

RESEARCH ARTICLE

Stability of the tangent bundle through conifold transitions

Tristan Collins¹ | Sebastien Picard² | Shing-Tung Yau³

¹Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Massachusetts, USA

²Department of Mathematics, The
University of British Columbia,
Vancouver, Canada

³Yau Mathematical Science Center,
Tsinghua University, Beijing, Haidian
District, China

Correspondence

Sebastien Picard, The University of
British Columbia, Vancouver, Canada.
Email: spicard@math.ubc.ca

Abstract

Let X be a compact, Kähler, Calabi-Yau threefold and suppose $X \mapsto \underline{X} \rightsquigarrow X_t$, for $t \in \Delta$, is a conifold transition obtained by contracting finitely many disjoint $(-1, -1)$ curves in X and then smoothing the resulting ordinary double point singularities. We show that, for $|t| \ll 1$ sufficiently small, the tangent bundle $T^{1,0}X_t$ admits a Hermitian-Yang-Mills metric H_t with respect to the conformally balanced metrics constructed by Fu-Li-Yau. Furthermore, we describe the behavior of H_t near the vanishing cycles of X_t as $t \rightarrow 0$.

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1 | INTRODUCTION

Let X be a compact, Kähler, Calabi-Yau threefold with trivial canonical bundle. Around 1985, Clemens described a general procedure for constructing new, possibly non-Kähler complex manifolds with trivial canonical bundle by contracting a collection of disjoint $(-1, -1)$ curves and then smoothing the resulting ordinary double point (ODP) singularities. Such a geometric transition is now called a conifold transition and we denote it by $X \mapsto \underline{X} \rightsquigarrow X_t$, where $\underline{X}$ is a singular variety with ODP singularities. Reid's fantasy conjectures [74] that all complex threefolds with trivial

canonical bundle can be connected by a sequence of conifold transitions. The goal of this paper, motivated in part by equations from heterotic string theory, is to show that the tangent bundle $T^{1,0}X_t$ admits a Hermitian-Yang-Mills metrics H_t with respect to a class of balanced metrics constructed by Fu-Li-Yau [35], and to study the geometry of these metrics as $|t| \rightarrow 0$. In order to put our work in context, we recall the origins of Calabi-Yau geometry in theoretical physics.

If X is a compact Kähler manifold with $c_1(X) = 0$ then the third author's solution of the Calabi conjecture [94] implies the existence of a unique Ricci-flat Kähler metric in any Kähler class on X . This result, which can be viewed as a higher dimensional analog of the Uniformization Theorem, yields a plethora of examples of compact Riemannian manifolds with zero Ricci curvature and holonomy contained in $SU(n)$.

Following the solution of the Calabi conjecture, Candelas-Horowitz-Strominger-Witten [9] showed that compact Kähler manifolds with holonomy $SU(3)$, in particular, Calabi-Yau manifolds of complex dimension 3, are fundamental building blocks in torsion-free superstring compactifications. Precisely, [9] constructed superstring compactifications with the Standard Model gauge group from a Calabi-Yau threefold together with a holomorphic vector bundle $E \rightarrow X$ admitting a Hermitian-Yang-Mills connection and satisfying the topological constraints $c_1(E) = 0$, and $c_2(E) = c_2(X)$. Of course, the natural choice to make is $E = T^{1,0}X$, but more generally the Donaldson-Uhlenbeck-Yau theorem [21, 91] implies that any slope stable vector bundle satisfying the topological constraints is admissible. In total, these works lead to an abundance of a priori distinct superstring compactifications.

Shortly thereafter, the possibility of superstring compactifications with torsion was investigated. In this case, the compactifying manifold is a complex threefold X with a non-vanishing holomorphic $(3,0)$ -form Ω (so that canonical bundle is trivial) equipped with a holomorphic vector bundle $E \rightarrow X$ satisfying the topological constraints $c_1(E) = 0 = c_1(X)$ and $c_2(E) = c_2(X)$. In order for this data to give rise to a supersymmetric compactification, X must admit a hermitian metric g , with associated $(1,1)$ -form ω , and E must admit a hermitian metric H solving the following system of equations, called the Strominger system [81]. The first equation is formulated as in [58].

$$d(\|\Omega\|_{\omega} \omega^2) = 0, \quad (1.1)$$

$$\omega^2 \wedge F_H = 0, \quad (1.2)$$

$$\sqrt{-1} \partial \bar{\partial} \omega - \frac{\alpha'}{4} (\text{Tr } Rm_g \wedge Rm_g - \text{Tr } F_H \wedge F_H) = 0. \quad (1.3)$$

Here $\alpha' > 0$ is the inverse string tension, F_H denotes the curvature of the Chern connection of (E, H) , and $\|\Omega\|_{\omega}^2$ is the norm of Ω with respect to g . It is natural to view Rm as the curvature of the Chern connection on $T^{1,0}X$, though other choices of connection are admissible as proposed by Hull [49] and further discussed in, for example [19, 28, 37]. Note that if g is Kähler, so that $d\omega = 0$, then (1.1) implies that ω solves the Monge-Ampère equation and is hence Ricci-flat, while (1.2) is the Hermitian-Yang-Mills equation for H , with respect to ω . Finally, in the Kähler case, the anomaly cancellation equation (1.3) dictates that

$$\text{Tr } Rm \wedge Rm - \text{Tr } F_H \wedge F_H = 0.$$

This coupling between the Hermitian-Yang-Mills equation and the Calabi-Yau equation is highly non-trivial, but it is automatically satisfied provided $E = T^{1,0}X$. Thus, we can view the system (1.1), (1.2), (1.3) as a generalization of the Calabi-Yau equation to the setting of non-Kähler complex manifolds. In particular, the Strominger system provides a set of equations for uniformizing non-Kähler complex threefolds with trivial canonical bundle which can be viewed as natural generalizations of the Calabi-Yau equation.

This system from heterotic string theory has recently generated a great deal of interest in mathematics, both for its applications to the study of non-Kähler Calabi-Yau manifolds and its connections to theoretical physics. Li-Yau [58] constructed solutions on Kähler Calabi-Yau threefolds by deforming the complex structure of $T^{1,0}X \oplus \mathbb{C}^r$. Deformations of Kähler solutions to more general bundles were considered by Andreas-Garcia-Fernandez [1, 2]. Fu-Yau [34] constructed solutions to the Strominger system on Calabi-Eckmann-Goldstein-Prokushkin fibrations by using a certain ansatz to reduce the system to a non-linear PDE of Monge-Ampère type on a $K3$ surface. Higher dimensional versions of the Fu-Yau construction have recently been considered in [17, 30, 48, 63, 64, 69, 70]. Further geometric constructions of solutions, in both the compact and non-compact cases, can be found in, for example [22–24, 27–29, 36, 38, 60]. Very recently, Phong, Zhang and the second author have introduced [66] a parabolic approach via the Anomaly flow and obtained a new proof of the Fu-Yau result; see [4, 25, 26, 65, 67, 68] and the references therein. We refer the reader to [37, 39, 40, 62, 90] and the references therein for more on this very active area of research.

While the plethora of solutions to (1.1), (1.2), (1.3) is interesting from a mathematical point of view, the lack of a unique vacuum for the heterotic string is a fundamental problem for the predictive power of string theory. A conjectural resolution of this problem was put forth by Reid [74], inspired by work of Clemens and Friedman. *Reid's fantasy* proposes that all complex three folds with trivial canonical bundle are connected by a sequence of contractions and smoothings. Recall that a conifold transition

$$X \rightarrow \underline{X} \rightsquigarrow X_t, \quad (1.4)$$

consists of a contraction followed by a smoothing, where the contraction map $X \rightarrow \underline{X}$ contracts a collection of disjoint rational curves $C_i \subset X$, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ (called $(-1, -1)$ curves) to ordinary double point (ODP) singularities, given locally by equations

$$\left\{ \sum_{i=1}^4 z_i^2 = 0 \right\} \subset \mathbb{C}^4. \quad (1.5)$$

By work of Friedman [31], under appropriate assumptions there is a smoothing

$$\mu : \mathcal{X} \rightarrow \Delta, \quad \Delta = \{t \in \mathbb{C} : |t| < 1\} \quad (1.6)$$

such that $\mu^{-1}(0) = \underline{X}$, and $\mu^{-1}(t) = X_t$ is a smooth complex threefold with trivial canonical bundle; see Section 2 for a more thorough discussion of conifold transitions. Locally near the ODP singularities, this smoothing is given by

$$\left\{ \sum_{i=1}^4 z_i^2 = t \right\} \subset \mathbb{C}^4 \times \mathbb{C}. \quad (1.7)$$

Note that in general, X_t will no longer be Kähler (even topologically), even if the initial manifold X is projective; see [31] for an example of this phenomenon with initial manifold a quintic in $\mathbb{P}^4$. Thus conifold transitions may take us out of Kähler geometry, and a theme of this work is to understand the geometric structures appearing on the other side. For an introduction to Calabi-Yau transitions, see [75]. Though non-Kähler, [33] proved that X_t generically satisfies the $\partial\bar{\partial}$ -Lemma.

Green-Hübsch [42, 43] and Candelas-Green-Hübsch [7, 8] argued that conifold transitions could be used to connect any two Calabi-Yau manifolds realized as complete intersections in products of projective spaces. As we move along in the moduli of string vacua, string physics should smoothly interpolate through topological changes. For Type II strings, this problem was studied in [44, 82]. Here we consider heterotic strings, and in order to resolve the vacuum degeneracy problem for these compactifications it is essential to understand the solvability of the system (1.1), (1.2), (1.3) through conifold transitions. In fact, the third author has advocated that the solvability of the Strominger system may provide a useful tool for studying Reid's fantasy as it provides a uniformization of non-Kähler Calabi-Yau threefolds.

The study of the Strominger system through conifold transitions was initiated by Fu-Li-Yau [35] who established the existence of metrics ω_t on X_t solving (1.1) assuming the input manifold X in (1.4) is a compact, Kähler Calabi-Yau. Chuan [16] showed that if $E \rightarrow X$ is a holomorphic vector bundle which is Mumford-Takemoto stable with respect to some Kähler class, holomorphically trivial in a neighborhood of the curves contracted by the map $X \rightarrow \underline{X}$, and $(\underline{X}, E)$ can be smoothed to a family of holomorphic bundles (X_t, E_t) , then stability can be passed through the transition in the following sense: there is a hermitian metric H_t on E_t solving the Hermitian-Yang-Mills equation (1.2) with respect to the Fu-Li-Yau metric. We remark that it is unclear whether such bundles $E \rightarrow X$ can be constructed so that, $c_2(E) = c_2(X)$ in $H^2(X, \mathbb{R})$ and in addition $c_2(E_t) = c_2(X_t)$ in $H^4(X_t, \mathbb{R})$ after the conifold transition. Such a situation would be necessary in order to pass solutions of the full Strominger system through conifold transitions. In any event, the metric H_t constructed by Chuan is approximately flat in a neighborhood of the vanishing cycles of the smooth $\underline{X} \rightsquigarrow X_t$, while the Fu-Li-Yau metric is modeled on a non-flat, Kähler Calabi-Yau cone metric.

In this work we initiate the study of the Strominger system through conifold transitions in the case when the gauge bundle E is taken to be $T^{1,0}X$. Our main theorem is the following:

Theorem 1.1. *Let X be a Kähler simply connected Calabi-Yau threefold. Let $X \rightarrow \underline{X} \rightsquigarrow X_t$ be a conifold transition, with X_t as in (1.6). Equip X_t with the Fu-Li-Yau balanced metric g_t with associated (1,1) form ω_t . Then, for all $|t| \ll 1$ sufficiently small there exists a hermitian metric H_t on $T^{1,0}X_t$ solving*

$$\omega_t^2 \wedge F_{H_t} = 0.$$

Furthermore, there exists $\lambda > 0$ such that if $p_i \in \underline{X}$ is a ODP singularity, then after identifying a neighborhood of $p_i \in \mathcal{X}$ with the model smoothing (1.7), there are constant $c_i, d_i > 0$ such that on a neighborhood of the vanishing cycles given by

$$\mathcal{R}_\lambda = \left\{ \|z\|^2 \leq |t|^{\frac{3}{3+\lambda}} \right\} \subset X_t,$$

for each $k \in \mathbb{Z}_{\geq 0}$ there is a constant C_k such that

$$\left| \nabla_{g_{co,t}}^k (g_t - c_i g_{co,t}) \right|_{g_{co,t}} \leq C_k |t|^\lambda \|z\|^{-\frac{2}{3}k},$$

$$\left| \nabla_{g_{co,t}}^k (H_t - d_i g_{co,t}) \right|_{g_{co,t}} \leq C_k |t|^\lambda \|z\|^{-\frac{2}{3}k}.$$

Here $\|z\|^2 = \sum_{i=1}^4 |z_i|^2$ and $g_{co,t}$ is the explicit Kähler Ricci-flat metric on the smoothing (1.7) constructed by Candelas-de la Ossa.

Remark 1.2. The existence of a metric H_t on $T^{1,0}X_t$ solving $\omega_t^2 \wedge F_{H_t} = 0$ implies that the tangent bundle $T^{1,0}X_t$ is stable with respect to the Fu-Li-Yau balanced class $[\omega_t]^2$ (Corollary 6.14). In the case when X_t is topologically $\#_k S^3 \times S^3$ [31, 45], the stability of the tangent bundle was noted in [5].

Remark 1.3. Since both g_t and H_t are locally modeled on the Candelas-de la Ossa explicit Kähler Ricci-flat metric $g_{co,t}$ near the vanishing cycles, we see that the local metric description of conifold transitions given by Candelas-de la Ossa [10] accurately describes global non-Kähler conifold transitions of heterotic strings near the vanishing cycles. Our estimates give convergence, after a suitable local rescaling, of the pair (g_t, H_t) to a solution of the anomaly cancellation equation (1.3) near the ordinary double points of $\underline{X}$ as $t \rightarrow 0$. For related work on Calabi-Yau metrics $(g_t = H_t)$ in the case when both sides are Kähler, see for example [47, 76, 78, 88] and references therein.

Remark 1.4. A similar result holds if we replace the balanced Fu-Li-Yau metric g_t with the conformally balanced metric $\check{g}_t$ obtained by conformally rescaling g_t . In particular, the pair $(\check{g}_t, H_t)$ simultaneously solves (1.1) and (1.2), and satisfies an estimate similar to Theorem 1.1; see Remark 6.13. This implies that near the ODP singularities, at a suitable scale, the pair $(\check{g}_t, H_t)$ converges to the Calabi-Yau solution of the Strominger system on the conifold as $|t| \rightarrow 0$.

The third author has conjectured [93] that if X is any complex threefold with trivial canonical bundle admitting a pair of metrics (ω, H) solving the conformally balanced equation (1.1) and the Hermitian-Yang-Mills equation (1.2), then there is a solution of the full Strominger system. We hope to return to this problem, in the setting of conifold transitions, in future work.

The outline of this paper is as follows. In Section 2 we discuss some background material, including the basic geometric properties of conifold transitions that will be important for our work. In Section 3 we construct a metric H_0 on the tangent bundle of $T\underline{X} \rightarrow \underline{X}_{reg}$. More precisely, the metric H_0 is Hermitian-Yang-Mills with respect to a smooth, balanced metric g_0 on $\underline{X}_{reg}$ and, near the singular points of $\underline{X}$, is uniformly equivalent to the Candelas-de la Ossa Ricci-flat Kähler metric on the conifold (1.5). Furthermore, we show that H_0 also satisfies scale invariant higher-order estimates. The metric H_0 serves as the model metric for the Hermitian-Yang-Mills metric on X_t , at least away from the vanishing cycles. In Section 4 we establish quantitative, polynomial decay of H_0 towards a multiple of the Candelas-de la Ossa metric on the conifold. In Section 5 we use H_0 to construct an approximately Hermitian-Yang-Mills metric H_t on X_t with an explicit estimate for the decay rate, with respect to $|t|$, towards a Hermitian-Yang-Mills

metric. Finally, in Section 6 we show that, for $|t| \ll 1$ sufficiently small, H_t can be perturbed to a genuine Hermitian-Yang-Mills metric on $T^{1,0}X_t$. For the reader's convenience we have provided an appendix detailing the aspects of the Fu-Li-Yau construction which are important for our work.

2 | THE GEOMETRY OF CONIFOLD TRANSITIONS

In this section we will discuss the basic geometry of conifold transitions. We begin with the following definition, which fixes the notion of Calabi-Yau manifold to be considered in this paper.

Definition 2.1. A smooth Calabi-Yau threefold is a smooth complex threefold with finite fundamental group and trivial canonical bundle

The primary aim of this paper is to understand the solvability of the Hermitian-Yang-Mills equation on the tangent bundle to a Calabi-Yau threefold as it passes through a conifold transition.

Definition 2.2. A $(-1, -1)$ curve $C \subset X$ is a smooth rational curve $C \simeq \mathbb{P}^1$ such that the normal bundle $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

From [41, Satz 7] there is an open neighborhood U of C in X such that U is biholomorphic to a neighborhood of the zero section in the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, this implies the existence of a contraction map

$$\pi_C : X \rightarrow \underline{X}$$

to a singular complex space $\underline{X}$ with an ordinary double point singularity at p such that $\pi_C : X \setminus C \rightarrow \underline{X} \setminus \{p\}$ and $\pi_C(C) = p$. Concretely, if $[X_1 : X_2]$ denote homogeneous coordinates on $\mathbb{P}^1$, then any non-zero point in the total space $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ can be written uniquely as $(w_1X_1, w_1X_2, w_2X_1, w_2X_2)$. This defines a biholomorphism from the complement of the zero section in $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ to the complement of the origin in the conifold

$$\hat{V}_0 := \{\hat{z}_1\hat{z}_2 - \hat{z}_3\hat{z}_4 = 0\} \subset \mathbb{C}^4. \quad (2.1)$$

This map can clearly be extended holomorphically over the zero section of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ by sending $\mathbb{P}^1$ to the origin in $\mathbb{C}^4$. After a unitary change of coordinates we can rewrite (2.1) as the standard conifold

$$V_0 := \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^4. \quad (2.2)$$

We now describe another realization of the affine variety (2.2). Consider the Fano surface $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by

$$p_i : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \quad \text{for } i = 1, 2$$

the projection onto the i -th factor. The conifold can be realized as the blow-down of the zero section in the total space of $p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$. Explicitly, the global holomorphic sections of

$p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ define an embedding

$$\iota : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$

which is precisely the Segre embedding

$$\iota(\mathbb{P}^1 \times \mathbb{P}^1) = \{X_1 X_2 - X_3 X_4 = 0 \mid [X_1 : X_2 : X_3 : X_4] \in \mathbb{P}^3\}.$$

Taking the cone over this projective variety yields (2.1).

The singular affine variety V_0 given in (2.2) admits an explicit smoothing given by

$$\mathcal{V} = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subset \mathbb{C}^4 \times \mathbb{C}. \quad (2.3)$$

We denote by $\mu : \mathcal{V} \rightarrow \mathbb{C}$ the projection $\mu(z, t) = t$, and let $V_t = \mu^{-1}(t)$ be the fiber over t .

Now suppose that X is a Calabi-Yau threefold and let $C_1, \dots, C_k \subset X$ be a collection of disjoint $(-1, -1)$ curves. Let $\pi : X \rightarrow \underline{X}$ be the map contracting the C_i , so that $\underline{X}$ is a compact complex space with ordinary double point singularities at $p_i = \pi(C_i)$. We have the following well-known result of Friedman [31, 32]

Theorem 2.3 Friedman, [31]. *There is a first order deformation of $\underline{X}$ smoothing p_i if and only if there is a relation*

$$\sum_{i=1}^k \lambda_i [C_i] = 0 \text{ in } H_2(X, \mathbb{R}) \quad (2.4)$$

where each $\lambda_i \neq 0$.

When X is Kähler (or more generally, satisfies the $\sqrt{-1}\partial\bar{\partial}$ -lemma) Kawamata [55], building on work of Ran [73], and independently Tian [87] showed that the first order smoothings in Theorem 2.3 integrate to genuine smoothings. Furthermore, by [32, Lemma 8.2] if X is a Kähler, Calabi-Yau threefold in the sense of Definition 2.1 then the fibers of smooth $\mu : \mathcal{X} \rightarrow \Delta$ are again Calabi-Yau. In particular, assuming (2.4) holds, there is a holomorphic family

$$\mu : \mathcal{X} \rightarrow \Delta := \{t \in \mathbb{C} : |t| < 1\}$$

such that $\mu^{-1}(t) = X_t$ is smooth for $t \neq 0$ and $\mu^{-1}(0) = \underline{X}$. By a result of Kas-Schlessinger [54] the family $\mathcal{X}_t$ is locally biholomorphic to the model smoothing $\mathcal{V}$ near each ordinary double point. We make the following definition.

Definition 2.4. Let X be a smooth, compact, complex three fold. A conifold transition of X , denoted $X \rightarrow \underline{X} \rightsquigarrow X_t$ consists of a holomorphic map $\pi : X \rightarrow \underline{X}$ and family $\mu : \mathcal{X} \rightarrow \Delta$ with $\mu^{-1}(0) = \underline{X}$ such that

- (1) $\pi : X \rightarrow \underline{X}$ contracts a collection of disjoint $(-1, -1)$ curves $C_1, \dots, C_k$ to isolated, ordinary double point singularities $p_1, \dots, p_k \in \underline{X}$, and π is a biholomorphism $X \setminus \cup_i C_i \rightarrow \underline{X} \setminus \{p_1, \dots, p_k\}$.

(2) $\mu : \mathcal{X} \rightarrow \Delta$ is a holomorphic smoothing of $\underline{X} = \mu^{-1}(0)$ and $X_t = \mu^{-1}(t)$.

Said informally, a conifold transition consists of contracting a collection of disjoint $(-1, -1)$ curves followed by smoothing the resulting double point singularities. At the level of topology, removing a neighborhood of the singular points of $\underline{X}$ leaves a boundary $S^2 \times S^3$, and either side of the transition corresponds topologically to gluing $S^2 \times B^4$ or $B^3 \times S^3$ along this boundary (see, e.g. [75] for details). Thus conifold transitions allow travel between Calabi-Yau threefolds of different topology by degenerating two-cycles and introducing three-cycles. By the above discussion, Theorem 2.3 gives necessary and sufficient conditions for the existence of conifold transitions.

2.1 | Metric geometry of the conifold

Let us turn now to the discussion of some aspects of the metric geometry of conifold transitions. Recall that the conifold (2.2) can be viewed as the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ in the negative line bundle $p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is Kähler-Einstein with positive Ricci curvature, it is well-known that the conifold admits a conical Calabi-Yau metric. Explicitly, let

$$h_{FS} = \sum_{i=1}^4 |X_i|^2$$

denote the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^3}(-1)$. By direct computation we have

$$\iota^* h_{FS} = h_{KE} \quad (2.5)$$

where $h_{KE} = p_1^* h_{FS} \otimes p_2^* h_{FS}$ and $p_i^* h_{FS}$ is the pull-back of the Fubini-Study metric on $\mathbb{P}^1$ for $i = 1, 2$. Define a function

$$r : p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathbb{R}_{\geq 0}$$

in the following way. If $x \in \mathbb{P}^1 \times \mathbb{P}^1$, and σ is a local section of

$$p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1), \text{ then define}$$

$$r(x, \sigma(x))^2 = \left(|\sigma|_{h_{KE}}^2 \right)^{2/3}.$$

Clearly $r^{-1}(0)$ is precisely the zero section of $p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$, and hence r defines a function on the conifold (2.2). From the observation (2.5), we can write this function in terms of the coordinates on $\mathbb{C}^4$ as

$$r^2 = \|z\|^{4/3}, \quad \|z\|^2 := \sum_{i=1}^4 |z_i|^2.$$

An easy calculation shows that

$$\omega_{co,0} := \frac{3}{2} \sqrt{-1} \partial \bar{\partial} r^2$$

defines a conical Calabi-Yau metric $g_{co,0}$ on V_0 ; note that the factor of $\frac{3}{2}$ is a harmless scaling, but we have included it to be consistent with [35]. More precisely, the Ricci-flat Kähler metric $g_{co,0}$ is a cone metric over the link $L := \{r = 1\} \subset V_0$ and can be written as

$$g_{co,0} = \frac{3}{2}(dr^2 + r^2 g_L)$$

where g_L is (the pullback of) a Sasaki-Einstein metric on $L := \{r = 1\} \subset V_0$. The cone $(V_0, g_{co,0})$ has a natural rescaling action generated by the vector field $r \frac{\partial}{\partial r}$. The vector field

$$\xi = J \left(r \frac{\partial}{\partial r} \right)$$

is tangent the level sets of r and defines the Reeb vector field of the Sasaki structure on the link. We will use frequently the holomorphic vector field

$$\xi_{\mathbb{C}} := r \frac{\partial}{\partial r} - \sqrt{-1} \xi.$$

Explicitly, the vector field $\xi_{\mathbb{C}}$ generates the $\mathbb{C}^*$ action on V_0 given by

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda^{3/2} z_1, \lambda^{3/2} z_2, \lambda^{3/2} z_3, \lambda^{3/2} z_4),$$

and one can easily check that the cone metric $g_{co,0}$ is homogeneous of degree 2 under this action. In particular, we have

Lemma 2.5. *The conical Calabi-Yau metric $g_{co,0}$ on the conifold has the following property: for every $k \in \mathbb{Z}_{\geq 0}$ there is a constant C_k so that*

$$|\nabla^k \text{Rm}|_{g_{co,0}}(p) \leq C_k r(p)^{-2-k}.$$

2.2 | Metric geometry of the local smoothings

Candelas-de la Ossa [10] and independently Stenzel [80] constructed Calabi-Yau metrics on the smoothings of the conifold (2.3) using ODE techniques. These metrics will play an important role for us.

