Collectanea Mathematica
https://doi.org/10.1007/s13348-024-00449-5

Polynomial growth of Betti sequences over local rings

Luchezar L. Avramov[1] **· Alexandra Seceleanu**[1] **· Zheng Yang**[2]

Received: 4 January 2024 / Accepted: 14 July 2024

Abstract
This is a study of the sequences of Betti numbers of finitely generated modules over a complete intersection local ring, R. The subsequences $(\beta_i^R(M))$ with even, respectively, odd i are known to be eventually given by polynomials in i with equal leading terms. We show that these polynomials coincide if $I^{\square}$, the ideal generated by the quadratic relations of the associated graded ring of R, satisfies height $I^{\square} \geq$ codim $R-1$, and that the converse holds if R is homogeneous or codim $R \leq 4$. Subsequently Avramov, Packauskas, and Walker proved that the terms of degree $j >$ codim $R -$ height $I^{\square}$ of the even and odd Betti polynomials are equal. We give a new proof of that result, based on an intrinsic characterization of residue rings of c.i. local rings of minimal multiplicity obtained in this paper. We also show that that bound is optimal.

Keywords Local ring · Complete intersection · Minimal multiplicity · Free resolution · Betti number · Poincaré series · Complexity · Granularity

Mathematics Subject Classification Primary: 13D02 · 13D40; Secondary: 14M10 · 16E45

Contents

Research partly supported by National Science Foundation grants DMS-1103176, DMS-210122 and National Science Foundation of China Grant 12001384.

✉ Alexandra Seceleanu
aseceleanu@unl.edu

Luchezar L. Avramov
avramov@unl.edu

Zheng Yang
zhengyang2018@scu.edu.cn

1 Department of Mathematics, University of Nebraska, Lincoln, NE 68588, USA

2 Sichuan University – Pittsburgh Institute, Chengdu, Sichuan, China

Published online: 04 August 2024

1 Introduction

This paper is concerned with free resolutions of finitely generated modules M over a commutative noetherian ring R with unique maximal ideal, $\mathfrak{m}$. Each such module has a unique up to isomorphism *minimal* free resolution. The rank $\beta_i^R(M)$ of the ith module in such a resolution is called the ith *Betti number* of M.

The asymptotic patterns of *Betti sequences* $(\beta_i^R(M))$ reflect and affect the singularity of R. This dynamic is best understood when the ring R is *complete intersection*, abbreviated to *c.i.*; that is, when the $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the residue ring of some regular local ring modulo an ideal generated by a regular set; the smallest cardinality of such a set is equal to codim R, the *codimension* of R.

Gulliksen [24] proved that if R is c.i., then for every M there exist *Betti polynomials*, $\beta_0^{R,M}$ and $\beta_1^{R,M} \in \mathbb{Q}[x]$ with $\deg(\beta_j^{R,M}) < \operatorname{codim} R$ (where $\deg(0) := -1$, by convention) such that $\beta_i^R(M) = \beta_j^{R,M}(i)$ for $i \gg 0$ and $i \equiv j \pmod 2$. The hypothesis on R cannot be relaxed, as $\beta_i^R(k) \le b(i)$ with $k := R/\mathfrak{m}$ and $b \in \mathbb{R}[x]$ implies R is c.i. (Gulliksen, [25]), nor can the conclusion on $\beta_j^{R,M}$ be tightened, for $\beta_{\mathsf{even}}^{R,k} = \beta_{\mathsf{odd}}^{R,k}$ and $\deg(\beta_{\mathsf{even}}^{R,k}) = \operatorname{codim} R - 1$ hold when R is c.i. (Tate [40]).

Eisenbud [18] showed that if R is c.i. and codim $R \le 1$, then $(\beta_i^R(M))$ is eventually constant for every M; this was an early sign of possible connections between $\beta_{\mathsf{even}}^{R,M}$ and $\beta_{\mathsf{odd}}^{R,M}$. The general property is that these polynomials have equal degrees and leading coefficients over every c.i. ring; see Avramov [3]. The present work is a study of the discrepancy between $\beta_{\mathsf{even}}^{R,M}$ and $\beta_{\mathsf{odd}}^{R,M}$ as measured by a number,

$$\operatorname{gn}_R(M) := \deg\left(\beta_{\mathsf{even}}^{R,M} - \beta_{\mathsf{odd}}^{R,M}\right) + 1,$$

that we call the *granularity* of M over R. The least value, $\operatorname{gn}_R(M) = 0$, is attained when $(\beta_i^R(M))$ is *eventually polynomial*; that is, when $\beta_{\mathsf{even}}^{R,M} = \beta_{\mathsf{odd}}^{R,M}$.

Our main results link the granularities of R-modules and the structure of R. Let R^{g} denote the associated graded ring of R and $\pi\colon \operatorname{Sym}_k(R_1^{\mathsf{g}}) \twoheadrightarrow R^{\mathsf{g}}$ the canonical map. We write codim $R^{\square}$ for the height of the ideal generated by the quadratic forms in $\operatorname{Ker}(\pi)$ and call that number the *quadratic codimension* of R.

Theorem 1.1 (Theorem 4.1) *Every module M over a c.i. local ring R satisfies*

$$\operatorname{gn}_R(M) \le \max\{\operatorname{codim} R - \operatorname{codim} R^{\square} - 1,\ 0\}.$$

This theorem subsumes a number of contributions, related in time and content as follows. It was proved in [5] for local rings with codim R = codim $R^{\square}$. When R and M are *homogeneous* (that is, localizations of R^{g} and of a graded R^{g}-module at the maximal ideal (R_1^{g})) and codim R = codim $R^{\square} + 1 = 2$, it was obtained by Avramov and Zheng [13] using methods not available in other cases. Theorem 1.1 was proved for all c.i. rings R with codim R = codim $R^{\square} + 1$ in unpublished joint work of the authors of this paper. Motivated by that result, Avramov et al. ([11], to appear) subsequently proved the full theorem by different techniques.

The proof of Theorem 1.1, given below, extends our original approach. It relies on the following structure theorem for rings of given quadratic codimension.

Theorem 1.2 (Part of Theorem 3.7) *A local ring R with infinite residue field has* codim $R^{\square} =$ *q if and only if $\widehat{R}$ is a homomorphic image of some c.i. local ring Q with* codim $Q = q$, *multiplicity 2^q, and* edim Q = edim R.

This result is of independent interest, as every local c.i. ring of codimension q has multiplicity at least 2^q, and those *of minimal multiplicity* are to regular local rings what complete intersections of quadrics are to polynomial rings.

In the second half of the paper we explore possibilities of relaxing the hypothesis or tightening the conclusion of Theorem 1.1, and completely settle the second issue:

Theorem 1.3 (Abstracted from Theorem 5.2) *The upper bound in the inequality in Theorem 1.1 is optimal: For every pair (c, q) of integers with $c \geq q \geq 0$ there exist a c.i. local ring R and a cyclic R-module S that satisfy*

$$(\operatorname{codim} R, \operatorname{codim} R^{\square}) = (c, q) \quad \textit{and} \quad \operatorname{gn}_R(S) = \max\{c - q - 1, 0\}.$$

In order to probe the tightness of the hypotheses of Theorem 1.1 we search for partial converses to its statement. Below we focus on those rings over which all finite modules have granularity zero and obtain the following result:

Theorem 1.4 (Contained in Theorems 6.1 and 6.2) *Let $(R, \mathfrak{m}, k)$ be a c.i. local ring such that the Betti sequence of each finite R-module is eventually polynomial.*

If R is homogeneous, or $\operatorname{codim} R \leq 4$ *and k is algebraically closed, then one has*

$$\operatorname{codim} R \leq \operatorname{codim} R^{\square} + 1.$$

The proofs of Theorems 1.3 and 1.4 hinge upon identifying and constructing families of residue rings S of R, where the import of invariants of the rings R and S on the values of $\operatorname{gn}_R(S)$ can be traced explicitly. The relevant arguments involve hard computations that draw on a number of different techniques.

The results in this work and in [11] open up a new narrative concerning the patterns of Betti sequences of modules over a given c.i. ring. The methods of proof in these papers suggest possible approaches and specific questions. Here is a sample.

Question 1.5 Let R be a c.i. local ring R and set $n := \operatorname{codim} R - \operatorname{codim} R^{\square}$.

Does R have modules with non-polynomial Betti sequences if $n \geq 2$?

Do R-modules of maximal granularity, equal to $n - 1$, always exist?

We thank Nicholas Packauskas and Mark Walker for numerous useful discussions.

2 Complexity and granularity

We first overview notation, constructions, and results that will be used throughout the main text of the paper. The statement that $(R, \mathfrak{m}, k)$ is a *local ring* here means that R is a commutative noetherian ring with unique maximal ideal $\mathfrak{m}$ and k is the residue field $R/\mathfrak{m}$. As usual, $\dim R$ denotes the (Krull) dimension of R and $\operatorname{edim} R$ its *embedding dimension* (that is, the minimal number of generators of $\mathfrak{m}$); the (*embedding*) *codimension* of R is the number $\operatorname{codim} R := \operatorname{edim} R - \dim R$, and the number $\operatorname{codepth} R := \operatorname{edim} R - \operatorname{depth} R$ is its (*embedding*) *codepth*.

When $(R', \mathfrak{m}', k')$ is a local ring, a ring homomorphism $\varphi\colon R \to R'$ is *local* if $\varphi(\mathfrak{m})$ lies in $\mathfrak{m}'$. The map φ is faithfully flat if and only if it is flat and local. Surjective homomorphisms are assumed to induce the identity map on the residue fields.

For our purposes, it is often convenient to introduce invariants through non-canonical presentations of modifications of R, or of its $\mathfrak{m}$-adic completion, $\widehat{R}$.

2.1. A *regular presentation* of R is a surjective ring map $R \twoheadleftarrow P : \rho$ with P local and regular; we use the same name for an isomorphism $R \cong P/I$, with P as above.

Every regular presentation ρ factors through one that is *minimal*, meaning $\operatorname{edim} P = \operatorname{edim} R$ or, equivalently, $I \subseteq \mathfrak{p}^2$, where $\mathfrak{p}$ is the maximal ideal of P. Indeed, one has $I/(I \cap \mathfrak{p}^2) \cong \operatorname{Ker}(\mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{m}/\mathfrak{m}^2)$. Lifting a k-basis of this kernel to a subset $\boldsymbol{t}$ of I yields a minimal presentation $R \cong \overline{P}/\overline{I}$, with $\overline{P} := P/P\boldsymbol{t}$; the ring $\overline{P}$ is regular because $\boldsymbol{t}$ extends to a regular system of parameters of P.

By Cohen's Structure Theorem, regular presentations $\widehat{R} \cong P/I$ exist, and such *Cohen presentations* also produce minimal ones. Minimal Cohen presentation need not be isomorphic, but $\operatorname{rank}_k(I/\mathfrak{p}I)$ is the same for all of them (see 2.2 below); we let $\operatorname{rel} R$ denote that common value and call it the *number of relations* of R.

Throughout the paper, M denotes a finite, that is, finitely generated R-module. We review some numerical invariants of minimal free resolutions of modules over local rings. For general information on free resolutions we refer to [6].

2.2. The ith *Betti number* $\beta_i^R(M)$ of M is the rank of the ith module in a(ny) minimal free R-resolution F of M. It can be computed in different ways:

$$\beta_i^R(M) = \operatorname{rank}_k(F \otimes_R k) = \operatorname{rank}_k \operatorname{Tor}_i^R(M, k) = \operatorname{rank}_k \operatorname{Ext}_R^i(M, k).$$

One measure of the growth of the *Betti sequence* $(\beta_i^R(M))$ is given by the number

$$\operatorname{cx}_R(M) := \inf\{n \in \mathbb{N}_0 \mid \beta_i^R(M) \leq a i^{n-1} \text{ for } i \gg 0 \text{ and some } a > 0\},$$

called the *complexity* of M over R. Thus $\operatorname{cx}_R(M) = 0$ means that $\operatorname{proj\,dim}_R M$ is finite and $\operatorname{cx}_R(M) = \infty$ that $i \mapsto \beta_i^R(M)$ cannot be bounded above by a polynomial.

The Betti numbers of M are handily packed into its *Poincaré series*, given by

$$P_M^R := \sum_{i \geqslant 0} \beta_i^R(M) z^i \in \mathbb{Z}[[z]].$$

If $\widehat{R} \cong P/I$ is a minimal Cohen presentation, then the series P_I^P lies in $\mathbb{N}_0[z]$ and it is an invariant of R; in particular, so is the number $\operatorname{rank}_k(I/\mathfrak{p}I)$; see [6, 4.1.3].

We study the asymptotic behavior of a Betti sequence in terms of its Poincaré series, complexity, and granularity—a new invariant that we introduce next.

2.3. The sequence $(\beta_i^R(M))$ is *linearly recursive* if and only if the series P_M^R is *rational*; that is, if and only if $p \cdot P_M^R$ lies in $\mathbb{Z}[z]$ for some nonzero $p \in \mathbb{Z}[z]$.

If P_M^R is rational and $\operatorname{cx}_R(M)$ is finite, then the poles of P_M^R are at roots of unity, that of highest non-negative order is at 1, and its order equals $\operatorname{cx}_R(M)$; see [2, 2.4].

We say that M has *granularity* g and write $\operatorname{gn}_R(M) = g$ if P_M^R is rational and has a pole of order $g \geq 0$ at -1. Formulas involving granularity are stated or used with the tacit assumption that the relevant modules have rational Poincaré series.

The definitions of complexity and granularity given in 2.3 and those used in the introduction will soon be reconciled; see 2.5.

2.4. The properties of Poincaré series and of complexity, listed below, hold without restrictions; the formulas for granularity follow from those for Poincaré series.

(1) If N is an nth syzygy module of M over R, then one has

$$P_M^R - z^n P_N^R \in \mathbb{Z}[z], \quad \operatorname{cx}_R(M) = \operatorname{cx}_R(N), \quad \text{and} \quad \operatorname{gn}_R(M) = \operatorname{gn}_R(N).$$

(2) If $R \to (R', \mathfrak{m}', k')$ is a local ring homomorphism, M' denotes the R'-module $R' \otimes_R M$, and $\operatorname{Tor}_i^R(R', M) = 0$ holds for $i \geq 1$, then one has

$$P_M^R = P_{M'}^{R'}, \quad \operatorname{cx}_R(M) = \operatorname{cx}_{R'}(M'), \quad \text{and} \quad \operatorname{gn}_R(M) = \operatorname{gn}_{R'}(M').$$

This is the case, in particular, if R' is flat over R, or if $R' = R/R\boldsymbol{g}$ for some R-regular set $\boldsymbol{g}$ that is also M-regular.

