^q(s)ds \le \mu_{r_{i_0}} + \frac{r_{i_0}^d}{d}. \tag{3.29}$$

Hence by the Monotone Convergence Theorem, as $j \to \infty$, we have

$$\int_\varepsilon^\infty s^{d-1}|u'(s)|^p ds + \int_\varepsilon^\infty s^{d-1}u^{q+1}(s)ds = \varepsilon^{d-1}u(\varepsilon)|u'(\varepsilon)|^{p-1} \tag{3.30}$$

$$\le \varepsilon^{d-1}|u'(\varepsilon)|^{p-1} = \int_\varepsilon^\infty s^{d-1}u^q(s)ds. \tag{3.31}$$

Notice that

$$\lim_{\varepsilon\to 0} \varepsilon^{d-1}u(\varepsilon)|u'(\varepsilon)|^{p-1} = \lim_{\varepsilon\to 0} \varepsilon^{d-1}|u'(\varepsilon)|^{p-1}$$

due to $\lim_{\varepsilon\to 0} u(\varepsilon) = 1$ in Step 2, we deduce (3.20).

Step 5. We prove that the limit function $u(r)$ is the required radial solution of (2.35) in the distribution sense.

In fact, for any $\phi \in C_c^1(0, +\infty)$, we have

$$-\int_0^{+\infty} \phi' r^{d-1}|u_i'|^{p-1}dr + \int_0^{+\infty} \phi r^{d-1}u_i^q dr = 0. \tag{3.32}$$

Using the uniform convergence property of u_i and u_i' from Step 1, we obtain

$$-\int_0^{+\infty} \phi' r^{d-1}|u'|^{p-1}dr + \int_0^{+\infty} \phi r^{d-1}u^q dr = 0, \tag{3.33}$$

i.e., the limit function $u(r)$ is the required radial solution of (2.35) in the distribution sense.

Step 6 (To prove the property (ii)). Since $u \in L^{q+1}(\mathbb{R}^d)$ and $\nabla u \in L^p(\mathbb{R}^d)$, from equation (2.35), we can obtain (2.13). Hence we have that $u'(r) < 0$ in the set $\{r|u(r) > 0\}$, i.e., the case (ii) holds.

This completes the proof of Proposition 8. □

REMARK 3.4. Proposition 8 proved existence of C^1 non-increasing solutions to (2.35)–(2.36) for the case $q > 0$. Existence for the case $q = 0$ will be established in Proposition 11 by giving an exact closed form solution.

3.2. *Uniqueness.* In this subsection, we prove uniqueness of solutions to (2.35)–(2.36) by following works [15, Theorem 1] given by Franchi, Lanconelli and Serrin. Let $u(r)$ and $v(r)$ be two C^1 non-increasing solutions to the problem (2.35)–(2.36). By (i) of Proposition 8, we have $u'(r) < 0$ when $u(r) > 0$, and hence both $u(r)$ and $v(s)$ possess inverse functions in those supports. We denote respectively by $r(u)$ and $s(v)$ the inverse functions of $u(r)$ and $v(s)$, defined on the interval $(0, 1]$.

Lemmas 3.5 and 3.6 are special cases from results in [7]. We supply a proof to show how their proof is used in our special cases.

LEMMA 3.5 ([15, Lemma 3.3.1]). Assume $q \geq 0$ and $d > 1$. If $r(u) > s(u)$ in some open interval $(0, 1)$, then $r(u) - s(u)$ can have at most one critical point in $(0, 1)$. Moreover if such a critical point exists, it must be a strict maximum point.

Proof. By equation (2.35), it is immediately verified that the function $r = r(u)$ satisfies the equation

$$(p-1)r_{uu} - \frac{d-1}{r} r_u^2 - |r_u|^{p+1} u^q = 0, \ 0 < u < 1,$$

and the same equation holds for $s(u)$. Hence by subtracting one from another, we get

$$(p-1)(r-s)_{uu} - (d-1)\left(\frac{r_u^2}{r} - \frac{s_u^2}{s}\right) - \left(|r_u|^{p+1} - |s_u|^{p+1}\right) u^q = 0. \tag{3.34}$$

Now we suppose that $u = u_* \in (0, 1)$ is a critical point of $r(u) - s(u)$, then $r_u = s_u < 0$ at $u = u_*$. Thus from (3.34), we have that

$$(p-1)(r-s)_{uu} = (d-1)r_u^2 \left(\frac{1}{r} - \frac{1}{s}\right) < 0, \text{ at } u = u_*,$$

where the last inequality used the fact $r(u) > s(u)$ in $(0, 1)$. Hence we get that all critical points must be maximum points, which implies that $r(u) - s(u)$ has at most one critical point in $(0, 1)$. □

LEMMA 3.6 ([15, Lemma 3.3.2]). Assume $q \geq 0$ and $d > 1$. If $r(u) - s(u)$ has two zero points in $(0, 1]$, denoting them as ξ_0 and ξ_1, then $r(u) = s(u)$ for all u between ξ_0 and ξ_1.

Proof. Inspired by [15, Lemma 3.3.2], use the contradiction method to prove this lemma. Without loss of generality, we assume that $\xi_0 < \xi_1$ and $r(u) > s(u)$ for all $u \in (\xi_0, \xi_1)$. By Lemma 3.5, we know that $r(u) - s(u)$ has at most one critical point in (ξ_0, ξ_1). Since $r(\xi_0) - s(\xi_0) = r(\xi_1) - s(\xi_1) = 0$, then there is at least one critical point in (ξ_0, ξ_1). Suppose that $\xi_2 \in (\xi_0, \xi_1)$ is a unique critical point satisfying $(r - s)'(\xi_2) = 0$.