Proposition 2.6 Candelas-de la Ossa, [10]. *Consider the smoothing of the conifold given by (2.3), and set $V_t = \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = t\} \subset \mathbb{C}^4$. Let $\|z\|^2 = \sum_{i=1}^4 |z_i|^2$, and, for each $t \in \overline{\Delta}$, set*

$$f_t(s) = 2^{-\frac{1}{3}} |t|^{2/3} \int_0^{\cosh^{-1}(\frac{s}{|t|})} (\sinh(2\tau) - 2\tau)^{\frac{1}{3}} d\tau \quad (2.6)$$

Then

$$\omega_{co,t} := \sqrt{-1} \partial \bar{\partial} f_t(\|z\|^2)$$

defines a smooth Calabi-Yau metric $g_{co,t}$ on V_t whose tangent cone at infinity is the conifold $(V_0, g_{co,0})$.

We note that $\|z\|^2 \geq |t|$ on V_t , and the set $\{\|z\|^2 = |t|\}$, which is topologically S^3 , will sometimes be called the vanishing cycle.

Let $\mathcal{V}$ denote the model smoothing (2.3) and let $\mu : \mathcal{V} \rightarrow \mathbb{C}$ be the projection to the t coordinate. There is a natural $\mathbb{C}^*$ action on the family $\mathcal{V}$, given by

$$S_\lambda(z_1, z_2, z_3, z_4, t) := (\lambda^{3/2}z_1, \lambda^{3/2}z_2, \lambda^{3/2}z_3, \lambda^{3/2}z_4, \lambda^3t). \quad (2.7)$$

so that $S_\lambda : V_t \rightarrow V_{\lambda^3t}$. Under this $\mathbb{C}^*$ action we have

$$S_\lambda^* f_{\lambda^3t_0}(\|z\|^2) = |\lambda|^2 f_{t_0}(\|z\|^2)$$

and so, in particular, we have

$$g_{co,t} = |t|^{2/3} (S_{t^{-1/3}})^* g_{co,1}. \quad (2.8)$$

Note that, strictly speaking, we should fix a branch of log in the above expression, but since the CO metric is manifestly S^1 invariant such a distinction is irrelevant. It follows that the CO metrics $g_{co,t}$ are generated by the $\mathbb{C}^*$ action on $\mathcal{V}$, up to rescaling. In particular, this shows (cf. [35, Lemma 5.1])

Lemma 2.7. *For each $k \in \mathbb{Z}_{\geq 0}$, and $A > 0$ there is a constant $C_{k,A} > 0$, independent of t , so that the Calabi-Yau metrics $g_{co,t}$ satisfy*

$$\sup_{\|z\|^2 \leq A|t|} |\nabla^k \text{Rm}(g_{co,t})|_{g_{co,t}} \leq C_{k,A} |t|^{-\frac{2}{3}(2+k)}.$$

It will be important for us to understand the rate at which the CO metric converges to its tangent cone at infinity. Consider the map

$$\begin{aligned} \Phi_t(z) : V_0 \setminus \left\{ \|z\|^2 \leq \frac{|t|}{2} \right\} &\longrightarrow V_t \setminus \{ \|z\|^2 = |t| \}, \\ z &\longmapsto z + \frac{t\bar{z}}{2\|z\|^2}. \end{aligned} \quad (2.9)$$

Tracing the definitions one can check that $\|z\|^2 \leq \|\Phi_t(z)\|^2 \leq 2\|z\|^2$ and

$$\Phi_t = S_{t^{1/3}} \circ \Phi_1 \circ S_{t^{-1/3}}. \quad (2.10)$$

We have

Lemma 2.8 Conlon-Hein [18], Proposition 5.9. *Under the identification Φ_1 we have that, for all $k \in \mathbb{Z}_{\geq 0}$, there is a constant C_k such that*

$$|\nabla_{g_{co,0}}^k (\Phi_1^* g_{co,1} - g_{co,0})|_{g_{co,0}} \leq C_k r^{-3-k}.$$

Combining Lemma 2.8 with (2.8) and (2.10) we obtain estimates for the decay rate of $g_{co,t}$ towards $g_{co,0}$.

Corollary 2.9. *For all $k \in \mathbb{Z}_{\geq 0}$ there is a uniform constant C_k , independent of t , so that*

$$|\nabla_{g_{co,0}}^k \Phi_t^* g_{co,t} - g_{co,0}|_{g_{co,0}} \leq C_k |t| r^{-3-k}.$$

Proof. From (2.8) and (2.10) we have

$$\Phi_t^* g_{co,t} = |t|^{2/3} (S_{t^{-1/3}})^* \Phi_1^* g_{co,1}.$$

It follows that, if $x \in V_0$, then, from Lemma 2.8 we get

$$\begin{aligned} |\nabla_{g_{co,0}}^k (\Phi_t^* g_{co,t} - g_{co,0})|_{g_{co,0}}(x) &= |t|^{2/3} |\nabla_{g_{co,0}}^k \Phi_t^* g_{co,t} - g_{co,0}|_{|t|^{2/3} g_{co,0}}(S_{t^{-1/3}}(x)) \\ &\leq C_k |t|^{2/3} |t|^{-\frac{2}{3} \cdot \frac{2+k}{2}} r (S_t^{-1/3}(x))^{-3-k} \\ &= C_k |t| r(x)^{-3-k} \end{aligned}$$

□

One application of this result will be to transplanting estimates for tensors on V_0 to estimates on the smooth varieties V_t . The following lemma follows from Corollary 2.9 and $\|z\|^2 \leq \|\Phi(z)\|^2 \leq 2\|z\|^2$.

Lemma 2.10. *There is a constant $R > 0$ depending only on the constant C_0 in Corollary 2.9, such that, if T is a contravariant tensor on some subset of $V_0 \cap \{r^3 > R \frac{|t|}{2}\}$ satisfying the estimate*

$$|\nabla_{g_{co,0}}^k T|_{g_{co,0}} \leq M_k r^{\lambda-k}$$

for some $k \in \mathbb{Z}_{\geq 0}$, constant $M_k > 0$ and some λ , then $(\Phi_t^{-1})^ T$ defines a tensor on $V_t \cap \{r^3 \geq \frac{R^2+1}{2R} |t|\}$ satisfying the estimate*

$$|\nabla_{g_{co,t}}^k (\Phi_t^{-1})^* T|_{g_{co,t}} \leq M'_k r^{\lambda-k}$$

for constants $M'_k > 0$ depending only on M_k and the constants C_k appearing in Corollary 2.9.

We also note that the estimate from Corollary 2.9 implies the following estimate on V_t

$$|\nabla_{g_{co,t}}^k [(\Phi_t^{-1})^* g_{co,0} - g_{co,t}]|_{g_{co,t}} \leq C |t| r^{-3-k}. \quad (2.11)$$

It will be useful later in the paper to have a well adapted system of coordinates in which to carry out our analysis. The following, which we refer to as “holomorphic cylindrical coordinates” were used in [16]. For completeness, we recall these coordinates and prove their existence in the following

Lemma 2.11. *There are uniform constants $\rho > 0$ and $C_k > 0, k \in \mathbb{Z}_{\geq 0}$ with the following effect: If $\hat{z} := (\hat{z}_1, \dots, \hat{z}_4) \in V_t$ and $\hat{z} \neq 0$, then there is an open neighborhood $U_{\hat{z}} \ni \hat{z}$ and a holomorphic embedding $\psi : B_\rho(0) \rightarrow U_{\hat{z}}$ such that, if $(w_1, \dots, w_3)$ denote the local holomorphic coordinates on $B_\rho(0)$, then, setting $\hat{r} = r(\hat{z})$ we have*

(i) *We have $\frac{1}{4}\hat{r} \leq r(w) \leq 4\hat{r}$ on B_ρ , and*

$$|\partial^k r|_{g_{\text{euc}}}(w) \leq C_k \hat{r}$$

where g_{euc} denotes the Euclidean metric on B_ρ .

(ii) *In these coordinates we have*

$$C_0^{-1} g_{\text{euc}} \leq \hat{r}^{-2} g_{\text{co},t} \leq C_0 g_{\text{euc}}$$

and

$$|\partial^k(\hat{r}^{-2} g_{\text{co},t})|_{g_{\text{euc}}} \leq C_k$$

Proof. We begin by constructing some candidate coordinates. Fix $\hat{z} := (\hat{z}_1, \dots, \hat{z}_4) \in V_t$. Clearly $|\hat{z}_i| > \frac{1}{100} \|\hat{z}\|$ for some $1 \leq i \leq 4$, and hence, without loss of generality we may assume $i = 4$. We claim that $w_i = z_i - \hat{z}_i$, for $1 \leq i \leq 3$ form a coordinate system near $\hat{z}$. Indeed, consider the function

$$F_t(z_1, z_2, z_3, z_4) = \sum_{i=1}^4 z_i^2 - t.$$

By the implicit function theorem, the coordinates (z_1, z_2, z_3) form local coordinates on $\{F_t = 0\}$ whenever $\frac{\partial F}{\partial z_4} = 2z_4 \neq 0$. Let us examine these coordinates. Since $r^2 = \|z\|^{4/3}$ one can easily show that $\frac{3}{2} \sqrt{-1} \partial \bar{\partial} r^2$ is a globally defined, smooth metric on $\mathbb{C}^4 \setminus \{0\}$, which we still denote by $g_{\text{co},0}$. We claim that, up to scaling and translating, the estimates in (i) always hold in these coordinates. Indeed, from the scaling relation $r(\lambda \cdot z) = |\lambda| r(z)$ we may as well assume that $\|\hat{z}\| = 1$. It is easy to see that, for any multi-index $\alpha = (k_1, k_2, k_3, k_4) \in \mathbb{Z}_{\geq 0}^4$ we have

$$\sup_{\{\frac{1}{8} < \|z\| < 8\}} \left| \frac{\partial^{|\alpha|}}{\partial z^\alpha} r \right|_{g_{\text{euc},z}} \leq C(\alpha).$$

Since $10^{-3} < |z_4| < 10$ on this region, the estimates in (i) will follow from comparing $g_{\text{euc},z} = \sum_{i=1}^4 |dz_i|^2$ to the Euclidean metric in the coordinates

$$(w_1, w_2, w_3) = (z_1 - \hat{z}_1, z_2 - \hat{z}_2, z_3 - \hat{z}_3).$$

Using the definition of the coordinates we have

$$g_{\text{euc},z} = \sum_{i=1}^3 |dw_i|^2 + \frac{|\sum_{i=1}^3 z_i(w) dw_i|^2}{|z_4(w)|^2}.$$

For clarity, let us denote by $g_{\text{euc},w} = \sum_{i=1}^3 |dw_i|^2$ the Euclidean metric in the w coordinates. Then, from the Cauchy-Schwarz inequality, together with $|z_4| \geq \frac{1}{100} \|\hat{z}\| = \frac{1}{100}$ we have

$$g_{\text{euc},w} \leq g_{\text{euc},z} \leq 10^3 g_{\text{euc},w}.$$

Noting that $\{\frac{1}{8} < \|z\| < 8\}$ implies that $\{\frac{\hat{r}}{4} < r(z) < 4\hat{r}\}$ we see that (i) holds in these coordinates.

Next, we address (ii) in the special case $t = 0$. In this case the desired estimates follow immediately from a scaling argument and the above estimates for r , since rescaling preserves V_0 .

It only remains to determine a bound for ρ such that $\{\sum_{i=1}^3 |w_i| < \rho\}$ implies $\{\frac{1}{8} < \|z(w)\| < 8\}$. From the definition of the coordinates we have

$$\|z(w)\|^2 = \sum_{i=1}^3 |w_i + \hat{z}_i|^2 + \left| \sum_{i=1}^3 (w_i + \hat{z}_i)^2 \right|.$$

On the one hand, we have

$$\sum_{i=1}^3 |w_i + \hat{z}_i|^2 \geq \frac{1}{2} \sum_{i=1}^3 |\hat{z}_i|^2 - \sum_{i=1}^3 |w_i|^2,$$

while on the other hand

$$\left| \sum_{i=1}^3 (w_i + \hat{z}_i)^2 \right| \geq \left| \sum_{i=1}^3 \hat{z}_i^2 \right| - 5 \sum_{i=1}^3 |w_i|^2 - \frac{1}{4} \sum_{i=1}^3 |\hat{z}_i|^2.$$

Thanks to the fact that $1 = \sum_{i=1}^3 |\hat{z}_i|^2 + |\sum_{i=1}^3 \hat{z}_i^2|$ we get

$$\|z(w)\|^2 \geq \frac{1}{4} - 6 \sum_{i=1}^3 |w_i|^2 > \frac{1}{64}$$

provided $\sum_{i=1}^3 |w_i|^2 < \rho \leq \frac{1}{8}$. The upper bound is similar.

Next we consider the case when $\hat{z} \in V_1$ and $\{\|\hat{z}\|^2 > R\}$ for some large constant $R \geq 2$ to be determined. In this region we will also use coordinates constructed from (z_1, z_2, z_3) , assuming as before that $|\hat{z}_4| > \frac{1}{100} \|\hat{z}\|$. Since we have already established the estimates (i) in these coordinates, it suffices to prove (ii). Let $\hat{y} \in V_0$ with $\|\hat{y}\|^2 > \frac{1}{2}$ be defined by

$$\Phi_1(\hat{y}) = \hat{z} \in V_1.$$

Note that from the definition (2.9) of Φ_1 we have $\|\hat{y}\|^2 > R/2$. Define coordinates (x_1, x_2, x_3) on V_1 , near $\hat{z}$ by

$$z_i = |\hat{z}|x_i + \hat{z}_i$$

and let (w_1, w_2, w_3) be the coordinates on V_0 , centered at $\hat{w}$ constructed above. Explicitly,

$$z_i = |\hat{y}|w_i + \hat{y}_i, \quad 1 \leq i \leq 3$$

where $\sum_{i=1}^3 |w_i|^2 < \frac{1}{8}$. The map Φ_1 is given in these coordinates by

$$w_i \mapsto \frac{\|\hat{y}\|}{\|\hat{z}\|} \left(w_i + \frac{\overline{w_i \|\hat{y}\|} + \hat{y}_i}{2\|\hat{y}\| \cdot \|z(w)\|^2} - \frac{\overline{\hat{y}_i}}{2\|\hat{y}\| \cdot \|\hat{z}\|^2} \right) \quad (2.12)$$

where

$$\|z(w)\|^2 = \sum_{i=1}^3 |\|\hat{y}\| \cdot w_i + \hat{y}_i|^2 + \left| \sum_{i=1}^3 (\|\hat{y}\| \cdot w_i + \hat{y}_i)^2 \right|.$$

When $\|y\|^2 > \frac{1}{2}$ we have $1 \leq \frac{\|\Phi_1(y)\|^2}{\|y\|^2} \leq 2$ and so

$$\frac{1}{2}r(\Phi_1(y)) < r(y) < r(\Phi_1(y)). \quad (2.13)$$

From (2.12) it follows that there is a constant $c \in [\frac{1}{2}, 1]$ such that

$$\frac{\partial}{\partial w_j} x_i \circ \Phi_1 = c\delta_j^i + O(R^{-1}), \quad \frac{\partial}{\partial \bar{w}_j} x_i \circ \Phi_1 = O(R^{-1}),$$

and, for all multi-indices α with $|\alpha| \geq 2$,

$$\frac{\partial^{|\alpha|}}{\partial w^\alpha} x_i \circ \Phi_1 = O(R^{-1}), \quad \frac{\partial^{|\alpha|}}{\partial \bar{w}^\alpha} x_i \circ \Phi_1 = O(R^{-1}). \quad (2.14)$$

Thus, by choosing R sufficiently large we can ensure that

$$\frac{1}{2}g_w < \Phi_1^*g_x < 2g_w. \quad (2.15)$$

where g_x denotes the Euclidean metric in x coordinates, and g_w denotes the Euclidean metric in w coordinates. Furthermore, (2.14) implies that, for all multi-indices $\alpha \neq 0$ we have

$$\frac{\partial^{|\alpha|}}{\partial w^\alpha} \Phi_1^*g_x = O(R^{-1}).$$

By Lemma 2.8 and (2.13) we can choose R sufficiently large so that

$$\frac{1}{2}g_{c0,0} \leq \Phi_1^*g_{c0,1} \leq 2g_{c0,0}. \quad (2.16)$$

Combining (2.15), (2.16) and (2.13) we deduce the estimates in (ii) from Lemma 2.8 and the estimates for $g_{co,0}$ in the w coordinates. For example, we have

$$\begin{aligned} |\nabla_{g_x} r(\hat{z})^{-2} g_{co,1}|_{g_x} &= |\nabla_{\Phi_1^* g_x} r(\hat{z})^{-2} \Phi_1^* g_{co,1}|_{\Phi_1^* g_x} \\ &\leq C(|\nabla_{g_w} r(\hat{z})^{-2} \Phi_1^* g_{co,1}|_{g_w} + |O(R^{-1})r(\hat{y})^{-2} g_{co,0}|_{g_w}) \\ &\leq C(|\nabla_{g_{co,0}}(\Phi_1^* g_{co,1} - g_{co,0})|_{g_{co,0}} + |r(\hat{y})^{-2} \partial g_{co,0}|_{g_w} + 1) \\ &\leq C_1. \end{aligned}$$

Higher order derivatives follow similarly by induction. Note that if R is sufficiently large then $\Phi_1(\{w : \sum_{i=1}^3 |w_i|^2 < \frac{1}{8}\}) \supset \{x : \sum_{i=1}^3 |x_i|^2 < \frac{1}{10}\}$ and so again, ρ can be chose uniformly.

The portion of V_1 given by $\{z \in V_1 : \|z\|^2 < 2R\}$ is compact and hence the desired coordinates can be constructed by a covering argument. It only remains to construct the coordinates on V_t for $0 < |t| < 1$. But for $t \neq 0$ we can use the holomorphic rescaling map (2.7) to induce holomorphic coordinates on V_t from those on V_1 . By (2.8) the metrics $g_{co,t}$ are generated, up to a scaling parameter, by the holomorphic rescaling map. On the other hand, the estimates in (i), (ii) are invariant under this rescaling, and hence the lemma follows. $\square$

Remark 2.12. Note that the construction in Lemma 2.11 shows that there is a constant $R > 0$ such that, $\hat{z} \in \{\|z\|^2 \geq R|t|\}$ and $|\hat{z}_4| > \frac{1}{100} \|\hat{z}\|$, then the holomorphic cylindrical coordinates can be taken to be

$$(w_1, w_2, w_3) = \frac{1}{\|\hat{z}\|} (z_1 - \hat{z}_1, z_2 - \hat{z}_2, z_3 - \hat{z}_3)$$

Before moving on from this local discussion we state a lemma regarding extending some of the local objects introduced above.

Lemma 2.13. *If $\mu : \mathcal{X} \rightarrow \Delta$ is a global smoothing of a Calabi-Yau variety $X_0 = \mu^{-1}(0)$ with ordinary double point singularities at points $\{p_1, \dots, p_k\}$, then there are disjoint open sets $U_i \subset \mathcal{X}$, with $p_i \in U_i$ such that U_i is biholomorphic to a neighborhood of 0 in the model smoothing (2.3) and,*

- (i) *there is a globally defined function $r : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ such that, after identifying U_i with the model smoothing, $r^2|_{U_i} = \|z\|^{4/3}$, and $r^{-1}(0) = \{p_1, \dots, p_k\}$.*
- (ii) *There is a collection of closed sets $U'_i \subset U_i$, and closed sets $C_i^t \subset X_t$ smooth map $\Phi_t : X_0 \setminus \cup_i U'_i \rightarrow \mu^{-1}(t) \setminus C_i^t$ such that, after identifying U_i with the model smoothing we have*

$$U'_i = \left\{ \|z\|^2 \leq \frac{|t|}{2} \right\}, \quad C_i^t = \{\|z\|^2 = |t|\}$$

and $\Phi_t|_{U_i}$ is the map $z \mapsto z + \frac{t\bar{z}}{2\|z\|^2}$.

Proof. We only sketch the proof, since it is straightforward. Given a choice of the open sets U_i we extend the locally defined functions $\|z\|^{4/3}$ on $\mathcal{X}$ to globally defined positive functions which only

vanish that the singular points p_i . This establishes (i). To prove (ii) we just observe that the locally defined maps are given by the flow of a vector field which lifts $\frac{\partial}{\partial t}$. We can extend this map globally by using a partition function to glue with any lift of $\frac{\partial}{\partial t}$ to $\mathcal{X} \setminus U_i$ (eg. by choosing a Riemannian metric on $\mathcal{X}$). $\square$

2.3 | Balanced metrics on conifold transitions

We now review the work of Fu-Li-Yau [35] who constructed balanced metrics on non-Kähler Calabi-Yau threefolds using conifold transitions and gluing.

Definition 2.14. Let (X, g) be a complex manifold of complex dimension n with a hermitian metric. The metric g is said to be balanced if the associated (1,1) form ω satisfies

$$d\omega^{n-1} = 0.$$

We have the following theorem

Theorem 2.15 Fu-Li-Yau [35], Theorem 1.2. *Let X be a smooth, Kähler, Calabi-Yau threefold, and suppose that $X \rightarrow \underline{X} \rightsquigarrow X_t$ is a conifold transition. Then, for $|t|$ sufficiently small X_t admits a balanced metric $\omega_{\text{FLY},t}$.*

We will need to recall some aspects of the proof of Theorem 2.15 as they will play an important role in subsequent sections. The first step [35] is to construct a balanced metric on $\underline{X}_{\text{reg}}$ by appropriately gluing a Calabi-Yau metric on X with the conical Calabi-Yau metric on the conifold. To fix notation, for each ordinary double point $p_i \in X$ we fix an identification of a neighborhood of p_i with a neighborhood of the singular point in the conifold. Define

$$U_i(\varepsilon) = \left\{ (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \sum_{i=1}^4 z_i^2 = 0, \text{ and } \|z\|^2 \leq \varepsilon \right\}$$

and let $U(\varepsilon) = \bigcup_i U_i(\varepsilon)$. We state the following result of Fu-Li-Yau [35]; for the reader's convenience we have given a self-contained proof in Appendix A.

Proposition 2.16 Fu-Li-Yau [35], Proposition 2.6. *With the above notation, for every $0 < \varepsilon \ll 1$ sufficiently small there is a constant $M_0 > 0$ such that there exists a hermitian metric g_0 on $\underline{X}_{\text{reg}}$ whose associated (1,1) form ω_0 has the following properties*

- (i) On $\underline{X} \setminus U(1)$ we have $\omega_0 = \omega_{CY}$ where ω_{CY} is a Calabi-Yau metric on X .
- (ii) On $U(\varepsilon) \cap \underline{X}_{\text{reg}}$ we have $\omega_0 = M_0^{1/2} \varepsilon^{-1/3} \omega_{\text{co},0}$.
- (iii) On $U(1) \setminus U(\varepsilon)$, ω_0^2 is $\sqrt{-1} \partial \bar{\partial}$ -exact.

In particular, g_0 is balanced on $\underline{X}_{\text{reg}}$.