(3) A (*codimension* n) *deformation* of R to Q is an isomorphism $R \cong Q/Q\boldsymbol{f}$, with $(Q, \mathfrak{q}, k)$ local and $\boldsymbol{f}$ a Q-regular set (of n elements); it is *embedded* if $\boldsymbol{f} \subseteq \mathfrak{q}^2$. We use the same name(s) also for the canonical homomorphism $R \twoheadleftarrow Q$.

Betti sequences whose asymptotic patterns are (almost) completely determined by complexity and granularity admit several descriptions:

2.5. Let R be a local ring and M a nonzero R-module.
The following conditions on an integer $c \geq 0$ are equivalent.

(i) There is an inclusion $(1 - z^2)^c \cdot P_M^R \in \mathbb{Z}[z]$.
(ii) There exists a unique $p_M^R \in \mathbb{Z}[z]$ with $p_M^R(1) > 0$ such that

$$P_M^R = \frac{p_M^R}{(1+z)^{\operatorname{gn}_R(M)}(1-z)^{\operatorname{cx}_R(M)}} \quad \text{and} \quad 0 \leq \operatorname{gn}_R(M) < \operatorname{cx}_R(M) \leq c.$$

(iii) There exist unique polynomials $\beta_j^{R,M} \in \mathbb{Q}[x]$ for $j = 0, 1$ that satisfying the following conditions, where the convention $\deg(0) = -1$ is used:

$$\beta_i^R(M) = \beta_j^{R,M}(i) \quad \text{for} \quad i \gg 0 \quad \text{when} \quad i \equiv j \pmod 2, \quad \text{and}$$
$$\operatorname{gn}_R(M) = \deg\left(\beta_0^{R,M} - \beta_1^{R,M}\right) + 1 < \deg\left(\beta_j^{R,M}\right) + 1 = \operatorname{cx}_R(M) \leq c.$$

Indeed, it is shown in [3, Proof of Theorem 4.1] that (i) implies (ii). Partial fraction decomposition yields the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

2.6. A collection of known results illustrates the conditions in 2.5.

(1) If $\operatorname{proj\,dim}_Q(\widehat{M})$ is finite for some codimension c deformation $\widehat{R} \cong Q/Q\boldsymbol{f}$, then 2.5(i) holds; see [24, 4.2(i)]; this is a major source of modules whose Poincaré series have poles only at ± 1. However, R-modules M with $P_M^R = 2/(1-z)$ may exist over rings R that admit no non-trivial deformations; see [9].
(2) The ring R is said to be *complete intersection*, or *c.i.*, if $\widehat{R}$ admits a deformation to some regular local ring. When $\widehat{R} \cong P/I$ is a minimal Cohen presentation, R is c.i. if and only if I can be minimally generated by some P-regular set, if and only if $\operatorname{rel} R = \operatorname{codim} R$ (i.e., I can be generated by codim R elements, see 2.1).
(3) The following conditions are equivalent: (i) R is c.i.; (ii) $(1 - z^2)^c \cdot P_M^R \in \mathbb{Z}[z]$ for every R-module M; (iii) $P_k^R = (1+t)^{\dim R}/(1-t)^{\operatorname{codim} R}$; (iv) $\operatorname{cx}_R(k) < \infty$.

See (1) for (i) $\Longrightarrow$ (ii), [40, Theorem 6] for (i) $\Longrightarrow$ (iii), and [25, 2.3] for (iv) $\Longrightarrow$ (i).

2.7. Maps of local rings $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ that are flat with $\mathfrak{m}R' = \mathfrak{m}'$ will be called *adjustments* of R; for every R-module M and $M' := R' \otimes_R M$ one has

$$H_{M'} = H_M, \quad \dim_{R'} M' = \dim_R M, \quad \text{and} \quad \operatorname{edim} R' = \operatorname{edim} R'.$$

Any adjustment $R \to (R', \mathfrak{m}', k')$, composed with the completion map $R' \to \widehat{R'}$ and some minimal Cohen presentation $\widehat{R'} \cong P/I$ yields an adjustment $R \to P/I$ with $(P, \mathfrak{p}, k')$

regular and edim P = edim R. In adjustments $R \to P/I$ with P regular, the presentation $P/I \twoheadleftarrow P$ is minimal if and only if edim P = edim R.

Grothendieck [23, 10.3.1] proved that every field extension $k \subseteq l$ occurs as the extension of residue fields induced by adjustments of R, called *inflations*.

3 Associated quadratic rings

Recall that $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module.

In this section we introduce and study invariants of R that are defined in terms of its associated graded ring and appear in the main results of the paper. We first record terminology and notation used when dealing with associated graded objects; for general background on graded rings and their modules; see 7.1.

3.1. Set $M^{\mathfrak{g}}_j = \mathfrak{m}^j M/\mathfrak{m}^{j+1}M$ for $j \in \mathbb{Z}$, and $M^{\mathfrak{g}} = \bigoplus_{j\in\mathbb{Z}} M^{\mathfrak{g}}_j$. Thus, $R^{\mathfrak{g}}$ is the *associated graded ring* of R and $M^{\mathfrak{g}}$ the *associated graded* $R^{\mathfrak{g}}$*-module* of M. For $x \in M \setminus \{0\}$ set $v(x) = \max\{j \mid x \in \mathfrak{m}^j\}$. The image of x in $M^{\mathfrak{g}}_{v(x)}$ is called the *initial form* of x and is denoted by x^*; in addition, we set $0^* = 0$.

We write H_M for $\sum_{j\geqslant 0} \operatorname{rank}_k M^{\mathfrak{g}}_j(z)$. Since $R^{\mathfrak{g}}$ is generated by $R^{\mathfrak{g}}_1$ over $R^{\mathfrak{g}}_0 = k$, the Hilbert-Serre Theorem yields $h^R_M \in \mathbb{Z}[z]$, with $h^R_M(1) \neq 0$, such that

$$H_M = h^R_M \cdot (1-z)^{-\dim R}.$$

The integer $h^R_M(1)$, called the *multiplicity* of M over R, is denoted by $e_R(M)$; one has $e_R(M) \geq 0$, with equality if and only if $\dim M < \dim R$; set $e(R) = e_R(R)$.

3.2. Let $\widehat{R} \twoheadleftarrow P$ ρ be a minimal Cohen presentation; see 2.1. It induces k-linear isomorphisms $\mathfrak{p}/\mathfrak{p}^2 \cong \widehat{\mathfrak{p}}/\widehat{\mathfrak{p}}^2 \cong \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2 \cong \mathfrak{m}/\mathfrak{m}^2$ that we use to identify these vector spaces, and hence their symmetric k-algebras. Thus we view $P^{\mathfrak{g}}$ (cf. 3.1) as the symmetric k-algebra of $\mathfrak{m}/\mathfrak{m}^2$ and we have a canonical surjection $R^{\mathfrak{g}} \twoheadleftarrow P^{\mathfrak{g}}$ $\rho^{\mathfrak{g}}$. Set $I^* := \operatorname{Ker}(\rho^{\mathfrak{g}})$ and call the isomorphism $R^{\mathfrak{g}} \cong P^{\mathfrak{g}}/I^*$ the *canonical presentation of* $R^{\mathfrak{g}}$; if a minimal regular presentation ρ (see 2.7) is at hand, then I^* is equal to the ideal of $P^{\mathfrak{g}}$ generated by the set of leading forms $\{f^*\}_{f\in I}$.

As P is regular and dim P = edim R, the following relations hold:

$$\text{height } I^* = \operatorname{codim} R^{\mathfrak{g}} = \operatorname{codim} R = \text{height } I \leq \operatorname{rel} R \leq \operatorname{rel} R^{\mathfrak{g}}. \tag{3.2.1}$$

3.3. We define the *associated quadratic ring* of R to be the graded k-algebra

$$R^{\square} := P^{\mathfrak{g}}/I^{\square}, \quad \text{where} \quad I^{\square} := P^{\mathfrak{g}} I^*_2. \tag{3.3.1}$$

It is an invariant of R, as is the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^{\square} & \longrightarrow & P^{\mathfrak{g}} & \longrightarrow & R^{\square} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & I^* & \longrightarrow & P^{\mathfrak{g}} & \xrightarrow{\rho^{\mathfrak{g}}} & R^{\mathfrak{g}} & \longrightarrow & 0 \end{array} \tag{3.3.2}$$

By definition, the ideal $I^{\square}$ is minimally generated by rel $R^{\square}$ quadrics.

If $R \cong P/I$ is a minimal regular presentation, it yields surjective homomorphisms

$$I \twoheadrightarrow I/\mathfrak{p}I \twoheadrightarrow I/(\mathfrak{p}^3 \cap I) \cong (I + \mathfrak{p}^3)/\mathfrak{p}^3 = I^*_2 = I^{\square}_2. \tag{3.3.3}$$

Letting $\overline{f}$ denote the class of $f \in I$ in $I/\mathfrak{p}I$ and $f^\square$ its class in $(I+\mathfrak{p}^3)/\mathfrak{p}^3 = I_2^\square$, one gets a k-linear surjection $\overline{f} \mapsto f^\square$ with $f^\square = f^*$ for $f \notin \mathfrak{p}^3$ and $f^\square = 0$ otherwise.

For ease of reference, we spell out a few formal properties of that construction.

Lemma 3.4 *The ring $R^\square$ and the ideal $I^\square$ from (3.3.1) satisfy the relations below.*

$$\operatorname{codim} R^\square = \operatorname{edim} R - \dim R^\square = \operatorname{height} I^\square \le \operatorname{rel} R^\square \le \operatorname{rel} R\,. \tag{3.4.1}$$

$$\operatorname{codim} R^\square - \operatorname{codim} R = \dim R - \dim R^\square = \operatorname{height} I^\square - \operatorname{height} I^* \le 0\,. \tag{3.4.2}$$

If 3.2 and $R \to (R', \mathfrak{m}', k')$ is an adjustment (see 2.7), then

$$\operatorname{codim} R^\square = \operatorname{codim} R'^\square \quad \textit{and} \quad \operatorname{rel} R^\square = \operatorname{rel} R'^\square. \tag{3.4.3}$$

If $R \twoheadleftarrow (Q, \mathfrak{q}, k)$ is a surjective ring homomorphism with kernel in $\mathfrak{q}^2$, then

$$\operatorname{codim} R^\square \ge \operatorname{codim} Q^\square \quad \textit{and} \quad \operatorname{rel} R^\square \ge \operatorname{rel} Q^\square. \tag{3.4.4}$$

Proof In (3.4.1) the equalities hold because $P^{\mathfrak{g}}$ is a polynomial ring; in (3.4.2) they follow from (3.4.1) and (3.2.1). The Principal Ideal Theorem, the surjection (3.3.3), and $I^\square \subseteq I^*$ yield the inequalities in (3.4.1) and (3.4.2).

Let $P'^{\mathfrak{g}}$ denote the symmetric k'-algebra of $\mathfrak{m}'/\mathfrak{m}'^2$; since $R \to R'$ induces isomorphisms $R^{\mathfrak{g}} \otimes_k k' \cong R'^{\mathfrak{g}}$ and $P^{\mathfrak{g}} \otimes_k k' \cong P'^{\mathfrak{g}}$ of graded k'-algebras, $(? \otimes_k k')$ turns (3.3.2) into the corresponding diagram for R', and (3.4.3) follows.

In particular, for (3.4.4) we may assume that R is complete. A minimal Cohen presentation $\widehat{Q} \cong P/J$ then yields such a presentation $R \cong P/I$. As $P^{\mathfrak{g}} \twoheadrightarrow R^{\mathfrak{g}}$ factors through $P^{\mathfrak{g}} \twoheadrightarrow Q^{\mathfrak{g}}$, we get $I^* \supseteq J^*$, whence $I^\square \supseteq J^\square$, and (3.4.4) follows. □

The largest value of $\operatorname{codim} R^\square$ allowed by (3.4.1) is $\operatorname{edim} R$: it is reached if and only if $I^\square = 0$; that is, if and only if I lies in $\mathfrak{p}^3$. We have no similar description of the rings with $\operatorname{codim} R^\square = \operatorname{codim} R$, the largest value allowed by (3.4.2), except if R is *c.i.*; see Proposition 3.6; we record a few facts used in its proof, and later on.

3.5. Let $(P, \mathfrak{p}, k)$ be a local ring and $Q := P/(g_1, \dots, g_s)$ with $g_i \in \mathfrak{p}^{n_i}$ for $1 \le i \le s$.

(1) The set $\{g_1^*, \dots, g_s^*\}$ is $P^{\mathfrak{g}}$-regular if and only if it generates $\operatorname{Ker}(P^{\mathfrak{g}} \to Q^{\mathfrak{g}})$ and $\{g_1, \dots, g_s\}$ is P-regular; see Valabrega and Valla [41, 2.7 and 1.1].

(2) When $\{g_1, \dots, g_s\}$ is part of a system of parameters, there is an inequality

$$e(Q) \ge n_1 \cdots n_s \cdot e(P);$$

equality holds if $\{g_1^*, \dots, g_s^*\}$ is $P^{\mathfrak{g}}$-regular; see [14, VIII, §7, Proposition 4].

(3) Assume $P^{\mathfrak{g}}$ is Cohen-Macaulay. The set $\{g_1^*, \dots, g_s^*\}$ is $P^{\mathfrak{g}}$-regular if and only if $\{g_1, \dots, g_s\}$ is P-regular and equality holds in (2) above; such an equivalence is proved by Rossi and Valla [36, 1.8] under the additional hypothesis that $\{g_1, \dots, g_s\}$ is regular, which is superfluous in one direction, due to (1) above.

(4) When Q is c.i. and $\widehat{Q} \twoheadleftarrow P$ is a minimal Cohen presentation, (2) above yields

$$e(Q) = e(\widehat{Q}) \ge v(g_1) \cdots v(g_s) \cdot e(P) \ge 2^{\operatorname{codim} \widehat{Q}} = 2^{\operatorname{codim} Q}.$$

When $e(Q) = 2^{\operatorname{codim} Q}$ holds the ring Q is said to be *c.i. of minimal multiplicity*.

(5) The ring Q is c.i. (of minimal multiplicity), if some, and only if all of its adjustments have the corresponding property. Indeed, the invariants used to define these notions do not change when Q is replaced by an adjustment; see 3.1.

Next we collect various characterizations of local c.i. rings of minimal multiplicity. They are used in upcoming proofs to produce and/or to recognize such rings. For notions and notation concerning graded rings we refer to Sect. 7.