From (3.34), we have

$$(p-1)(r''(u)-s''(u)) = (d-1)(r'(u))^2\left(\frac{1}{r(u)}-\frac{1}{s(u)}\right) < 0, \quad \text{at } u=\xi_2, \tag{3.35}$$

where the last inequality used the fact $r(\xi_2) > s(\xi_2)$. Hence by the continuity of $r'(u)$, we know that there is a $\delta > 0$ such that $(r-s)'(u) < 0$ in $(\xi_2, \xi_2+\delta)$. Since the critical point of $r(u)-s(u)$ is unique in (ξ_0, ξ_1), we have

$$(r-s)'(u) < 0, \quad \text{in } (\xi_2, \xi_1), \quad \text{i.e. } |r'(u)| > |s'(u)|. \tag{3.36}$$

Denote $r_1 = r(\xi_1)$ and $r_2 = r(\xi_2)$. Multiplying r^{d-1} to equation (2.35) and integrating the resulting equation from r_1 to r_2, we deduce

$$\int_{r_1}^{r_2} \frac{d}{dr}\left(r^{d-1}|u'|^{p-2}u'(r)\right)dr = \int_{r_1}^{r_2} r^{d-1}u^q(r)dr = \int_{\xi_2}^{\xi_1} r(u)^{d-1}\frac{u^q}{|u'(r(u))|}du. \tag{3.37}$$

Hence it holds that

$$r_2^{d-1}|u'(r_2)|^{p-2}u'(r_2) - r_1^{d-1}|u'(r_1)|^{p-2}u'(r_1) = \int_{\xi_2}^{\xi_1} r(u)^{d-1}\frac{u^q}{|u'(r(u))|}du. \tag{3.38}$$

Similar for v, denote $s_1 = s(\xi_1)$ and $s_2 = s(\xi_2)$. We have the same formula

$$s_2^{d-1}|v'(s_2)|^{p-2}v'(s_2) - s_1^{d-1}|v'(s_1)|^{p-2}v'(s_1) = \int_{\xi_2}^{\xi_1} s(u)^{d-1}\frac{u^q}{|v'(s(u))|}du. \tag{3.39}$$

Due to $r_1 = r(\xi_1) = s(\xi_1) = s_1$ and $r'(\xi_2) = s'(\xi_2)$, subtracting (3.39) from (3.38) gives that

$$\begin{aligned}\left(s_2^{d-1}-r_2^{d-1}\right)|u'(r_2)|^{p-1} &+ r_1^{d-1}\left(|u'(r_1)|^{p-1}-|v'(s_1)|^{p-1}\right)\\ &= \int_{\xi_2}^{\xi_1} u^q\left(\frac{r(u)^{d-1}}{|u'(r(u))|}-\frac{s(u)^{d-1}}{|v'(s(u))|}\right)du.\end{aligned} \tag{3.40}$$

Since (3.36), $s_2 < r_2$ and $r(u) > s(u)$ for all $u \in (\xi_0, \xi_1)$, we directly verify that both terms on the left side of (3.40) are strictly negative, while the right side of (3.40) is non-negative. This is a contradiction. Hence the assumption is not true, i.e., $r(u) = s(u)$ in (ξ_0, ξ_1).

□

PROPOSITION 9 (Uniqueness). Assume $q \geq 0$ and $p > d > 1$. Let u and v be two C^1 non-increasing solutions of the problem (2.35)–(2.36). Then $u(r) \equiv v(r)$ for any $0 \leq r < \infty$.

Proof. We use a contradiction method to prove this proposition. If not, then $u(r) \not\equiv v(r)$ on $[0,\infty)$. Equivalently their inverse functions $r(u) \not\equiv r(u)$ in $(0,1]$. Hence there is $u_* \in (0,1)$ such that $r(u_*) \neq s(u_*)$. Then Lemma 3.6 implies that $r(u), s(u)$ satisfy either $r(u) > s(u)$ or $r(u) < s(u)$ in $(0,1)$.

For the case $q \geq p-1$, without loss of generality, we suppose $r(u) > s(u)$ for $u \in (0,1)$, then $u(r) > v(r)$ for $r > 0$. Multiplying r^{d-1} to equation (2.35) and integrating the resulting equation from r to ∞ and using (2.25), we have

$$r^{d-1}|u'|^{p-1} = \int_r^\infty s^{d-1}u^q(s)ds. \tag{3.41}$$

The same process for $v(r)$ gives

$$r^{d-1}|v'|^{p-1} = \int_r^\infty s^{d-1}v^q(s)ds. \tag{3.42}$$

Subtracting (3.42) from (3.41) gives that

$$r^{d-1}|u'(r)|^{p-1} - r^{d-1}|v'(r)|^{p-1} = \int_r^\infty s^{d-1}(u^q(s) - v^q(s))ds.$$

Since $u(r) > v(r)$ and $q \geq p-1 > 0$, then we have from the above equation

$$v'(r) > u'(r), \quad \text{for } r > 0. \tag{3.43}$$

Integrating (3.43), we obtain $\lim_{r\to 0^+} v(r) < \lim_{r\to 0^+} u(r)$, which is a contradiction with $\lim_{r\to 0^+} v(r) = \lim_{r\to 0^+} u(r) = 1$.

For the case $0 \leq q < p-1$, we suppose $r(u) > s(u)$ for $u \in (0,1)$. Then $u(r) > v(r)$ for $0 < r < R_v$, and $u > 0$, $v = 0$ for $R_v < r < R_u$. Multiplying r^{d-1} to equation (2.35) and integrating the resulting equation from r to R_v, we have

$$\begin{aligned} r^{d-1}|v'|^{p-1} &= \int_r^{R_v} s^{d-1}v^q(s)ds \\ &< \left(\int_r^{R_v} + \int_{R_v}^{R_u}\right) s^{d-1}u^q(s)ds = r^{d-1}|u'|^{p-1} \quad \text{for } 0 < r < R_v. \end{aligned}$$

Thus $v'(r) > u'(r)$ for $0 < r < R_v$. In (R_v, R_u), $u'(r) < 0 = v'(r)$. Hence we have

$$0 < \int_0^{R_u} (v'(r) - u'(r))dr = \int_0^{R_v} v'(r)dr - \int_0^{R_u} u'(r)dr = 0,$$

which is a contradiction. □

REMARK 3.7. For the case $d = 1$, uniqueness of C^1 non-increasing solutions to the problem (2.35)–(2.36) is given by a direct computation in Proposition 12. Hence the proof of Theorem 1.1 will be given in Subsection 4.3.