For the remainder of the paper, $\varepsilon > 0$ is fixed and we will use the corresponding metric g_0 . It will be useful to use to have a comparison of ω_{CY} with ω_0 on X , away from the contracted rational curves. In a neighborhood of the curves, the metric ω_{CY} is uniformly equivalent to the smooth reference metric $\omega_{sm} = \sqrt{-1}\partial\bar{\partial}r^3 + \pi_{\mathbb{P}^1}^*\omega_{FS}$ (where we recall that $r^3 = \|z\|^2$). For $0 < \lambda < 1$, we can rescale by $S_\lambda : \{1 \leq r \leq 2\} \rightarrow \{\lambda \leq r \leq 2\lambda\}$ with $S_\lambda(z) = \lambda^{3/2}z$. This gives

$$S_\lambda^*\omega_{sm} = \lambda^3\sqrt{-1}\partial\bar{\partial}r^3 + \pi_{\mathbb{P}^1}^*\omega_{FS}, \quad S_\lambda^*\omega_{co,0} = \lambda^2\omega_{co,0},$$

and by the uniform equivalence of metrics $C^{-1}\omega_{sm} \leq \omega_{co,0} \leq C\omega_{sm}$ on $\{1 \leq r \leq 2\}$, we obtain $C^{-1}\lambda^2 S_\lambda^*\omega_{sm} \leq S_\lambda^*\omega_{co,0} \leq C\lambda^{-1}S_\lambda^*\omega_{sm}$. Thus

$$C^{-1}r^2\omega_{CY} \leq \omega_0 \leq Cr^{-1}\omega_{CY} \quad (2.17)$$

on $\{0 < r < 1\}$.

The next step in the proof of Theorem 2.15 is to construct approximately balanced metrics on the smooth fibers X_t . Let $U(\frac{1}{2}) \subset \mathcal{X}$ be a small open set containing the singular points p_i . Note that $\underline{X} \setminus U(\frac{1}{2})$ is diffeomorphic to $X_t \setminus U(\frac{1}{\sqrt{2}})$ by the map Φ_t constructed in Lemma 2.13. If ω_0^2 is the metric from Proposition 2.16, then $(\Phi_t^*\omega_0^2)^{(2,2)}$ is positive definite for $|t|$ sufficiently small and can be glued to the Candelas-de la Ossa metric $\omega_{co,t}^2$ to obtain a positive (2,2) form. The following result can be extracted from Fu-Li-Yau [35, Section 3], see, for example [35, equation (3.4)].

Proposition 2.17 Fu-Li-Yau [35]. *With notation as above, for $\varepsilon, |t|$ sufficiently small and M_0 sufficiently large there is a hermitian metric g_t on X_t such that the associated (1,1) form ω_t has*

- (i) $\omega_t = M_0^{1/2}\varepsilon^{-1/3}\omega_{co,t}$ is Kähler Ricci-flat in $U(\varepsilon) \cap X_t$.
- (ii) There is a constant C_k , independent of $|t|$ so that $|d\omega_t^2|_{C^k(X_t, \omega_t)} \leq C_k|t|$.
- (iii) As $|t| \rightarrow 0$, $\Phi_t^*\omega_t$ converges smoothly, in compact subsets of $\underline{X} \setminus \{p_1, \dots, p_k\}$ to the balanced metric ω_0 of Proposition 2.16.

With this result, Theorem 2.15 is obtained by solving a fourth -th order linear equation with estimates in order to perturb the approximately balanced metric ω_t of Proposition 2.17 to a genuine balanced metric $\omega_{FLY,t}$ for $|t|$ sufficiently small.

2.4 | Balanced metrics on the small resolution

The space X can be viewed as a small resolution of the singular space $\underline{X}$, obtained by replacing the ordinary double points with $(-1, -1)$ rational curves. It will be useful to us to have a sequence of degenerating balanced metrics on X .

Consider the total space $\pi_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1$. Let h_{FS} denote the standard Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$. If $z \in \mathbb{P}^1$, and $(u, v) \in \pi_{\mathbb{P}^1}^{-1}(z)$ denotes a point in the fiber over z , we define

$$r(u, v, z)^2 = \left(|u|_{h_{FS}}^2 + |v|_{h_{FS}}^2 \right)^{2/3}.$$

It is easy to check that r^2 is the pull-back to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ of the potential for the conical Calabi-Yau metric on the conifold. Candelas-de la Ossa [10] constructed asymptotically conical Calabi-Yau metrics on the resolved conifold via the ansatz

$$\omega_{co,a} := \sqrt{-1}\partial\bar{\partial}f_a(r^3) + 4a^2\pi_{\mathbb{P}^1}^*\omega_{FS}$$

where $\omega_{FS} = \sqrt{-1}\partial\bar{\partial}\log h_{FS}$ is the Fubini-Study metric on $\mathbb{P}^1$. This metric is Calabi-Yau provided $f_a(x)$ satisfies the differential equation

$$(xf'_a(x))^3 + 6a^2(xf'_a(x))^2 = x^2, \quad f'_a(x) \geq 0$$

for $x \geq 0$. From this expression it is straightforward to check that

$$f_a(x) = a^2 f_1\left(\frac{x}{a^3}\right).$$

In particular we see that

$$\omega_{co,a} = a^2 S_{a^{-1}}^* \omega_{co,a=1} \quad (2.18)$$

where $S_a(u, v, z) = (a^{3/2}u, a^{3/2}v, z)$ is the scaling generated by the holomorphic Reeb vector field on the conifold. The argument of Fu-Li-Yau carries over to give the following proposition; for the reader's convenience, we have provided the details in Appendix A.

Proposition 2.18 Fu-Li-Yau [35], Proposition 2.6. *For every $0 < \varepsilon \ll 1$ sufficiently small there is a constant $M_0, M_1 > 0$ such that there exists a sequence of balanced hermitian metric ω_a on X with $a \rightarrow 0$ which has the following properties:*

- (i) On $X \setminus U(1)$ we have $\omega_a = \omega_{CY}$ where ω_{CY} is a Calabi-Yau metric on X .
- (ii) On $U(\varepsilon) \cap X$ we have $\omega_a = M_0^{1/2} \varepsilon^{-1/3} \omega_{co,a}$.
- (iii) On $U(1) \setminus U(\varepsilon)$, ω_a^2 is $\sqrt{-1}\partial\bar{\partial}$ -exact.
- (iv) As $a \rightarrow 0$, ω_a converges smoothly to ω_0 on compact subsets of $X \setminus \{C_1, \dots, C_k\}$.
- (v) There is an estimate $M_1^{-1} \leq \text{Vol}(X, g_a) \leq M_1$.
- (vi) We have $[\omega_a^2] = [\omega_{CY}^2] \in H^4(X, \mathbb{R})$.

Remark 2.19. Regarding notation, following the conventions of [16], we will use $\omega_{co,a}$ to denote the Candelas-de la Ossa metric on the small resolution of the conifold, while reserving $\omega_{co,t}$ for the Candelas-de la Ossa metric on the smoothing of the conifold.

2.5 | Notation

Before beginning the construction, we establish notation and conventions that will be used throughout the paper. Throughout the paper $T^{1,0}X$ will denote the holomorphic tangent bundle to X , and we will write the components of a hermitian metric H on $T^{1,0}X$ as $H = H_{\bar{k}j} dz^j \otimes d\bar{z}^k$,

and we will denote the inner product by

$$\langle V, W \rangle_H = H_{\bar{k}j} V^j \overline{W^k}, \quad V = V^i \frac{\partial}{\partial z^i}, \quad W = W^i \frac{\partial}{\partial z^i}.$$

The hermitian condition in this convention is $\overline{H_{\bar{k}j}} = H_{j\bar{k}}$. The inverse of H will be denoted $H^{p\bar{q}}$, so that matrix multiplication is $H^{j\bar{p}} H_{\bar{p}k} = \delta^j_k$. A hermitian metric H induces metrics on all the associated bundles in the usual fashion.

An endomorphism $A : T^{1,0}X \rightarrow T^{1,0}X$ has an adjoint $A^\dagger$ defined by

$$\langle AV, W \rangle_H = \langle V, A^\dagger W \rangle_H.$$

When the dependence on the metric is emphasized, we will write this as $A^{\dagger H}$. An endomorphism is H -self-adjoint when $A^\dagger = A$. Note that, in this notation, the inner-product on endomorphisms is given by

$$\langle A, B \rangle_H = \text{Tr}(AB^\dagger)$$

The curvature F_H of the Chern connection of the metric H will follow the convention

$$(F_H)_{j\bar{k}}{}^p{}_q = -\partial_{\bar{k}}(H^{p\bar{r}} \partial_j H_{\bar{r}q}).$$

From two hermitian metrics $\hat{H}$ and H we can form the relative endomorphism denoted $h = \hat{H}^{-1}H$, or in index notation denoted

$$h^p{}_q = \hat{H}^{p\bar{r}} H_{\bar{r}q}.$$

A formula which is the starting point for many computations is for the difference of the curvature tensors of $\hat{H}$ and H .

$$(F_H)_{j\bar{k}} - (F_{\hat{H}})_{j\bar{k}} = -\partial_{\bar{k}}(h^{-1} \nabla_j^{\hat{H}} h). \quad (2.19)$$

Here the p, q endomorphism indices are omitted for ease of notation, and $\nabla^{\hat{H}}$ denotes the Chern connection of $\hat{H}$ acting on $h \in \Gamma(\text{End } T^{1,0}X)$ by

$$\nabla_j^{\hat{H}} h = \partial_j h + [\hat{H}^{-1} \partial_j \hat{H}, h].$$

In this paper, g will typically denote a balanced hermitian metric with associated form $\omega = \sqrt{-1} g_{\bar{k}j} dz^j \wedge d\bar{z}^k$. The contraction operator Λ_ω acts on F_H by

$$\left(\sqrt{-1} \Lambda_\omega F_H \right)^p{}_q = g^{j\bar{k}} (F_H)_{j\bar{k}}{}^p{}_q.$$

The curvature of g will be denoted $(R_g)_{j\bar{k}}{}^p{}_q = -\partial_{\bar{k}}(g^{p\bar{r}} \partial_j g_{\bar{r}q})$. Since g is not Kähler, it will have non-zero torsion, which we denote by

$$(T_g)^r{}_{ij} = (A_g)_i{}^r{}_j - (A_g)_j{}^r{}_i \quad (2.20)$$

where $(A_g)_i^p{}_q = g^{p\bar{n}} \partial_i g_{\bar{n}q}$ is the Chern connection of g . We will denote the (n, n) form corresponding to g by

$$d\text{vol}_g = \frac{\omega^n}{n!}.$$

We will denote the complex Laplacian acting on functions by $\Delta_g f = g^{j\bar{k}} \partial_j \partial_{\bar{k}} f$. The balanced condition $d\omega^{n-1} = 0$ allows us to integrate by parts so that

$$\int_X \psi \Delta_g \varphi d\text{vol}_g = \int_X \varphi \Delta_g \psi d\text{vol}_g$$

for any $\psi, \varphi \in C^\infty(X)$. Lastly, we note that when obtaining estimates, we will use the convention where C denotes a positive constant depending on known quantities which may vary line-by-line.

3 | HERMITIAN-YANG-MILLS METRICS ON THE CENTRAL FIBER

In this section, we construct a Hermitian-Yang-Mills metric on the tangent bundle of the singular space $\underline{X}$ with respect to the Fu-Li-Yau balanced metric ω_0 . We will prove:

Theorem 3.1. *There exists a hermitian metric H_0 on $T\underline{X}_{\text{reg}}$ satisfying*

$$F_{H_0} \wedge \omega_0^2 = 0$$

where ω_0 is the Fu-Li-Yau metric of Proposition 2.16 and F_{H_0} is the curvature of the Chern connection of H_0 . For each $k \in \mathbb{Z}_{\geq 0}$, there is a constant $C_k > 0$ such that the metric H_0 satisfies the estimates

$$|H_0|_{g_0} + |H_0^{-1}|_{g_0} \leq C_0, \quad |\nabla_{g_0}^k H_0|_{g_0} \leq C_k r^{-k}. \quad (3.1)$$

Here $r : \underline{X} \rightarrow \mathbb{R}_{\geq 0}$ is as in Lemma 2.13.

We will produce H_0 by extracting a limit from a sequence $\{(\omega_a, H_a)\}$ of Hermitian-Yang-Mills metrics with respect to the degenerating sequence of background metrics $\{\omega_a\}$ from Proposition 2.18. A similar approach is taken in [16].

Let X be a simply-connected, compact Kähler manifold of dimension $n = 3$ with trivial canonical bundle. By Yau's theorem [94], the bundle $T^{1,0}X$ has a Ricci-flat metric ω_{CY} , and therefore $T^{1,0}X$ is polystable with respect to $[\omega_{CY}]$. In fact, $T^{1,0}X$ is stable because it cannot holomorphically split. As noted in [95], the de Rham decomposition theorem implies that if the tangent bundle to Calabi-Yau manifold splits holomorphically, then the manifold itself splits holomorphically and metrically as a product. In dimension $n = 3$, at least one factor in this decomposition must be 1-dimensional, and hence a torus. When X is simply connected, this is impossible.

Thus, $(X, [\omega_{CY}])$ satisfies the stability condition

$$\frac{1}{\text{rk}(F)} \int_X c_1(F) \wedge \omega_{CY}^2 < 0$$

for all torsion-free, coherent subsheaves $F \subseteq T^{1,0}X$ of rank 1 or 2. It follows from Proposition 2.18 that the same inequality holds if we replace ω_{CY} by ω_a for any $a > 0$.

Therefore $T^{1,0}X$ is stable with respect to the balanced classes $[\omega_a^2] \in H^{2,2}(X, \mathbb{R})$. By the Li-Yau [57] generalization to Gauduchon metrics of the Donaldson-Uhlenbeck-Yau theorem [21, 91], there exists a family of metrics H_a on $T^{1,0}X$ such that

$$F_{H_a} \wedge \omega_a^2 = 0.$$

Our goal is to obtain a limiting metric H_0 as $a \rightarrow 0$.

3.1 | Reference metrics

To study the sequence (g_a, H_a) , we will use a sequence of reference metrics $\hat{H}_a$, given by

$$\hat{H}_a = e^{\psi_a} g_a$$

where ψ_a satisfies

$$\Delta_{g_a} \psi_a = \frac{1}{3} \text{Tr} \sqrt{-1} \Lambda_{\omega_a} F_{g_a}, \quad \int_X \psi_a \, d\text{vol}_{g_a} = 0. \quad (3.2)$$

The solvability of (3.2) follows from the balanced condition since $\text{Tr} F_{g_a}$ is exact. The advantage of $\hat{H}_a$ is that these metrics now have the property that

$$\text{Tr} \Lambda_{\omega_a} F_{\hat{H}_a} = 0.$$

This follows from

$$\sqrt{-1} \Lambda_{\omega_a} F_{\hat{H}_a} = -(\Delta_{g_a} \psi_a) I + \sqrt{-1} \Lambda_{\omega_a} F_{g_a}. \quad (3.3)$$

If we form the relative endomorphism $\hat{h}_a = \hat{H}_a^{-1} H_a$, then (2.19) implies

$$\Delta_{g_a} \log \det \hat{h}_a = \text{Tr} \sqrt{-1} \Lambda_{\omega_a} F_{H_a} - \text{Tr} \sqrt{-1} \Lambda_{\omega_a} F_{\hat{H}_a} = 0.$$

Therefore, $\det \hat{h}_a$ is constant, and we will choose a normalization for H_a such that

$$\det \hat{h}_a \equiv 1.$$

We prove a uniform C^0 estimate for the conformal factor.

Lemma 3.2. *The sequence $\{\psi_a\}$ satisfies a uniform bound*

$$\|\psi_a\|_{L^\infty(X)} \leq C.$$

Proof. Let $U(\delta) = \{r < \delta\}$. Since $g_a = Rg_{co,a}$ on $U(\varepsilon)$, we have $\Lambda_{\omega_a} F_{g_a} = 0$ on $U(\varepsilon)$ and

$$\Delta_{g_a} \psi_a = 0 \text{ on } U(\varepsilon).$$

By the maximum principle,

$$\sup_{U(\varepsilon)} |\psi_a| \leq \sup_{\partial U(\varepsilon)} |\psi_a|.$$

Define

$$\varphi_a(z) = \sup_X \psi_a - \psi_a(z).$$

Let $p \in X$ be a point where $0 = \varphi_a(p) = \inf_X \varphi_a$. By the above estimate, we may assume $p \in X \setminus U(\varepsilon)$. Fix a finite open cover $\{V_i\}$ of coordinate charts of $K = X \setminus U(\varepsilon)$ such that the eigenvalues of $(g_a)^{i\bar{j}}$ are uniformly bounded above and below on each chart. Denote by B_i coordinate balls each compact contained in V_i which still cover K . Note that since the metrics g_a converge smoothly and uniformly on $X \setminus U(\varepsilon)$ to the metric g_0 the sets B_i, V_i can be chosen independent of a .

Suppose $p \in B_1$ so that $\inf_{B_1} \varphi = 0$. By the Harnack inequality for elliptic PDE (e.g. [46, Theorem 5.10]),

$$\sup_{B_1} \varphi_a \leq C \left(\inf_{B_1} \varphi_a + \|\Delta_{g_a} \varphi_a\|_{L^\infty} \right) = C \|\Delta_{g_a} \varphi_a\|_{L^\infty}.$$

From Proposition 2.18, the function $\text{Tr } \Lambda_{\omega_a} F_{g_a}$ is supported on $U(1) \setminus U(\varepsilon)$ and is bounded uniformly, independent of a . Therefore

$$\sup_X |\Delta_{g_a} \varphi_a| \leq C.$$

It follows that $\sup_{B_1} \varphi_a \leq C$. Let B_2 be another coordinate ball with $B_1 \cap B_2 \neq \emptyset$. By the Harnack inequality,

$$\sup_{B_2} \varphi_a \leq C \left(\inf_{B_1 \cap B_2} \varphi_a + \|\Delta_{g_a} \varphi_a\|_{L^\infty} \right) \leq C \left(\sup_{B_1} \varphi_a + 1 \right) \leq C.$$

Continuing this process for each B_i , we conclude

$$\sup_K \varphi_a \leq C,$$

and hence $\sup_X \varphi_a \leq C$. This gives a bound on the oscillation

$$\sup_X \psi_a - \inf_X \psi_a \leq C.$$

Since $\int_X \psi_a \, d\text{vol}_{g_a} = 0$, there is a point $q \in X$ where $\psi_a(q) = 0$. Therefore

$$\sup_X |\psi_a| \leq C$$

where C is independent of $a > 0$. □

We record some corollaries of this estimate.

Corollary 3.3. *Let $H_a, \hat{H}_a, g_a, h_a, \psi_a$ be as above, for $0 < a \ll 1$. Then we have*

- (i) *Let $Z = \cup_i C_i$ be the union of all $(-1, -1)$ curves contracted during the conifold transition. On compact sets $K \subset X \setminus Z$, the metrics $\hat{H}_a$ converge smoothly to a limiting metric $\hat{H}_0 = e^{\psi_0} g_0$.*
- (ii) *For all $0 \leq a \ll 1$, the metrics $\hat{H}_a$ are uniformly equivalent to the background metrics g_a . That is, there is a uniform constant C , independent of a such that*

$$C^{-1}g_a \leq \hat{H}_a \leq Cg_a.$$

In particular, since $\det \hat{H}_a^{-1} H_a = 1$, the endomorphism $h_a = g_a^{-1} H_a = e_a^{\psi} \hat{H}_a^{-1} H_a$ satisfies

$$C^{-1} \leq \det h_a \leq C$$

for a uniform constant C , independent of $0 \leq a \ll 1$.

- (iii) *The Hermitian-Yang-Mills tensor is bounded along the sequence*

$$\sup_X |\sqrt{-1} \Lambda_{\omega_a} F_{\hat{H}_a}|_{\hat{H}_a} \leq C.$$

The full curvature of the limiting $\hat{H}_0$ satisfies $|F_{\hat{H}_0}|_{\hat{H}_0} \leq Cr^{-2}$ on X_{reg} .

Proof. To prove convergence of $\hat{H}_a$ on a compact set K , we cover K by finitely many coordinate charts and apply interior estimates for the Laplace equation (3.2) to ψ_a . On K , the metrics g_a converge uniformly to g_0 , and hence after a subsequence $\hat{H}_a = e^{\psi_a} g_a$ converges to a limiting metric $\hat{H}_0$. The uniform bounds for $\hat{H}_a$ and $\det h_a$ follow from the C^0 estimate $\|\psi_a\|_{L^\infty} \leq C$.

On a neighborhood $U(\varepsilon)$ containing the holomorphic curves, we have $\Lambda_{\omega_a} F_{g_a} = 0$ and $\Lambda_{\omega_a} F_{\hat{H}_a} = 0$. Outside of $U(\varepsilon)$, the metrics $(\hat{H}_a, \omega_a)$ are uniformly bounded, hence $|\Lambda_{\omega_a} F_{\hat{H}_a}|_{\hat{H}_a} \leq C$.

The full curvature $F_{\hat{H}_0}$ does not vanish on $U(\varepsilon)$, however in this neighborhood $|Rm_{g_0}|_{g_0} \leq Cr^{-2}$. In $U(\varepsilon)$ we have $\Delta_{g_0} \psi_0 = 0$, and estimates for the Laplacian in cylindrical coordinates imply

$$|\partial^2 \psi_0|_{g_{\text{euc}}} \leq C$$

and hence $|\nabla_{g_0}^2 \psi_0|_{g_0} \leq Cr^{-2}$ by Lemma 2.11. It follows that $|F_{\hat{H}_0}|_{\hat{H}_0} \leq Cr^{-2}$ since g_0 and $\hat{H}_0$ are uniformly equivalent. □

3.2 | Uhlenbeck-Yau C^0 estimate

In this section, we derive the following estimate:

Proposition 3.4. *Along the sequence of endomorphisms $h_a = g_a^{-1}H_a$, we have the uniform C^0 estimate*

$$C^{-1}I \leq h_a \leq CI \quad (3.4)$$

where I denotes the identity endomorphism.

We will prove this by following the argument of Uhlenbeck-Yau [91]. Thanks to the estimate $C^{-1} \leq \det h_a \leq C$, it suffices to show

$$\mathrm{Tr} h_a \leq C. \quad (3.5)$$

Rather than $h_a = g_a^{-1}H_a$, we will work with the reference metric $\hat{H}_a = e^{\psi_a}g_a$ from the previous section and relative endomorphism $\hat{h}_a = \hat{H}_a^{-1}H_a$. The estimate $\|\psi_a\|_{L^\infty(X)} \leq C$ in Lemma 3.2 shows that a bound $\mathrm{Tr} \hat{h}_a \leq C$ implies (3.5).

To prove (3.5), suppose on the contrary that $\mathrm{Tr} \hat{h}_a \rightarrow \infty$ as $a \rightarrow 0$. Let

$$\tilde{h}_a = \frac{\hat{h}_a}{\sup_X \mathrm{Tr} \hat{h}_a}.$$

The starting point in the proof of the C^0 estimate of Uhlenbeck-Yau is the following inequality (see [91, equation (4.6)]):

Lemma 3.5. *Fix $0 < \sigma \leq 1$, and any two metrics $\hat{H}, H$ on $T^{1,0}X \rightarrow X$. Let $h = \hat{H}^{-1}H$, and let g be a Hermitian metric on X . Then we have*

$$|h^{-\sigma/2} \hat{\nabla} h^\sigma|_{\hat{H},g}^2 \leq g^{j\bar{k}} \langle h^{-1} \hat{\nabla}_j h, \hat{\nabla}_k h^\sigma \rangle_{\hat{H}} \quad (3.6)$$

where $\hat{\nabla}$ is the Chern connection of $\hat{H}$.