Proposition 3.6 *The following conditions on a local ring Q are equivalent.*

(i) *Q is c.i. of minimal multiplicity (see 3.5(5)).*
(ii) *Q is c.i. and* $\operatorname{codim} Q = \operatorname{codim} Q^{\square}$.
(iii) *Q is c.i. and $Q^{\mathsf{g}} \cong Q^{\square}$ as graded k-algebras.*
(iv) *Q is c.i. and the graded k-algebra Q^{g} is Koszul (see 7.1(2)).*
(v) *Q^{g} is a graded complete intersection of quadrics (see 7.1(3)).*
(vi) *If $\widehat{Q} \cong P/J$ is a minimal Cohen presentation and $\{g_1, \dots, g_s\}$ minimally generates J, then $\{g_1^{\square}, \dots, g_s^{\square}\}$ minimally generates J^* and is P^{g}-regular.*

Proof We set $d := \operatorname{edim} Q$ and assume, as we may (see 3.5(4)) that Q is complete.

(i) $\Longrightarrow$ (vi). With $n_i := v(g_i)$ for $1 \le i \le s$ in 3.5(4), we get $n_i = 2$, and hence $g_i^* = g_i^{\square}$. Thus $\{g_1^{\square}, \dots, g_s^{\square}\}$ is P^{g}-regular by 3.5(3) and generates J^* by 3.5(1).

(vi) $\Longrightarrow$ (v). This implication follows from the hypothesis, as $Q^{\mathsf{g}} = P^{\mathsf{g}}/J^*$.

(v) $\Longrightarrow$ (iv). The ring Q is c.i., by 3.5(1). From $P_k^{Q^{\mathsf{g}}} = (1+yz)^e/(1-y^2z^2)^s$ (see 7.1(3)), we get $\beta_{i,j}^{Q^{\mathsf{g}}}(k) = 0$ for $j \ne i$; therefore Q^{g} is a Koszul algebra.

(iv) $\Longrightarrow$ (i). As Q^{g} is Koszul, $\sum_j \beta_{i,j}^{Q^{\mathsf{g}}}(k) = \beta_i^Q(k)$ holds for every integer i; see Şega [37, 2.3]. With $s := \operatorname{codim} Q$, this result yields the third equality in the string

$$\frac{1}{H_{Q^{\mathsf{g}}}(-z)} = \frac{H_k(-z)}{(-z)^0 H_{Q^{\mathsf{g}}}(-z)} = P_k^{Q^{\mathsf{g}}}(1,z) = P_k^Q(z) = \frac{(1+z)^{\dim Q}}{(1-z)^s}.$$

The second one comes from 7.2(1), and the fourth from 2.6(3). Thus we get equalities $H_Q(y) = H_{Q^{\mathsf{g}}}(y) = (1+y)^s/(1-y)^{\dim Q}$, whence $e(Q) = 2^{\operatorname{codim} Q}$.

(vi) $\Longrightarrow$ (iii). This implication is given by 3.5(1).

(iii) $\Longrightarrow$ (ii). This implication holds because $\operatorname{codim} Q = \operatorname{codim} Q^{\mathsf{g}}$.

(ii) $\Longrightarrow$ (i). The hypothesis and Formulas (3.2.1) and (3.4.1) yield (in)equalities

$$\operatorname{codim} Q^{\mathsf{g}} = \operatorname{codim} Q = \operatorname{codim} Q^{\square} \le \operatorname{rel} Q^{\square} \le \operatorname{rel} Q = \operatorname{codim} Q$$

that force equalities throughout. In particular, $Q^{\square}$ is a graded complete intersection of $s := \operatorname{codim} Q$ quadrics; from the surjection $Q^{\square} \twoheadrightarrow Q^{\mathsf{g}}$ and 3.5(5), we obtain

$$2^{\operatorname{codim} Q} = 2^s = e(Q^{\square}) \ge e(Q^{\mathsf{g}}) = e(Q) \ge 2^{\operatorname{codim} Q}.$$ □

We are ready for the main results in this section, which concern general local rings. In the special case when $k = k'$ and $R' = R'' = \widehat{R}$, the first theorem below yields a structure theorem for local rings with prescribed quadratic codimension, stated in the introduction as Theorem 1.2. The second theorem provides, under manageable additional hypotheses, families of local rings with prescribed quadratic codimension parametrized by dense subsets of affine spaces.

Theorem 3.7 *Let $(R, \mathfrak{m}, k)$ be a local ring, and set $r :=$* $\operatorname{rel} R$ *and $q :=$* $\operatorname{codim} R^{\square}$.

(1) *For each field extension $k \hookrightarrow k'$ with k' infinite there exists an adjustment $R \to P/I$ with $(P, \mathfrak{p}, k')$ regular, $I = (f_1, \dots, f_r)$, and $Q := P/(f_1, \dots, f_q)$ c.i. of minimal multiplicity; every such adjustment satisfies* $\operatorname{edim} P = \operatorname{edim} R$.

(2) *If $R \to R'$ is an adjustment, $R' \twoheadleftarrow R'' \twoheadleftarrow Q$ are surjective ring maps, and Q is local c.i. of minimal multiplicity with* $\operatorname{codim} Q = q$ *and* $\operatorname{edim} Q = \operatorname{edim} R$*, then*

$$\operatorname{codim} R^{\square} = \operatorname{codim} R''^{\square} = \operatorname{codim} Q^{\square} = \operatorname{codim} Q.$$

Proof (1) Referring to 2.7, choose an adjustment $R \to R' = P/I$ with $k \hookrightarrow k'$ the induced residue field extension and $P \twoheadrightarrow P/I$ a minimal regular presentation. Due to the equalities height $I^{\square} = \operatorname{codim} R'^{\square} = \operatorname{codim} R^{\square} = q$ and $\operatorname{rel} R'^{\square} = \operatorname{rel} R = r$ (see 3.4.1 and 3.4.3), the ideal $I^{\square}$ is minimally generated by r elements and contains $P^{\mathfrak{g}}$-regular sets of q forms. As $I^{\square}$ is generated by quadrics and k' is infinite, $I_2^{\square}$ contains a $P^{\mathfrak{g}}$-regular set of q elements. In view of (3.3.3), it can be chosen in the form $\{f_1^{\square}, \dots, f_q^{\square}\}$ with $f_i \in I$; by (vi) $\Longrightarrow$ (i) in Proposition 3.6, the ring $P/(f_1, \dots, f_q)$ is c.i. of minimal multiplicity. Since $\{f_1^{\square}, \dots, f_q^{\square}\}$ is k-independent, $\{f_1, \dots, f_q\}$ can be extended to a minimal set of generators of I.

(2) The hypothesis provides the equalities that bookend the following string:

$$q = \operatorname{codim} R^{\square} = \operatorname{codim} R'^{\square} \geq \operatorname{codim} R''^{\square} \geq \operatorname{codim} Q^{\square} = \operatorname{codim} Q = q.$$

For the rest, use (3.4.3), (3.4.4), and (i) $\Longrightarrow$ (ii) in Proposition 3.6. □

Theorem 3.8 *Let $(P, \mathfrak{p}, k)$ be a local ring; let $\overline{a}$ denote the image in k of $a \in P$. Given $(f_1, \dots, f_r) \in (\mathfrak{p}^2)^r$ and $\boldsymbol{a} := (a_1, \dots, a_{r-1}) \in P^{r-1}$ set $f_i^{\boldsymbol{a}} := f_i - a_i f_r$ for $1 \leq i \leq r-1$. Put $\overline{\boldsymbol{a}} := (\overline{a}_1, \dots, \overline{a}_{r-1})$, where $\overline{a}$ is the image of a in k.*

If k is algebraically closed, $\{f_1, \dots, f_q, f_r\}$ is P-regular for some $q < r$, and $\{f_1^{\square}, \dots, f_q^{\square}\}$ is $P^{\mathfrak{g}}$-regular, then the following set is Zariski-open and not empty:

$$\mathcal{U} := \{\overline{\boldsymbol{a}} \in \mathbb{A}_k^{r-1} \mid P/(f_1^{\boldsymbol{a}}, \dots, f_q^{\boldsymbol{a}}) \text{ is c.i. of minimal multiplicity}\}.$$

Proof Let $k[\boldsymbol{x}]$ be the polynomial ring with indeterminates $x_1, \dots, x_q$.

The ring $P/(f_1^{\boldsymbol{a}}, \dots, f_q^{\boldsymbol{a}})$ is c.i. of minimal multiplicity if and only if the set $\boldsymbol{f}_{\boldsymbol{a}}^{\square} := \{f_1^{\square} - \overline{a}_1 f_r^{\square}, \dots, f_q^{\square} - \overline{a}_q f_r^{\square}\}$ is $P^{\mathfrak{g}}$-regular; see (i) $\Longleftrightarrow$ (vi) in Proposition 3.6. The set $\boldsymbol{f}_{\boldsymbol{a}}^{\square}$ is regular if and only if $\dim P_{\boldsymbol{a}}^{\mathfrak{g}} \leq d - q$ holds with $P_{\boldsymbol{a}}^{\mathfrak{g}} := P^{\mathfrak{g}}/P^{\mathfrak{g}} \boldsymbol{f}_{\boldsymbol{a}}^{\square}$ and $d := \dim P$. The algebra $P_{\boldsymbol{a}}^{\mathfrak{g}}$ is the fiber of the canonical map

$$k[\boldsymbol{x}] \to (k[\boldsymbol{x}] \otimes_k P^{\mathfrak{g}})/(1 \otimes f_1^{\square} - x_1 \otimes f_r^{\square}, \dots, 1 \otimes f_q^{\square} - x_q \otimes f_r^{\square})$$

at the maximal ideal $\mathfrak{n}_{\boldsymbol{a}} := (x_1 - \overline{a}_1, \dots, x_q - \overline{a}_q)$. Since fiber dimension is upper semicontinuous (see [19, 14.8.b]), $V := \{\overline{\boldsymbol{a}} \in \mathbb{A}_k^q \mid \dim(P_{\boldsymbol{a}}) > d - q\}$ is closed in $\mathbb{A}_k^q$. Thus the set $\mathcal{U}$ is open, as it equals $A_k^q \setminus V$, and $\mathcal{U}$ contains $\mathbf{0}$ by hypothesis. □

Free resolutions over c.i. rings of minimal multiplicity are known to have special properties. We note two, which will be promptly applied in the next section.

3.9. If Q is a local c.i. ring of minimal multiplicity and N a Q-module, then one has $P_N^Q = p_N^Q(z) \cdot (1 - z)^{-\operatorname{cx}_Q(N)}$ with $p_N^Q(z) \in \mathbb{Z}[z]$ and $p_N^Q(1) \neq 0$; see [5, 2.3].

Proposition 3.10 *With notation as in 3.9,* $\dim N < \dim Q$ *implies* $p_N^Q(-1) = 0$.

Proof As $Q^{\mathfrak{g}}$ is Koszul (see Proposition 3.6), the module N has finite linearity defect; see Herzog and Iyengar, [27, 5.10]. Thus Şega' s result [38, 6.2] applies and, in view of the expression for P_N^Q in 3.9, it yields the first equality in the string

$$p_N^Q(-1)e(Q) = (1 - (-1))^{\operatorname{cx}_Q(N)} e_Q(N) = 2^{\operatorname{cx}_Q(N)} e_Q(N).$$

Since $\dim N < \dim Q$ means $e_Q(N) = 0$, we obtain $p_N^Q(-1) = 0$, as desired. □

4 An upper bound on granularity

Here our goal is to give a concise proof of Theorem 1.1.

Theorem 4.1 *Every finite module M over a c.i. local ring $(R, \mathfrak{m}, k)$ satisfies*

$$\mathrm{gn}_R(M) \leq \begin{cases} \operatorname{codim} R - \operatorname{codim} R^{\square} - 1 & \text{if } \operatorname{codim} R \geq \operatorname{codim} R^{\square} + 2; \\ 0 & \text{if } \operatorname{codim} R \leq \operatorname{codim} R^{\square} + 1. \end{cases} \tag{4.1.1}$$

As noted in the introduction, a different proof Theorem 4.1 was obtained in [11]. Our argument, presented in 4.8, proceeds by induction on $n := \operatorname{codim} R - \operatorname{codim} R^{\square}$; see (3.4.2). For this we need yet another way of factoring embedded deformations through deformations with specified properties, provided by the next result.

Theorem 4.2 *Let $(P, \mathfrak{p}, k)$ be a local ring, $\boldsymbol{f} := \{f_1, \dots, f_c\}$ a P-regular set contained in $\mathfrak{p}^2$, and put $R := P/P\boldsymbol{f}$. For every $\boldsymbol{a} := (a_1, \dots, a_{c-1}) \in P^{c-1}$, set $f_j^{\boldsymbol{a}} := f_j - a_j f_c$ for $1 \leq j \leq c-1$ and put $\boldsymbol{f}_{\boldsymbol{a}} := (f_1^{\boldsymbol{a}}, \dots, f_{c-1}^{\boldsymbol{a}})$ and $R_{\boldsymbol{a}} := P/P\boldsymbol{f}_{\boldsymbol{a}}$.*

Let N be an nth R-syzygy module of a finite R-module M that satisfies

$$\operatorname{proj\,dim}_P(M) < \infty = \operatorname{proj\,dim}_R(M).$$

There exist an integer $\operatorname{cr\,deg}_R(M) \geq -1$ and a finite set $Z(M)$ of linear varieties in $\mathbb{A}_k^{c-1}$, defined in (4.6.2), such that the following conditions are equivalent:

(i) $P_N^R = P_N^{R_{\boldsymbol{a}}} \cdot (1 - z^2)^{-1}$.

(ii) $n > \operatorname{cr\,deg}_R(M)$ *and* $\overline{\boldsymbol{a}} \notin \bigcup_{V \in Z(M)} V$, *where $\overline{\boldsymbol{a}}$ is the image of $\boldsymbol{a}$ in $\mathbb{A}_k^{c-1}$.*

If k is infinite, then $\bigcup_{V \in Z(M)} V \neq \mathbb{A}_k^{c-1}$.

Remark 4.3 The prototype of Theorem 4.2 is [18, Theorem 3.1], and both proofs utilize rings of cohomology operators defined by the deformation $R \twoheadleftarrow Q$. Such a structure was introduced by Gulliksen [24], and theories with similar properties were produced in [3, 8, 10, 12, 18, 33] from a priori incomparable constructions.

We present the proof in detail because the argument for [18, 3.1] is incomplete and references to several sources are needed to fill in the gaps. The facts that we use are listed below, along with pointers to the earliest published proof.