3.3. *Thomas-Fermi type equation.* This subsection shows that the non-increasing solution of the Euler Lagrange equation obtained above is equivalent to the radial non-increasing solution to a Thomas-Fermi type equation.

DEFINITION 3.8. We call a function $u(|x|)$ a radial non-increasing weak solution to the Thomas-Fermi type equation (1.20)–(1.21) if $u(|x|)$ satisfies

(i) $u(|x|)$ is a non-increasing function in $|x|$ and $\lim_{|x|\to 0^+} u(|x|) = 1$,
(ii) $\nabla u \in L^p$, $u \in L^{q+1}$, and denote $a := \|\nabla u\|_{L^p}^p + \|u\|_{L^{q+1}}^{q+1}$,
(iii) for any $\phi(|x|) \in C_c^\infty(\mathbb{R}^d)$, it holds that $(\nabla\phi, |\nabla u|^{p-2}\nabla u) + (\phi, u^q) = a(\phi, \delta_{x=0})$.

PROPOSITION 10. Assume $p > d \geq 1$, then

(i) For the case $q \geq p-1$,
   (a) if $u(r)$ is the weak solution to the problem (1.15)–(1.16) in Y^*_{rad}, then $u(|x|)$ is a radial non-increasing weak solution to a Thomas-Fermi type equation (1.20)–(1.21).
   (b) if $u(|x|) \in W^{1,p}(\mathbb{R}^d)$ is a radial non-increasing weak solution to the Thomas-Fermi type equation (1.20)–(1.21), then $u(r)$ is also the solution of (1.15)–(1.16) in Y^*_{rad}.

(ii) For the case $q < p-1$,
   (a) if $u(r)$ is the solution to the free boundary problem (1.13)–(1.14) in Y^*_{rad}, then $u(|x|)$ is a radial non-increasing solution to a Thomas-Fermi type equation (1.17)–(1.19).
   (b) if $u(|x|) \in W^{1,p}(\mathbb{R}^d)$ is a radial non-increasing solution to the Thomas-Fermi type equation (1.17)–(1.19), then $u(r)$ is also the solution of (1.13)–(1.14) in Y^*_{rad}.

In particular, for $d = 1$ we have $a = 2\left(\frac{p}{(q+1)(p-1)}\right)^{\frac{p-1}{p}}$.

Proof. We first prove the case (i). For the case (a), suppose that $u(r)$ is the solution to (1.15)–(1.16) in Y^*_{rad}. Hence we know that $u(|x|) \in W^{1,p}(\mathbb{R}^d)$ is radial non-increasing and satisfies $u(0) = 1$ and $u(|x|) \to 0$ as $|x| \to \infty$. Hence the boundary condition (1.21) holds.

For any test function $\phi(|x|) \in C_c^\infty(\mathbb{R}^d)$, it holds that

$$-(\nabla\phi, |\nabla u|^{p-2}\nabla u) - (\phi, u^q) = S_d \int_0^\infty \left(\phi'(r)|u'(r)|^{p-1} - \phi(r)u^q(r)\right) r^{d-1} dr.$$

From Proposition 1, we have that the solution is classical in $(0, \infty)$. Hence by integration by parts we have

$$\begin{aligned}(\nabla\phi, |\nabla u|^{p-2}\nabla u) + (\phi, u^q) &= S_d\phi(0) \lim_{r\to 0^+} |u'(r)|^{p-1} r^{d-1} \\ &\quad + S_d \int_0^\infty \phi\left(\left(|u'(r)|^{p-1}r^{d-1}\right)' + u^q r^{d-1}\right) dr. \qquad (3.44)\end{aligned}$$

On the other hand, from Proposition 8 we know that

$$\lim_{r\to 0^+} S_d r^{d-1}|u'(r)|^{p-1} = S_d \int_0^\infty r^{d-1}|u'|^p dr + S_d \int_0^\infty r^{d-1}u^{q+1} dr = a. \qquad (3.45)$$

Using (3.45) and equation (2.35), from (3.44) we obtain

$$(\nabla\phi, |\nabla u|^{p-2}\nabla u) + (\phi, u^q) = a\phi(0) = a(\phi, \delta_{x=0}).$$

Hence we have that the following equation holds in the distribution sense

$$\Delta_p u + a\delta_{x=0} = u^q.$$

Therefore, $u(|x|)$ is a radial non-increasing weak solution to a Thomas-Fermi type equation (1.20)–(1.21).

Now we prove the case (b), assume that $u(|x|) \in W^{1,p}(\mathbb{R}^d)$ is a radial non-increasing weak solution of (1.20)–(1.21) in Definition 3.8, then $u(r) := u(|x|)$ satisfies (1.16) and for any test function $\phi(|x|) \in C_c^\infty(\mathbb{R}^d)$ satisfying $\phi(0) = 0$, it holds that

$$\begin{aligned}0 = -(\nabla\phi, |\nabla u|^{p-2}\nabla u) - (\phi, u^q) &= S_d \int_0^\infty \left(\phi'(r)|u'(r)|^{p-1} - \phi(r)u^q(r)\right) r^{d-1} dr \\ &= S_d \int_0^\infty \phi\left(\left(r^{d-1}u'|u'|^{p-2}\right)' - u^q r^{d-1}\right) dr.\end{aligned}$$

Hence $u(r)$ satisfies (1.15). This completes the proof of the case (i).

The proof of the case (ii) is exactly the same as the case (i). Here we omit the details.