We rewrite (3.6) using the identity

$$\partial_j \mathrm{Tr} h^\sigma = \sigma \langle h^{-1} \hat{\nabla}_j h, h^\sigma \rangle_{\hat{H}}, \quad (3.7)$$

which implies

$$\frac{1}{\sigma} \Delta_g \mathrm{Tr} h^\sigma = g^{j\bar{k}} \partial_{\bar{k}} \langle h^{-1} \hat{\nabla}_j h, h^\sigma \rangle_{\hat{H}}.$$

Therefore, (3.6) is equivalent to

$$\begin{aligned} |h^{-\sigma/2} \hat{\nabla} h^\sigma|_{\hat{H},g}^2 - \frac{1}{\sigma} \Delta_g \mathrm{Tr} h^\sigma &\leq g^{j\bar{k}} \langle h^{-1} \hat{\nabla}_j h, \hat{\nabla}_k h^\sigma \rangle_{\hat{H}} - g^{j\bar{k}} \partial_{\bar{k}} \langle h^{-1} \hat{\nabla}_j h, h^\sigma \rangle_{\hat{H}} \\ &= -g^{j\bar{k}} \langle \partial_{\bar{k}} (h^{-1} \hat{\nabla}_j h), h^\sigma \rangle_{\hat{H}}. \end{aligned} \quad (3.8)$$

We will make use of inequality (3.8) by relating the right-hand side to the curvature tensor. With the same notation as Lemma 3.5, the difference between curvatures of the Chern connections (2.19) defined by $H, \hat{H}$ shows that the key inequality (3.8) can be written as

$$|h^{-\sigma/2} \hat{\nabla} h^\sigma|_{\hat{H},g}^2 - \frac{1}{\sigma} \Delta_g \operatorname{Tr} h^\sigma \leq \langle (\sqrt{-1} \Lambda_\omega F - \sqrt{-1} \Lambda_\omega \hat{F}), h^\sigma \rangle_{\hat{H}}.$$

In our case, applying this inequality to the Hermitian-Yang-Mills metric H_a and the reference metric $\hat{H}_a$, we obtain

$$|\tilde{h}_a^{-\sigma/2} \hat{\nabla} \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 - \frac{1}{\sigma} \Delta_{\omega_a} \operatorname{Tr} \tilde{h}_a^\sigma \leq -\langle \sqrt{-1} \Lambda_{\omega_a} F_{\hat{H}_a}, \tilde{h}_a^\sigma \rangle_{\hat{H}_a}. \quad (3.9)$$

Corollary 3.3 gives the bound $|\Lambda_{\omega_a} F_{\hat{H}_a}|_{\hat{H}_a} \leq C$ which together with and $0 < \tilde{h}_a \leq I$ yields the estimate

$$|\tilde{h}_a^{-\sigma/2} \nabla \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 - \frac{1}{\sigma} \Delta_{g_a} \operatorname{Tr} \tilde{h}_a^\sigma \leq C, \quad (3.10)$$

where C is independent of a, σ . Integrating both sides using the balanced condition gives

$$\int_X |\tilde{h}_a^{-\sigma/2} \hat{\nabla} \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 d\operatorname{vol}_{g_a} \leq C \quad (3.11)$$

by Proposition 2.18 (v). Since $0 \leq \tilde{h}_a \leq I$, this implies

$$\int_X |\hat{\nabla} \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 d\operatorname{vol}_{g_a} \leq C. \quad (3.12)$$

Let K be a compact set which is the closure of an open set K^o satisfying $K^o \cap Z = \emptyset$, where $Z = \cup_{i=1}^k C_i$ is the union of all $(-1, -1)$ curves being contracted. The metrics $(g_a, \hat{H}_a)$ are uniformly equivalent to $(g_1, \hat{H}_1)$ on K . Then

$$\int_K |\hat{\nabla} \tilde{h}_a^\sigma|_{\hat{H}_1, g_1}^2 d\operatorname{vol}_{g_1} \leq C_K, \quad (3.13)$$

where ∇ is with respect to g_1 . For each $0 < \sigma \leq 1$, we have weak convergence $\tilde{h}_{a_k}^\sigma \rightharpoonup \tilde{h}_\infty^\sigma$ in $W^{1,2}(K)$ along a subsequence; here $W^{1,2}(K)$ denotes the Sobolev space defined by $(g_1, \hat{H}_1)$. By a diagonal argument, there is a subsequence $a_i \rightarrow 0$ satisfying

$$\tilde{h}_{a_i}^\sigma \rightharpoonup \tilde{h}_\infty^\sigma$$

in $W^{1,2}(K)$ for all $\sigma \in \{1/n : n \in \mathbb{N}\}$. By semicontinuity of weak convergence, we have the estimate

$$\int_K |\nabla \tilde{h}_\infty^\sigma|_{g_1}^2 d\operatorname{vol}_{g_1} \leq \limsup_i \int_K |\nabla \tilde{h}_{a_i}^\sigma|_{g_1}^2 d\operatorname{vol}_{g_1} \leq C_K.$$

Let $\sigma_i = 1/i$, and define $\pi \in W^{1,2}(K)$ by

$$(I - h_\infty^{\sigma_i}) \rightharpoonup \pi$$

in the weak limit $i \rightarrow \infty$ in $W^{1,2}(K)$. Exhausting $X \setminus Z$ with compact sets K , we obtain an endomorphism $\pi \in \Gamma(X \setminus Z, \text{End } T^{1,0}X)$ with regularity $\pi \in W_{loc}^{1,2}(X \setminus Z)$. The definition of π is such that it is the projection onto $\text{Ker } h_\infty$, and it satisfies $\pi = \pi^{\dagger_{H_0}} = \pi^2$ almost everywhere.

We need to verify

Lemma 3.6. *The projection π is not trivial, in the sense that it is neither the identity nor the zero projection.*

Proof. We show that h_∞ is not identically zero. We will use repeatedly the following inequality, which is a consequence of (3.9) with $\sigma = 1$;

$$\Delta_{g_a} \text{Tr } \tilde{h}_a \geq \langle \sqrt{-1} \Lambda_{\omega_a} F_{\tilde{H}_a}, \tilde{h}_a \rangle_{\tilde{H}_a}. \quad (3.14)$$

By its normalization, we have $\text{Tr } \tilde{h}_a \leq 1$ and there exists $x_a \in X$ such that $(\text{Tr } \tilde{h}_a)(x_a) = 1$. In $U(\varepsilon)$, we have $\Lambda_{\omega_a} F_{\tilde{H}_a} = 0$ and hence

$$\Delta_{g_a} \text{Tr } \tilde{h}_a \geq 0, \quad \text{in } U(\varepsilon).$$

In particular, by the maximum principle, $\sup_{U(\varepsilon)} \text{Tr } \tilde{h}_a \leq \sup_{\partial U(\varepsilon)} \text{Tr } \tilde{h}_a$. Thus, we may assume $x_a \in \{r \geq \varepsilon\}$.

The metrics g_a are uniformly equivalent on $\{r \geq \varepsilon/2\}$. In particular, we can fix a uniform number $0 < \delta \ll 1$ such that there is a coordinate ball $B_\delta(x_a) \subseteq \{r \geq \varepsilon/2\}$ and, in local coordinates on $B_\delta(x_a)$ there is a uniform constant M , independent of a , such that the eigenvalues of $(g_a)_{\bar{k}j}$ are bounded above by M and below by M^{-1} .

From (3.14) we obtain the estimate

$$\Delta_{g_a} \text{Tr } \tilde{h}_a - C \text{Tr } \tilde{h}_a \geq 0.$$

Applying the Moser iteration (e.g. [46, Theorem 4.1]) gives

$$1 = \sup_{B_\delta(x_a)} \text{Tr } \tilde{h}_a \leq C \|\text{Tr } \tilde{h}_a\|_{L^1(\{r \geq \varepsilon/2\}, d\text{vol}_{g_1})}$$

for a uniform constant C . Let $K = \{r \geq \varepsilon/2\}$. By (3.13) and Rellich's theorem, we have $\tilde{h}_a \rightarrow h_\infty$ in $L^1(K, d\text{vol}_{g_1})$ and

$$\|\text{Tr } h_\infty\|_{L^1(K, d\text{vol}_{g_1})} \geq C^{-1},$$

therefore h_∞ is not identically zero.

Finally, note that since $\tilde{h}_a$ converges to h_∞ pointwise almost everywhere on K , and $\det \tilde{h}_a = (\sup_X \text{Tr } \tilde{h}_a)^{-3} \rightarrow 0$ we see that h_∞ has a non-trivial kernel almost everywhere on K . Hence $\pi \neq 0$. $\square$

We have thus constructed a nontrivial projection π which projects onto the kernel of h_∞ . To obtain a holomorphic subbundle, we need a further holomorphic condition on π . Following Uhlenbeck-Yau [91] we have,

Lemma 3.7. *The projection π satisfies $(I - \pi)\bar{\partial}\pi = 0$ on $X \setminus Z$.*

Proof. Following [91], rather than work with $(I - \pi)\bar{\partial}\pi$, we differentiate $(I - \pi)\pi = 0$ to obtain

$$|(I - \pi)\bar{\partial}\pi|_{\hat{H}_0, g_0}^2 = |\bar{\partial}(I - \pi)\pi|_{\hat{H}_0, g_0}^2.$$

Taking the adjoint with respect to $\hat{H}_0$ and using $\pi^\dagger = \pi$ and $(\hat{\nabla}_i s)^\dagger = \partial_i s$ for self-adjoint endomorphisms s , we obtain

$$|(I - \pi)\bar{\partial}\pi|_{\hat{H}_0, g_0}^2 = |\pi\hat{\nabla}(I - \pi)|_{\hat{H}_0, g_0}^2$$

where $\hat{\nabla}$ is the covariant derivative with respect to $\hat{H}_0$. We approximate the integral of the quantity on the right-hand side by

$$\int_X |(I - \tilde{h}_a^s)\nabla_{\hat{H}_a} \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 d\text{vol}_{g_a}.$$

The elementary inequality

$$\tilde{h}^{-\sigma/2} \geq \frac{2s + \frac{1}{2}\sigma}{s}(I - \tilde{h}_a^s)$$

and inequality (3.11) implies

$$\begin{aligned} & \int_X |(I - \tilde{h}_a^s)\nabla_{\hat{H}_a} \tilde{h}_a^\sigma|_{\hat{H}_a, g_a}^2 d\text{vol}_{g_a} \\ & \leq \left(\frac{s}{2s + \frac{1}{2}\sigma} \right)^2 \int_X |h_a^{-\sigma/2} \nabla_{\hat{H}_a} h_a^\sigma|_{\hat{H}_a, g_a}^2 d\text{vol}_{g_a} \leq C \left(\frac{s}{2s + \frac{1}{2}\sigma} \right)^2. \end{aligned} \quad (3.15)$$

Let $U_\delta = \{r > \delta\}$. Then since $\tilde{h}_a^s \rightarrow \tilde{h}_\infty^s$ in $L^2(U_\delta)$ by (3.13) and Rellich's theorem, and $\tilde{h}_a^s \rightharpoonup \tilde{h}_\infty^s$ weakly in $W^{1,2}(U_\delta)$, we have that

$$(I - \tilde{h}_a^s)\nabla_{\hat{H}_a} \tilde{h}_a^\sigma \rightharpoonup (I - \tilde{h}_\infty^s)\hat{\nabla} \tilde{h}_\infty^\sigma \quad \text{weakly in } L^2(U_\delta).$$

We let $a \rightarrow 0$ and use semi-continuity of weak convergence to obtain

$$\int_{U_\delta} |(I - \tilde{h}_\infty^s)\hat{\nabla} \tilde{h}_\infty^\sigma|_{\hat{H}_0, g_0}^2 d\text{vol}_{g_0} \leq C \left(\frac{s}{2s + \frac{1}{2}\sigma} \right)^2.$$

We now let $s \rightarrow 0$, which implies

$$\int_{U_\delta} |\pi \hat{\nabla} \tilde{h}_\infty^\sigma|^2_{\hat{H}_0, g_0} d\text{vol}_{g_0} \leq 0,$$

and then taking $\sigma \rightarrow 0$, we conclude

$$\int_{U_\delta} |\pi \hat{\nabla} (I - \pi)|^2_{\hat{H}_0, g_0} d\text{vol}_{g_0} = 0,$$

using semi-continuity of weak convergence. $\square$

Altogether, we have produced an endomorphism $\pi \in \Gamma(\text{End } T^{1,0}X|_{X \setminus Z})$ such that

- $\pi \in W_{loc}^{1,2}(X \setminus Z)$ with respect to the norms $(\hat{H}_0, g_0)$.
- $\pi = \pi^\dagger = \pi^2$, where $\dagger$ is with respect to $\hat{H}_0$.
- $(I - \pi)\bar{\partial}\pi = 0$

We will need a more precise L^2 bound on $|\hat{\nabla}\pi|^2$.

Lemma 3.8. *For any $\delta > 0$, we can estimate*

$$\int_{\{r > \delta\}} |\hat{\nabla}\pi|_{\hat{H}_0, g_0}^2 d\text{vol}_{g_0} \leq \int_{\{r > \delta\}} (\text{Tr } \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\text{vol}_{g_0}.$$

Proof. We work on the set $U_\delta = \{r > \delta\}$ where $I - \tilde{h}_{a_i}^{\sigma_i}$ converges weakly as $i \rightarrow \infty$ to π in $W^{1,2}(U_\delta)$ and $\hat{H}_a$ converges to $\hat{H}_0$ in $C^\infty(U_\delta)$.

$$\begin{aligned} \int_{U_\delta} (\text{Tr } \sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\text{vol}_{g_0} &= \int_{U_\delta} \text{Tr} \left[\sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}_0} (\pi - I) \right] d\text{vol}_{g_0} \\ &= - \lim_{i \rightarrow \infty} \int_{U_\delta} \text{Tr} \left(\sqrt{-1} \Lambda_{\omega_i} F_{\hat{H}_i} \tilde{h}_i^{\sigma_i} \right) d\text{vol}_{g_i} \end{aligned} \quad (3.16)$$

In the first equality we used $\text{Tr } \Lambda_{\omega_0} F_{\hat{H}_0} = 0$. Using the formula (2.19) for the difference between the curvature tensors $F_{\hat{H}}$ and F_H , we obtain

$$\int_{U_\delta} (\text{Tr } \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\text{vol}_{g_0} = - \lim_{i \rightarrow \infty} \int_{U_\delta} \text{Tr } g^{j\bar{k}} \partial_{\bar{k}} (\tilde{h}_i^{-1} \hat{\nabla}_j \tilde{h}_i) \tilde{h}_i^{\sigma_i} d\text{vol}_{g_i}.$$

The inequality (3.6) can be written as

$$|h^{-\sigma/2} \hat{\nabla} h^\sigma|_{\hat{H}, g}^2 - \frac{1}{\sigma} \Delta_g \text{Tr } h^\sigma \leq -g^{j\bar{k}} \langle \partial_{\bar{k}} (h^{-1} \hat{\nabla}_j h), h^\sigma \rangle_{\hat{H}}.$$

Since $h^\dagger = h$ (with respect to $\hat{H}$) and $\langle u, v \rangle_{\hat{H}} = \text{Tr}(uv^\dagger)$, we obtain

$$\int_{U_\delta} (\text{Tr } \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\text{vol}_{g_0} \geq \lim_{i \rightarrow \infty} \int_{U_\delta} |\tilde{h}_i^{-\sigma/2} \hat{\nabla} \tilde{h}_i^{\sigma_i}|_{\hat{H}_i, g_i}^2 d\text{vol}_{g_i} - \lim_{i \rightarrow \infty} \int_{U_\delta} \frac{1}{\sigma} \Delta_{\omega_i} \text{Tr } \tilde{h}_i^{\sigma_i} d\text{vol}_{g_i}.$$

By the balanced condition, this is

$$\int_{U_\delta} (\mathrm{Tr} \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\mathrm{vol}_{g_0} \geq \lim_{i \rightarrow \infty} \int_{U_\delta} |\tilde{h}_i^{-\sigma/2} \hat{\nabla} \tilde{h}_i^{\sigma_i}|^2_{\hat{H}_i, g_i} d\mathrm{vol}_{g_i} + \lim_{i \rightarrow \infty} \int_{\{r < \delta\}} \frac{1}{\sigma} \Delta_{\omega_i} \mathrm{Tr} \tilde{h}_i^\sigma d\mathrm{vol}_{g_i}.$$

Let $0 < \delta < \varepsilon$, where ε is the transition radius in the construction of ω_a , so that $\Lambda_{\omega_a} F_{\hat{H}_a} = 0$ on $\{r < \delta\}$. It follows from (3.9) that

$$\frac{1}{\sigma} \Delta_{\omega_i} \mathrm{Tr} \tilde{h}_i^\sigma \geq 0, \quad \text{on } \{r < \delta\}.$$

Combining this with $|\hat{\nabla} \tilde{h}^\sigma|^2 \leq |\tilde{h}^{-\sigma/2} \hat{\nabla} \tilde{h}^\sigma|^2$, we obtain

$$\int_{U_\delta} (\mathrm{Tr} \Lambda_{\omega_0} F_{\hat{H}_0} \pi) d\mathrm{vol}_{g_0} \geq \lim_{i \rightarrow \infty} \int_{U_\delta} |\hat{\nabla} \tilde{h}_i^{\sigma_i}|^2_{\hat{H}_i, g_i} d\mathrm{vol}_{g_i}.$$

We conclude by semi-continuity of weak convergence. $\square$

We now apply the work of Uhlenbeck-Yau [91] (see also [72]) to conclude that, at least over $X \setminus Z$, the projection π defines a coherent subsheaf $\mathcal{E} \subset T^{1,0}X|_{X \setminus Z}$, which is locally free outside of a complex codimension 2 set. Let $k > 0$ be the generic rank of $\mathcal{E}$. We can view $\mathcal{E}$ as defining a meromorphic map

$$\mu_{\mathcal{E}} : X \setminus Z \rightarrow \mathrm{Gr}(k, T^{1,0}X)$$

to the Grassmann bundle of k -planes in $T^{1,0}X$. Locally near a point in Z we can trivialize $T^{1,0}X$ and, by taking Plücker coordinates on $\mathrm{Gr}(k, T^{1,0}X)$, we view $\mu_{\mathcal{E}}$ as a collection of meromorphic functions defined on the complement of Z . On the other hand, since Z has complex codimension 2, a classical result of Levi [56] (see also [20, Chapter 2]) implies that μ extends over Z . It follows that $\mathcal{E}$ extends over Z to a coherent sheaf (also denoted by $\mathcal{E}$) by taking the direct image of the tautological bundle over $\mathrm{Gr}(k, T^{1,0}X)$. We have thus produced a coherent sheaf $\mathcal{E} \subset T^{1,0}X$, locally free outside a codimension 2 set Z' . We will show that this sheaf contradicts the stability of $T^{1,0}X$.

To contradict stability, we need to show that

$$c_1(\mathcal{E}) \cdot [\omega_{\mathrm{CY}}]^2 \geq 0.$$

The only reason this does not follow immediately from the standard argument is that the metrics $\hat{H}_0$ and ω_0 are not smooth on X . Thus we need to show that the singularities do not contribute. Denote by $\hat{H}'_0$ the metric induced by $\hat{H}_0$ on the subbundle $\mathcal{E}|_{X \setminus Z'} \subset T^{1,0}X|_{X \setminus Z'}$. We begin by computing the slope defined by $\hat{H}'_0$ and ω_0 . Let us introduce the notation

$$c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2 = \int_{X \setminus Z'} \mathrm{Tr} \sqrt{-1} F_{\hat{H}'_0} \wedge \frac{\omega_0^2}{2}.$$

The identity for the curvature of the induced connection on a subbundle defined by a projection π is (see, e.g. [91, equation (4.16)])

$$\mathrm{Tr} \sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}'_0} = \mathrm{Tr} \sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}_0} \pi - |\hat{\nabla} \pi|_{\hat{H}_0, g_0}^2.$$

Therefore

$$c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2 = \int_{X \setminus Z'} \left[\left(\text{Tr} \sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}'_0} \pi \right) - |\hat{\nabla} \pi|_{\hat{H}_0, g_0}^2 \right] d\text{vol}_{g_0}.$$

By letting $\delta \rightarrow 0$ in Lemma 3.8, we see that

$$c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2 \geq 0.$$

We note that this quantity is finite. We can estimate the endomorphism $\Lambda_{\omega_0}(F_{\hat{H}'_0} \pi)$ by using that $|F_{\hat{H}'_0}|_{\hat{H}_0, g_0} \leq Cr^{-2}$, thanks to Corollary 3.3, and $|\pi|_{\hat{H}_0} \leq C$ since $\pi = \pi^{\dagger \hat{H}_0} = \pi^2$. Therefore

$$\left| \int_{X \setminus Z} \left(\text{Tr} \sqrt{-1} \Lambda_{\omega_0} F_{\hat{H}'_0} \pi \right) d\text{vol}_{g_0} \right| \leq C \int_{X \setminus Z} r^{-2} d\text{vol}_{g_0} \leq C,$$

using that, near $\{r = 0\}$, the metric g_0 is a cone over a five-dimensional link.

The next step is to show that $c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2$ is equal to $c_1(\mathcal{E}) \cdot [\omega_{CY}]^2$. Recall that $\mathcal{E}$ defines a line bundle $L = \det \mathcal{E}$, and if e^φ is a smooth metric on L then $\beta = -\sqrt{-1} \partial \bar{\partial} \varphi$ is a representative of $c_1(\mathcal{E})$. For concreteness, we let β be the curvature form associated to the metric $g_{CY}|_{\mathcal{E}}$. We write the difference as

$$\begin{aligned} & c_1(\mathcal{E}) \wedge [\omega_{CY}]^2 - c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2 \\ &= \int_{X \setminus Z'} \left(\beta - \sqrt{-1} \text{Tr} F_{\hat{H}'_0} \right) \wedge \omega_0^2 + \int_{X \setminus Z} \beta \wedge (\omega_{CY}^2 - \omega_0^2) \\ &=: \text{(I)} + \text{(II)}. \end{aligned}$$

We will treat each term individually. Recall Z' is a codimension 2 analytic set containing the singularities of $\mathcal{E}$ (which contains $Z = \cup C_i$). Let η_δ be a cutoff function such that $\eta_\delta \equiv 1$ on $\{\text{dist}_{g_{CY}}(Z', \cdot) > 2\delta\}$ and $\eta_\delta \equiv 0$ on $\{\text{dist}_{g_{CY}}(Z', \cdot) < \delta\}$ with $|\sqrt{-1} \partial \bar{\partial} \eta_\delta|_{g_{CY}} \leq C\delta^{-2}$.

- Term (I). Working near a point where $\mathcal{E}$ is locally free we have

$$\beta - \sqrt{-1} \text{Tr} F_{\hat{H}'_0} = -\sqrt{-1} \partial \bar{\partial} \log \left(\frac{\det g_{CY}|_{\mathcal{E}}}{\det \hat{H}_0|_{\mathcal{E}}} \right).$$

Note that by the AM-GM inequality we have

$$\left(\frac{\det g_{CY}|_{\mathcal{E}}}{\det \hat{H}_0|_{\mathcal{E}}} \right)^{\frac{1}{\text{rk}(\mathcal{E})}} \leq \frac{1}{\text{rk}(\mathcal{E})} \text{Tr} \left(\hat{H}_0|_{\mathcal{E}}^{-1} g_{CY}|_{\mathcal{E}} \right) \leq \frac{1}{\text{rk}(\mathcal{E})} \text{Tr} \left(\hat{H}_0^{-1} g_{CY} \right)$$

where in the second inequality we used that $\hat{H}_0^{-1} g_{CY}$ is positive definite. Similarly we have

$$\left(\frac{\det \hat{H}_0|_{\mathcal{E}}}{\det g_{CY}|_{\mathcal{E}}} \right)^{\frac{1}{\text{rk}(\mathcal{E})}} \leq \frac{1}{\text{rk}(\mathcal{E})} \text{Tr} \left(g_{CY}^{-1} \hat{H}_0 \right).$$

On the other hand from (2.17) we have

$$\text{dist}_{g_{CY}}(Z, \cdot)^{4/3} g_{CY} \leq g_0 \leq \text{dist}_{g_{CY}}(Z, \cdot)^{-2/3} g_{CY}$$

since $r^3 = \|z\|^2 \sim \text{dist}_{g_{CY}}^2(Z, \cdot)$ near the singular points. Hence the same estimates hold for $\hat{H}_0$ and so we have

$$\left| \log \left(\frac{\det g_{CY}|_{\mathcal{E}}}{\det \hat{H}_0|_{\mathcal{E}}} \right) \right| \leq -C \log \text{dist}_{g_{CY}}(Z, \cdot) + C$$

for a uniform constant C . Integrating by parts gives

$$\int_{X \setminus Z'} \eta_{\delta} \left(\beta - \sqrt{-1} \text{Tr} F_{\hat{H}'_0} \right) \wedge \omega_0^2 = \int_{X \setminus Z'} \log \left(\frac{\det g_{CY}|_{\mathcal{E}}}{\det \hat{H}_0|_{\mathcal{E}}} \right) \sqrt{-1} \partial \bar{\partial} \eta_{\delta} \wedge \omega_0^2.$$

From the definition of η_{δ} and the bound $\omega_0 \leq \text{dist}_{g_{CY}}(Z, \cdot)^{-2/3} \omega_{CY}$ we get

$$\left| \int_{X \setminus Z'} \eta_{\delta} \left(\beta - \sqrt{-1} \text{Tr} F_{\hat{H}'_0} \right) \wedge \omega_0^2 \right| \leq C \delta^{4-2-4/3} (-\log(\delta))$$

since $\text{Vol}_{g_{CY}}(\{x \in X : \delta < \text{dist}_{g_{CY}}(Z', x) < 2\delta\}) \sim \delta^4$. It follows that term (I) vanishes.