4.4. The hypotheses in the opening sentence of Theorem 4.2 are in force.

Let X denote the k-vector space $P\boldsymbol{f}/\mathfrak{m}\boldsymbol{f}$ and $\overline{\boldsymbol{f}}$ its basis $\{\overline{f}_1, \dots, \overline{f}_c\}$, consisting of the images of f_j for $j = 1, \dots, c$. Let $\{\chi_1, \dots, \chi_c\}$ be the dual basis of the vector space $\mathcal{X} := \operatorname{Hom}_k(X, k)$ and let $\mathcal{R}$ be the symmetric algebra of the graded vector space that has $\mathcal{X}$ in degree 2 and 0 in all other degrees. We identify $\mathcal{R}$ and the graded polynomial ring $k[\chi_1, \dots, \chi_c]$ with indeterminates of degree 2.

The graded vector space $\operatorname{Ext}_R(M, k) := \bigoplus_{i \geqslant 0} \operatorname{Ext}_R^i(M, k)$ supports a structure of graded left $\mathcal{R}$-modules that has the following properties:

(1) The assignment $? \rightsquigarrow \operatorname{Ext}_R(?, k)$ is a contravariant additive functor from the category of R-modules to that of graded $\mathcal{R}$-modules; see [24, Theorem 3.1(i)].
(2) The connecting maps in cohomology sequences induced by short exact sequences of R-modules commute with the actions of $\mathcal{R}$; this holds as [12, Theorem, p. 700] shows that the action of $\mathcal{R}$ factors through Yoneda products.

(3) For $R' := P/(f_1, \ldots, f_{c-1})$ and for every $i \in \mathbb{Z}$ there is an exact sequence

$$\mathrm{Ext}_R^{i-2}(M,k) \xrightarrow{\chi_c} \mathrm{Ext}_R^i(M,k) \to \mathrm{Ext}_{R'}^i(M,k) \to \mathrm{Ext}_R^{i-1}(M,k) \xrightarrow{\chi_c} \mathrm{Ext}_R^{i+1}(M,k)$$

See [3, Theorem 2.3]; it is implicit in [24, Formula (8) on p. 178].

(4 The $\mathcal{R}$-module $\mathrm{Ext}_R(M,k)$ is finitely generated if the R-module M is finite with $\mathrm{proj\,dim}_P\, M < \infty$; see [24, Theorem 3.1(ii)]. The proof of [18, Theorem 3.1] is flawed: it uses [18, Proposition 1.6] whose proof is invalid; see [12, Remark 4.2].

The properties of $\mathcal{R}$-modules described above can be used concurrently because the results in [12, Section 4] show that the sets of operators produced by the constructions in [3, 8, 10, 12, 18, 33] differ at most by some sign.

At a final stop before the proof in 4.6 we introduce notation to describe $Z(M)$.

4.5. Let k be a field, $\mathcal{X}$ a k-vector space, and $\{\chi_j\}_{1\leqslant j\leqslant c}$ a basis of $\mathcal{X}$.

Let $\mathcal{V} \subseteq \mathcal{X}$ be a subspace of rank d. If $\mathcal{V} \neq 0$ let $\left\{\sum_{j=1}^{c} a_{l,j}\chi_j\right\}_{1\leqslant l\leqslant d}$ with $a_{l,j} \in k$ be a basis of $\mathcal{V}$; let $k[\boldsymbol{x}] := k[x_1, \ldots, x_{c-1}]$ be a polynomial ring, put

$$A(\boldsymbol{x}) := \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,d} & x_1 \\ a_{2,1} & a_{2,2} & \ldots & a_{2,d} & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{c-1,1} & a_{c-1,2} & \ldots & a_{c-1,d} & x_{c-1} \\ a_{c,1} & a_{c,2} & \ldots & a_{c,d} & 1 \end{pmatrix}$$

and let $\mathcal{V}^\perp$ be the zero set in $\mathbb{A}_k^{c-1}$ of the maximal minors of $A(\boldsymbol{x})$ that contain the last column; also, set $0^\perp := \emptyset$. For $\boldsymbol{u} := (u_1, \ldots, u_{c-1}) \in \mathbb{A}_k^{c-1}$, one has:

$$\left[\sum_{j=1}^{c-1} u_i\chi_j + \chi_c \in \mathcal{V}\right] \iff \left[\mathrm{rank}_k\, A(\boldsymbol{u}) = d\right] \iff \left[\boldsymbol{u} \in \mathcal{V}^\perp\right]. \qquad (4.5.1)$$

If these conditions hold, then $\mathcal{V}^\perp$ is a linear subvariety of $\mathbb{A}_k^{c-1}$.

4.6. *Proof of Theorem* 4.2. We keep the notation in the statement of the theorem.

A minimal free resolution F of M yields an exact sequence of R-modules

$$0 \to N \to F_{n-1} \to \cdots \to F_0 \to M \to 0 \qquad (4.6.1)$$

finite free F_i's. Thus N satisfies $\mathrm{proj\,dim}_R(N) = \infty > \mathrm{proj\,dim}_P(N)$, and hence

$$\mathcal{M} := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}_R^i(M,k) \quad \text{and} \quad \mathcal{N} := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}_R^i(N,k)$$

are finitely generated graded left $\mathcal{R}$-modules; see 4.4(4). As $\boldsymbol{f_a} \cup \{f_c\}$ minimally generates $P\boldsymbol{f}$, the set $\{\overline{f_1^{\boldsymbol{a}}}, \ldots, \overline{f_{c-1}^{\boldsymbol{a}}}\} \cup \{\overline{f_c}\}$ is a k-basis of the vector space X defined in 4.4. The dual basis of the space $\mathrm{Hom}_k(X,k) = \mathcal{R}_2$ is the set $\{\chi_1^{\overline{\boldsymbol{a}}}, \ldots, \chi_c^{\overline{\boldsymbol{a}}}\}$ of the $\mathcal{X} := \mathcal{R}_2$ has $\chi_j^{\overline{\boldsymbol{a}}} = \chi_j$ for $j \le i \le c-1$ and $\chi_c^{\overline{\boldsymbol{a}}} = \sum_{j=1}^{c-1} \overline{a}_j\chi_j + \chi_c$.

Put $R_{\boldsymbol{a}} := P/P\boldsymbol{f}_{\boldsymbol{a}}$. From 4.4(3) we get an exact sequences of k-vector spaces

$$0 \to \mathcal{K}_{i-2} \to \mathrm{Ext}_R^{i-2}(N,k) \to \mathrm{Ext}_R^i(N,k) \to \mathrm{Ext}_{R_{\boldsymbol{a}}}^i(N,k) \to \mathcal{K}_{i-1} \to 0$$

with $\mathcal{K}_i := \{v \in \mathrm{Ext}_R^i(N,k) \mid \chi_c^{\overline{\boldsymbol{a}}} v = 0\}$ for $i \in \mathbb{Z}$. The resulting equality

$$(1-z^2)P_N^R = P_N^{R_{\boldsymbol{a}}} - (1+z)\sum_{i\geqslant 0} \mathrm{rank}_k\, \mathcal{K}^i z^i$$

shows that condition (i) in the theorem is equivalent to the following condition:

(i′) $\chi_c^{\overline{\boldsymbol{a}}} = \sum_{j=1}^{c-1} \overline{a}_j \chi_j + \chi_c$ is $\mathcal{N}$-regular.

The iterated connecting maps $\operatorname{Ext}_R^i(N,k) \to \operatorname{Ext}_R^{i+n}(M,k)$ defined by the exact sequence (4.6.1) are bijective. In view of 4.4(2) they coalesce into an isomorphism $\mathcal{N} \cong \mathcal{M}_{\geqslant n}(n)$ of graded $\mathcal{R}$-modules. Put $\operatorname{Ass}^\circ_{\mathcal{R}}(\mathcal{M}) := \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}) \setminus \{\mathcal{R}_{>0}\}$, and also

$$\begin{aligned} \operatorname{cr}\deg_R(M) &:= \sup\{i \in \mathbb{Z} \mid \operatorname{Ann}_{\mathcal{R}}(\mu) = \mathcal{R}_{>0} \text{ for some } \mu \in \mathcal{M}_i\}; \\ Z(M) &:= \{(\mathcal{P}_2)^\perp \subseteq \mathbb{A}_k^{c-1} \mid \mathcal{P} \in \operatorname{Ass}^\circ_{\mathcal{R}}(\mathcal{M})\} \text{ with } ?^\perp \text{ defined in 4.5.} \end{aligned} \tag{4.6.2}$$

We complete the proof of the theorem by showing that (i′) is equivalent to (ii).

The number $t := \operatorname{cr}\deg_R(M)$ is an integer because $(0 :_{\mathcal{M}} \mathcal{R}_{>0})$ is a homogeneous subspace of $\mathcal{R}$ and has finite k-rank. The following statements are equivalent:

$$\big[\operatorname{Ass}_{\mathcal{R}}(\mathcal{M}_{\geqslant n}) = \operatorname{Ass}^\circ_{\mathcal{R}}(\mathcal{M})\big] \iff \big[\mathcal{R}_{>0} \notin \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}_{\geqslant n})\big] \iff \big[n > t\big]. \tag{4.6.3}$$

Indeed, we have $\operatorname{Ass}_{\mathcal{R}}(\mathcal{M}_{\geqslant n}) \subseteq \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}) \subseteq \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}_{\geqslant n}) \cup \{\mathcal{R}_{>0}\}$ as $\mathcal{M}/\mathcal{M}_{\geqslant n}$ is of finite length; the implication $\Leftarrow$ on the left follows, the rest hold by definition.

As $\mathcal{M}/\mathcal{M}_{<t}$ is not zero, we have $Z := Z(M) \neq \emptyset$. Condition (i′) is equivalent to $\chi^{\overline{\boldsymbol{a}}} \notin \bigcup_{\mathcal{P} \in \operatorname{Ass}_{\mathcal{R}}(\mathcal{M}_{\geqslant n})} \mathcal{P}$, and hence to $n > t$ and $\chi^{\overline{\boldsymbol{a}}} \notin \bigcup_{\mathcal{P} \in Z} \mathcal{P}$, due to (4.6.2). The last exclusion is equivalent to $\chi^{\overline{\boldsymbol{a}}} \notin \bigcup_{\mathcal{P} \in Z} \mathcal{P}_2$ (as each $\mathcal{P}$ is homogeneous), which can be rewritten as $\overline{\boldsymbol{a}} \notin \bigcup_{\mathcal{P} \in Z} (\mathcal{P}_2)^\perp$, by (4.5.1). Finally, recall that affine spaces over infinite fields are not unions of finitely many proper linear subvarieties. □

Remark 4.7 A *critical degree* is defined in [10, 7.1] for every nonzero finite module over any local ring in terms of chain endomorphisms of its minimal free resolutions; in the context of Theorem 4.2 it is equal to the integer in (4.6.2); see [10, 7.2(1)].

A priori estimates for the critical degree are known in case $\operatorname{proj\,dim}_Q M$ is finite for some deformation $R \twoheadleftarrow Q$; they involve the number $g := \operatorname{depth} R - \operatorname{depth}_R M$:

- $\operatorname{cr}\deg_R(M) = g$ if $\operatorname{cx}_R M \leq 0$, by the Auslander-Buchsbaum Equality.
- $\operatorname{cr}\deg_R(M) \leq g$ if $\operatorname{cx}_R M = 1$; see [18, 5.3 and 6.1] and [10, 7.3(1)].
- $\operatorname{cr}\deg_R(M) \leq g + \max\{2\beta_g^R - 2, 2\beta_{g+1}^R - 1\}$ if $\operatorname{cx}_R M = 2$; see [8, 7.6]).

The last assertion of Theorem 4.2 may fail when k is finite; see [3, 6.7].

4.8. *Proof of Theorem* 4.1. Set $c := \operatorname{codim} R$, $q := \operatorname{codim} R^\square$, and $n := c - q$.

We argue by induction on n. When $n = 0$ the ring R has minimal multiplicity (see Proposition 3.6), and then 3.9 yields $\operatorname{gn}_R M = 0$; this is the desired result.

Now we assume $n \geq 1$ and set out to prove $\operatorname{gn}_R(M) < n$. The invariants in play do not change under adjustments of R; see 2.4(2) and (3.4.3). Due to Theorem 3.7(1) we may assume k algebraically closed and $R = P/P\boldsymbol{f}$ for some regular local ring $(P, \mathfrak{p}, k)$ and P-regular sequence $\boldsymbol{f} := (f_1, \ldots, f_c)$ contained in $\mathfrak{p}^2$ such that $Q := P/(f_1, \ldots, f_q)$ is a c.i. ring of minimal multiplicity and $\operatorname{codim} Q^\square = q < c$.

For every $\boldsymbol{a} := (a_1, \ldots, a_{c-1}) \in P^{c-1}$ and $1 \leq i \leq c-1$, put $f_i^{\boldsymbol{a}} := f_i - a_i f_c$. The deformation $R \twoheadleftarrow P$ factors as a composition of deformations

$$R \twoheadleftarrow R_{\boldsymbol{a}} := P/(f_1^{\boldsymbol{a}}, \ldots, f_{c-1}^{\boldsymbol{a}}) \twoheadleftarrow Q_{\boldsymbol{a}} := P/(f_1^{\boldsymbol{a}}, \ldots, f_q^{\boldsymbol{a}}) \twoheadleftarrow P.$$

Let $\overline{\boldsymbol{a}}$ be the image of $\boldsymbol{a}$ in $\mathbb{A}_k^{c-1}$. Theorem 4.2 yields an R-syzygy module N of M and a non-empty Zariski-open set $\mathcal{U}_1$ of $\mathbb{A}_k^{c-1}$ such that $\overline{\boldsymbol{a}} \in \mathcal{U}_1$ implies

$$P_N^R = P_N^{R_{\boldsymbol{a}}} \cdot (1 - z^2)^{-1}. \tag{4.8.1}$$

Theorem 3.8 produces a non-empty Zariski-open set $\mathcal{U}_2 \neq \emptyset$ of $\mathbb{A}_k^{c-1}$ such that for each $\overline{\boldsymbol{a}} \in \mathcal{U}_2$ the ring $Q_{\boldsymbol{a}}$ is c.i. of codimension q and minimal multiplicity. Note that $\mathcal{U}_1 \cap \mathcal{U}_2$ is not empty and choose $\boldsymbol{a}$ with $\overline{\boldsymbol{a}} \in \mathcal{U}_1 \cap \mathcal{U}_2$.