Finally, we determine the value of a for $d = 1$. Since $u' < 0$, (3.45) implies $a = S_d \lim_{r\to 0^+} r^{d-1}|u'(r)|^{p-1}$. Thus multiplying u' to equation (1.15) with $d = 1$, and integrating it from r to ∞, and using the boundary condition (1.16), we deduce

$$\frac{p-1}{p}|u'|^p = \frac{u^{q+1}}{q+1}.$$

Noticing the fact $\lim_{r\to 0^+} u(r) = 1$, we have

$$\lim_{r\to 0^+} |u'(r)| = \left(\frac{p}{(p-1)(q+1)}\right)^{\frac{1}{p}}.$$

Thus

$$a = 2\lim_{r\to 0^+} |u'(r)|^{p-1} = 2\left(\frac{p}{(q+1)(p-1)}\right)^{\frac{p-1}{p}}.$$

This completes the proof of Proposition 10. □

4. Best constant for L^∞-type G-N inequality. This section is divided into three subsections. We give some closed form solutions for the case $q = 0$ in Subsection 4.1 or for the case $d = 1$ in Subsection 4.2. In Subsection 4.3, we will complete the proofs of Theorem 1.1 and Theorem 1.2.

4.1. *Existence, uniqueness and closed form solution for $q = 0$, $p > d \geq 1$.* In Proposition 7, we require the condition $q > 0$. For the case $q = 0$, we use the closed form solution to prove existence and uniqueness in Proposition 11.

PROPOSITION 11. Suppose $d \geq 1$, $p > d$ and $q = 0$. Then there is a unique non-negative solution $u_{c,\infty}$ to the free boundary problem (1.13)–(1.14) and $u_{c,\infty}$ has the following closed form

$$u_{c,\infty}(r) = d^{-\frac{p}{p-1}} R^{\frac{p}{p-1}} \left(\mathcal{B}\left(\frac{p-d}{d(p-1)}, \frac{p}{p-1}\right) - B\left(\left(\frac{r}{R}\right)^d; \frac{p-d}{d(p-1)}, \frac{p}{p-1}\right)\right), \tag{4.1}$$

$$R = d\left(\mathcal{B}\left(\frac{p-d}{d(p-1)}, \frac{p}{p-1}\right)\right)^{-\frac{p-1}{p}}. \tag{4.2}$$

Proof. Let $r = Rs$, and $v(s) = u(Rs)$. Hence we have $v(1) = u(R) = 0$ and $v(0) = u(0) = 1$. Then from (1.13) with $q = 0$, we obtain that $v(s)$ satisfies the following equation

$$-v'(s)/R = d^{-\frac{1}{p-1}}\left((R^d - R^d s^d)(Rs)^{1-d}\right)^{\frac{1}{p-1}} = d^{-\frac{1}{p-1}} R^{\frac{1}{p-1}} (1-s^d)^{\frac{1}{p-1}} s^{\frac{1-d}{p-1}}.$$

Hence

$$-v'(s) = d^{-\frac{1}{p-1}} R^{\frac{p}{p-1}} (1-s^d)^{\frac{1}{p-1}} s^{\frac{1-d}{p-1}}. \tag{4.3}$$

Integrating (4.3) from s to 1, we deduce

$$\begin{aligned} v(s) =& d^{-\frac{1}{p-1}} R^{\frac{p}{p-1}} \int_s^1 (1-r^d)^{\frac{1}{p-1}} r^{\frac{1-d}{p-1}} dr \\ =& d^{-\frac{p}{p-1}} R^{\frac{p}{p-1}} \left(\mathcal{B}\left(\frac{p-d}{d(p-1)}, \frac{p}{p-1}\right) - B\left(s^d; \frac{p-d}{d(p-1)}, \frac{p}{p-1}\right)\right). \end{aligned}$$

Hence we obtain the closed form solution $u_{c,\infty}$ to the free boundary problem (1.13)–(1.14) given by (4.1).

With condition $v(0) = 1$, we have explicit formula of R:

$$\begin{aligned}1 =& d^{-1/(p-1)} R^{p/(p-1)} \int_0^1 (1-r^d)^{\frac{1}{p-1}} r^{\frac{1-d}{p-1}} dr \\ =& d^{-\frac{p}{p-1}} R^{\frac{p}{p-1}} \mathcal{B}\left(\frac{p-d}{d(p-1)}, \frac{p}{p-1}\right),\end{aligned}$$

which means that (4.2) holds.

□

4.2. *The closed form solution for $q \geq 0$ and $p > d = 1$.* In this subsection, we present a result in the one dimensional case, for which there is a closed form solution and deduce the best constant $C_{q,\infty,p}$ of the inequality (1.6) for $d = 1$.

PROPOSITION 12. Suppose $p > d = 1$, and $q \geq 0$. Then the solution $u_{c,\infty}$ of the problem (2.35)–(2.36) possesses the following closed form:

(i) for $q = p - 1$,

$$u_{c,\infty}(r) = e^{-(p-1)^{-\frac{1}{p}} r}; \tag{4.4}$$

(ii) for $q < p - 1$,

$$u_{c,\infty}(r) = \left(1 - \frac{r}{R}\right)_+^{\frac{p}{p-q-1}}, \quad R = \frac{(p-1)^{1/p}(q+1)^{1/p}}{p^{1/p-1}(p-q-1)}, \text{ for } r > 0; \tag{4.5}$$

(iii) for $q > p - 1$,

$$u_{c,\infty}(r) := \left(1 + \frac{p^{1/p-1}(q+1-p)}{(p-1)^{1/p}(q+1)^{1/p}} r\right)^{-\frac{p}{q+1-p}}, \text{ for } r > 0. \tag{4.6}$$

The best constant is given by $C_{q,\infty,p} = \left(\frac{p+(p-1)(q+1)}{2p}\right)^{\frac{p}{p+(p-1)(q+1)}}$.

Proof. For $q = p-1$, multiplying u' on both sides of equation (2.35) with $d = 1$ gives that

$$\frac{d}{dr}\left(\frac{p-1}{p}|u'|^p - \frac{u^{q+1}}{q+1}\right) = 0. \tag{4.7}$$

Noticing $u \in W^{1,p}(\mathbb{R})$ and integrating (4.7) from r to ∞, we have

$$\frac{p-1}{p}|u'|^p - \frac{u^{q+1}}{q+1} = 0, \tag{4.8}$$

which implies that

$$u(r) = e^{\pm(p-1)^{-\frac{1}{p}} r}.$$

Hence $u(r) = e^{-(p-1)^{-\frac{1}{p}} r}$ is the unique solution satisfying $u(0) = 1$ and $\lim_{r\to\infty} u(r) = 0$.