- Term (II). By Proposition 2.18,

$$(II) = \int_{U \setminus Z} \beta \wedge (\omega_{CY}^2 - \omega_0^2)$$

where U is a tubular neighborhood of Z , which is a disjoint union of tubular neighborhoods of the $(-1, -1)$ rational curves. Since U retracts onto Z , which has complex dimension 1 (and hence $H^4(U) = 0$) we can write

$$\omega_{CY}^2 = d\Phi$$

for a smooth 3-form Φ . On the other hand, by Proposition 2.18 $\omega_{co}^2 = \sqrt{-1} \partial \bar{\partial} \Phi'$ where, Φ' is smooth in $U \setminus Z$ and near each $(-1, -1)$ curve C_i there is a constant $\lambda_i > 0$ such that we have

$$\Phi' = \lambda_i r^2 \sqrt{-1} \partial \bar{\partial} r^2 = \lambda_i r^2 \omega_0$$

where we recall that $r^2 = \|z\|^{4/3}$. Thus, we have

$$\int_{X \setminus Z} \eta_{\delta} \beta \wedge (\omega_{CY}^2 - \omega_0^2) = \int_U d\eta_{\delta} \wedge \beta \wedge \Phi - \int_{U \setminus Z} \sqrt{-1} \partial \bar{\partial} \eta_{\delta} \wedge \beta \wedge \Phi'.$$

The first integral is easily seen to be of order δ^3 . For the second integral, we use the bound $g_0 \leq \text{dist}_{g_{CY}}(Z, \cdot)^{-2/3} g_{CY}$ together with the definition of η_{δ} and r to conclude

$$\left| \int_{U \setminus Z} \sqrt{-1} \partial \bar{\partial} \eta_{\delta} \wedge \beta \wedge \Phi' \right| \leq C \delta^{4-2+\frac{4}{3}-\frac{2}{3}}.$$

It follows that term (II) vanishes, and hence

$$c_1(\mathcal{E}, \hat{H}'_0) \cdot \omega_0^2 = c_1(\mathcal{E}) \cdot [\omega_{CY}]^2$$

and hence $c_1(\mathcal{E}) \cdot [\omega_{CY}]^2 \geq 0$, which contradicts the stability of $T^{1,0}X$. We conclude that $\sup_X \text{Tr } \hat{h}_a \leq C$ as $a \rightarrow 0$, which proves the C^0 estimate (3.4).

3.3 | Gradient estimate

In this section, we will show that, along the sequence (g_a, H_a) , there holds an estimate of the form

$$|\nabla_{g_a} H_a|_{g_a} \leq Cr^{-1}. \quad (3.17)$$

We will use the ideas from Calabi's C^3 estimate [6], as applied in complex geometry by Yau [94] and further developed by Phong-Sesum-Sturm [71] (for other applications of this technique, see, e.g. [26, 77, 89]). We will work with the quantity

$$S = |\nabla_{H_a} h_a h_a^{-1}|_{g_a, H_a}^2,$$

where $h_a = g_a^{-1}H_a$ and by the mixed norms we mean

$$|\nabla h_a h_a^{-1}|_{g_a, H_a}^2 = (g_a)^{j\bar{k}} (H_a)_{\bar{\beta}\alpha} (H_a)^{\mu\bar{\nu}} (\nabla_j h_a h_a^{-1})^\alpha_\mu \overline{(\nabla_{\bar{k}} h_a h_a^{-1})^\beta_\nu}.$$

The quantity S can be understood as the difference of connections by the formula

$$\nabla_H h h^{-1} = A_H - A_g, \quad (3.18)$$

where

$$A_H = H^{-1}\partial H, \quad A_g = g^{-1}\partial g.$$

For ease of notation, in this section we omit the sequence subscript a . We will obtain the following estimate.

Proposition 3.9. *Let (X, g) be a compact Hermitian complex manifold with smooth function $r : X \rightarrow [0, \infty)$ satisfying $|\nabla r|_g \leq \Lambda$ for $\Lambda > 0$. Let H be a second Hermitian metric on $T^{1,0}X$ satisfying $g^{j\bar{k}}(F_H)_{j\bar{k}}^\alpha_\beta = 0$ and $C_0^{-1}g \leq H \leq C_0g$. Let $\varepsilon > 0$. Suppose that on $\{r \leq \varepsilon\}$, the metric g is Kähler Ricci-flat, and satisfies*

$$|Rm_g|_g \leq C_1 r^{-2}.$$

Suppose on the set $\{r > \varepsilon\}$, we have the estimate

$$|T_g| + |Rm_g| + |\nabla^g Rm_g| \leq \Lambda,$$

where T_g is the torsion of g and Rm_g is the curvature of g . Then

$$r^2 |\nabla_H h h^{-1}|_{g,H}^2 \leq C(C_0, C_1, \Lambda, \varepsilon) \quad (3.19)$$

where $h = g^{-1}H$ and ∇_H is the Chern connection of H .

The sequence (g_a, H_a) satisfies the hypothesis of the proposition on X with a function r which is an extension of $\|z\|^{2/3}$ from $U(1/2)$ to all of X with $r^{-1}(0) = \cup C_i$, and the constants are uniform in a . Indeed, the uniform bounds

$$|Rm_{g_a}|_{g_a} \leq Cr^{-2}, \quad |\nabla r|_{g_a} \leq C$$

can be seen by a scaling argument. First, the bounds hold on $\{r \geq \varepsilon\}$ since the geometry of g_a is uniform there. Second, on $\{r < \varepsilon\}$ these bounds hold when $a = 1$, and to obtain uniform bounds for all a we work in coordinates (u, v, z) used previously on $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ and use the scaling map $S_{a^{-1}}(u, v, z) = (a^{-3/2}u, a^{-3/2}v, z)$. Since $r^3 = (1 + |z|^2)(|u|^2 + |v|^2)$, we have $S_{a^{-1}}^* r = a^{-1}r$, and we also have $S_{a^{-1}}^* \omega_{co,1} = a^{-2} \omega_{co,a}$, see, for example (2.18). Pulling back $|Rm_{g_1}|_{g_1} \leq Cr^{-2}$ gives the uniform estimate in a , and similarly for $|\nabla r|$.

Therefore, by proving Proposition 3.9, we can conclude the gradient estimate (3.17). Indeed

$$|\nabla_{g_a} H_a|_{g_a} = |(\nabla_{g_a} - \nabla_{H_a})H_a|_{g_a} = |(\nabla_{H_a} h_a h_a^{-1})H_a|_{g_a} \leq Cr^{-1},$$

since the C^0 estimate (3.4) is $C^{-1}g_a \leq H_a \leq Cg_a$.

3.3.1 | Laplacian of S

The proof of Proposition 3.9 will occupy the remainder of this section. To estimate $S = |\nabla h h^{-1}|_{g,H}^2$, we start by differentiating it once

$$\nabla_{\bar{k}} S = \langle \nabla_{\bar{k}} (\nabla h h^{-1}), \nabla h h^{-1} \rangle_{g,H} + \langle \nabla h h^{-1}, \nabla_{\bar{k}} (\nabla h h^{-1}) \rangle_{g,H}.$$

The covariant derivative ∇_k here is the Chern connection of H on indices measured with H , and the Chern connection of g on indices measured with g . Concretely, we mean

$$\begin{aligned} \nabla_k (\nabla_j h h^{-1})^\alpha_\beta &= \partial_k (\nabla_j h h^{-1})^\alpha_\beta - (\nabla_r h h^{-1})^\alpha_\beta (A_g)_k^r_j \\ &\quad + (A_H)_k^\alpha_\gamma (\nabla_j h h^{-1})^\gamma_\beta - (\nabla_j h h^{-1})^\alpha_\gamma (A_H)_k^\gamma_\beta. \end{aligned} \quad (3.20)$$

Differentiating S twice gives

$$\begin{aligned} g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} S &= |\nabla (\nabla h h^{-1})|_{g,H}^2 + |\bar{\nabla} (\nabla h h^{-1})|_{g,H}^2 \\ &\quad + g^{j\bar{k}} \langle \nabla_j \nabla_{\bar{k}} (\nabla h h^{-1}), \nabla h h^{-1} \rangle_{g,H} \\ &\quad + \langle \nabla h h^{-1}, g^{k\bar{j}} \nabla_{\bar{j}} \nabla_k (\nabla h h^{-1}) \rangle_{g,H} \end{aligned}$$

where

$$|\nabla(\nabla hh^{-1})|_{g,H}^2 = g^{j\bar{k}} H^{p\bar{q}} H^{\mu\bar{\nu}} H_{\bar{\beta}\alpha} \nabla_j (\nabla_p hh^{-1})^\alpha{}_\mu \overline{\nabla_k (\nabla_q hh^{-1})^\beta{}_\nu}.$$

Our curvature conventions imply the commutator relations

$$[\nabla_j, \nabla_{\bar{k}}]V^\alpha = F_{j\bar{k}}{}^\alpha{}_\gamma V^\gamma, \quad [\nabla_j, \nabla_{\bar{k}}]V_\alpha = -V_\gamma F_{j\bar{k}}{}^\gamma{}_\alpha,$$

which gives

$$\begin{aligned} \nabla_{\bar{j}} \nabla_k (\nabla_r hh^{-1})^\alpha{}_\beta &= \nabla_k \nabla_{\bar{j}} (\nabla_r hh^{-1})^\alpha{}_\beta + (\nabla_s hh^{-1})^\alpha{}_\beta (R_g)_{k\bar{j}}{}^s{}_r \\ &\quad - F_{k\bar{j}}{}^\alpha{}_\gamma (\nabla_r hh^{-1})^\gamma{}_\beta + (\nabla_r hh^{-1})^\alpha{}_\gamma F_{k\bar{j}}{}^\gamma{}_\beta. \end{aligned} \quad (3.21)$$

Since $\Lambda_\omega F_H = 0$ and we write $g^{j\bar{k}}(R_g)_{\bar{k}j}{}^s{}_r = (R_g)^s{}_r$, we have

$$g^{k\bar{j}} \nabla_{\bar{j}} \nabla_k (\nabla_r hh^{-1})^p{}_q = g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla_r hh^{-1})^p{}_q + (\nabla_s hh^{-1})^\alpha{}_\beta (R_g)^s{}_r.$$

Therefore

$$\begin{aligned} \Delta_g S &= 2\operatorname{Re} \langle g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla hh^{-1}), \nabla hh^{-1} \rangle + |\nabla(\nabla hh^{-1})|_{g,H}^2 + |\bar{\nabla}(\nabla hh^{-1})|_{g,H}^2 \\ &\quad + g^{s\bar{r}} H_{\bar{\beta}\alpha} H^{\mu\bar{\nu}} (\nabla_s hh^{-1})^\alpha{}_\mu \overline{(\nabla_r hh^{-1})^\beta{}_\nu} (R_g)^p{}_r. \end{aligned}$$

We relate the highest order terms of order $\nabla^3 h$ to the curvatures R_g, F_H and their derivatives.

Lemma 3.10. *The following identity holds:*

$$\begin{aligned} g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla_i hh^{-1})^\alpha{}_\beta &= g^{j\bar{k}} \nabla_j^g (R_g)_{i\bar{k}}{}^\alpha{}_\beta + g^{j\bar{k}} (\nabla_j hh^{-1})^\alpha{}_\gamma (R_g)_{i\bar{k}}{}^\gamma{}_\beta \\ &\quad - g^{j\bar{k}} (R_g)_{i\bar{k}}{}^\alpha{}_\gamma (\nabla_j hh^{-1})^\gamma{}_\beta \\ &\quad + g^{j\bar{k}} \partial_{\bar{k}} (\nabla_r hh^{-1})^\alpha{}_\beta (T_g)^r{}_{ij} - g^{j\bar{k}} (R_g)_{r\bar{k}}{}^\alpha{}_\beta (T_g)^r{}_{ij}, \end{aligned} \quad (3.22)$$

where $\nabla^g R_g$ is the covariant derivative of the curvature tensor of g with respect to the Chern connection of g (in particular the connection of H is not involved in this term).

Proof. By (3.18),

$$\partial_{\bar{k}} (\nabla_i hh^{-1}) = (R_g)_{i\bar{k}} - (F_H)_{i\bar{k}}. \quad (3.23)$$

Therefore

$$g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla_i hh^{-1})^\alpha{}_\beta = g^{j\bar{k}} \nabla_j (R_g)_{i\bar{k}}{}^\alpha{}_\beta - g^{j\bar{k}} \nabla_j (F_H)_{i\bar{k}}{}^\alpha{}_\beta. \quad (3.24)$$

Recall that our notation (3.20) is such that ∇ acts by the Chern connection of H on α, β, γ indices and acts by the Chern connection of g on i, j, k indices. The Bianchi identity is

$$\nabla_j(F_H)_{i\bar{k}}^\alpha{}_\beta = \nabla_i(F_H)_{j\bar{k}}^\alpha{}_\beta + (F_H)_{r\bar{k}}^\alpha{}_\beta (T_g)^r{}_{ij} \quad (3.25)$$

where T_g is the torsion (2.20) of the metric g .

Contracting (3.25) and using $\Lambda_\omega F_H = 0$, we obtain

$$g^{j\bar{k}} \nabla_j(F_H)_{i\bar{k}}^\alpha{}_\beta = g^{j\bar{k}} (F_H)_{r\bar{k}}^\alpha{}_\beta (T_g)^r{}_{ij}.$$

Substituting this into (3.24) gives

$$g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla_i h h^{-1})^\alpha{}_\beta = g^{j\bar{k}} \nabla_j (R_g)_{i\bar{k}}^\alpha{}_\beta - g^{j\bar{k}} (F_H)_{r\bar{k}}^\alpha{}_\beta (T_g)^r{}_{ij}. \quad (3.26)$$

Our notation (3.20) means that $\nabla_j(R_g)_{i\bar{k}}^\alpha{}_\beta$ involves connection terms $H^{-1}\partial H$ on the α, β indices. We will now convert the Chern connection of H into the Chern connection of g via

$$\begin{aligned} \nabla_j(R_g)_{i\bar{k}}^\alpha{}_\beta &= \nabla_j^g(R_g)_{i\bar{k}}^\alpha{}_\beta + [(A_H)_j{}^\alpha{}_\gamma - (A_g)_j{}^\alpha{}_\gamma](R_g)_{i\bar{k}}^\gamma{}_\beta \\ &\quad - (R_g)_{i\bar{k}}^\alpha{}_\gamma [(A_H)_j{}^\gamma{}_\beta - (A_g)_j{}^\gamma{}_\beta]. \end{aligned}$$

By (3.18), equation (3.26) becomes

$$\begin{aligned} g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} (\nabla_i h h^{-1})^\alpha{}_\beta &= g^{j\bar{k}} \nabla_j^g(R_g)_{i\bar{k}}^\alpha{}_\beta + g^{j\bar{k}} (\nabla_j h h^{-1})^\alpha{}_\gamma (R_g)_{i\bar{k}}^\gamma{}_\beta \\ &\quad - g^{j\bar{k}} (R_g)_{i\bar{k}}^\alpha{}_\gamma (\nabla_j h h^{-1})^\gamma{}_\beta - g^{j\bar{k}} (F_H)_{r\bar{k}}^\alpha{}_\beta (T_g)^r{}_{ij}. \end{aligned} \quad (3.27)$$

Using (3.23), we obtain the statement in the lemma. $\square$

Altogether, the Laplacian of S is

$$\begin{aligned} \Delta_g S &= |\nabla(\nabla h h^{-1})|_{g,H}^2 + |\bar{\nabla}(\nabla h h^{-1})|_{g,H}^2 \\ &\quad + 2\operatorname{Re}[(I) + (II) + (IIIa) + (IIIb)] \end{aligned} \quad (3.28)$$

where

$$\begin{aligned} (I) &= g^{i\bar{\ell}} H_{\bar{\mu}\alpha} H^{\beta\bar{\nu}} g^{j\bar{k}} \partial_{\bar{k}} (\nabla_r h h^{-1})^\alpha{}_\beta (T_g)^r{}_{ij} \overline{(\nabla_\ell h h^{-1})^{\mu\nu}}, \\ (II) &= g^{i\bar{\ell}} H_{\bar{\mu}\alpha} H^{\beta\bar{\nu}} [g^{j\bar{k}} \nabla_j^g(R_g)_{i\bar{k}}^\alpha{}_\beta - g^{j\bar{k}} (R_g)_{r\bar{k}}^\alpha{}_\beta (T_g)^r{}_{ij}] \overline{(\nabla_\ell h h^{-1})^{\mu\nu}}, \\ (IIIa) &= g^{i\bar{\ell}} H_{\bar{\mu}\alpha} H^{\beta\bar{\nu}} [g^{j\bar{k}} (\nabla_j h h^{-1})^\alpha{}_\gamma (R_g)_{i\bar{k}}^\gamma{}_\beta] \overline{(\nabla_\ell h h^{-1})^{\mu\nu}} \\ &\quad - g^{j\bar{k}} (R_g)_{i\bar{k}}^\alpha{}_\gamma (\nabla_j h h^{-1})^\gamma{}_\beta \overline{(\nabla_\ell h h^{-1})^{\mu\nu}}, \\ (IIIb) &= g^{i\bar{\ell}} H_{\bar{\mu}\alpha} H^{\beta\bar{\nu}} (\nabla_i h h^{-1})^\alpha{}_\beta (R_g)_{\bar{\ell}}{}^{\bar{\rho}} \overline{(\nabla_{\bar{p}} h h^{-1})^{\mu\nu}} \end{aligned}$$

Recall $\varepsilon > 0$ divides the manifold into two regions $\{r \leq \varepsilon\}$ and $\{r > \varepsilon\}$.

- On $\{r > \varepsilon\}$, the geometry (X, g) is uniformly bounded by a constant Λ . The C^0 estimate $C_0^{-1}g \leq H \leq C_0g$ allows us to use norms with respect to g or H interchangeably up to the cost of constant. Hence on $\{r > \varepsilon\}$ we have

$$\begin{aligned} & \left| (I) + (II) + (IIIa) + (IIIb) \right| \\ & \leq C_{\Lambda, C_0} \left[|\bar{\nabla}(\nabla h h^{-1})|_{g, H}^2 + |\nabla h h^{-1}|_{g, H}^2 + |\nabla h h^{-1}|_{g, H}^2 \right] \end{aligned}$$

and

$$\begin{aligned} & \left| (I) + (II) + (IIIa) + (IIIb) \right| \\ & \leq \frac{1}{4} \left(|\nabla(\nabla h h^{-1})|_{g, H}^2 + |\bar{\nabla}(\nabla h h^{-1})|_{g, H}^2 \right) + C(S + 1) \end{aligned} \quad (3.29)$$

where C depends on C_0 and Λ .

- On $\{r < \varepsilon\}$, the metric g is Kähler Ricci-flat. Term (I) vanishes since the torsion $T_g = 0$. Term (II) vanishes by the Bianchi identity

$$g^{j\bar{k}} \nabla_j R_{i\bar{k}}^{\alpha\beta} = g^{j\bar{k}} \nabla_i R_{j\bar{k}}^{\alpha\beta} = 0$$

combined with the Ricci-flat condition. Term (IIIb) also vanishes and we are left with

$$\Delta_g S = |\nabla(\nabla h h^{-1})|_{g, H}^2 + |\bar{\nabla}(\nabla h h^{-1})|_{g, H}^2 + 2\operatorname{Re}(IIIa).$$

By uniform equivalence of the metrics g and H , we may estimate this as

$$|(IIIa)| \leq C(C_0) |Rm_g|_g S.$$

By the estimate $|Rm_g| \leq C_1 r^{-2}$, we see that on the entirety of X we can estimate

$$\begin{aligned} & \left| (I) + (II) + (IIIa) + (IIIb) \right| \\ & \leq \frac{1}{4} (|\nabla(\nabla h h^{-1})|_{g, H}^2 + |\bar{\nabla}(\nabla h h^{-1})|_{g, H}^2) + C r^{-2} (S + 1) \end{aligned} \quad (3.30)$$

where C depends on $C_0, C_1, \Lambda, \varepsilon$.

3.3.2 | Test function

To construct a test function to control S , we will use $\operatorname{Tr} h$. Contracting the formula (2.19) for the difference of curvature tensors and using $\Lambda_\omega F_H = 0$, we see that

$$\sqrt{-1} \Lambda_\omega Rm_g = -g^{j\bar{k}} \partial_{\bar{k}} (h \nabla_j^H h^{-1}) = g^{j\bar{k}} \partial_{\bar{k}} (\nabla_j^H h h^{-1}).$$

Therefore

$$\mathrm{Tr}(\sqrt{-1}\Lambda_\omega Rm_g)h = \Delta_g \mathrm{Tr} h - g^{j\bar{k}} \mathrm{Tr} \nabla_j h h^{-1} \nabla_{\bar{k}} h.$$

We note that since $\Lambda_\omega Rm_g = 0$ in $\{r \leq \varepsilon\}$, we have the bound

$$|\Lambda_\omega Rm_g| \leq C(\Lambda).$$

Let $\delta > 0$. Let $\zeta(s) : [0, \infty) \rightarrow [0, 1]$ be a cutoff function satisfying $\zeta(s) \equiv 1$ when $s \geq 2$ and $\zeta(s) \equiv 0$ when $s \leq 1$, and $|\zeta'|^2 \leq 9\zeta$. We will use the test function

$$P(z) = \zeta_\delta(z)S(z) + \frac{A}{\delta^2} \mathrm{Tr} h(z), \quad \zeta_\delta(z) = \zeta\left(\frac{r(z)}{\delta}\right),$$

for $A(\Lambda, C_0, C_1) \gg 1$ to be chosen later. Then

$$\Delta_g P = \zeta_\delta \Delta_g S + S \Delta_g \zeta_\delta + 2\mathrm{Re} g^{j\bar{k}} \partial_j \zeta_\delta \partial_{\bar{k}} S + A\delta^{-2} \Delta_g \mathrm{Tr} h.$$

By (3.28),

$$\begin{aligned} \Delta_g P &= \zeta_\delta (|\nabla(\nabla h h^{-1})|_{g,H}^2 + |\bar{\nabla}(\nabla h h^{-1})|_{g,H}^2) + A\delta^{-2} g^{j\bar{k}} \mathrm{Tr} \nabla_j h h^{-1} \nabla_{\bar{k}} h \\ &\quad + 2\zeta_\delta \mathrm{Re} [(I) + (II) + (III)] + (IV) + (V) + (VI) \end{aligned} \quad (3.31)$$

where

$$(IV) = S \Delta_g \zeta_\delta$$

$$(V) = 2\mathrm{Re} g^{j\bar{k}} \partial_j \zeta_\delta [\langle \nabla_{\bar{k}}(\nabla h h^{-1}), \nabla h h^{-1} \rangle + \langle \nabla h h^{-1}, \nabla_{\bar{k}}(\nabla h h^{-1}) \rangle]$$

$$(VI) = A\delta^{-2} \mathrm{Tr}(\sqrt{-1}\Lambda Rm_g)h$$

We want to show that P is bounded by $C(C_0, C_1, \Lambda)\delta^{-2}$. If P attains a maximum on $\{r \leq \delta\}$, then $\zeta_\delta = 0$ and P is bounded by $A\delta^{-2} \sup_X \mathrm{Tr} h$. Suppose P attains a maximum at a point $x \in \{r > \delta\}$. Our good term will be

$$A\delta^{-2} g^{j\bar{k}} \mathrm{Tr} \nabla_j h h^{-1} \nabla_{\bar{k}} h \geq \frac{A}{\delta^2 C(C_0)} |\nabla h h^{-1}|_{g,H}^2 = \frac{A}{C\delta^2} S \quad (3.32)$$

using $C_0^{-1}g \leq H \leq C_0g$ and $h^\dagger = h$. For the term (V),

$$|(V)| \geq -CS^{1/2} |\nabla \zeta_\delta|_g (|\nabla(\nabla h h^{-1})|_{g,H} + |\bar{\nabla}(\nabla h h^{-1})|_{g,H}).$$

We have

$$|\nabla \zeta_\delta|_g = \delta^{-1} |\zeta'| |\nabla r|_g \leq C(\Lambda) \delta^{-1} |\zeta'|.$$

Since $|\zeta'| \leq 3|\zeta|^{1/2}$, we have

$$|(\mathbf{V})| \geq -C\delta^{-1}S^{1/2}|\zeta_\delta|^{1/2}(|\nabla(\nabla hh^{-1})| + |\bar{\nabla}(\nabla hh^{-1})|)$$

Using $2ab \leq a^2 + b^2$, we can estimate

$$|(\mathbf{V})| \geq -\frac{1}{4}|\zeta_\delta|(|\nabla(\nabla hh^{-1})|^2 + |\bar{\nabla}(\nabla hh^{-1})|^2) - C\delta^{-2}S.$$

In (3.30), we showed the estimate

$$2\zeta_\delta \operatorname{Re}[(\text{I}) + (\text{II}) + (\text{III})] \geq -\frac{1}{4}|\zeta_\delta|(|\nabla(\nabla hh^{-1})|^2 + |\bar{\nabla}(\nabla hh^{-1})|^2) - C|\zeta_\delta|r^{-2}S.$$

We are working in the region where $r^{-2} < \delta^{-2}$, hence we obtain at x the inequality

$$\begin{aligned} 0 &\geq \Delta P(x) \\ &\geq \frac{\zeta_\delta}{2}(|\nabla(\nabla hh^{-1})|_{g,H}^2 + |\bar{\nabla}(\nabla hh^{-1})|_{g,H}^2) + \frac{A}{C}\delta^{-2}S - C\delta^{-2}S - CA\delta^{-2}. \end{aligned}$$

For $A \gg 1$ depending on C_0, C_1, Λ , we obtain

$$S(x) \leq C.$$

It follows that $P(x)$ is bounded by $C\delta^{-2}$. From the bound of P , we obtain a bound for S on $\{r \geq 2\delta\}$ since $\zeta \equiv 1$ there.

$$\sup_{\{r \geq 2\delta\}} S \leq C\delta^{-2}.$$

It follows that for any point $x \in X$, we have the estimate

$$S(x) \leq Cr(x)^{-2}.$$

This completes the proof of Proposition 3.9.