If $n = 1$, then $R_{\boldsymbol{a}} = Q_{\boldsymbol{a}}$ and $\dim N < \dim R_{\boldsymbol{a}}$ hold and we obtain

$$P_N^R = \frac{P_N^{R_{\boldsymbol{a}}}}{(1-z^2)} = \frac{(1+z)\cdot p(z)}{(1-z)^{\operatorname{cx}_{R_{\boldsymbol{a}}}(N)}\cdot(1-z^2)} = \frac{p(z)}{(1-z)^{\operatorname{cx}_R(N)+1}}$$

from (4.8.1) and Proposition 3.10. This gives $\operatorname{gn}_R N = 0$, and hence $\operatorname{gn}_R M = 0$ (see 2.4(1)); thus (4.1.1) holds for $n = 1$. When $n \geq 2$ we may suppose, by induction, that (4.1.1) holds for local rings S with $\operatorname{codim} S - \operatorname{codim} S^{\square} < n$. Referring to 2.4(1), (4.8.1), the induction hypothesis, and Theorem 3.7(2) we get

$$\operatorname{gn}_R M = \operatorname{gn}_R N \leq \operatorname{gn}_{R_{\boldsymbol{a}}} N + 1 < \operatorname{codim} R_{\boldsymbol{a}} - \operatorname{codim}(R_{\boldsymbol{a}})^{\square} + 1 = c - q\,.$$

The induction step is complete, and with it the proof of (4.1.1). □

5 The upper bound is optimal

In this section we prove that the upper bound on granularity, established in Theorem 4.1, cannot be tightened in general; see Theorem 5.2 below.

5.1. Let $\varphi : (R, \mathfrak{m}, k) \twoheadrightarrow (S, \mathfrak{n}, k)$ be a surjective homomorphism of local rings. Choose a minimal Cohen presentation $\rho : (P, \mathfrak{p}, k) \twoheadrightarrow \widehat{R}$. Put $I := \operatorname{Ker}(\rho)$ and $\widetilde{J} := \operatorname{Ker}(\widehat{\varphi}\rho)$, and choose a subset $\boldsymbol{t}$ of $\widetilde{J}$ that is mapped bijectively onto some k-basis of $\widetilde{J}/\widetilde{J} \cap \mathfrak{p}^2$. Put $(Q, \mathfrak{q}, k) := (P/P\boldsymbol{t}, \mathfrak{p}/P\boldsymbol{t}, k)$ and choose in $\mathfrak{p}$ a subset that is is mapped bijectively onto some minimal set of generators $\mathfrak{q}$. The exact sequence

$$0 \to \widetilde{J}/\widetilde{J} \cap \mathfrak{p}^2 \to \mathfrak{p}/\mathfrak{p}^2 \to \mathfrak{q}/\mathfrak{q}^2 \to 0$$

of k-vector spaces shows that $\boldsymbol{t} \sqcup \boldsymbol{u}$ minimally generates $\mathfrak{p}$. Thus $\widehat{S} \cong Q/J$ with $J := P/I$ is a minimal Cohen presentation. As P_J^Q is an invariant of S (cf. 2.2), so is the first integer defined below; the second one is an invariant of φ (see [7]):

$$m(S) := \max\{n \in \mathbb{N}_0 : (1+z)^n \mid (z^2 P_J^Q - 1)\} \ \text{ and } \ a(\varphi) := \operatorname{rank}_k(I/I \cap \mathfrak{p}\widetilde{J}),$$

Theorem 5.2 *If d, c, q, a are integers that satisfy $d \geq c \geq q, a \geq 0$, then there exist a c.i. local ring $(R, \mathfrak{m}, k)$ and a residue ring S of R with $\mathfrak{m}^3 S = 0$ such that*

$$(\operatorname{edim} R, \operatorname{codim} R, \operatorname{codim} R^{\square}, a(\varphi)) = (d, c, q, a) \quad \text{and} \tag{5.2.1}$$

$$\operatorname{gn}_R(S) = \max\{c - q - 1, 0\}. \tag{5.2.2}$$

As a consequence, the upper bound in Theorem 4.1 is optimal.

The proof of the theorem, presented in 5.5, has two crucial ingredients. The first one is a closed formula for the Poincaré series of Golod residue rings of c.i. rings.

5.3. We assign nicknames to invariants of R, S, and φ defined in 2.1, 2.2, and 5.1:

$$\begin{aligned} d &:= \operatorname{edim} R, & c &:= \operatorname{codim} R, & q &:= \operatorname{codim} R^{\square}, & r &:= \operatorname{rel} R;\\ e &:= \operatorname{edim} S, & m &:= m(S); & a &:= a(\varphi). \end{aligned} \tag{5.3.1}$$

These numbers compare as follows:

$$0 \leq q \leq c \leq d \geq e \geq m \geq 0 \leq a \leq r \geq c \ \text{ and } \ r = c \iff R \ \text{ is c.i.} \tag{5.3.2}$$

Recall that the ring S is said to be *Golod* if it satisfies the relation

$$P_k^S = (1+z)^e/(1-z^2P_J^Q)$$

for some (and hence, for every – see 2.2) minimal Cohen presentation $\widehat{S} \cong Q/J$.

If R is c.i. and S is Golod, then the following equality holds; see [7]:

$$P_S^R = \frac{(1+z)^{a+1}(1-z)^a + z^2P_J^Q - 1}{z(1+z)^{c-d+e}(1-z)^c}\,. \tag{5.3.3}$$

The very special case $S = R/\mathfrak{m}^2$ of this result first appeared as [5, Theorem 2.1].

The second ingredient is the next theorem, where we identify families of Golod residue rings S of an *arbitrary* c.i. ring R and express their granularities in terms of the numbers in (5.3.1). The construction of the rings S and the computation of their invariants utilize a different set of techniques; they are deferred to Sect. 7.

Theorem 5.4 *Let $(Q, \mathfrak{q}, k)$ be a regular local ring of dimension $e \geq 1$, $\boldsymbol{u}$ a regular system of parameters, U a $2\times(h+1)$ matrix with $h \geq 1$ and entries in $\boldsymbol{u} \cup \{0\}$, $I_2(U)$ the ideal generated by the 2×2 minors of U, and $J := I_2(U) + \mathfrak{q}^3$. Put $S := Q/J$, let $\varphi\colon R \twoheadrightarrow S$ be a surjective ring map, and let d, c, a, m be as in (5.3.1).*

If U is adequate for $\boldsymbol{u}$ (see 7.4), then $h \leq e$ holds and S is Golod. If, furthermore, R is c.i., then $\operatorname{gn}_R(S)$ depends on the position of a respective to h and e, as follows:

(a) *$h = e$; this is equivalent to $S = Q/\mathfrak{q}^2$ and it implies $m = e$ and*

$$\operatorname{gn}_R(S) = \begin{cases} \max\{c-d+e-a-1, 0\} & \text{if } a \leq e-2; \\ 0 & \text{if } a \geq e-1. \end{cases} \tag{5.4.1}$$

(b) *$h \leq e-1$; this implies $m = h+1$ and*

$$\operatorname{gn}_R(S) = \begin{cases} \max\{c-d+e-a-1, 0\} & \text{if } a \leq h-1; \\ \max\{c-d+e-h-1, 0\} & \text{if } a \geq h. \end{cases} \tag{5.4.2}$$

Proof The case $s = 2$ of Theorem 7.9(2) shows that S is Golod, $h \leq e$ holds, and $h = e$ is equivalent to $J = \mathfrak{q}^2$; in addition, it yields an equality

$$z^2P_J^Q - 1 = \frac{(1+z)^e}{z}\left(1 - ez + \frac{(e+h+1)(e-h)}{2}z^2\right) + \frac{(1+z)^{h+1}}{z}(hz-1).$$

To compute $\operatorname{gn}_R(S)$ we feed the above expression into Formula (5.3.1), write P_S^R as a rational function, evaluate the order of its pole at -1, and refer to 2.5(ii).

(a) When $h = e$ holds, we get $z^2P_J^Q - 1 = (1+z)^e(ez-1)$, whence $m = e$ and

$$P_S^R = \frac{(1+z)^{a+1}(1-z)^a + (1+z)^e(ez-1)}{z(1+z)^{c-d+e}(1-z)^c}.$$

If $a \leq e-2$ the highest power of $(1+z)$ that divides the numerator is $a+1$; this verifies the order of the pole of P_S^R at -1 announced in (5.4.1). If $a \geq e-1$, then that highest power is $e+1$ when $(a, e) = (2, 3)$, and e otherwise; in neither case does P_S^R have a pole at -1, as $c - d \leq 0$ holds. Now the proof of (5.4.1) is complete.

(b) When $h \leq e-1$ holds, we have $z^2P_J^Q - 1 = (1+z)^{h+1}\cdot p(z)$ with

$$p(z) := \frac{(1+z)^{e-h-1}}{z}\left(1 - ez + \frac{(e+h+1)(e-h)}{2}z^2\right) + \frac{1}{z}(hz-1).$$

The equalities $p(-1) = h + 1$ if $h \le e - 2$ and $p(-1) = -(h + 2)$ if $h = e - 1$ show that $m = h + 1$ holds in both cases. Therefore (5.3.3) takes the form

$$P^R_S = \begin{cases} \dfrac{(1-z)^a + (1+z)^{h-a} \cdot p(z)}{z(1+z)^{c-d+e-a-1}(1-z)^c} & \text{if } a \le h-1; \\ \dfrac{(1+z)^{a-h}(1-z)^a + p(z)}{z(1+z)^{c-d+e-h-1}(1-z)^c} & \text{if } a \ge h. \end{cases}$$

Evaluating the numerators of P^R_S at $z = -1$ yields 2^a if $a \le h - 1$ and $p(-1) \ne 0$ if $a > h$; therefore (5.4.2) holds when $a \ne h$. When $a = h$ the formula above becomes

$$P^R_S = \frac{(1-z)^h + p(z)}{z(1+z)^{c-d+e-h-1}(1-z)^c}.$$

At $z = -1$ the numerator equals $2^h + h + 1$ if $h \le e - 2$ and $2^h - h - 2$ if $h = e - 1$; this settles (5.4.2) except if $(h, e) = (2, 3)$, where $(1 - z)^2 + p(z) = z(z + 1)$ yields

$$\operatorname{gn}_R(S) = \max\{c-d-1, 0\} = 0 = \max\{c-d+e-h-1, 0\}. \qquad \square$$

Appropriate choices, in that order, of a matrix U and of a ring R in Theorem 5.4 provide the setup for proving that the upper bound in (4.1.1) is optimal.

5.5. *Proof of Theorem* 5.2. Let $(P, \mathfrak{p}, k)$ be a d-dimensional regular local ring, e an integer satisfying $0 \le e \le d$, and $\{t_1, \dots, t_{d-e}\} \sqcup \{u_1, \dots, u_e\}$ a regular system of parameters for P. Thus $Q := P/(t_1, \dots, t_{d-e})$ is regular and the canonical map $P \twoheadrightarrow Q$ sends $\{u_1, \dots, u_e\}$ bijectively onto a minimal set of generators of $\mathfrak{q} := \mathfrak{p}Q$.

If $q > a$ the hypothesis yields $d \ge c \ge q > a \ge 0$, and hence the number $e := d - q + a$ satisfies $d - e = q - a > 0$; define residue rings of P by setting

$$\begin{aligned} R &:= P/I \text{ with } I := (t_1^2, \dots, t_{q-a}^2) + (u_1^2, \dots, u_a^2) + (u_{q+1}^4, \dots, u_c^4); \\ S &:= P/\widetilde{J} \text{ with } \widetilde{J} := (t_1, \dots, t_{q-a}) + (u_1, \dots, u_q)^2 + (u_1, \dots, u_e)^3. \end{aligned}$$

If $q \le a$ holds, then the line-up is $d \ge c \ge a \ge q \ge 0$. Choose $e = d$ and set

$$\begin{aligned} R &:= P/I \text{ with } I := (u_1^2, \dots, u_q^2) + (u_{q+1}^3, \dots, u_a^3) + (u_{a+1}^4, \dots, u_c^4); \\ S &:= P/\widetilde{J} \text{ with } \widetilde{J} := (u_1, \dots, u_q)^2 + (u_1, \dots, u_d)^3. \end{aligned}$$

It is clear that R is c.i with $(\operatorname{edim} R, \operatorname{codim} R, \operatorname{codim} R^{\square}) = (d, c, q)$. Choose $\varphi\colon R \twoheadrightarrow S$ to be the homomorphism defined by $I \subseteq \widetilde{J}$; in both cases it is easy to see that $\{u_1, \dots, u_a\}$ is a k-basis of $I/I \cap \mathfrak{p}\widetilde{J}$, and this yields $a(\varphi_*) = a$. The ideal

$$\widetilde{J} = I_2(U) + \mathfrak{q}^3 \quad \text{with} \quad U := \begin{bmatrix} u_1 & u_2 & \dots & u_q & 0 \\ 0 & u_1 & \dots & u_{q-1} & u_q \end{bmatrix}$$

of Q satisfies $\widetilde{J} := JQ$ and $Q/\widetilde{J} = S$. Applying Theorem 5.4 with $h = q$ yields

$$\operatorname{gn}_R(S) = \begin{cases} \max\{c-d+(d-q+a)-a-1, 0\} = \max\{c-q-1, 0\} & \text{if } a \le q-1; \\ \max\{c-d+d-q-1, 0\} = \max\{c-q-1, 0\} & \text{if } a \ge q. \end{cases}$$

The proof of Theorem 5.2 is complete. $\square$

As another application of Theorem 5.4, we show that the existence of residue rings S with $\mathfrak{n}^2 = 0$ and $\operatorname{gn}_R(S) = 0$ imposes upper bounds on $\operatorname{codim} R$.

Proposition 5.6 *Let* $(R, \mathfrak{m}, k)$ *be a c.i. local ring,* $\widehat{R} \cong P/I$ *a minimal Cohen presentation,* $\mathfrak{p}$ *the maximal ideal of* P*, and* L *a proper ideal of* P *satisfying* $L_2^* \supseteq I_2^{\square}$.