For $q \neq p-1$, solving (4.8) and using boundary conditions $u(0) = 1$ and $\lim_{r\to\infty} u(r) = 0$, we can obtain (4.5) and (4.6).

Hence plugging $u_{c,\infty}$ into M_c in (1.23) below, we deduce

$$M_c(u_{c,\infty}) = \int_{\mathbb{R}} u_{c,\infty}^{q+1}(x) dx = \frac{2(p-1)^{1/p}(q+1)^{1/p}}{p^{1/p-1}(p+(p-1)(q+1))},$$

and thus it holds that

$$C_{q,\infty,p} = \left(\frac{(p-1)(q+1)}{p}\right)^{\frac{1}{p+(p-1)(q+1)}} M_c^{-\frac{p}{p+(p-1)(q+1)}}$$
$$= \left(\frac{p+(p-1)(q+1)}{2p}\right)^{\frac{p}{p+(p-1)(q+1)}}. \tag{4.9}$$

This completes the proof of Proposition 12. □

4.3. *The proofs of main theorems.* In this subsection, we utilize the results from above sections to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. By Proposition 8, Proposition 9 and Proposition 12, we know that the problem (2.35)–(2.36) has a unique solution $u(r)$ for the case $q > 0$. While for the case $q = 0$, from Proposition 11 we know that there is a unique closed form solution $u_{c,\infty}(r)$ satisfying the free boundary problem (1.13)–(1.14). That completes the proof of Theorem 1.1. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. From Proposition 1, Proposition 3 and Lemma 2.3, we have that any critical point of $G(u)$ in Y^*_{rad} satisfies the problem (2.35)–(2.36) up to a re-scaling. Conversely, any non-negative solution u of the problem (2.35)–(2.36) is also a critical point of $G(u)$ in Y^*_{rad}. Moreover any re-scaling function of u is still a critical point of $G(u)$ in Y^*_{rad} by Proposition 6.

By Theorem 1.1, we know that the critical point of $G(u)$ in Y^*_{rad} is unique up to a re-scaling.

Hence any re-scaling function set of $u_{c,\infty}$, $\{u_\lambda\}_{\lambda>0}$ contains all critical points of $G(u)$ and $G(u_\lambda) \equiv G(u_{c,\infty})$. Notice that $G(u)$ does not have maximum. Hence all critical points $\{u_\lambda\}_{\lambda>0}$ are minimizers of $G(u)$.

Next we derive the best constant $C_{q,\infty,p}$ for $q \geq 0$. Since the solution $u_{c,\infty}(r)$ to the problem (2.35)–(2.36) is a minimizer of $G(u)$. Hence from the problem (1.12) and the formula (2.17), we have

$$\alpha = \|u_{c,\infty}\|^{1-\theta}_{L^{q+1}} \|\nabla u_{c,\infty}\|^{\theta}_{L^p} = \left(\frac{\theta}{1-\theta}\right)^{\frac{\theta}{p}} \left(\|u_{c,\infty}(r)\|^{q+1}_{L^{q+1}}\right)^{\frac{\theta}{d}}.$$

Notice $C_{q,\infty,p} = \alpha^{-1}$, thus we have (1.23). □

5. The large m limit behavior in the best constant problem for 1-D. In this section, we show the limit behavior as $m \to \infty$ in the best constant problem for $d = 1$, which implies the closed relation between the functional inequalities (1.26) and (1.22) in the one-dimension case. Now we show the proof of Theorem 1.3.

Proof of Theorem 1.3. The proof of Theorem 1.3 is divided into the two steps.

Step 1. In this step, we prove

$$\lim_{m\to\infty} u_{c,m}(r) = u_{c,\infty}(r) \quad \text{for any } r > 0. \tag{5.1}$$

For any fixed $0 < u < 1$, denote $r_{c,m}(u)$ as an inverse function of $u_{c,m}(r)$. Let the energy functional

$$H(r) := \frac{p-1}{p}|u'_{c,m}(r)|^p + \frac{u_{c,m}^{m+1}(r)}{m+1} - \frac{u_{c,m}^{q+1}(r)}{q+1}. \tag{5.2}$$

Then by multiplying $u'_{c,m}(r)$ to (1.28), we have the following energy-dissipation relation

$$\frac{dH(r)}{dr} + \frac{d-1}{r}|u'_{c,m}(r)|^p = 0. \tag{5.3}$$

Hence we know that for $d = 1$, (1.28) possesses a first integral and it is a constant, i.e.,

$$\frac{p-1}{p}|u'_{c,m}|^p + \frac{u_{c,m}^{m+1}}{m+1} - \frac{u_{c,m}^{q+1}}{q+1} = C. \tag{5.4}$$

Due to $u_{c,m}(R_m) = u'_{c,m}(R_m) = 0$, then $C = 0$. Hence the conditions $u_{c,m}(0) = \alpha_c$ and $u'_{c,m}(0) = 0$ imply

$$\frac{\alpha_c^{m+1}}{m+1} - \frac{\alpha_c^{q+1}}{q+1} = 0, \text{ i.e. } , \alpha_c = \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}} > 1. \tag{5.5}$$

Solving (5.4) gives

$$u'_{c,m}(r) = -\left(\frac{p}{p-1}\right)^{1/p}\left(\frac{u_{c,m}^{q+1}(r)}{q+1} - \frac{u_{c,m}^{m+1}(r)}{m+1}\right)^{1/p}. \tag{5.6}$$

From (5.6), we deduce

$$\left(\frac{p}{p-1}\right)^{1/p} r_{c,m} = \frac{(q+1)^{1/p}}{m-q}\left(\frac{m+1}{q+1}\right)^{\frac{p-q-1}{p(m-q)}} \int_0^{1-\frac{q+1}{m+1}u_{c,m}^{m-q}} y^{-1/p}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy. \tag{5.7}$$

For the case $q = p-1$, we have $\frac{p-q-1}{p(m-q)} = 0$, (5.7) becomes

$$\left(\frac{1}{p-1}\right)^{1/p} r_{c,m}(u) = \frac{1}{m-q}\int_0^{1-\frac{q+1}{m+1}u^{m-q}} y^{-1/p}(1-y)^{-1}dy. \tag{5.8}$$

Since $\left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}} \to 1$ as $m \to \infty$, for m sufficient large, we have that $0 < u_{c,m} < \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}}$.