3.4 | Higher estimates

By combining the C^0 and C^1 estimates along the degenerating sequence (g_a, H_a) , we can apply regularity theory of elliptic equations to obtain higher order estimates and obtain a limit as $a \rightarrow 0$.

Proof of Theorem 3.1. To extract a limit from (g_a, H_a) , we fix $\delta > 0$ and work on the set $U_\delta = \{r > \delta\}$. Since $g_a \rightarrow g_0$ smoothly uniformly on U_δ , we can cover U_δ by finitely many charts where the matrices representing the metrics g_a satisfy

$$\Lambda_\delta^{-1}\delta_{kj} \leq (g_a)_{\bar{k}j} \leq \Lambda_\delta\delta_{kj}, \quad \|(g_a)_{\bar{k}j}\|_{C^k} \leq \Lambda_{\delta,k}.$$

The C^0 estimate (3.4) for H_a implies that on this cover of U_δ , the local matrices satisfy

$$C^{-1}\delta_{kj} \leq (H_a)_{\bar{k}j} \leq C\delta_{kj}.$$

The C^1 estimate (3.17) gives a uniform bound

$$\|\partial(H_a)_{\bar{\mu}\nu}\|_{L^\infty(U_\delta)} \leq C$$

uniform in a on the local matrices $(H_a)_{\bar{\mu}\nu}$. The equation $\Lambda_{\omega_a} F_{H_a} = 0$ is given in local charts as

$$(g_a)^{j\bar{k}} \partial_j \partial_{\bar{k}} (H_a)_{\bar{\mu}\nu} = (g_a)^{j\bar{k}} \partial_{\bar{k}} (H_a)_{\bar{\mu}\gamma} (H_a)^{\gamma\bar{\alpha}} \partial_j (H_a)_{\bar{\alpha}\nu}.$$

The right-hand side is bounded in L^∞ . By the local $C^{1,\alpha}$ estimate for elliptic PDE, the local matrices $(H_a)_{\bar{\mu}\nu}$ satisfy

$$\|(H_a)_{\bar{\mu}\nu}\|_{C^{1,\alpha}(U_\delta)} \leq C.$$

By the local Schauder estimates, we can take a smooth limit of the sequence (g_a, H_a) on U_δ as $a \rightarrow 0$. The limiting metric H_0 satisfies

$$\Lambda_{\omega_0} F_{H_0} = 0$$

on U_δ . We can now let $\delta \rightarrow 0$ to obtain a limiting metric H_0 on $\underline{X}_{reg}$. The C^0 and C^1 estimates imply

$$|H_0|_{g_0} + |H_0^{-1}|_{g_0} + r|\nabla_{g_0} H_0|_{g_0} \leq C. \quad (3.33)$$

To obtain higher estimates, we work near the singularities of $\underline{X}$, which can be identified with a neighborhood of V_0 with $g_0 = g_{co,0}$. In holomorphic cylindrical coordinates (see Lemma 2.11), we have $r^2 g_{co,0}^{-1} = O(I)$ (notation $O(I)$ is used for a matrix uniformly equivalent to the identity) and the equation

$$r^2 (g_{co,0})^{j\bar{k}} \partial_j \partial_{\bar{k}} (H_0)_{\bar{\mu}\nu} = r^2 (g_{co,0})^{j\bar{k}} \partial_{\bar{k}} (H_0)_{\bar{\mu}\gamma} (H_0)^{\gamma\bar{\alpha}} \partial_j (H_0)_{\bar{\alpha}\nu}.$$

Estimate (3.33) in these coordinates is $H = r^2 O(I)$ and $\partial H = r^2 O(1)$. Therefore this local equation is of the form

$$a^{i\bar{j}} \partial_i \partial_{\bar{j}} H_0 = f, \quad f = r^2 O(1), \quad a^{i\bar{j}} = O(I).$$

Local $C^{1,\alpha}$ estimates for elliptic PDE imply

$$\|(H_0)_{\bar{\mu}\nu}\|_{C^{1,\alpha}(B_{1/2}, g_{euc})} \leq C(\|H_{\bar{\mu}\nu}\|_{L^\infty(B_1, g_{euc})} + \|f\|_{L^\infty(B_1)}) \leq Cr^2.$$

Local Schauder estimates then imply $\|(H_0)_{\bar{\mu}\nu}\|_{C^{k,\alpha}(B_{1/2}, g_{\text{euc}})} \leq C_k r^2$. Converting these local estimates to norms using $g_{c_0,0}$ gives estimates of the form

$$|\nabla_{g_{c_0,0}}^k H_0|_{g_{c_0,0}} \leq C_k r^{-k}$$

for each $k \in \mathbb{Z}_{\geq 0}$. This completes the estimate.

4 | QUANTITATIVE CONVERGENCE TO THE TANGENT CONE

In this section we show that the estimates for the Hermitian-Yang-Mills metric H_0 constructed in Theorem 3.1 can be improved to obtain the decay of H_0 towards the Candelas-de la Ossa metric. This will be an essential ingredient in the perturbation argument later in the paper. The main goal of this section is to prove

Theorem 4.1. *Let $(V_0, g_{c_0,0})$ denote the conifold equipped with Candelas-de la Ossa Ricci-flat Kähler cone metric, and let $0 \in V_0$ denote the tip of the cone. Suppose H is a Hermitian-Yang-Mills metric on $T^{1,0}V_0$ over $B_1(0) \setminus \{0\}$. Assume that there is a constant $C > 0$ so that H satisfies*

$$C^{-1}g_{c_0,0} < H < Cg_{c_0,0}.$$

Then there are constants $c_0 > 0, \lambda \in (0, 1)$, and for each $k \in \mathbb{Z}_{\geq 0}$ a constant $C_k > 0$, such that the following estimate holds

$$|\nabla_{g_{c_0,0}}^k (H - c_0 g_{c_0,0})|_{g_{c_0,0}} \leq C_k r^{\lambda-k},$$

where, as usual, $r(x) = dg_{c_0,0}(x, 0)$ is the distance to $0 \in V_0$ with respect to $g_{c_0,0}$.

The proof of Theorem 4.1 follows closely the work of Jacob-Sá Earp-Walpuski [51] who studied related quantitative convergence results in the case of punctured balls in $\mathbb{C}^n$. Related results for stationary Yang-Mills connections were obtained by Yang using a Łojasiewicz inequality [92]. Chen-Sun [12–15] obtained a general characterization tangent cones of Hermitian-Yang-Mills connections on reflexive sheaves on the ball in $\mathbb{C}^n$ without estimates for the convergence rate. For our applications, the polynomial decay rate, as well as the convergence at the level of metrics (rather than connections) obtained in Theorem 4.1 is crucial.

The first step towards establishing Theorem 4.1 is to prove the following Poincaré inequality.

Lemma 4.2. *Let $(V_0, g_{c_0,0})$ be the conifold equipped with the Candelas-de la Ossa metric. There is a uniform constant $C > 0$ with the following property: for any $\rho \in (0, 1]$ and any $s \in C^\infty(\{r = \rho\}, \sqrt{-1}\mathfrak{su}(T^{1,0}V_0, g_{c_0,0}))$ we have*

$$\int_{\{r=\rho\}} |s|_{g_{c_0,0}}^2 dS(\rho)_{g_{c_0,0}} \leq C\rho^2 \int_{\{r=\rho\}} |\nabla_{g_{c_0,0}} s|_{g_{c_0,0}}^2 dS(\rho)_{g_{c_0,0}}$$

where $dS(\rho)_{g_{c_0,0}}$ denotes the surface measure on $\{r = \rho\}$ induced by $g_{c_0,0}$.

Proof. The result follows from standard elliptic theory and scaling provided we can show that there are no parallel sections of $\sqrt{-1}\mathfrak{Z}\mathbf{u}(T^{1,0}V_0, g_{co,0})$ on the link of the cone $\{r = 1\}$.

To begin, recall from Section 2 that V_0 can be identified with the complement of the zero section in $\iota^*\mathcal{O}_{\mathbb{P}^3}(-1)$ where

$$\iota : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$$

is the Segre embedding. In particular, there is a projection $\pi : V_0 \setminus \{0\} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ whose fibers are orbits of the holomorphic Reeb vector field. Let $E \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the holomorphic vector bundle generated by the invariant sections of $T^{1,0}V_0$, so that $T^{1,0}V_0 = \pi^*E$. We can describe E explicitly; if $\mathcal{L} \subset T^{1,0}V_0$ denotes the trivial line bundle generated by the non-vanishing holomorphic Reeb field, then we have an exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow T^{1,0}V_0 \rightarrow \pi^*T^{1,0}(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow 0.$$

Note that the $\mathcal{L}$ -valued $(1,0)$ form on V_0 given by $\xi \otimes \partial \log r$ is precisely the orthogonal projection $T^{1,0}V_0 \rightarrow \mathcal{L}$ given by the Calabi-Yau metric $g_{co,0}$. Therefore the second fundamental form of $\mathcal{L} \subset T^{1,0}V_0$ is represented by

$$\xi \otimes \sqrt{-1}\partial\bar{\partial} \log r = \xi \otimes \pi^*\omega_{KE}$$

where ω_{KE} is the Kähler-Einstein metric on $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying

$$\text{Ric}(\omega_{KE}) = 3\omega_{KE}.$$

Since $\mathcal{L}$ is trivial, we can view E as the bundle corresponding to $\frac{1}{3}c_1(\mathbb{P}^1 \times \mathbb{P}^1)$ under the isomorphisms

$$\text{Ext}^1(T^{1,0}(\mathbb{P}^1 \times \mathbb{P}^1), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}) \cong H^1(T^{1,0}(\mathbb{P}^1 \times \mathbb{P}^1)^\vee) \cong H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{C})$$

Thus E sits in an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow E \rightarrow T^{1,0}(\mathbb{P}^1 \times \mathbb{P}^1) \rightarrow 0. \quad (4.1)$$

Furthermore, since $g_{co,0}$ is Calabi-Yau on the cone, one can easily show that E admits a natural Hermitian-Yang-Mills metric with respect to the Kähler class $c_1(p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1))$, see for example [86] for related discussion.

It is easy to show, by direct calculation, that any parallel section of $s \in C^\infty(\{r = \rho\}, \sqrt{-1}\mathfrak{Z}\mathbf{u}(T^{1,0}V_0, g_{co,0}))$ descends to a trace-free, global holomorphic section $s_0 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \text{End}(E))$. Thus, it suffices to show that the only global holomorphic endomorphisms of E are multiples of the identity map. This will follow from the usual result for stable vector bundles provided we can show that E is indecomposable [83]. Let H_E denote the Hermitian-Yang-Mills connection on E and let $\omega = p_1^*\omega_{FS} + p_2^*\omega_{FS}$ where ω_{FS} denotes the Fubini-Study metric on $\mathbb{P}^1$. By the exact sequence (4.1), the slope of E is given by

$$\mu(E) = \frac{c_1(E) \cup [\omega]}{\text{rk}(E)} = \frac{2 \int_{\mathbb{P}^1 \times \mathbb{P}^1} \omega^2}{3} = \frac{4}{3}$$

Now suppose that E can be holomorphically decomposed as $E = E_1 \oplus E_2$ where $\text{rk}(E_1) = 1, 2$. A standard computation shows that the decomposition $E = E_1 \oplus E_2$ is orthogonal with respect to the Hermitian-Yang-Mills metric and the restriction $H_E|_{E_1}$ is Hermitian-Yang-Mills with slope $\mu(E_1) = \mu(E)$ [83]. Thus we have

$$c_1(E_1) \cup [\omega] = \frac{4}{3} \text{rk}(E_1).$$

However, since $c_1(E_1), [\omega] \in H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ this implies that $\frac{4}{3} \text{rk}(E_1) \in \mathbb{Z}$, which is impossible since $\text{rk}(E_1) = 1, 2$. Therefore E is indecomposable and hence stable. The result follows. $\square$

In the remainder of this section we will show that Lemma 4.2 implies Theorem 4.1. Much of the argument is based on the following well-known formula: if $H, \hat{H}$ are Hermitian metrics on a holomorphic vector bundle E , then the positive definite, hermitian (with respect to either $H, \hat{H}$) endomorphism $h = \hat{H}^{-1}H$ satisfies

$$F_{j\bar{k}} - \hat{F}_{j\bar{k}} = -\partial_{\bar{k}}(h^{-1}\hat{\nabla}_j h) \quad (4.2)$$

where F (resp. $\hat{F}$) denotes the curvature of the Chern connection ∇ (resp. $\hat{\nabla}$) defined with respect to H (resp. $\hat{H}$). In our case we will take $\hat{H} = g$ to be the Calabi-Yau cone metric on V_0 . We begin with the following lemma, which shows that, at least at the level of the determinant, the metric H decays towards $g_{\text{co},0}$.

Lemma 4.3. *Let (V_0, g) be a Calabi-Yau cone of real dimension $n > 2$. Suppose H is a Hermitian-Yang-Mills metric on $T^{1,0}V_0 \rightarrow B_1(0) \setminus \{0\}$ with slope 0. Suppose there is a constant $C > 0$ such that the $h = g^{-1}H$ satisfies $C^{-1}Id < h < CId$. Then, there are constants $C_k, C_*, \gamma > 0$, with γ depending only on $(V_0, g_{\text{co},0})$ so that, for each $k \in \mathbb{N}$ we have*

$$\left| \nabla_g^k (\log(\det h) - C_*) \right|_g \leq C_k r^{\gamma-k}$$

Proof. Since g is Ricci flat, and H is Hermitian-Yang-Mills with slope 0, it follows from (4.2) that

$$g^{j\bar{k}} \partial_j \partial_{\bar{k}} \log \det h = -g^{j\bar{k}} F_{j\bar{k}} = 0.$$

On the other hand, since $\log \det h$ is bounded the result follows from separation of variables. To see this recall that if φ_λ is a function on the link $L := \partial B_1(0) \subset V_0$ satisfying

$$\Delta_{g_L} \varphi_\lambda + \lambda \varphi_\lambda = 0$$

with $\lambda \geq 0$, then we can produce harmonic functions $u_\lambda^\pm$ on V_0 given by

$$u_\lambda^\pm = r^{a(\lambda) \pm} \varphi_\lambda$$

where $2a(\lambda)_\pm = -(n-2) \pm \sqrt{(n-2)^2 + 4\lambda}$ (since $n > 2$). Fixing an orthonormal basis of eigenfunctions $\{\varphi_\lambda\}_{\lambda \in \text{Spec}(\Delta_{g_L})}$, standard elliptic theory says we can write

$$\log \det h|_L = \sum_{\lambda \in \text{Spec}(\Delta_{g_L})} c_\lambda \varphi_\lambda$$

for constant c_λ . Now, we claim that since $\log \det h$ is bounded we have

$$\log \det h = \sum_{\lambda \in \text{Spec}(\Delta_{g_L})} c_\lambda u_\lambda^+ := u.$$

This follows from the Caccioppoli inequality; let η be a standard, smooth cut-off function on $\mathbb{R}$ with $\eta(x) = 0$ for $x \leq 1$, and $\eta(x) = 1$ for $x \geq 2$. Consider $\eta_\varepsilon(y) = \eta(\varepsilon^{-1}d_g(y, 0))$. Since $\hat{u} := \log \det h - u$ is harmonic, bounded and vanishes on the link L we have

$$0 = - \int_{B_1} \eta_\varepsilon^2 \hat{u} \Delta_g \hat{u} d\text{vol}_g = \int_{B_1} \langle \nabla(\eta_\varepsilon^2 \hat{u}), \nabla \hat{u} \rangle d\text{vol}_g$$

Applying Cauchy-Schwarz we obtain

$$\int_{B_1} \eta_\varepsilon^2 |\nabla \hat{u}|^2 \leq C \int_{B_{2\varepsilon} \setminus B_\varepsilon} |\nabla \eta_\varepsilon|^2 \hat{u}^2.$$

Now we have $|\nabla \eta_\varepsilon|^2 \leq C\varepsilon^{-2}$, while $|B_{2\varepsilon} \setminus B_\varepsilon| \leq C\varepsilon^n$ and so, since $n > 2$ we can take the limit as $\varepsilon \rightarrow 0$ to obtain $|\nabla \hat{u}| \equiv 0$ and the claim follows.

Now the result for $k = 0$ follows from the fact that the only harmonic function on the link $\{r = 1\}$ is a constant. Combining this with standard estimates for harmonic functions and scaling, we obtain the result for $k \geq 1$. $\square$

Next we prove that the relative endomorphism $h = g_{co,0}^{-1}H$ is $W^{1,2}$ on the cone.

Lemma 4.4. *With $h = g_{co,0}^{-1}H$ as above, we have $|\nabla_{g_{co,0}} h|_{g_{co,0}} \in L^2(B_1(0), g_{co,0})$.*

Proof. To ease notation let us denote $g = g_{co,0}$ and $\nabla = \nabla_{g_{co,0}}$. To prove the L^2 bound, observe that (4.2) implies

$$g^{j\bar{k}} \partial_j \partial_{\bar{k}} \text{Tr } h = g^{j\bar{k}} \text{Tr} (\partial_{\bar{k}} h h^{-1} \nabla_j h).$$

On the other hand, since h is hermitian and bounded we have

$$C^{-1} |\nabla h|_g^2 \leq g^{j\bar{k}} \text{Tr} (\partial_{\bar{k}} h h^{-1} \nabla_j h) \leq C |\nabla h|_g^2.$$

This implies an L^2 estimate for $|\nabla h|_g$. To see this, let $\eta(x)$ be a standard cut-off function such that $\eta(x) \equiv 1$ for $x \in [0, 1]$, $\eta(x) \equiv 0$ for $x \in [2, \infty)$ and $|\eta'| + |\eta''| < 10$. For $y > 0$ we set $\eta_y = \eta(y^{-1}r)$.

Since $\text{Tr } h$ is uniformly bounded we have

$$\begin{aligned} \int_{B_1} g^{j\bar{k}} \text{Tr}(\partial_{\bar{k}} h h^{-1} \nabla_j h) d\text{vol}_g &\leq \lim_{\varepsilon \rightarrow 0} \int_{B_2} (1 - \eta_\varepsilon)^2 (\eta_1)^2 g^{j\bar{k}} \text{Tr}(\partial_{\bar{k}} h h^{-1} \nabla_j h) d\text{vol}_g \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_2} (1 - \eta_\varepsilon)^2 \eta_1^2 (\Delta_g \text{Tr } h) d\text{vol}_g \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_2} (\Delta_g (1 - \eta_\varepsilon)^2 \eta_1^2) \text{Tr } h d\text{vol}_g \\ &\leq C \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \text{Vol}_g(B_{2\varepsilon} \setminus B_\varepsilon) + C \end{aligned}$$

and the result follows since $\varepsilon^{-2} \text{Vol}_g(B_{2\varepsilon} \setminus B_\varepsilon) \leq C\varepsilon^4$. $\square$

The next step is to establish some decay for the endomorphism $h = g_{co,0}^{-1} H$.

Lemma 4.5. *Define a hermitian endomorphism s by*

$$e^s = \left(\frac{\det g_{co,0}}{\det H} \right)^{\frac{1}{3}} \cdot g_{co,0}^{-1} H.$$

Then are constants $C > 0, \alpha \in (0, 1)$ depending on $(V_0, g_{co,0})$ and $|s|_{L^\infty(B_1(0))}$ such that

$$\mu(\tau) := \int_{B_\tau} r^{2-2n} |\nabla_{g_{co,0}} s|_{g_{co,0}}^2 d\text{vol}_{g_{co,0}} \leq C\tau^{2\alpha}$$

for all $\tau \leq 1$.

Proof. Again, we denote $g = g_{co,0}$ and $\nabla = \nabla_{g_{co,0}}$ to ease notation. It is not hard to check that

$$\langle e^{-s} \nabla_j e^s, s \rangle_g = \langle \nabla_j s, s \rangle_g.$$

Therefore, from (4.2) we have

$$\begin{aligned} g^{j\bar{k}} \partial_{\bar{k}} |s|_g^2 &= g^{j\bar{k}} \partial_{\bar{k}} (\langle \nabla_j s, s \rangle_g + g^{j\bar{k}} \langle s, \partial_{\bar{j}} s \rangle_g) \\ &= g^{j\bar{k}} \partial_{\bar{k}} (\langle e^{-s} \nabla_j e^s, s \rangle_g + g^{j\bar{k}} \langle s, e^{-s} \partial_{\bar{j}} e^s \rangle_g) \\ &= g^{j\bar{k}} \langle e^{-s} \nabla_j e^s, \nabla_k s \rangle_g + g^{j\bar{k}} \langle \partial_{\bar{k}} s, e^{-s} \partial_{\bar{j}} e^s \rangle_g. \end{aligned}$$

Note that the bound $C^{-1}g < H < Cg$ implies that $|s| < C$. Thanks to [91, Lemma 2.1] there is a uniform constant $A > 0$ depending only on C so that

$$g^{j\bar{k}} \langle e^{-s} \nabla_j e^s, \nabla_k s \rangle_g + g^{j\bar{k}} \langle \partial_{\bar{k}} s, e^{-s} \partial_{\bar{j}} e^s \rangle_g \geq A^{-1} |\nabla s|^2.$$

In summation, we have show that

$$\Delta|s|_g^2 \geq A^{-1}|\nabla s|_g^2. \quad (4.3)$$

We now show that $\mu(\tau) < C$ for some constant C independent of τ . Let $\eta_\tau(r)$ be the cut-off function from Lemma 4.3. First note that, for any $\varepsilon > 0$, integration by parts using (4.3), $\Delta_g r^{2-2n} = 0$ and $|\Delta_g \eta_\tau| \leq C\tau^{-2}$, together with the bound for $|s|$ yields

$$\begin{aligned} \int_{B_\tau \setminus B_{2\varepsilon}} r^{2-2n} |\nabla s|_g^2 d\text{vol}_g &\leq A \int_{B_{2\varepsilon} \setminus B_\varepsilon} (1 - \eta_\varepsilon) \eta_\tau r^{2-2n} \Delta|s|_g^2 d\text{vol}_g \\ &\leq A' \left(\varepsilon^{-2n} \int_{B_{2\varepsilon} \setminus B_\varepsilon} |s|^2 dV_g + \int_{B_{2\varepsilon} \setminus B_\tau} r^{-2n} |s|^2 d\text{vol}_g \right) \end{aligned} \quad (4.4)$$

which is bounded thanks to the L^∞ bound for s . To improve the estimate we decompose the second integral appearing on the right of (4.4) as

$$\int_{B_{2\varepsilon} \setminus B_\tau} |s|^2 d\text{vol}_g \leq \left(\int_\tau^{2\varepsilon} dr \cdot \int_{\partial B_r(0)} |s|^2 dS_g(r) \right)$$

where $dS_g(r)$ denotes the surface measure on ∂B_r . Since s is trace free we can apply the Poincaré inequality in Lemma 4.2 to get

$$\int_{\partial B_r} |s|^2 dS_g(r) \leq Cr^2 \int_{\partial B_r(0)} |\nabla^T s|^2 dS_g(r) \leq Cr^2 \int_{\partial B_r} |\nabla s|^2 dS_g(r)$$

where we wrote ∇^T for the covariant derivative tangent to ∂B_r . Thus, we have

$$\int_{B_{2\varepsilon} \setminus B_\tau} r^{-2n} |s|^2 d\text{vol}_g \leq C \int_{B_{2\varepsilon} \setminus B_\tau} r^{2-2n} |\nabla s|^2 d\text{vol}_g = C(\mu(2\varepsilon) - \mu(\tau))$$

Arguing similarly for the first term yields

$$\int_{B_{2\varepsilon} \setminus B_\varepsilon} \varepsilon^{-2n} |s|^2 d\text{vol}_g \leq C\mu(2\varepsilon)$$

All together this implies

$$\mu(\tau) \leq \frac{C}{C+1} (\mu(2\varepsilon) + \mu(2\varepsilon)).$$

Since this estimate holds for all $\varepsilon > 0$ and, thanks to Lemma 4.4, $\mu(\varepsilon) \rightarrow 0$ by the dominated convergence theorem, we conclude

$$\mu(\tau) \leq \frac{C}{C+1} \mu(2\varepsilon).$$

The lemma follows by a standard iteration argument. □

Theorem 4.1 will now follow from Lemma 4.5 together with the regularity theory for the Hermitian-Yang-Mills equation, which we recall below. The regularity theory is originally due to Bando-Siu [3, Proposition 1], but we refer the reader to the paper of Jacob-Walpuski [50, Theorem C.1] for the precise statement which implies the one below.