If $S := P/(L + \mathfrak{p}^2)$ *has* $\mathrm{gn}_R(S) = 0$*, then the following inequalities hold:*

$$\operatorname{codim} R - 1 \le \operatorname{rank}_k L_1^* + \operatorname{rank}_k \left(I_2^{\square}/(I_2^{\square} \cap P_1^{\mathsf{g}} L_1^*)\right). \tag{5.6.1}$$

$$\operatorname{codim} R - 1 \le \operatorname{rel} R^{\square} \quad \text{if} \quad L \subseteq \mathfrak{p}^2 . \tag{5.6.2}$$

Proof Choose a subset $\boldsymbol{t}$ of L that is mapped bijectively onto some k-basis of $L/L \cap \mathfrak{p}^2$. The ideal $\widetilde{J} := P\boldsymbol{t} + \mathfrak{p}^2$ satisfies $\widetilde{J} = L + \mathfrak{p}^2$, $\widetilde{J}_1^* = L_1^*$, and

$$\frac{I_2^{\square}}{I_2^{\square} \cap P_1^{\mathsf{g}} J_1^*} \cong \frac{I_2^{\square} + P_1^{\mathsf{g}} J_1^*}{P_1^{\mathsf{g}} J_1^*} \cong \frac{(I + \mathfrak{p}(P\boldsymbol{t} + \mathfrak{p}^2))/\mathfrak{p}^3}{\mathfrak{p}(P\boldsymbol{t} + \mathfrak{p}^2)/\mathfrak{p}^3} \cong \frac{I + \mathfrak{p}J}{\mathfrak{p}J} \cong \frac{I}{I \cap \mathfrak{p}J}.$$

With notation from 5.3, we get $\operatorname{rank}_k(I_2^{\square}/I_2^{\square} \cap P_1^{\mathsf{g}} J_1^*) = a$, and hence the right-hand side of (5.6.1) equals $d - e + a$. As $\mathfrak{q}^2 S = 0$, Theorem 5.4(a) applies to $R \twoheadrightarrow S$, and (5.4.1) yields $c - 1 \le d - e + a$ when $a \le e - 2$. On the other hand, when $a \ge e - 1$ we get $c - 1 \le d - 1 \le d - e + a$ from the relation $c \le d$; see (5.3.2). Now Formula (5.6.1) has been proved. Formula (5.6.2) records the special case $L_1^* = 0$. □

The proposition yields a stronger and sharper version of [5, Theorem B].

Corollary 5.7 *If* $(R, \mathfrak{m}, k)$ *is a local ring with* $\mathrm{cx}_R(\mathfrak{m}^2) < \infty$ *and* $\mathrm{gn}_R(\mathfrak{m}^2) = 0$*, then* R *is c.i. and* $\operatorname{codim} R \le \operatorname{rel} R^{\square} + 1$ *holds.*

Proof Put $L := R/\mathfrak{m}^2$. From 2.4(1) and [4, Theorem 4 and Proposition 2], one gets $\mathrm{cx}_R(k) = \mathrm{cx}_R(L) = \mathrm{cx}_R(\mathfrak{m}^2) < \infty$; thus R is c.i. and $\mathrm{gn}_R(L) = \mathrm{gn}_R(\mathfrak{m}^2) = 0$ holds; see 2.6(3) and 2.4(1). Now Formula (5.6.2) yields $\operatorname{codim} R \le \operatorname{rel} R^{\square} + 1$. □

6 Eventually polynomial Betti sequences

Recall that $(R, \mathfrak{m}, k)$ denotes a local ring and M a finite R-module.

We say that the Betti sequence $(\beta_i^R(M))$ is *eventually polynomial* if there exists $\beta^{R,M} \in \mathbb{Q}[z]$ such that $\beta_i^R(M) = \beta^{R,M}(i)$ holds for $i \gg 0$. In this section we look for conditions on the structure of R that imply or follow from the property that the Betti sequence of *every* finite R-module is eventually polynomial.

The next result yields the homogeneous case in Theorem 1.4 in the introduction; it answers, in the positive, a question raised at the end of the introduction of [5].

Theorem 6.1 *If* A *is a standard graded* k*-algebra and* R *is its localization at the maximal ideal* (A_1)*, then the following conditions are equivalent.*

(i) *The Betti sequence of* $R/\mathfrak{m}^2$ *is eventually polynomial.*
(ii) *The Betti sequence of each finite* R*-module is eventually polynomial.*
(iii) *The ring* R *is c.i. and satisfies* $\operatorname{codim} R \le \operatorname{codim} R^{\square} + 1$.
(iv) *The graded algebra* R^{g} *is c.i. and has at most one non-quadratic relation.*

Proof Let $\pi\colon \operatorname{Sym}_k(A_1) \twoheadrightarrow A$ be the canonical map of graded k-algebras. Localizing π at the maximal ideal (A_1) yields a minimal regular presentation $R \cong P/I$ with $I = \operatorname{Ker}(\pi)P$ and isomorphisms $R^{\mathsf{g}} \cong A \cong P^{\mathsf{g}}/I^*$ of graded k-algebras; cf. 3.2. They induces isomorphisms $R^{\square} \cong P^{\mathsf{g}}/I_2^*$ of graded algebras and $I/\mathfrak{p}I \cong I^*/P_1^{\mathsf{g}} I^*$ of k-vector spaces, where $\mathfrak{p}$ is the maximal ideal of P.

(i) $\Longrightarrow$ (iv). In view of Corollary 5.7, R is c.i. with $\operatorname{codim} R - \operatorname{rel} R^{\square} \le 1$. Choose $f_1, \dots, f_r \in I$ such that $\{f_1^*, \dots, f_r^*\}$ minimally generates I^* and $\{f_1^*, \dots, f_b^*\}$ is a k-basis of I_2^*. From $I/\mathfrak{p}I \cong I^*/P_1^{\mathsf{g}} I^*$ we get $r = \operatorname{codim} R$ and $I = (f_1, \dots, f_r)$, and hence $\{f_1, \dots, f_r\}$ is P-regular; therefore $\{f_1^*, \dots, f_r^*\}$ is P^{g}-regular; see 3.5(1). From $R^{\square} \cong P^{\mathsf{g}}/I_2^*$ we get $\operatorname{rel} R^{\square} = \operatorname{rank}_k I_2^* = b$, whence $r - b \le 1$, as desired.

(iv) $\Longrightarrow$ (iii). Choose $f_1, \dots, f_c \in I$ such that $\{f_1^*, \dots, f_c^*\}$ is P^{g}-regular, generates I^*, and $\deg(f_j^*) = 2$ for $1 \le j \le c-1$. As R is isomorphic to the localization of R^{g} at (P^*) (due to $A \cong P^{\mathsf{g}}/I^*$), the image of $\{f_1^*, \dots, f_c^*\}$ in P is a regular set that generates I, and $\{f_1^*, \dots, f_{c-1}^*\}$ is k-independent in $(I_2 + (P_1)^2)/(P_1)^3 \cong I_2^*$.

(iii) $\Longrightarrow$ (ii). This implication follows from Theorem 4.1.

(ii) $\Longrightarrow$ (i). This implication is a tautology. □

The part of Theorem 1.4 concerning c.i. rings of low codimension comes from

Theorem 6.2 *When $(R, \mathfrak{m}, k)$ is a local ring whose cyclic modules S with $\mathfrak{m}^j S = 0$ for some $j \ge 1$ have eventually polynomial Betti sequences, then R is c.i., and*

$$\operatorname{codim} R - \operatorname{codim} R^{\square} \le \max\{\operatorname{codim} R - i, 1\} \tag{6.2.1}$$

holds for $i \le 2$ if $j = 2$, and also for $i = 3$ if $j = 3$ and k is algebraically closed.

This theorem is proved in 6.4. The argument draws upon a classical description of the homogeneous prime ideals of codimension two and minimal multiplicity in polynomial rings over algebraically closed fields; Huneke, Mantero, McCullough, and Seceleanu [31] used it to bound the projective dimensions of those ideals. We review the relevant parts, using notation that will facilitate the references in 6.4.

6.3. Let k be an algebraically closed field, P^{g} a polynomial ring over k with variables $\{u_1^*, \dots, u_d^*\}$ of degree one, and D a homogeneous prime ideal of P^{g}. The ideal D is said to be *degenerate* if $D_1 \neq 0$, and *non-degenerate* otherwise; in the latter case, a well known inequality involves the multiplicity of P^{g}/D (see [20, Proposition 0]):

$$e(P^{\mathsf{g}}/D) \ge \operatorname{height} D + 1. \tag{6.3.1}$$

Homogeneous prime ideals of height two admit explicit descriptions, possibly after some change of variables. The ideal D is degenerate if and only if $D = (u_1^*, g_2^*)$ with g_2^* an irreducible form in $k[u_2^*, \dots, u_d^*]$; in this case, $e(P^{\mathsf{g}}/D) = \deg(g_2^*)$.

Non-degenerate D belong to one of two types. If $e(P^{\mathsf{g}}/D) = 3$, then D is the ideal generated by the 2×2 minors of one of the matrices U^*, displayed below:

$$\begin{bmatrix} u_1^* & u_2^* & u_3^* \\ u_4^* & u_1^* & u_2^* \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} u_1^* & u_2^* & u_3^* \\ u_4^* & u_5^* & u_2^* \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} u_1^* & u_2^* & u_3^* \\ u_4^* & u_5^* & u_6^* \end{bmatrix}. \tag{6.3.2}$$

If $e(P^{\mathsf{g}}/D) \neq 3$, then $D = (g_1^*, g_2^*)$ for some P^{g}-regular set $\{g_1^*, g_2^*\}$ of forms and $e(P^{\mathsf{g}}/D) = \deg(g_1^*)\deg(g_2^*)$. This classification was obtained in [42] and [39, Theorem 3] (see also [20, Theorem 1]); it is described as above in [21, p. 63].

6.4. *Proof of Theorem* 6.2. The Betti sequence of k is eventually polynomial, by assumption, and therefore R is c.i.; see 2.6(3). Replacing R with $\widehat{R}$ does not change the hypothesis of the theorem (as both rings have the same modules of prescribed Loewy length), nor its conclusion (see (3.4.3)). Thus we may assume $R = P/I$ with $(P, \mathfrak{p}, k)$ a regular local and I generated by a regular sequence in $\mathfrak{p}^2$.

Put $c := \operatorname{codim} R$ and $q := \operatorname{codim} R^{\square}$, and hence $q = \operatorname{height} I^{\square}$; see (3.4.1). We show by (a very short!) induction on i that if certain cyclic R-modules have eventually polynomial Betti sequences, then $c \geq i+1$ implies $q \geq i$ for $0 \leq i \leq 3$.

There is nothing to prove when $i = 0$. Suppose that the cyclic R-modules S with $\mathfrak{m}^2 S = 0$ have $\operatorname{gn}_R(S) = 0$. If $i = 1$, then from Corollary 5.7 we get $\operatorname{rel} R^{\square} \geq c - 1 \geq 1$, whence $I^{\square} \neq 0$, and hence $q \geq 1$. When $i = 2$, the claim is that $I^{\square} \neq 0$ and $c \geq 3$ implies $q \geq 2$. Indeed, $q = 1$ means that $I^{\square}$ is contained in $P^{\mathfrak{g}} g^*$ for some $g \in \mathfrak{p}$ with $\deg(g^*) = 1$ or $\deg(g^*) = 2$, and then Formula (5.6.1) applied with $L := Pg$ yields $2 \leq c - 1 \leq 1$, which is absurd.

Now assume that k is algebraically closed and the cyclic R-modules S annihilated by $\mathfrak{m}^3$ have $\operatorname{gn}_R(S) = 0$. We claim that $I^{\square} \neq 0$ and $c \geq 4$ implies $q \geq 3$. For the sake of contradiction, suppose $q = 2$; then $P^{\mathfrak{g}}$ has prime ideals of height 2 that contain I, and they are homogeneous; let D be one of them. As k is infinite, there exist $f_1, f_2 \in I$ such that $\{f_1^{\square}, f_2^{\square}\}$ is $P^{\mathfrak{g}}$-regular. The surjective ring homomorphisms $P^{\mathfrak{g}}/(f_1^{\square}, f_2^{\square}) \twoheadrightarrow P^{\mathfrak{g}}/I^{\square} \twoheadrightarrow P^{\mathfrak{g}}/D$ yield inequalities of multiplicities, to wit

$$4 = e(P^{\mathfrak{g}}/(f_1^{\square}, f_2^{\square})) \geq e(P^{\mathfrak{g}}/I^{\square}) \geq e(P^{\mathfrak{g}}/D) \geq 1.$$

We rule out every admissible value of $e(P^{\mathfrak{g}}/D)$ by using the classification in 6.3.

When $e(P^{\mathfrak{g}}/D) \neq 3$, one has $D = (g_1^*, g_2^*)$ for some regular set $\{g_1^*, g_2^*\}$ of forms with $n_i := \deg(g_i^*)$ satisfying $1 \leq n_1 \leq n_2 \leq 2$. We prove that the existence of such a set implies $c \leq 3$, which is ruled out by our hypothesis. Indeed, D equals L^* for $L := (g_1, g_2) \subset P$; see 3.5(1). If $n_2 = 1$, then $P_1^{\mathfrak{g}} L_1^* = L_2^* \supseteq I_2^{\square}$ holds and (5.6.1) yields $c - 1 \leq 2 + 0$. If $n_1 < n_2$, then we have $L_1^* = k g_1^*$ and $L_2^* = P_1^{\mathfrak{g}} g_1^* \oplus k g_2^*$; as $I_2^{\square}$ contains a $P^{\mathfrak{g}}$-regular set of two elements, we get $I_2^{\square} \not\subseteq P_1^{\mathfrak{g}} g_1^*$, whence $P_1^{\mathfrak{g}} g_1^* + I_2^{\square} = L_2^*$, and hence $I_2^{\square}/(I_2^{\square} \cap P_1^{\mathfrak{g}} L_1^*) \cong k g_2^*$; now (5.6.1) gives $c - 1 \leq 1 + 1$. Finally, $n_1 = 2$ implies $c - 1 \leq 0 + 2$, again by (5.6.1).

If $e(P^{\mathfrak{g}}/D) = 3$, then $D = (y_1, y_2, y_3)$, where the y_js are the 2×2 minors of a 2×3 matrix U^* in Formula (6.3.2) and u_i^* is the initial form of u_i, where $\boldsymbol{u} := \{u_1, \ldots, u_d\}$ of $\mathfrak{p}$ is a minimal generating set. Let U be the matrix obtained from U^* by replacing each u_i^* with u_i, and let g_j be the minor of U that corresponds to y_j. As $g_j^* = y_j$ holds for $j = 1, 2, 3$, the ideal $L := (g_1, g_2, g_3)$ of P has $L_1^* = 0$ and $L_2^* = D_2$; also, I lies in $\widetilde{J} := L + \mathfrak{p}^3$, as seen from the relations

$$\frac{I + \mathfrak{p}^3}{\mathfrak{p}^3} = I_2^{\square} \subseteq D_2 = L_2^{\square} = \frac{L + \mathfrak{p}^3}{\mathfrak{p}^3} = \frac{\widetilde{J}}{\mathfrak{p}^3}.$$

Put $Q := P$ and $S := P/\widetilde{J}$, and let d, c, a, e be the numbers assigned in 5.3 to the canonical map $R \twoheadrightarrow S$. The matrix U is adequate for $\boldsymbol{u}$ (cf. 7.4), so Theorem 5.4 applies. Here we have $e = d$ and $h = 2$, and therefore the granularity of S is given by Formula (5.4.2). When $\operatorname{gn}_R(S) = 0$ this formula yields $c - a - 1 \leq 0$ if $a \leq 1$, and $c - 3 \leq 0$ if $a \geq 2$. We end up with $4 \leq c \leq 3$ and therefore $q \geq 3$ holds. □

7 A family of Golod homomorphisms

This section does not rely on material in earlier parts of the paper. The goal is Theorem 7.9, which contains results crucial to the proofs in Sects. 5 and 6.