Taking the limit for (5.8) as $m \to \infty$ and using the L'Hôpital's rule, we deduce that

$$\begin{aligned}\left(\frac{1}{p-1}\right)^{1/p} \lim_{m\to\infty} r_{c,m}(u) &= \lim_{m\to\infty} \frac{\int_0^{1-\frac{q+1}{m+1}u^{m-q}} y^{-1/p}(1-y)^{-1}dy}{m-q} \\ &= \lim_{m\to\infty}\left(1 - \frac{q+1}{m+1}u^{m-q}\right)^{-1/p}\left(\frac{1}{m+1} - \ln u\right) \\ &= -\ln u.\end{aligned} \tag{5.9}$$

Hence

$$\lim_{m\to\infty} r_{c,m}(u) = -(p-1)^{1/p}\ln u.$$

We denote the above limit function as $r_{c,\infty} : (0,1) \mapsto (0,\infty)$, $u \to r_{c,\infty}(u) := -(p-1)^{1/p}\ln u$. Denote inverse function as $u_{c,\infty} : (0,\infty) \mapsto (0,1)$, $r \to u_{c,\infty}(r) = e^{-(p-1)^{-\frac{1}{p}}r}$, and $u_{c,\infty}(r)$ is the solution to equation (1.15) by (4.4). Noticing $u_{c,m}(0) \to 1$ as $m \to \infty$ and $u_{c,m}(r) > 0$ is continuous and strictly decreasing in $(0,\infty)$, hence we have

$$\lim_{m\to\infty} u_{c,m}(r) = u_{c,\infty}(r) \quad \text{ for any } r \in (0,+\infty).$$

For the case $q < p-1$, let $\tilde{u}_{c,m}(r) = \frac{u_{c,m}(r)}{u_{c,m}(0)}$, $u_{c,m}(0) = \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}}$. Then $\tilde{u}_{c,m}(0) = 1$, and $\tilde{u}_{c,m}(r)$ satisfies the following equation

$$(|\tilde{u}'|^{p-2}\tilde{u}')' + \left(\frac{m+1}{q+1}\right)^{\frac{m+1-p}{m-q}} \tilde{u}^m = \left(\frac{m+1}{q+1}\right)^{\frac{q+1-p}{m-q}} \tilde{u}^q,\ 0 < r < R_m, \tag{5.10}$$

$$\tilde{u}(0) = 1, \quad \tilde{u}(R_m) = \tilde{u}'(R_m) = 0. \tag{5.11}$$

For any fixed $0 < \tilde{u} < 1$, denote the inverse function of $\tilde{u}_{c,m}(r)$ as $r_{c,m}(\tilde{u})$. A direct computation gives that

$$\frac{p-1}{p}\left|\frac{1}{r'_{c,m}(\tilde{u})}\right|^p + \left(\frac{m+1}{q+1}\right)^{\frac{m+1-p}{m-q}} \frac{1}{m+1}\tilde{u}^{m+1} - \left(\frac{m+1}{q+1}\right)^{\frac{q+1-p}{m-q}} \frac{1}{q+1}\tilde{u}^{q+1} = 0. \tag{5.12}$$

From (5.12), we deduce that

$$-\left(\frac{p}{p-1}\right)^{\frac{1}{p}} r'_{c,m}(\tilde{u}) = \frac{1}{\left(\frac{u_{c,m}(0)^{q+1-p}}{q+1}\tilde{u}^{q+1} - \frac{u_{c,m}(0)^{m+1-p}}{m+1}\tilde{u}^{m+1}\right)^{1/p}}. \tag{5.13}$$

Furthermore, integrating (5.13) with respect to $\tilde{u}$ from $\tilde{u}$ to 1 and plugging $u_{c,m}(0) = \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}}$ give that

$$\left(\frac{p}{p-1}\right)^{\frac{1}{p}} r_{c,m}(\tilde{u}) = \int_{\tilde{u}}^1 \frac{1}{\left(\frac{u_{c,m}(0)^{q+1-p}}{q+1}s^{q+1}\right)^{1/p}(1-s^{m-q})^{1/p}}ds. \tag{5.14}$$

Making variable substitution $y = 1 - s^{m-q}$, we have

$$\left(\frac{p}{p-1}\right)^{\frac{1}{p}} r_{c,m}(\tilde{u}) = \frac{(q+1)^{1/p}}{m-q}\left(\frac{m+1}{q+1}\right)^{\frac{p-q-1}{p(m-q)}} \int_0^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy. \tag{5.15}$$

Noticing that $\left(\frac{m+1}{q+1}\right)^{\frac{p-q-1}{p(m-q)}} \to 1$ as $m \to \infty$, thus it holds

$$\left(\frac{p}{(p-1)(q+1)}\right)^{1/p} \lim_{m\to\infty} r_{c,m}(\tilde{u}) = \lim_{m\to\infty} \frac{\int_0^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q}. \tag{5.16}$$

Since $0 < \tilde{u} < 1$, we get $1 - \tilde{u}^{m-q} > \frac{1}{2}$ if m is large. Hence

$$\begin{aligned}\lim_{m\to\infty} \frac{\int_0^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} &= \lim_{m\to\infty} \frac{\left(\int_0^{1/2} + \int_{1/2}^{1-\tilde{u}^{m-q}}\right) y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} \\ &= \lim_{m\to\infty} \frac{\int_{1/2}^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} \\ &= \lim_{m\to\infty} \frac{-p\int_{1/2}^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}d(1-y)^{\frac{p-q-1}{p(m-q)}}}{p-q-1}. \qquad (5.17)\end{aligned}$$