Proposition 4.6. *Let (Y, g, J) be a Kähler manifold of dimension n with bounded geometry, and let $E \rightarrow Y$ be a holomorphic vector bundle. If H_0, H are hermitian metrics on E , H is Hermitian-Yang-Mills and $s := \log(H_0^{-1}H) \in C^\infty(Y, \sqrt{-1}\mathfrak{su}(E, H_0))$, then, for all $k \in \mathbb{N}$ and $p \in (1, \infty)$ there is a function $f_{k,p}(y) > 0$ depending only on k, p and the geometry of (Y, g) such that $f_{k,p}(0) = 0$ and*

$$r^{k+2-\frac{2n}{p}} \|\nabla_{H_0}^{k+2} s\|_{L^p(B_r(x))} \leq f_{k,p} \left(\|s\|_{L^\infty(B_{2r}(x))} + \sum_{i=0}^k r^{2+i} \|\nabla_{H_0}^i F_{H_0}\|_{L^\infty(B_{2r}(x))} \right).$$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let H be as in the statement of the theorem, and set $g = g_{co,0}$. Throughout the proof C will denote a constant which can change from line to line, but depends only on (V_0, g) and the positive upper and lower bounds for $g^{-1}H$. Define s by

$$e^s = \left(\frac{\det g_{co,0}}{\det H} \right)^{\frac{1}{3}} \cdot g_{co,0}^{-1} H.$$

so that $s \in C^\infty(V_0, \sqrt{-1}\mathfrak{su}(T^{1,0}V_0, g))$.

Fix $0 < R \ll 1$ and consider the annulus $B_{4R} \setminus B_{R/4}$. Let $m_R : V_0 \rightarrow V_0$ be the map $m_R(p) = R^{-1} \cdot p$ where $\cdot$ denotes the natural scaling action on the cone. Let $\hat{s} = m_R^* s$. From the scale invariance of $\mu(\tau)$ we have

$$\int_{B_4 \setminus B_{\frac{1}{4}}} |\nabla \hat{s}|_g^2 d\text{vol}_g \leq CR^{2\alpha}$$

while the Poincaré inequality proved in Lemma 4.2 implies

$$\int_{B_4 \setminus B_{\frac{1}{4}}} |\hat{s}|^2 \leq C \int_{B_4 \setminus B_{\frac{1}{4}}} |\nabla \hat{s}|^2 \leq CR^{2\alpha}. \quad (4.5)$$

Note that the L^∞ bound for $\hat{s}$ together with the interior estimates, Proposition 4.6, yield

$$\|\nabla^{k+2} \hat{s}\|_{L^p(B_4 \setminus B_{1/4})} \leq C_{k,p}$$

and hence, by the Sobolev imbedding theorem we get

$$\|\hat{s}\|_{C^k(B_3 \setminus B_{1/2})} \leq C_k. \quad (4.6)$$

To improve this bound to a decay estimate we appeal to the Hermitian-Yang-Mills equation. From (4.2), we have

$$g^{j\bar{k}} \nabla_{\bar{k}} \nabla_j \hat{s} + B(\nabla \hat{s} \otimes \nabla \hat{s}) = 0 \quad (4.7)$$

where $B(\cdot)$ is linear with coefficients depending on $\hat{s}$, but not on any of its derivatives. The result now follows from standard elliptic regularity and bootstrapping. By elliptic regularity we have

$$\|\hat{s}\|_{W^{2,2}(B_{2.5} \setminus B_{3/4})} \leq C \left(\|\Delta \hat{s}\|_{L^2(B_3 \setminus B_{1/2})} + \|\hat{s}\|_{L^2(B_3 \setminus B_{1/2})} \right).$$

On the other hand, from (4.7) we have

$$\|\Delta \hat{s}\|_{L^2(B_3 \setminus B_{1/2})} \leq C \left(\int_{B_3 \setminus B_{1/2}} |\nabla \hat{s}|^4 \right)^{\frac{1}{2}} \leq C \left(\int_{B_3 \setminus B_{1/2}} |\nabla \hat{s}|^2 \right)^{\frac{1}{2}}$$

where, in the second inequality, we used (4.6) with $k = 2$. From (4.5) we conclude that

$$\|\hat{s}\|_{W^{2,2}(B_{2.5} \setminus B_{3/4})} \leq CR^\alpha,$$

for a uniform constant $C > 0$. By differentiating (4.7) a straightforward boot-strapping argument yields

$$\|\hat{s}\|_{W^{k,2}(B_2 \setminus B_1)} \leq C_k R^\alpha$$

for uniform constants $C_k > 0$. All together we obtain

$$|\hat{s}|_{C^k(B_2 \setminus B_1)} \leq C_k R^\alpha$$

from the Sobolev imbedding theorem. Rescaling yields

$$|s|_{C^k(B_{2R} \setminus B_R)} \leq C_k R^{\alpha-k}. \quad (4.8)$$

Finally, Theorem 4.1 follows from this estimate together with Lemma 4.3. $\square$

Remark 4.7. In the setting where we apply Theorem 4.1, where the metric H_0 obtained as a limit of (H_a, g_a) , we can apply Theorem 3.1 to obtain (4.6) bypassing the Bando-Siu regularity theorem.

5 | APPROXIMATE HERMITIAN-YANG-MILLS METRICS

In the previous section, we started from a Calabi-Yau metric ω_{CY} on a simply connected Kähler Calabi-Yau threefold (X, Ω) and constructed a pair of hermitian metrics (g_0, H_0) on a singular space $\underline{X}$ obtained by contracting $(-1, -1)$ curves C_i . These metrics satisfy $d\omega_0^2 = 0$ and $\Lambda_{\omega_0} F_{H_0} =$

0. Let $\{p_i\}$ denote the nodal points of X_0 . There are constants $c_i, R_i, \lambda > 0$ such that we have the following local description near the nodes:

- The Fu-Li-Yau construction gives

$$g_0 = R_i g_{co,0}, \quad \text{near node } p_i.$$

- By (4.8) and Lemma 4.3, for $k \in \mathbb{Z}_{\geq 0}$ there exists $M_k > 1$ such that

$$|\nabla_{g_{co,0}}^k (H_0 - c_i g_{co,0})|_{g_{co,0}} \leq M_k r^{\lambda-k}, \quad \text{near node } p_i. \quad (5.1)$$

For ease of notation, in this section we will work at a single node point with scale constants $c_1 = R_1 = 1$. The metric H_0 on $\underline{X}$ has conical singularities which we will desingularize by gluing in the asymptotically conical metric $g_{co,t}$ on V_t . For other work in geometry using this technique, see, for example [11, 52, 53, 61]. The glued metric H_t will approximately solve the Hermitian-Yang-Mills equation on the smoothings X_t for t sufficiently small.

Recall that under the assumption of Theorem 2.3 there is a smoothing $\mu : \mathcal{X} \rightarrow \Delta$ with $\mu^{-1}(t) = X_t$, $\mu^{-1}(0) = \underline{X}$, and the family $\mathcal{X}$ is locally described by $\{(z, t) : z \in V_t\}$ near the nodes, with $V_t = \{\sum_{i=1}^4 z_i^2 = t\} \subset \mathbb{C}^4$. We will show there exists $\gamma, \varepsilon \in (0, 1)$ and $C > 1$ such that the approximate solution H_t satisfies

$$\|\Lambda_{\omega_t} F_{H_t}\|_{C_{\beta-2}^{0,a}} \leq C|t|^\gamma$$

for all $0 < |t| \leq \varepsilon$, and for suitably defined weighted Hölder spaces with $\beta \in [-2, 0]$, and $0 < a < 1$; see Section 5.2 below for a precise definition. We recall that we denote by ω_t the Hermitian metric from Proposition 2.17 constructed by Fu-Li-Yau, which satisfies $\omega_t = \omega_{co,t}$ near the nodes and converges back to the balanced metric ω_0 on compact sets as $t \rightarrow 0$.

5.1 | Definition of the approximate solution

To construct a Hermitian metric H_t on X_t which approximately solves the Hermitian-Yang-Mills equation, we will glue $g_{co,t}$ to a deformation of the singular metric H_0 on the annulus region

$$\{|t|^\alpha \leq \|z\|^2 \leq 2|t|^\alpha\} \subset V_t.$$

Here $0 < \alpha < 1$, and specifically we will take $\alpha = (1 + \lambda/3)^{-1}$ where $\lambda > 0$ is the rate in (5.1). Let

$$\chi(z) = \zeta(|t|^{-\alpha} \|z\|^2)$$

be a cutoff function on this annulus region, that is, the function $\zeta : [0, \infty) \rightarrow [0, 1]$ satisfies $\zeta \equiv 1$ on $[0, 1]$ and $\zeta \equiv 0$ on $[2, \infty)$. Our glued Hermitian metric on X_t is

$$H_t = \chi g_{co,t} + (1 - \chi) K_t, \quad (5.2)$$

where

$$K_t = [(\Phi_t^{-1})^* H_0]^{1,1}$$

is the J_t -invariant part of the pullback $(\Phi_t^{-1})^* H_0$. Explicitly, we define $(A^{1,1})_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} + J^\mu_\alpha A_{\mu\nu} J^\nu_\beta)$ for a symmetric 2-tensor A . Recall Φ_t is defined in Lemma 2.13, and note that K_t is defined on $X_t \setminus \{\|z\|^2 = |t|\}$ and so H_t is defined on all of X_t .

We will need estimates on the glued metric H_t which are uniform in t .

Lemma 5.1. *There exists $\varepsilon > 0$ such that for any $k \in \mathbb{Z}_{\geq 0}$ there exists $C_k > 1$ such that for all $0 < |t| < \varepsilon$ we have*

$$C_0^{-1} g_t \leq H_t \leq C_0 g_t, \quad |\nabla_{g_t}^k H_t|_{g_t} \leq C_k r^{-k}. \quad (5.3)$$

Proof. We work region-by-region.

- Region $\{\|z\|^2 \leq |t|^\alpha\}$. Here $H_t = g_t = g_{co,t}$ so the estimates are trivial.
- Region $\{|t|^\alpha \leq \|z\|^2 \leq 1\}$. Here $g_t = g_{co,t}$ and $\|z\|^2 \gg |t|$ for all t small enough. The estimate (3.1) reads $|H_0|_{g_{co,0}} \leq C$ and $|H_0^{-1}|_{g_{co,0}} \leq C$, and so pulling back by Φ_t^{-1} and using Lemma 2.10 gives

$$C^{-1} g_{co,t} \leq (\Phi_t^{-1})^* H_0 \leq C g_{co,t}.$$

Since $|[(\Phi_t^{-1})^* H_0]^{1,1}|_{g_{co,t}} \leq |(\Phi_t^{-1})^* H_0|_{g_{co,t}}$ and similarly for H_0^{-1} , this proves that $C^{-1} g_{co,t} \leq H_t \leq C g_{co,t}$. Next, pulling back $|\nabla_{g_{co,0}}^k H_0| \leq C_k r^{-k}$ by Lemma 2.10 gives

$$|\nabla_{g_{co,t}}^k K_t|_{g_{co,t}} \leq C r^{-k}.$$

This proves (5.3) in the region $\{2|t|^\alpha \leq \|z\|^2 \leq 1\}$ where $H_t = K_t = [(\Phi_t^{-1})^* H_0]^{1,1}$. In the transition region $\{|t|^\alpha \leq \|z\|^2 \leq 2|t|^\alpha\}$ we have,

$$\nabla H_t = (\nabla \chi)(g_{co,t} - K_t) + (1 - \chi)\nabla K_t$$

and

$$|\nabla_{g_t} H_t|_{g_t} \leq C |\nabla \chi|_{g_t} + C |\nabla_{g_t} K_t|_{g_t}.$$

We estimate

$$|\nabla \chi|_{g_t} \leq C |t|^{-\alpha} |\nabla r^3|_{g_t} \leq C |t|^{-\alpha} r^2 \leq C r^{-1}. \quad (5.4)$$

Here we used $|\nabla r|_{g_t} \leq C$ and $r^3 = \|z\|^2 \leq 2|t|^\alpha$ in the transition region. Thus

$$|\nabla_{g_{co,t}} H_t|_{g_{co,t}} \leq C r^{-1} \quad (5.5)$$

and the higher order estimates are similar.

- Region $\{1 \leq r\}$. Here $H_t = [(\Phi_t^{-1})^* H_0]^{1,1}$ and the metrics $\Phi_t^* g_t$ converge smoothly uniformly to g_0 as $t \rightarrow 0$ by Lemma 2.17. The estimates for H_t follow from pulling back the estimates for H_0 obtained in Theorem 3.1. $\square$

5.2 | Weighted Hölder spaces

In the upcoming analysis we will work in weighted Hölder spaces on X_t using the weight function r , the metric g_t which is equal to the model metric $g_{co,t}$ near the nodes, and the glued metric H_t . We will use the connection ∇_{H_t} when differentiating. For endomorphisms $h \in \Gamma(\text{End } T^{1,0} X_t)$, we use the norm

$$\|h\|_{C_\beta^k(g_t, H_t)} = \sum_{i=0}^k \sup_{X_t} |r^{-\beta+i} \nabla_{H_t}^i h|_{g_t}.$$

For $\Phi \in \Gamma((TX_t)^p \otimes (T^* X_t)^q)$, we define the semi-norm

$$[\Phi]_{C_\beta^{0,a}} = \sup_{x \neq y} \left[\min(r(x), r(y))^{-\beta} \frac{|\Phi(x) - \Phi(y)|_{g_t}}{d(x, y)^a} \right]$$

where the sup is taken over points x, y with distance less than the injectivity radius and $\Phi(x) - \Phi(y)$ is understood by ∇_{g_t} -parallel transport along the minimal g_t geodesic connecting x and y . The weighted Hölder norms are then

$$\|h\|_{C_\beta^{k,a}(g_t, H_t)} = \|h\|_{C_\beta^k(g_t, H_t)} + \left[\nabla_{H_t}^k h \right]_{C_{\beta-k-a}^{0,a}}.$$

This definition is well adapted to work on annuli $U_{\hat{r}} = \{(1/2)\hat{r} \leq r(z) \leq 2\hat{r}\}$ at a given scale $\hat{r} > 0$. The norm over $U_{\hat{r}}$ is equivalent to

$$\|h\|_{C_\beta^{0,a}(U_{\hat{r}})} = \hat{r}^{-\beta} \left[\sup_{U_{\hat{r}}} \|h\|_{C^0(\hat{r}^{-2} g_t)} + \sup_{x, y \in U_{\hat{r}}} \frac{|h(x) - h(y)|_{\hat{r}^{-2} g_t}}{d_{\hat{r}^{-2} g_t}(x, y)^a} \right], \quad (5.6)$$

where norms on the endomorphism $h \in \Gamma(\text{End } T^{1,0} X_t)$ are now all with respect to the rescaled metric $\hat{r}^{-2} g_t$. We will often estimate global Hölder norms by estimating them on local annuli $U_{\hat{r}}$.

Lemma 5.2. *Let $\beta \leq 0$ and $h \in \Gamma(\text{End } T^{1,0} X_t)$. Suppose there is a uniform bound on the local estimates $\|h\|_{C_\beta^{0,a}(U_{\hat{r}})} \leq K$ for all $\hat{r} > 0$, where $U_{\hat{r}} = \{(1/2)\hat{r} \leq r \leq 2\hat{r}\} \subset X_t$. Then $\|h\|_{C_\beta^{0,a}(X_t)} \leq CK$ for a constant $C > 1$ which is independent of t .*

Proof. The local bounds imply $\|h\|_{C_\beta^0(X_t)} \leq K$, so we need to estimate the global Hölder semi-norm. Let $x, y \in X_t$ and suppose $r(x) \leq r(y)$. If y lies in the set $U = \{(1/2)r(x) < r < 2r(x)\}$, the estimate is assumed. If $r(x) \geq \varepsilon > 0$, the geometry is uniform in t and the estimate holds for $C(\varepsilon)$. In the remaining case, we assume $x, y \in V_t$ with $2r(x) < r(y)$ and $g_t = g_{co,t}$ and we claim that

$$d_{g_t}(x, y) \geq C^{-1} r(x)$$

for $C > 1$ independent of t . Indeed, this inequality holds at $t = 1$ for a constant $C > 1$, and the uniform bound in t follows from the bound when $t = 1$ by scaling $S(z) = t^{-1/2}z$ with $S^*g_{co,1} = |t|^{-2/3}g_{co,t}$ and $S^*r = |t|^{-1/3}r$. Therefore

$$r(x)^{-\beta+a} \frac{|h(x) - h(y)|}{d_{g_t}(x, y)^a}$$

is bounded by $C\|h\|_{C_\beta^0(X_t)}$ since $\beta \leq 0$. $\square$

We end this discussion with the following remark: if

$$r^{-\beta}|h|_{g_t} + r^{-\beta+1}|\nabla_{g_t} h|_{g_t} \leq K,$$

then for $0 < a < 1$ we can estimate $\|h\|_{C_\beta^{0,a}(X_t)} \leq CK$ where C is independent of t . This can be seen for example from expression (5.6), since $\hat{r}^{-2}g_t$ is uniformly (in t) equivalent to the Euclidean metric in holomorphic cylindrical coordinates (Lemma 2.11). Also, the difference of connections satisfies the bound $r|A_{g_t} - A_{H_t}|_{g_t} \leq C$ by Lemma 5.1, so to estimate $\|h\|_{C_\beta^{0,a}(X_t)}$ we could equivalently estimate $r^{-\beta+1}|\nabla_{H_t} h|_{H_t}$ instead of $r^{-\beta+1}|\nabla_{g_t} h|_{g_t}$.

5.3 | Smallness of the approximate solution

The main objective of this section is to show that the glued metric H_t has small Hermitian-Yang-Mills tensor.

Proposition 5.3. *Let H_t be the glued metric as in (5.2). There exists $C > 0$ and $\varepsilon > 0$ such that for any $0 < a < 1$, and any $t \in \mathbb{C}^*$ with $|t| \leq \varepsilon$ we have*

$$\|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}} \leq C|t|^{\frac{\alpha\lambda}{3}}, \quad (5.7)$$

where $\lambda > 0$ is the rate in (5.1) (see Theorem 4.1), and $\alpha = (1 + \frac{\lambda}{3})^{-1}$.

The approximate solution will be estimated in four regions.

- Region $\{\|z\|^2 \leq |t|^\alpha\}$. Here $H_t = g_{co,t}$ and $\Lambda_{\omega_t} F_{H_t} = 0$.
- Region $\{\frac{1}{4}|t|^\alpha \leq \|z\|^2 \leq 4|t|^\alpha\}$. This contains the transition region, and we will show that here

$$\|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}} \leq C|t|^{(\alpha\lambda)/3}$$

in Lemma 5.4 below.

- Region $\{2|t|^\alpha \leq \|z\|^2 \leq 2\}$. In this region, $H_t = [(\Phi_t^{-1})^* H_0]^{1,1}$ and we need to control the Hermitian-Yang-Mills tensor of $[(\Phi_t^{-1})^* H_0]^{1,1}$. We will estimate

$$\|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}} \leq C|t|^{1-\alpha}$$

in this region in Lemma 5.5 below.

- Region $\{r \geq 1\}$. In this region the geometry is smoothly varying, and so since $\Lambda_{\omega_0} F_{H_0} = 0$ then

$$\|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}} \leq C|t|.$$

By Lemma 5.2, it suffices to check the Hölder estimate on these local pieces to obtain the global estimate. We start by estimating the Hermitian-Yang-Mills tensor in the transition region.

Lemma 5.4. *With notation as in Proposition 5.3, the estimate $\|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}(U)} \leq C|t|^{\frac{\alpha\lambda}{3}}$ holds in the region $U = \{\frac{1}{4}|t|^\alpha \leq \|z\|^2 \leq 4|t|^\alpha\}$.*

Proof. If we decompose

$$H_0 = g_{co,0} + E_0,$$

then the glued metric is

$$H_t = g_{co,t} + (1 - \chi) \left[[(\Phi_t^{-1})^* E_0]^{1,1} + [(\Phi_t^{-1})^* g_{co,0}]^{1,1} - g_{co,t} \right].$$

Since $\Lambda_{\omega_t} F_{g_{co,t}} = 0$, the formula (2.19) for the difference of curvature tensors gives

$$\sqrt{-1} \Lambda_{\omega_t} F_{H_t} = -(g_{co,t})^{j\bar{k}} \partial_{\bar{k}} (h_t^{-1} (\nabla_{g_{co,t}})_j h_t) \quad (5.8)$$

where

$$\begin{aligned} h_t &= I + (1 - \chi) \left[g_{co,t}^{-1} [(\Phi_t^{-1})^* E_0]^{1,1} + g_{co,t}^{-1} \left([(\Phi_t^{-1})^* g_{co,0}]^{1,1} - g_{co,t} \right) \right] \\ &:= I + (1 - \chi) \mathcal{E}. \end{aligned} \quad (5.9)$$

During this proof, we simply write $\nabla = \nabla_{g_{co,t}}$. We claim that

$$C^{-1}I \leq h_t \leq CI, \quad |\nabla^k h_t|_{g_{co,t}} \leq Cr^{-k} |t|^{\lambda\alpha/3}. \quad (5.10)$$

Assuming this, (5.8) and $|t|^{\lambda\alpha/3} < 1$ imply

$$|\Lambda_{\omega_t} F_{H_t}|_{g_{co,t}} \leq |h^{-1} \nabla h|_{g_{co,t}}^2 + C |h^{-1} \nabla^2 h_t|_{g_{co,t}} \leq Cr^{-2} |t|^{\lambda\alpha/3}.$$

Similarly

$$|\nabla \Lambda_{\omega_t} F_{H_t}|_{g_{co,t}} \leq Cr^{-3} |t|^{\lambda\alpha/3}$$

and this proves the estimate $|\Lambda_{\omega_t} F_{H_t}|_{C_{-2}^a(U)} \leq C|t|^{\lambda\alpha/3}$.

We now prove the claim (5.10). Estimate (5.1) implies $|\nabla^k E_0|_{g_{co,0}} \leq C_k r^{\lambda-k}$, which by Lemma 2.10 and (2.11) yields

$$|\nabla^k(\Phi_t^{-1})^* E_0|_{g_{co,t}} + |\nabla^k[(\Phi_t^{-1})^* g_{co,0} - g_{co,t}]|_{g_{co,t}} \leq C r^{-k}(r^\lambda + |t|r^{-3}).$$

Since $r^3 \sim |t|^\alpha$, this implies

$$|\nabla^k \mathcal{E}|_{g_{co,t}} \leq C r^{-k}(|t|^{\lambda\alpha/3} + |t|^{1-\alpha}).$$

We choose α such that $\lambda\alpha/3 = 1 - \alpha$, so that $|\mathcal{E}| \leq C|t|^{\lambda\alpha/3}$ and

$$|h_t - I|_{g_{co,t}} \leq C|t|^{\lambda\alpha/3} \ll 1$$

which implies $C^{-1}I \leq h_t \leq CI$. Taking a derivative gives

$$\nabla h_t = -\mathcal{E}\nabla\chi + (1 - \chi)\nabla\mathcal{E}.$$

Since $|\mathcal{E}| \leq C|t|^{\lambda\alpha/3}$, $|\nabla\mathcal{E}| \leq Cr^{-1}|t|^{\lambda\alpha/3}$ and $|\nabla\chi| \leq Cr^{-1}$ (e.g. (5.4)), we obtain

$$|\nabla h_t| \leq Cr^{-1}|t|^{\lambda\alpha/3}.$$

The higher order estimates in the claim (5.10) are similar. □

We now consider the next region past the transition zone.