7.1. In this section k denotes a field, $\boldsymbol{x}$ a finite set of indeterminates of degree one, and A a k algebra isomorphic to $k[\boldsymbol{x}]/I$, where I is a homogeneous ideal in $(\boldsymbol{x})^2$. Furthermore, N

denotes a graded A-module; we set $\inf N := \inf\{j \in \mathbb{Z} \mid N_j \neq 0\}$ if $N \neq 0$ and $\inf 0 = \infty$; abusing notation, we write k for $A/(A_1)$.

A blanket hypothesis is that all A-modules are graded and finitely generated, their submodules are homogeneous, and their homomorphisms preserve degrees.

Natural gradings $\operatorname{Tor}_i^A(N, k) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Tor}_i^A(N, k)_j$ are inherited from resolutions by free graded A-modules. The *graded Betti numbers* $\beta_{i,j}^A(N) := \operatorname{rank}_k \operatorname{Tor}_i^A(N, k)_j$ satisfy the conditions $\beta_{i,j}^A(N) = 0$ for $i \notin [0, \operatorname{proj\,dim}_A N]$, $j < i + \inf N$, and $j \gg i$.

We write $P_N^A(y, z)$, or P_N^A, for the *graded Poincaré series* of N, defined to be

$$P_N^A(y, z) := \sum_{i\geqslant 0}\sum_{j\in\mathbb{Z}} \beta_{i,j}^A(N) y^j z^i \in \mathbb{Z}[y^{\pm 1}][[z]].$$

(1) Localization at (A_1), denoted here by $? \rightsquigarrow ?^\ell$, is a faithfully exact functor from graded A-modules to A^ℓ-modules; it preserves freeness and minimality, whence

$$P_{N^\ell}^{A^\ell}(z) = P_N^A(1, z).$$

Three relevant properties of graded algebras are defined in terms of P_k^A.

(2) The algebra A is said to be *Koszul* if it satisfies the condition

$$H_A(-yz) \cdot P_k^A(y, z) = 1;$$

see also 7.2(1). In particular, $k[\boldsymbol{x}]$ is Koszul and $P_k^{k[\boldsymbol{x}]}(y, z) = (1 + yz)^{|\boldsymbol{x}|}$.

(3) The algebra A is said to be a *graded complete intersection* if $I = k[\boldsymbol{x}]\,\boldsymbol{g}$ for some $k[\boldsymbol{x}]$-regular set of forms, $\boldsymbol{g}$; by a graded version of 2.6(3), this is equivalent to

$$P_k^A(y, z) \cdot \prod_{g\in\boldsymbol{g}} (1 - y^{\deg(g)} z^2) = (1 + yz)^{|\boldsymbol{x}|}.$$

(4) The algebra A is said to be *Golod* if it satisfies the condition

$$P_k^A(y, z) \cdot \left(1 - z^2 P_I^{k[\boldsymbol{x}]}(y, z)\right) = (1 + yz)^{|\boldsymbol{x}|}.$$

(Note: This equality differs from that in [28], where Formula (2.2) is incorrect.)

In view of 7.1(1), the algebra A is c.i., respectively, Golod if and only if the local ring A^ℓ has the corresponding property; cf. 2.6, respectively, 5.3.

The focus in this section is on specific properties of polynomial ideals.

7.2. Let $B := k[\boldsymbol{x}]$ be a polynomial ring, I an ideal of B, and t an integer.

We say that I is *t-linear* (or, *I has an t-linear resolution*) if $\beta_{i,j}^B(I) = 0$ holds for $j \neq i+t$; for instance, $(\boldsymbol{x})$ is 1-linear. The ideal I is *linear* if it is t-linear for some t; when I is linear, it is $(\inf I)$-linear if $I \neq 0$, and t-linear for each $t \in \mathbb{Z}$ if $I = 0$.

Following Herzog and Hibi [26], we say I is *componentwise linear* if the ideal $I_{\langle j\rangle} := BI_j$ is j-linear for each $j \in \mathbb{Z}$. We list a few relevant properties.

(1) If I is t-linear, then the following equality holds:

$$(-z)^t P_I^B(y, z) = (1 + yz)^{|\boldsymbol{x}|} H_I(-yz)\,.$$

(2) If I is linear, then it is componentwise linear.

(3) If I is componentwise linear, then the following equality holds:

$$\beta_{i,j}^B(I) = \sum_{h\in\mathbb{Z}} \beta_{i,h}(I_{\langle j\rangle}) - \sum_{h\in\mathbb{Z}} \beta_{i,h}(B_1 I_{\langle j-1\rangle}) \quad \text{for} \quad i, j \in \mathbb{Z}.$$

Part (1) can be read off Polishchuk and Positselski [35, Proof of Proposition 2.2]. (2) is well known; e.g., [35, 1.1] or [28, Lemma 1]. Part (3) is proved in [26, 1.3].

We reduce computation of Poincaré series of componentwise linear ideals to a potentially simpler task—computing Hilbert series of finitely many residue rings.

Proposition 7.3 *Let $B := k[\boldsymbol{x}]$ be a polynomial ring, I an ideal of B, and put*

$$e := |\boldsymbol{x}|,\ n_j := \operatorname{rank}_k I_j,\ \mathbb{J} := \{j \in \mathbb{Z} \mid I_j \neq B_1 I_{j-1}\},\ \textit{and}\ t := \max \mathbb{J}. \qquad (7.3.1)$$

If I is componentwise linear, then $P_I^B(z, y)$ is given by the following formula:

$$\begin{aligned} P_I^B(y,z) &= (1+yz)^e \sum_{j\in\mathbb{J}} (-z)^{-j} \big(H_{B/I_{\langle j-1\rangle}}(-yz) - H_{B/I_{\langle j\rangle}}(-yz)\big) \\ &\quad + (1+yz)^e \sum_{j\in\mathbb{J}} (-z)^{-j} n_{j-1} (-yz)^{j-1}. \end{aligned} \qquad (7.3.2)$$

Proof Both $I_{\langle j\rangle}$ and $B_1 I_{\langle j-1\rangle}$ are j-linear, by the assumption on I and by 7.2(2), respectively; thus, the formula in 7.2(3) is shorthand for a family of equalities:

$$\beta_{i,j}^B(I) = \begin{cases} \beta_{i,i+j}^B(I_{\langle j\rangle}) - \beta_{i,i+j}^B(B_1 I_{\langle j-1\rangle}) & \text{for } j \in \mathbb{J}; \\ 0 & \text{otherwise.} \end{cases}$$

Multiply the ith equality by $y^{i+j} z^i$ and sum up over $i \in \mathbb{Z}$; the result is

$$\sum_{i\in\mathbb{Z}} \beta_{i,j}^B(I) y^{i+j} z^i = \begin{cases} P_{I_{\langle j\rangle}}^B - P_{B_1 I_{\langle j-1\rangle}}^B & \text{for } j \in \mathbb{J}; \\ 0 & \text{otherwise.} \end{cases}$$

Now multiply each power series by $(-z)^t$, and aggregate the products:

$$(-z)^t P_I^B = (-z)^t \sum_{i,j} \beta_{i,j}^B(I) y^{i+j} z^i = \sum_{j\in\mathbb{J}} (-z)^{t-j} \big((-z)^j P_{I_{\langle j\rangle}}^B - (-z)^j P_{B_1 I_{\langle j-1\rangle}}^B\big).$$

Multiply the last equality by $H_B(-yz)$ and invoke 7.2(1) to get

$$\begin{aligned} (-z)^t H_B(-yz) P_I^B &= \sum_{j\in\mathbb{J}} (-z)^{t-j} \big((-z)^j H_B(-yz) P_{I_{\langle j\rangle}}^B - (-z)^j H_B(-yz) P_{B_1 I_{\langle j-1\rangle}}^B\big) \\ &= \sum_{j\in\mathbb{J}} (-z)^{t-j} (H_{I_{\langle j\rangle}}(-yz) - H_{B_1 I_{\langle j-1\rangle}}(-yz)) \\ &= \sum_{j\in\mathbb{J}} (-z)^{t-j} \big(H_{I_{\langle j\rangle}}(-yz) - (H_{I_{\langle j-1\rangle}}(-yz) - n_{j-1}(-yz)^{j-1})\big) \\ &= \sum_{j\in\mathbb{J}} (-z)^{t-j} (H_{B/I_{\langle j-1\rangle}}(-yz) - H_{B/I_{\langle j\rangle}}(-yz) + n_{j-1}(-yz)^{j-1}) \end{aligned}$$

The preceding equalities, multiplied by $(-z)^{-t}(1+yz)^e$, yield (7.3.2); see 7.1(2). □

7.4. Let Q be a noetherian ring and L a nonzero Q-module. The maximal length of Q-regular sequences in $\operatorname{Ann}_Q(L)$ is called the *grade* of L; it is denoted by grade L and satisfies grade $L \leq \operatorname{proj\,dim}_Q L$. If equality holds (and grade $L = g$), then L is said to be *perfect* (of grade g). When Q is regular, grade $L =$ height $\operatorname{Ann}_Q(L)$ holds, and L is perfect of grade g if and only if L is Cohen-Macaulay and $\dim L = \dim B - g$.

Let $\boldsymbol{x} := \{x_1, \dots, x_e\}$ be a set of nonzero elements of Q. Let $X = \left[x_{i,j}\right]$ be an $s \times (s+h-1)$ matrix with entries from $\boldsymbol{x} \cup \{0\}$, and put

$$\boldsymbol{x}' := \{x_1, \dots, x_h\}, \ \Delta_n^X := \{x_{i,j}\}_{j-i+1=n} \text{ for } n \in [1, h], \text{ and } \Delta^X := \bigcup_{n=1}^{h} \Delta_n^X.$$

We say that X is *adequate for* $\boldsymbol{x}$ if the following conditions are satisfied:

$$\Delta^X \subseteq \boldsymbol{x}; \quad x_l \in \Delta_n^X \iff l = n \in [1, h]; \quad \left|\{(i, j) : x_{i,j} = x_n \in \boldsymbol{x} \setminus \boldsymbol{x}'\}\right| \le 1.$$

Let $I(X, Q)$ denote the ideal of Q generated by the $s \times s$ minors of X.

Eagon and Northcott [16, Theorem 2] proved that $I(X)$ is perfect of grade h if the entries of X are distinct indeterminates. In the results that follow we describe families of ideals with similar properties that are parametrized by fewer variables.

Lemma 7.5 *Let k be a field, $\boldsymbol{x} := \{x_1, \dots, x_e\}$ a set of indeterminates, and put $B := k[\boldsymbol{x}]$. Let X be an $s \times (s + h - 1)$ matrix that is adequate for $\boldsymbol{x}$ (see 7.4).*

The module $A(X) := B/I(X)$ is perfect of grade h, the ideal $I(X)$ is s-linear, and

$$(-z)^s P^B_{I(X)}(-yz) = 1 - (1 + yz)^h \sum_{i=0}^{s-1} \binom{h-1+i}{i} (-yz)^i. \tag{7.5.1}$$

In the special case $h = e$ one has $I(X) = (\boldsymbol{x})^s$.

Proof Put $A(X) := k[\boldsymbol{x}]/I(X)$. For each $n \in [1, h]$ choose $i \in [1, s]$ such that $x_{i,i+n-1} = x_n$. Let $\sigma\colon k[\boldsymbol{x}] \to k[\boldsymbol{x}]$ be the k-algebra map that swaps $x_{1,i+n-1}$ and $x_{1,n}$ for $i \in [1, h]$ and fixes the rest of $\boldsymbol{x}$. As σ induces an isomorphism of k-algebras $A(X) \cong A(\sigma(X))$, we may suppose that $x_{1,n} = x_n$ holds for $n \in [1, h]$.

Let $Y = \left[y_{i,j}\right]$ be an $s \times (s+h-1)$ matrix with distinct entries from a set $\boldsymbol{y}$ of $s \times (s+h-1)$ indeterminates such that $\boldsymbol{y} \cap \boldsymbol{x} = \emptyset$; put $\boldsymbol{y}' := \{y_{1,1}, \dots, y_{1,h}\}$ and $C := k[\boldsymbol{y}']$. The module $A(Y)$ is perfect of grade h, the following is $A(Y)$-regular

$$\boldsymbol{z}_Y := \{y_{i,i+n-1} - y_{1,n}\}_{i \ne 1, n \in [1,h]} \cup \{y_{i,j} \notin \Delta_Y\},$$

and $A(Y)/\boldsymbol{z}_Y A(Y) \cong C/(C_1)^s$ holds; see Eagon [15, Proof of Theorem 1].

Let $\varkappa\colon k[\boldsymbol{y}] \to k[\boldsymbol{x}] = B$ be the k-algebra map with $y_{i,j} \mapsto x_{i,j}$. Since $\boldsymbol{x}$ is adequate for X, the ideal $\operatorname{Ker}(\varkappa)$ is generated by the following set of linear forms:

$$\boldsymbol{z}_X := \{\, y_{i,i+n-1} - y_{1,n} \mid x_{i,i+n-1} = x_n \,\}_{i \ne 1, n \in [1,h]} \cup \{\, y_{i,j} \mid x_{i,j} = 0 \,\}$$

The set $\boldsymbol{z}_X$ is $A(Y)$-regular (as it is a part of $\boldsymbol{z}_Y$) and $A(X) \cong A(Y)/\boldsymbol{z}_X A(Y)$ holds; thus $A(X)$ is perfect of grade h. In addition, the set of linear forms $\boldsymbol{z} := \varkappa(\boldsymbol{z}_Y \setminus \boldsymbol{z}_X)$ is $A(X)$-regular with $A(X)/\boldsymbol{z}A(X) \cong C/(C_1)^s$. Therefore $A(X)$ is perfect of grade h and $P^B_{A(X)} = P^C_{C/(C_1)^s}$ holds. This implies $P^B_{I(X)} = P^C_{(C_1)^s}$; in particular, $I(X)$ is B-linear, by 7.2(2). Setting $\boldsymbol{y}'' := \boldsymbol{y} \setminus \boldsymbol{y}'$ and applying (7.3.2) with $\mathbb{J} = \{s\}$ yields

$$(-z)^{-s} P^B_I(y, z) = (1 + yz)^e \big(H_{k[\boldsymbol{y}]}(-yz) - H_{k[\boldsymbol{y}'']}(-yz) H_{C/(C_1)^s}(-yz)\big).$$

It remains to plug in the well known expressions of the Hilbert series involved. □

7.6. Let $\boldsymbol{x} = \{x_1, \dots, e_e\}$ be a set of indeterminates, $D := \mathbb{Z}[\boldsymbol{x}]$, I a homogeneous ideal, and $g := \operatorname{grade} D/I$. When Q is a noetherian ring and $\boldsymbol{u} = \{u_1, \dots, u_e\} \subset Q$, put $I(\boldsymbol{u}, Q) := Q\phi(I)$, where $\phi\colon D \to Q$ is the ring map with $\phi(x_i) = u_i$ for $i \in [1, e]$.