Using the integration by parts, we know that

$$\begin{aligned}\int_{1/2}^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}d(1-y)^{\frac{p-q-1}{p(m-q)}} =& y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}}\Big|_{1/2}^{1-\tilde{u}^{m-q}} \\ &- \int_{1/2}^{1-\tilde{u}^{m-q}} \left(y^{-\frac{1}{p}}\right)'(1-y)^{\frac{p-q-1}{p(m-q)}}dy. \qquad (5.18)\end{aligned}$$

We can directly check

$$\lim_{m\to\infty}\int_{1/2}^{1-\tilde{u}^{m-q}} \left(y^{-\frac{1}{p}}\right)'(1-y)^{\frac{p-q-1}{p(m-q)}}dy = \lim_{m\to\infty}\int_{1/2}^{1-\tilde{u}^{m-q}} \left(y^{-\frac{1}{p}}\right)'dy = 1 - 2^{\frac{1}{p}}. \qquad (5.19)$$

Plugging (5.18) and (5.19) into (5.17), we obtain

$$\lim_{m\to\infty} \frac{\int_0^{1-\tilde{u}^{m-q}} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} = \frac{p}{p-q-1}\left(1-\tilde{u}^{\frac{p-q-1}{p}}\right). \qquad (5.20)$$

Therefore, (5.16) and (5.20) imply that

$$\begin{aligned}\lim_{m\to\infty} r_{c,m}(\tilde{u}) =& \left(\frac{p}{(p-1)(q+1)}\right)^{-1/p} \frac{p}{p-q-1}\left(1-\tilde{u}^{\frac{p-q-1}{p}}\right) \\ =& R_\infty\left(1-\tilde{u}^{\frac{p-q-1}{p}}\right), \qquad (5.21)\end{aligned}$$

where R_∞ is the same as R defined in (4.5).

We denote the above limit function as $r_{c,\infty} : (0,1) \mapsto (0, R_\infty)$:

$$\tilde{u} \to r_{c,\infty}(\tilde{u}) := R_\infty\left(1-\tilde{u}^{\frac{p-q-1}{p}}\right).$$

Denote its inverse function as $u_{c,\infty} : (0, R_\infty) \mapsto (0,1)$. Then it is given by

$$r \to u_{c,\infty}(r) = \left(1 - \frac{r}{R}\right)^{\frac{p}{p-q-1}} \quad \text{for } 0 < r < R_\infty. \qquad (5.22)$$

Next we show that $\lim_{m\to\infty} R_m = R_\infty$. Indeed,

$$\begin{aligned}\lim_{m\to\infty} R_m =& \lim_{m\to\infty}\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{(m+1)^{\frac{p-q-1}{p(m-q)}}(q+1)^{\frac{1}{p}-\frac{p-q-1}{p(m-q)}}}{m-q}\int_0^1 y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy \\ =& \left(\frac{(p-1)(q+1)}{p}\right)^{\frac{1}{p}} \lim_{m\to\infty}\frac{\int_0^1 y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q}. \qquad (5.23)\end{aligned}$$

Due to $q < p-1$ we have

$$\lim_{m\to\infty}\frac{\int_{1-\tilde{u}^{m-q}}^{1} y^{-\frac{1}{p}}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} = \lim_{m\to\infty}\frac{\int_{1-\tilde{u}^{m-q}}^{1}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy}{m-q} = \frac{p}{p-q-1}\tilde{u}^{\frac{p-q-1}{p}}. \tag{5.24}$$

Combining (5.20) and (5.24), we have

$$\lim_{m\to\infty} R_m = \left(\frac{p}{(p-1)(q+1)}\right)^{-1/p}\frac{p}{p-q-1} = R_\infty.$$

Noticing that $u_{c,m}(0) \to 1$ as $m \to \infty$ and $u_{c,m}(r) > 0$ is continuous and strictly decreasing in $(0, R_m)$, hence we have from (5.21)

$$\lim_{m\to\infty} u_{c,m}(r) = u_{c,\infty}(r) \quad \text{for any } r \in (0, R_\infty).$$

For the case $q > p-1$, the proof of (5.1) is exactly the same as the case $q < p-1$. We omit the details.

Step 2. We prove decay properties of $u_{c,m}$, i.e., for $m > 2q+1$ there is $r_* > 0$ (it is independent of m) such that $u_{c,m}$ satisfies the following estimates

- for $q > p-1$, it holds that

$$u(r) + r|u'(r)| \le Cr^{-\frac{p}{q+1-p}}, \quad \text{for } r \ge 2r_*, \tag{5.25}$$

- for $q = p-1$, it holds that

$$u(r) \le e^{-Cr}, \quad \text{for } r \ge 2r_*, \tag{5.26}$$

where C is a constant independent of m.

In fact, since $u(r) \to 0$ as $r \to \infty$, then there exists a $0 < r_0 < \infty$ such that $u(r_0) = 1$. From (5.7), we have

$$\left(\frac{p}{p-1}\right)^{1/p} r_0 = \frac{(q+1)^{1/p}}{m-q}\left(\frac{m+1}{q+1}\right)^{\frac{p-q-1}{p(m-q)}}\int_0^{1-\frac{q+1}{m+1}} y^{-1/p}(1-y)^{\frac{p-q-1}{p(m-q)}-1}dy,$$

which means that

$$0 < r_0 \le \left(\frac{p}{(p-1)(q+1)}\right)^{1-1/p}\left(\frac{m+1}{m-q}\right)^{1/p}. \tag{5.27}$$

By (5.27), we deduce if $m > 2q+1$, then $0 < r_0 < 2\left(\frac{p}{(p-1)(q+1)}\right)^{1-1/p} =: r_*$. Since $r'(r) < 0$ for $r > 0$, we have $0 < u(r) < 1$ for $r \ge r_*$, hence $\frac{u^{q+1}(r)}{q+1} - \frac{u^{m+1}(r)}{m+1} \ge u^{q+1}(r)\left(\frac{1}{q+1} - \frac{1}{m+1}\right)$ for $r \ge r_*$. Therefore from (5.6), we have that

$$-u'(r) \ge \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1} - \frac{1}{m+1}\right)^{\frac{1}{p}} u^{\frac{q+1}{p}}(r). \tag{5.28}$$

Using the method of separation of variable for (5.28) and integrating the resulting inequality from r_* to r for any $r > r_*$, we obtain

$$u^{\frac{p-q-1}{p}}(r) \ge 1 + \frac{q+1-p}{p}\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1} - \frac{1}{m+1}\right)^{\frac{1}{p}}(r - r_*).$$

Taking $r > 2r_*$, we have $r - r_* \geq \frac{r}{2}$. Therefore, for $r > 2r_*$ we know that

$$u(r) \leq \left(1 + \frac{q+1-p}{p}\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{q+1} - \frac{1}{m+1}\right)^{\frac{1}{p}}\frac{r}{2}\right)^{-\frac{p}{q+1-p}}$$
$$\leq \left(\frac{q+1-p}{2p}\right)^{-\frac{p}{q+1-p}}\left(\frac{p}{p-1}\right)^{-\frac{1}{q+1-p}}\left(\frac{1}{q+1} - \frac{1}{m+1}\right)^{-\frac{1}{q+1-p}} r^{-\frac{p}{q+1-p}}.$$

Denoting

$$C(q,m,p) := \left(\frac{q+1-p}{2p}\right)^{-\frac{p}{q+1-p}}\left(\frac{p}{p-1}\right)^{-\frac{1}{q+1-p}}\left(\frac{1}{q+1} - \frac{1}{m+1}\right)^{-\frac{1}{q+1-p}},$$

then when $m > 2q+1$, we have

$$C(q,m,p) \leq \left(\frac{q+1-p}{2p}\right)^{-\frac{p}{q+1-p}}\left(\frac{p}{p-1}\right)^{-\frac{1}{q+1-p}}\left(\frac{1}{2(q+1)}\right)^{-\frac{1}{q+1-p}} =: C(q,p).$$

Hence we obtain

$$u(r) \leq C(q,m,p) r^{-\frac{p}{q+1-p}} \leq C(q,p) r^{-\frac{p}{q+1-p}}, \quad \text{for } r \geq 2r_*. \tag{5.29}$$

Again from (5.6), we obtain for any $r > 0$

$$|u'(r)| = \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{u^{q+1}}{q+1} - \frac{u^{m+1}}{m+1}\right)^{\frac{1}{p}} \leq \left(\frac{p}{(p-1)(q+1)}\right)^{\frac{1}{p}} u^{\frac{q+1}{p}}. \tag{5.30}$$

Combining (5.29) and (5.30), we can deduce

$$r|u'(r)| \leq \left(\frac{p}{(p-1)(q+1)}\right)^{\frac{1}{p}} C(q,p)^{\frac{q+1}{p}} r^{-\frac{p}{q+1-p}} \quad \text{for } r \geq 2r_*. \tag{5.31}$$

Hence (5.29) and (5.31) give (5.25).

For the case $q = p-1$, integrating (5.28) from r_* to r for any $r > r_*$, we obtain

$$-\ln u(r) \geq \left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{p} - \frac{1}{m+1}\right)^{\frac{1}{p}}(r - r_*).$$

Taking $r > 2r_*$, we have $r - r_* \geq \frac{r}{2}$. Therefore, for $r > 2r_*$ we know that

$$\ln u(r) \leq -\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{p} - \frac{1}{m+1}\right)^{\frac{1}{p}}\frac{r}{2}.$$

Let

$$C_{m,p} := \frac{1}{2}\left(\frac{p}{p-1}\right)^{\frac{1}{p}}\left(\frac{1}{p} - \frac{1}{m+1}\right)^{\frac{1}{p}}. \tag{5.32}$$

Since $m > 2q+1 > 2p-1$, we have $C_{m,p} > \frac{1}{2}\left(\frac{1}{2(p-1)}\right)^{\frac{1}{p}} =: C(p)$. Then we obtain

$$u(r) \leq e^{-C_{m,p}r} \leq e^{-C(p)r} \quad \text{for } r \geq 2r_*, \tag{5.33}$$

i.e., (5.26) holds.

Step 3. We prove that

$$\lim_{m\to\infty} C_{q,m,p} = C_{q,\infty,p}. \tag{5.34}$$

For the case $q = p-1$, define

$$U(r) := \begin{cases} \left(\dfrac{m+1}{q+1}\right)^{\frac{1}{m-q}}, & \text{if } 0 < r \le r'_*, \\ e^{-C(p)r}, & \text{if } r > r'_*. \end{cases}$$

From (5.26), we have $u_{c,m}(r) \le U(r)$ for any $r > 0$. We directly compute $\|U(|x|)\|_{L^{q+1}} < \infty$.

For the case $q > p-1$, define

$$U(r) := \begin{cases} \left(\dfrac{m+1}{q+1}\right)^{\frac{1}{m-q}}, & \text{if } 0 < r \le \bar{r}_*, \\ C(p,q) r^{-\frac{p}{q+1-p}}, & \text{if } r > \bar{r}_*. \end{cases} \tag{5.35}$$

We have $u_{c,m}(r) \le U(r)$ for any $r > 0$ due to (5.25) and $u_{c,m}(r) \le u_{c,m}(0) = \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}}$. A direct computation gives $\|U(r)\|_{L^{q+1}} < \infty$.

For the case $q < p-1$, we know that the solution $u_{c,m}(r)$ has a finite support $(0, R_m)$, $R_m \to R_\infty$ as $m \to \infty$, where R_m is defined by (1.32). Noticing that $u_{c,m}(r) \le \left(\frac{m+1}{q+1}\right)^{\frac{1}{m-q}} \le 2$ for m large, we have $\int_0^{R_\infty} 2^{q+1} dr \le C(p,q)$. Then for the above three cases, the dominated convergence theorem implies that

$$\lim_{m\to\infty} M_c(u_{c,m}) = \lim_{m\to\infty} \int_{\mathbb{R}} u_{c,m}^{q+1}(x)dx = \int_{\mathbb{R}} u_{c,\infty}^{q+1}(x)dx = M_c(u_{c,\infty}). \tag{5.36}$$

Hence from (1.23) and (1.27), we obtain (5.34). □

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