(1) When D/I is $\mathbb{Z}$-free the following conditions are equivalent: (i) D/I is perfect of grade g; (ii) $K[\boldsymbol{x}]/I(\boldsymbol{x}, K[\boldsymbol{x}])$ is perfect of grade g for every finite prime field K; (iii) $K[\boldsymbol{x}]/I(\boldsymbol{x}, K[\boldsymbol{x}])$ is perfect of grade g for every noetherian ring K.

The ideal I called *generically perfect* of grade g if it satisfies the conditions in (1). In case it does and $I(u, Q) \neq Q$ holds, $Q/I(u, Q)$ has the following properties:

(2) grade $Q/I(u, Q) \leq g$, and Q/I is perfect if equality holds.
(3) If $(Q, \mathfrak{q}, k)$ is a local ring, $\boldsymbol{u} \subset \mathfrak{q}$, and grade $Q/I(\boldsymbol{u}, Q) \geq g$ holds, then one has

$$P^{Q}_{Q/I(\boldsymbol{u},Q)}(z) = P^{B^{\ell}}_{B^{\ell}/I(\boldsymbol{x},B)^{\ell}}(z) = P^{B}_{B/I(\boldsymbol{x},B)}(1, z) \quad \text{with} \quad B := k[\boldsymbol{x}].$$

See Hochster [29, Theorem 1], complemented by [30, Proposition 20] for Part (1); [17, Proposition 4] for Part (2); [1, Proof of Theorem 6.2] and 7.1(1) for Part (3).

Corollary 7.7 *Let $\boldsymbol{x}$ be a set of indeterminates, X an $s \times (s + h - 1)$ matrix that is adequate for $\boldsymbol{x}$ (see (7.4)), and put $D := \mathbb{Z}[\boldsymbol{x}]$.*

(1) *The ideal $I := I(X, D)$ is generically perfect of grade h (see 7.6).*
(2) *The ideal $J := I(X, D) + (\boldsymbol{x})^{t+1}$ is generically perfect of grade e.*

Proof (1) One has $k \otimes_{\mathbb{Z}} I = I(X, k[\boldsymbol{x}])$ for every field k. From Lemma 7.5 we know that $I(X, k[\boldsymbol{x}])$ is perfect of grade g and that that $H_{k[\boldsymbol{x}]/I(X,k[\boldsymbol{x}])}(y)$ does not depend on k; thus for each $j \in \mathbb{Z}$ and every prime number p the $\mathbb{Z}/p\mathbb{Z}$-rank of $(D/I)_j/p(D/I)_j$ equals the $\mathbb{Q}$-rank of $(D/I)_j \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore D/I is $\mathbb{Z}$-free.

(2) As $(D/J)_j = (D/I)_j$ for $j \leq s$ and $(D/J)_j = 0$ for $j > s$, Part (1) shows that D/J is $\mathbb{Z}$-free. It is clear that $k[\boldsymbol{x}]/(I(X, k[\boldsymbol{x}]) + (\boldsymbol{x})^{s+1})$ is perfect of grade e. ◻

7.8. Recall that a ring homorphism $Q \twoheadrightarrow Q/J$ is said to be *Golod* if $J \subseteq \mathfrak{q}^2$ and $P^{Q/J}_k(z) = P^Q_k(z)/(1 - z^2 P^Q_J)$ holds; see Levin [32]. Thus a local ring S is Golod if and only a minimal Cohen presentation $Q \twoheadrightarrow \widehat{S}$ is a Golod homomorphism; cf. 5.3.

Theorem 7.9 *Let $(Q, \mathfrak{q}, k)$ be a local ring, $\boldsymbol{u} = \{u_1, \ldots, u_e\}$ a Q-regular subset of Q, and U an $s \times (s + h - 1)$ matrix with entries in $\{\boldsymbol{u}\} \cup \{0\}$ that is adequate for $\boldsymbol{u}$; see 7.4. Let $I(U)$ be the ideal of Q generated by the $s \times s$ minors of U.*

(1) *The ideal $I := I(U)$ is perfect of grade h, and $P^Q_I(z) = P^B_{I(X)}(1, z)$; see (7.5.1).*
(2) *The ideal $J := I(U) + (\boldsymbol{u})^{s+1}$ is perfect of grade e, and*

$$z^2 P^Q_J = \frac{1}{(-z)^{s-2}} + \frac{(1+z)^{h+1}}{(-z)^{s-1}} \left(\sum_{i=0}^{s-1} \binom{h-1+i}{i} (-z)^i \right) - \frac{(1+z)^e}{(-z)^{s-1}} \left(\sum_{i=0}^{s} \binom{e-1+i}{i} (-z)^i - \binom{h-1+s}{s} (-z)^s \right).$$

(3) *When $s \geq 2$ the canonical maps $Q/I \twoheadleftarrow Q \twoheadrightarrow Q/J$ are Golod homomorphisms.*

Proof (1) Let S denote the associated graded ring of the ideal $(\boldsymbol{u})$ and a^* the initial form of $a \in Q$. Note that $\boldsymbol{u}^* := \{u^*_1, \ldots, u^*_e\}$ is algebraically independent over the subring $K := Q/(u_1, \ldots, u_e)$; we identify S and $K[\boldsymbol{x}]$, write X for the matrix $U^* = \left[u^*_{i,j}\right]$, and do not distinguish between $I(U^*, S)$ and $I(X, K[\boldsymbol{x}])$, see 7.4. It is clear that X is adequate for $\boldsymbol{x}$; then $I(\boldsymbol{x}, \mathbb{Z}[\boldsymbol{x}])$ is generically perfect, by Corollary 7.7(1), and hence $I(X, K[\boldsymbol{x}])$ is perfect of grade h, by 7.6(1). Therefore so is $I(U)$ (see Northcott, [34, Proposition 3]), and 7.6(3) yields $P^Q_I(z) = P^B_{I(X)}(1, z)$.

(2) The ideal J is perfect of grade e, by Corollary 7.7(2) and 7.6(2), so we have $P_J^Q = P_{J(\boldsymbol{x},k[\boldsymbol{x}])}^{k[\boldsymbol{x}]}(1,z)$, by 7.6(3). The ideal $J(\boldsymbol{x},k[\boldsymbol{x}])$ is componentwise linear, with $J(\boldsymbol{x},k[\boldsymbol{x}])_{\langle s\rangle} = I(X)$ and $J(\boldsymbol{x},k[\boldsymbol{x}])_{\langle s+1\rangle} = (\boldsymbol{x})^{s+1}$. Applying Proposition 7.3 with $\mathbb{J} = \{s, s+1\}$ and substituting the expressions for $P_{k[\boldsymbol{x}/I(\boldsymbol{x})}^{k[\boldsymbol{x}]}$ and $P_{k[\boldsymbol{x}]/(\boldsymbol{x})^{s+1}}^{k[\boldsymbol{x}]}(y,z)$, from Lemma 7.5, into Formula (7.3.2) yields $P_{J(\boldsymbol{x},k[\boldsymbol{x}])}^{k[\boldsymbol{x}]}(y,z)$. Now refer to 7.1(1).

(3) The map $Q \twoheadrightarrow Q/J$ is Golod if and only if the ring $(k[\boldsymbol{x}]/J(\boldsymbol{x},k[\boldsymbol{x}]))^{\ell}$ is Golod (see [1, Theorem 6.2]), if and only if the algebra $k[\boldsymbol{x}]/J(\boldsymbol{x},k[\boldsymbol{x}])$ is Golod (see 7.8); this algebra is Golod because the ideal $J(\boldsymbol{x},k[\boldsymbol{x}])$ is componentwise linear; therefore Herzog, Reiner, and Welker, [28, Theorem 4] applies. □

References

1. Avramov, L.L.: Small homomorphisms of local rings. J. Algebra **50**, 400–453 (1978)
2. Avramov, L.L.: Homological asymptotics of modules over local rings. In: Commutative Algebra. Math. Sci. Res. Inst. Publ., 15, pp. 33–62. Springer, New York (1987)
3. Avramov, L.L.: Modules of finite virtual projective dimension. Invent. Math. **96**, 71–101 (1989)
4. Avramov, L.L.: Modules with extremal resolutions. Math. Res. Lett. **3**, 319–328 (1993)
5. Avramov, L.L.: Local rings over which all modules have rational Poincaré series. J. Pure Appl. Algebra **91**, 29–48 (1994)
6. Avramov, L.L.: Infinite free resolutions, Six lectures on commutative algebra (Bellaterra,: Progr. Math. 166), pp. 1–118. Birkhäuser, Basel (1996)
7. Avramov, L.L.: Betti numbers of Golod residue rings (in preparation)
8. Avramov, L.L., Buchweitz, R.-O.: Homological algebra modulo a regular sequence with special attention to codimension two. J. Algebra **230**, 24–67 (2000)
9. Avramov, L.L., Gasharov, V.N., Peeva, I.N.: A periodic module of infinite virtual projective dimension. J. Pure Appl. Algebra **62**, 1–5 (1989)
10. Avramov, L.L., Gasharov, V.N., Peeva, I.N.: Complete intersection dimension. Publ. Math. IHES **86**, 67–114 (1997)
11. Avramov, L.L., Packauskas, N., Walker, M.E.: Quasi-polynomial growth of Betti sequences over complete intersection rings (in preparation)
12. Avramov, L.L., Sun, L.-C.: Cohomology operators defined by a deformation. J. Algebra **204**, 684–710 (1998)
13. Avramov, L.L., Yang, Z.: Betti sequences over standard graded commutative algebras with two relations, Homological and computational methods in commutative algebra, INdAM Series. vol. 20, pp. 1–31. Springer, Cham (2017)
14. Bourbaki, N.: Algèbre commutative. Chapitres 8 et 9. Springer, Berlin (2006)
15. Eagon, J.A.: Examples of Cohen-Macaulay rings that are not Gorenstein. Math. Z. **109**, 109–111 (1962)
16. Eagon, J.A., Northcott, D.J.: Ideals defined by matrices and a certain complex associated with them. Proc. Roy. Soc. Ser. A **269**, 188–204 (1962)
17. Eagon, J.A., Northcott, D.J.: Generically acyclic complexes and generically perfect ideals. Proc. Roy. Soc. Ser. A **299**, 147–172 (1967)
18. Eisenbud, D.: Homological algebra on a complete intersection, with an application to group representations. Trans. Am. Math. Soc. **260**, 35–64 (1980)
19. Eisenbud, D.: Commutative algebra. In: With a View Toward Algebraic Geometry, Graduate Texts in Mathematics, vol. 150. Springer, New York (1995)
20. Eisenbud, D., Harris, J.: On varieties of minimal degree (a centennial account). In: Algebraic Geometry, Proceedings of Symposia in Pure Mathematics, Bowdoin, vol. 46, pp. 3–13. American Mathematical Society, Providence (1985)
21. Engheta, B.: A bound on the projective dimension of three cubics. J. Symb. Comput. **45**(1), 60–73 (2010)
22. Golod, E.S.: On the homologies of certain local rings, Soviet Math. Doklady **3**, 745–448 (1962)
23. Grothendieck, A.: Éléments de Géometrie Algèbrique. Chapitre 0_{III}, Publ. Math. IHES **11** , 349–423 (1961)
24. Gulliksen, T.H.: A change of ring theorem with applications to Poincaré series and intersection multiplicity. Math. Scand. **34**, 167–183 (1974)
25. Gulliksen, T.H.: On the deviations of a local ring. Math. Scand. **47**, 5–20 (1980)

26. Herzog, J., Hibi, T.: Componentwise linear ideals. Nagoya Math. J. **153**, 141–153 (1999)
27. Herzog, J., Iyengar, S.: Koszul modules. J. Pure Appl. Algebra **201**, 154–188 (2005)
28. Herzog, J., Reiner, V., Welker, V.: Componentwise linear ideals and Golod rings. Michigan Math. J. **46**, 211–223 (1999)
29. Hochster, M.: Generically perfect modules are strongly generically perfect. Proc. Lond. Math. Soc. (3) **23**, 477–88 (1971)
30. Hochster, M., Eagon, J.: Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. Am. J. Math. **93**, 1020–1058 (1971)
31. Huneke, C., Mantero, P., McCullough, J., Seceleanu, A.: The projective dimension of codimension two algebras presented by quadrics. J. Algebra **393**, 170–186 (2013)
32. Levin, G.: Local rings and Golod homomorphisms. J. Algebra **37**, 266–289 (1975)
33. Mehta, V.: Endomorphisms of complexes and modules over Golod rings, Ph.D. Thesis, University of California, Berkeley (1976)
34. Northcott, D.J.: Some remarks on the theory of ideals defined by matrices. Quart. J. Math. Oxford (2) **14**, 193–204 (1963)
35. Polishchuk, A., Positselski, L.: Quadratic Algebras, University Lecture Series, 37. American Mathematics Soceity, Providence (2005)
36. Rossi, M.E., Valla, G.: Multiplicity and t-isomultiple ideals. Nagoya Math. J. **64**, 93–101 (1980)
37. Şega, L.M.: Homological properties of powers of the maximal ideal of a local ring. J. Algebra **241**, 827–858 (2001)
38. Şega, L.M.: On the linearity defect of the residue field. J. Algebra **384**, 276–290 (2013)
39. Swinnerton-Dyer, H.P.F.: An enumeration of all varieties of degree 4. Am. J. Math. **95**, 403–418 (1973)
40. Tate, J.: Homology of Noetherian rings and local rings. Illinois J. Math. **1**, 14–27 (1957)
41. Valabrega, P., Valla, G.: Form rings and regular sequences. Nagoya Math. J. **72**, 93–101 (1978)
42. X.X.X., Correspondence, Amer. J. Math. 79,: 951–952. See also A. Weil, Appendix I: Corresponence, Œuvres Scientifiques, Volume II (1951–1964). Springer-Verlag, New York **1979**, 555–556 (1957)