Lemma 5.5. *Let F_{K_t} be the curvature of $K_t = [(\Phi_t^{-1})^* H_0]^{1,1}$. Then on $D = \{|t|^\alpha \leq \|z\|^2 \leq 2\}$, we can estimate*

$$\|\Lambda_{\omega_t} F_{K_t}\|_{C_{-2}^{0,a}(D)} \leq C|t|^{1-\alpha}. \quad (5.11)$$

Proof. Let $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4, t)$ be a point in $\mathcal{X} = \{(z, t) : \sum_{i=1}^4 z_i^2 = t\}$ with $\|\hat{z}\|^2 \geq |t|^\alpha$, and suppose without loss of generality that $\hat{z}_4 \neq 0$. Let $\lambda = \|\hat{z}\|$ and $\hat{r} = \lambda^{2/3}$. We take local coordinates on $U_{\hat{r}} = \{\frac{1}{2}\lambda \leq \|z\| \leq 2\lambda\} \subset X_t$ given by $w^i = \frac{1}{\lambda} z_i$. These coordinates land in $\{\frac{1}{4} \leq |w| \leq 4\} \subset \mathbb{C}^3$ where $|w| = |(w^1, w^2, w^3)|$ is the Euclidean norm on $\mathbb{C}^3$. The formula for the curvature on V_t in coordinates is

$$(F_{K_t})_{j\bar{k}} = -K_t^{-1} \partial_j \partial_{\bar{k}} K_t + K_t^{-1} \partial_j K_t K_t^{-1} \partial_{\bar{k}} K_t.$$

We showed in Lemma 2.11 that in $\{w^i\}$ coordinates, we have $g_t = \hat{r}^2 O(I)$, and by Lemma 5.1 the metric $K_t = \hat{r}^2 O(I)$. Here we write $O(I)$ for a matrix which is positive-definite with positivity and derivative bounds independent of t, λ . Therefore

$$|F_{K_t}|_{g_t} = \hat{r}^{-2} O(1), \quad (5.12)$$

where $O(1)$ denotes a function with smooth bounds independent of t, λ .

- We claim:

$$\frac{\partial}{\partial t} \left[[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}}(F_{K_t})_{j\bar{k}} \right] = \hat{r}^{-5} O(1). \quad (5.13)$$

Here $[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}}$ are the local matrix entries of the inverse of $(\Phi_t^{-1})^* g_{co,0}$ (not the raised indices with respect to g_t). Assuming this for now, we complete the proof of the lemma. Since at $t = 0$ we have $[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}}(F_{K_t})_{j\bar{k}} = 0$, this implies

$$[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}}(F_{K_t})_{j\bar{k}} = t \hat{r}^{-5} O(1).$$

We can then write

$$\sqrt{-1} \Lambda_{\omega_t} F_{K_t} = [(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}}(F_{K_t})_{j\bar{k}} + \left[g_{co,t}^{j\bar{k}} - [(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} \right] (F_{K_t})_{j\bar{k}},$$

and by (2.11) and (5.12) we have

$$\sqrt{-1} \Lambda_{\omega_t} F_{K_t} = t \hat{r}^{-5} O(1)$$

in $\{w^i\}$ coordinates. Therefore

$$|\nabla \left(\sqrt{-1} \Lambda_{\omega_t} F_{K_t} \right)|_{g_t} = |t| \hat{r}^{-6} O(1).$$

Then for any $0 < a < 1$,

$$\|\Lambda_{\omega_t} F_{K_t}\|_{C_{-2}^{0,a}(U_{\hat{r}})} \leq C |t| \hat{r}^{-3} \leq C |t|^{1-\alpha}$$

using that $|t|^\alpha \leq \lambda^2 \leq 1$ and $\lambda^2 = \hat{r}^3$. By Lemma 5.2, this gives the Hölder estimate on all of $D = \{|t|^\alpha \leq \|z\|^2 \leq 1\}$.

- We now prove the claimed (5.13). We start with the variation of K_t . The metric H_0 is defined on $V_0 = \{\sum_i x_i^2 = 0\}$ and here we use coordinates (x^1, x^2, x^3) given by $x^i = \frac{1}{\|\hat{x}\|} x_i$ where $\hat{x} \in V_0$ is the point such that $\Phi_t(\hat{x}) = \hat{z}$. The map Φ_t (defined in (2.9)) appears in coordinates $\{x^i\}$ and $\{w^i\}$ as

$$\Phi_t^i(x) = \left(x^i + \frac{t}{2\|\hat{x}\|^2} \frac{\bar{x}^i}{\sum_{i=1}^3 |x^i|^2 + |\sum_{i=1}^3 (x^i)^2|} \right) \frac{\|\hat{x}\|}{\lambda}. \quad (5.14)$$

Recall that $\|x\|^2 \leq \|\Phi_t(x)\|^2 \leq 2\|x\|^2$, and so $\|\hat{x}\| \sim \lambda$ and coordinates z^i are in the range $\{\frac{1}{4} \leq |z| \leq 4\} \subset \mathbb{C}^3$. We may assume the coordinates x^i on V_0 are in the range $\{\frac{1}{2} \leq |x| \leq 2\} \subset \mathbb{C}^3$. Abusing notation, we simply write $w^i = w^i \circ \Phi_t(x)$. The change of coordinates is of the form

$$\frac{\partial w^i}{\partial x^j} = \frac{\|\hat{x}\|}{\lambda} \delta_j^i + \frac{t}{\lambda^2} O(1)$$

and hence

$$\frac{\partial}{\partial t} \frac{\partial w^i}{\partial x^j} = \lambda^{-2} O(1), \quad \frac{1}{4} \delta^i_j \leq \frac{\partial w^i}{\partial x^j} \leq 4 \delta^i_j. \quad (5.15)$$

Differentiating the inverse Jacobian then also gives

$$\frac{\partial}{\partial t} \frac{\partial x^i}{\partial w^j} = \lambda^{-2} O(1).$$

We now compute the variation of K_t in these coordinates. In components $K_t = (K_t)_{\bar{k}j} dw^j \otimes d\bar{w}^k$, we have

$$\frac{\partial}{\partial t} (K_t)_{\bar{k}j}(w) = \frac{\partial}{\partial t} \left[\overline{\frac{\partial x^p}{\partial w^k}} (H_0)_{\bar{p}q}(x(w)) \frac{\partial x^q}{\partial w^j} \right].$$

We note

$$\frac{\partial}{\partial t} [(H_0)(x(w))] = \frac{\partial H_0}{\partial x^i}(x(w)) \frac{\partial x^i}{\partial w^p} \frac{\partial w^p}{\partial t} = O(\lambda^{-2}) \frac{\partial H_0}{\partial x^i}.$$

Recall that in these coordinates, we have that $H_0 = r(\hat{x})^2 O(I)$, and we noted earlier that $r(\hat{x}) \sim r(\hat{z}) = \hat{r}$. Putting everything together, we have

$$\frac{\partial}{\partial t} (K_t)_{\bar{k}j}(w) = \hat{r}^2 \lambda^{-2} O(1). \quad (5.16)$$

Since $K_t = \hat{r}^2 O(I)$ in these coordinates, it follows that

$$\frac{\partial}{\partial t} (F_{K_t})_{j\bar{k}} = \lambda^{-2} O(1).$$

Thus

$$\begin{aligned} & \frac{\partial}{\partial t} \left[[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} (F_{K_t})_{j\bar{k}} \right] \\ &= \left[\frac{\partial}{\partial t} [(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} (F_{K_t})_{j\bar{k}} \right] + \left[[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} \frac{\partial}{\partial t} (F_{K_t})_{j\bar{k}} \right] \\ &= \left[\frac{\partial}{\partial t} [(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} (F_{K_t})_{j\bar{k}} \right] + \lambda^{-2} \hat{r}^{-2} O(1). \end{aligned} \quad (5.17)$$

Here we used (2.11) and $g_{co,t}^{-1} = \hat{r}^{-2} O(I)$ in these coordinates. The same computation as (5.16) gives

$$\frac{\partial}{\partial t} ((\Phi_t^{-1})^* g_{co,0})_{\bar{k}j}(w) = \hat{r}^2 \lambda^{-2} O(1)$$

and therefore

$$\frac{\partial}{\partial t} \left[[(\Phi_t^{-1})^* g_{co,0}]^{j\bar{k}} (F_{K_t})_{j\bar{k}} \right] = \lambda^{-2} \hat{r}^{-2} O(1).$$

Since $\lambda^2 = \hat{r}^3$, this completes the proof of (5.13). $\square$

6 | PERTURBATION

At this stage in the construction, we have a pair of metrics (g_t, H_t) on the smoothing X_t such that both of these metrics agree with a scaling of $g_{co,t}$ near the vanishing cycles $\{\|z\|^2 = t\}$. The metric ω_t is not balanced on all of X_t and the metric H_t is not Hermitian-Yang-Mills with respect to ω_t away from the vanishing cycles, but by the construction they are close to solving these equations. In this section we will perturb (g_t, H_t) to a pair $(g_{\text{FLY},t}, \check{H}_t)$ solving the Hermitian-Yang-Mills equation. We will prove:

Theorem 6.1. *There exists $\varepsilon > 0$ such that for all $0 < |t| < \varepsilon$, there exists on X_t a pair of hermitian metrics $(g_{\text{FLY},t}, \check{H}_t)$ solving*

$$d\omega_{\text{FLY},t}^2 = 0, \quad F_{\check{H}_t} \wedge \omega_{\text{FLY},t}^2 = 0.$$

Near the vanishing cycles, these metrics have the following local description. There exists $\lambda, c_i, d_i > 0$ such that for any $k \in \mathbb{Z}_{\geq 0}$, there exists $C_k > 0$ such that for all $|t| < \varepsilon$

$$|\nabla_{g_{co,t}}^k (g_{\text{FLY},t} - c_i g_{co,t})|_{g_{co,t}} \leq C_k |t|^{2/3} r^{-k} \quad (6.1)$$

and

$$|\nabla_{g_{co,t}}^k (\check{H}_t - d_i g_{co,t})|_{g_{co,t}} \leq C_k |t|^\lambda r^{-k} \quad (6.2)$$

in the region

$$\mathcal{R}_\lambda = \left\{ |t| \leq \|z\|^2 \leq |t|^{\frac{3}{3+\lambda}} \right\}.$$

Remark 6.2. The decay estimates (6.1) and (6.2) imply that, at an appropriate scale, the Hermitian-Yang-Mills metrics H_t converge smoothly to a multiple of the CO metric $g_{co,0}$ as $|t| \rightarrow 0$.

6.1 | The θ -perturbed Fu-Li-Yau metric

We recall the construction of Fu-Li-Yau [35] which perturbs ω_t to a balanced metric $\omega_{\text{FLY},t}$. The Fu-Li-Yau balanced metric is obtained via the ansatz

$$\omega_{\text{FLY},t}^2 = \omega_t^2 + \theta_t + \bar{\theta}_t. \quad (6.3)$$

The (2,2) form θ_t is constructed to satisfy

$$\partial\theta_t = 0, \quad \bar{\partial}\theta_t = -\bar{\partial}\omega_t^2.$$

More specifically, the correction θ_t is of the form

$$\theta_t = \partial\bar{\partial}^\dagger\partial^\dagger\gamma_t$$

where adjoints are with respect to g_t and $\gamma_t \in \Lambda^{2,3}(X_t)$ satisfies $\partial\gamma_t = 0$. Estimates for γ_t and θ_t were obtained by Fu-Li-Yau [35]. We will use the versions stated in [16]. [16, Proposition 3.8] states that

$$|\theta_t|_{g_t} \leq C\|z\|^{-2/3}|t|,$$

which using $\|z\|^2 \geq |t|$ implies

$$|\theta_t|_{g_t} \leq C|t|^{2/3}. \quad (6.4)$$

The proof of [16, Proposition 3.8] uses

$$\int_{X_t} |\gamma_t|_{g_t}^2 \leq C|t|^2. \quad (6.5)$$

We will need the following higher estimate on $\nabla\theta$.

Lemma 6.3.

$$|\nabla^k\theta_t|_{g_t} \leq C_k|t|^{2/3}r^{-k}.$$

Proof. This is similar to [16, Proposition 3.7]. We first show the estimate on a compact set K which does not intersect the vanishing cycles. The operator E_t given by

$$E_t = \partial\bar{\partial}\bar{\partial}^\dagger\partial^\dagger + \partial^\dagger\bar{\partial}\bar{\partial}^\dagger\partial + \partial^\dagger\partial,$$

where $\dagger$ is with respect to g_t , is a 4th order elliptic operator. The form γ_t satisfies

$$E_t(\gamma) = \bar{\partial}\omega_t^2.$$

In fact, it is obtained in [35] by solving this equation. On K , the geometry is uniform in t , hence by elliptic estimates we have

$$\|\gamma\|_{C^4(K)} \leq C(\|\gamma\|_{L^2(K)} + \|\bar{\partial}\omega_t^2\|_{W^{k,p}(K)})$$

for some $k, p > 1$. As noted in Lemma 2.17, the construction of ω_t is such that

$$|\bar{\partial}\omega_t^2|_{C^k(X_t, g_t)} \leq C_k|t|. \quad (6.6)$$

By (6.6) and (6.5), we have $\|\gamma\|_{C^4(K)} \leq C|t|$ and hence $\|\nabla\theta_t\|_{L^\infty(K)} \leq C|t|$. Similarly, $\|\nabla^k\theta_t\|_{L^\infty(K)} \leq C_k|t|$.

We now prove the estimate stated in the lemma on a set $U(\varepsilon) \cap X_t$ containing the vanishing cycles and assume $g_t = R_0 g_{co,t}$. Here $\bar{\partial}\theta = \bar{\partial}\omega_{co,t}^2 = 0$, and we also have

$$\bar{\partial}^\dagger\theta = \bar{\partial}^\dagger\partial\bar{\partial}^\dagger\partial^\dagger\gamma_t = 0$$

since ∂ and $\bar{\partial}^\dagger$ commute because g_t is Kähler on this set. Therefore

$$\Delta_{\bar{\partial}}\theta_t = 0.$$

Working in holomorphic cylindrical coordinates (see Lemma 2.11), we can verify that the coefficients of the equation $r^2\Delta_{\bar{\partial}}\theta_t = 0$ are uniformly bounded in C^α . Indeed, by the Bochner-Kodaira formula,

$$\Delta_{\bar{\partial}}\theta = -g_t^{i\bar{j}}\partial_i\partial_{\bar{j}}\theta + \Gamma * \partial\theta + \partial\Gamma * \theta + \Gamma * \Gamma * \theta + Rm_{g_t} * \theta,$$

and the uniform boundedness of the coefficients of $r^2\Delta_{\bar{\partial}}$ follows from Lemma 2.11. By the Schauder estimates in this coordinate chart, we obtain

$$\sup_{B_{1/2}}|\partial\theta_t|_{g_{euc}} \leq C \sup_{B_1}|\theta_t|_{g_{euc}}.$$

Using $g_{co,t} = r^2O(I)$ in these coordinates, we obtain

$$|\nabla\theta_t|_{g_t} \leq Cr^{-1} \sup_{X_t}|\theta_t|_{g_t}.$$

Since $|\theta_t|_{g_t} \leq C|t|^{2/3}$, we obtain the lemma for $k = 1$ and higher $k \geq 1$ are similar. $\square$

We now note some general facts on (2,2) forms constructed via the ansatz $\tilde{\omega}^2 = \omega^2 + \theta + \bar{\theta}$.

Lemma 6.4. *On a complex manifold of dimension n , the equation*

$$\omega^{n-1} = \Psi > 0$$

has solution

$$g_{\bar{k}j} = (\det \Psi)^{1/(n-1)}(\Psi^{-1})_{\bar{k}j},$$

where $\omega = \sqrt{-1}g_{\bar{k}j}dz^j \wedge d\bar{z}^k$ and Ψ is written as

$$\sum_{k,j} c_{kj} \Psi^{k\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \widehat{dz^k} \wedge d\bar{z}^k \wedge \cdots \wedge dz^j \wedge \widehat{d\bar{z}^j} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$$

with $c_{kj} = (\sqrt{-1})^{n-1}(n-1)! \operatorname{sgn}(k, j)$.

Proof. See for example [59] or [66]. Direct computation of ω^{n-1} gives $(\det g)g^{j\bar{k}} = \Psi^{j\bar{k}}$ and the result follows from taking the determinant of both sides and solving for g . $\square$

Lemma 6.5. *Let ∇ be the Chern connection with respect to a Hermitian metric ω . Let $\eta = \sqrt{-1}\eta_{\bar{k}j} dz^j \wedge d\bar{z}^k$ be a positive (1,1) form solving $\eta^2 = \omega^2 + \theta + \bar{\theta}$, where*

$$\theta = \frac{1}{4}\theta_{s\bar{r}j\bar{k}} dz^s \wedge d\bar{z}^r \wedge dz^j \wedge d\bar{z}^k.$$

Then

$$\nabla_i \eta_{\bar{k}j} = -\frac{1}{2}\eta^{s\bar{r}}(\nabla_i \theta_{s\bar{r}j\bar{k}} + \nabla_i \bar{\theta}_{s\bar{r}j\bar{k}}) + \frac{1}{8}[\eta^{p\bar{q}}\eta^{s\bar{r}}(\nabla_i \theta_{s\bar{r}j\bar{k}} + \nabla_i \bar{\theta}_{s\bar{r}p\bar{q}})]\eta_{\bar{k}j}.$$

Proof. A similar computation can be found in [67]. In components, the equation $\eta^2 = \omega^2 + \theta + \bar{\theta}$ is

$$-2\eta_{\bar{r}s}\eta_{\bar{k}j} + 2\eta_{\bar{r}j}\eta_{\bar{k}s} = (\omega^2 + \theta + \bar{\theta})_{s\bar{r}j\bar{k}}.$$

Differentiating this equation leads to

$$-\nabla_i \eta_{\bar{r}s}\eta_{\bar{k}j} - \eta_{\bar{r}s}\nabla_i \eta_{\bar{k}j} + \nabla_i \eta_{\bar{r}j}\eta_{\bar{k}s} + \eta_{\bar{r}j}\nabla_i \eta_{\bar{k}s} = \frac{1}{2}(\nabla_i \theta_{s\bar{r}j\bar{k}} + \nabla_i \bar{\theta}_{s\bar{r}j\bar{k}})$$

Contracting by $\eta^{s\bar{r}}$ gives

$$-(\eta^{s\bar{r}}\nabla_i \eta_{\bar{r}s})\eta_{\bar{k}j} - \nabla_i \eta_{\bar{k}j} = \frac{1}{2}\eta^{s\bar{r}}(\nabla_i \theta_{s\bar{r}j\bar{k}} + \nabla_i \bar{\theta}_{s\bar{r}j\bar{k}})$$

Contracting again by $\eta^{j\bar{k}}$ gives

$$-4\eta^{s\bar{r}}\nabla_i \eta_{\bar{r}s} = \frac{1}{2}\eta^{j\bar{k}}\eta^{s\bar{r}}(\nabla_i \theta_{s\bar{r}j\bar{k}} + \nabla_i \bar{\theta}_{s\bar{r}j\bar{k}}).$$

Combining the previous two identities proves the lemma. $\square$

Using what we have obtained so far in this subsection, we can derive the main estimate of this subsection which shows that the difference between g_t^{-1} and $(g_{\text{FLY},t})^{-1}$ is small.

Lemma 6.6. *There exists $C_k > 0$ and $\varepsilon > 0$ with the following property. For all $0 < |t| < \varepsilon$, the θ -perturbed Fu-Li-Yau metric $g_{\text{FLY},t}$ satisfies the estimates:*

$$2^{-1}g_t \leq g_{\text{FLY},t} \leq 2g_t,$$

and

$$|\nabla_{g_t}^k (g_{\text{FLY},t} - g_t)|_{g_t} \leq C_k |t|^{2/3} r^{-k},$$

for $k \in \mathbb{Z}_{\geq 0}$. Furthermore, we have

$$\|\omega_{\text{FLY},t}^{-1} - \omega_t^{-1}\|_{C_0^{0,\alpha}} \leq C_0 |t|^{2/3}.$$

Proof. If $|\theta_t + \bar{\theta}_t|_{g_t} \leq \frac{1}{100}$, then by Lemma 6.4 we have

$$|g_{\text{FLY},t}|_{g_t} \leq 2, \quad |g_{\text{FLY},t}^{-1}|_{g_t} \leq 2.$$

Next, we write the difference of metrics as

$$\omega_{\text{FLY},t} - \omega_t = \int_0^1 \frac{d}{ds} \eta_s ds$$

where η_s solves $\eta_s^2 = \omega_t^2 + s(\theta_t + \bar{\theta}_t)$. By the variation formula in Lemma 6.5, we have

$$\frac{d}{ds}(\eta_s)_{\bar{k}j} = -\frac{1}{2}\eta_s^{s\bar{r}}(\theta + \bar{\theta})_{s\bar{r}jk} + \frac{1}{8}\left[\eta_s^{p\bar{q}}\eta_s^{s\bar{r}}(\theta + \bar{\theta})_{s\bar{r}p\bar{q}}\right](\eta_s)_{\bar{k}j}.$$

The same argument as above shows that $|\eta_s|_{g_t} \leq C$ and $|\eta_s^{-1}|_{g_t} \leq C$ for $s \in [0, 1]$. Therefore

$$|g_{\text{FLY},t} - g_t|_{g_t} \leq C|\theta_t|_{g_t} \leq C|t|^{2/3}$$

by (6.4). Next, by Lemma 6.5 and Lemma 6.3, we have

$$|\nabla_{g_t}(g_{\text{FLY},t})|_{g_t} \leq C|\nabla_{g_t}\theta|_{g_t} \leq C|t|^{2/3}r^{-1}.$$

Higher order estimates for $|\nabla_{g_t}^k(g_{\text{FLY},t})|_{g_t}$ are similar.

It remains to estimate the difference, which can be done by:

$$\begin{aligned} \|\omega_{\text{FLY},t}^{-1} - \omega_t^{-1}\|_{C_0^{0,\alpha}} &= \|\omega_{\text{FLY},t}^{-1}(\omega_{\text{FLY},t} - \omega_t)\omega_t^{-1}\|_{C_0^{0,\alpha}} \\ &\leq \|\omega_{\text{FLY},t}^{-1}\|_{C_0^{0,\alpha}} \|\omega_{\text{FLY},t} - \omega_t\|_{C_0^{0,\alpha}} \|\omega_t^{-1}\|_{C_0^{0,\alpha}}. \end{aligned} \quad (6.7)$$

Since $|\nabla\omega_{\text{FLY},t}^{-1}|_{g_t} \leq C|t|^{2/3}r^{-1}$, we have

$$\|\omega_{\text{FLY},t}^{-1}\|_{C_0^{0,\alpha}} \leq C(1 + r|\nabla\omega_{\text{FLY},t}^{-1}|_{g_t}) \leq C.$$

The estimate

$$|\omega_{\text{FLY},t} - \omega_t| + r|\nabla_{g_t}(\omega_{\text{FLY},t} - \omega_t)| \leq C|t|^{2/3}$$

implies

$$\|\omega_{\text{FLY},t} - \omega_t\|_{C_0^{0,\alpha}} \leq C|t|^{2/3}$$

which proves the lemma. $\square$

6.2 | Uniform weighted Hölder estimates

Let g_t be the metric constructed by Fu-Li-Yau which satisfies $g_t = g_{co,t}$ near the vanishing cycles (for ease of notation, in this section we assume that the constant $M_0^{1/2}\varepsilon^{-1/3}$ in Proposition 2.17 is equal to 1). Let H_t be the metric on X_t constructed in the previous section, that is, the glued approximate solution to the Hermitian-Yang-Mills equation. We will use a linear operator L_t which acts on endomorphisms $h \in \Gamma(\text{End } T^{1,0}X_t)$ by

$$L_t h = (g_t)^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_t} h + \frac{1}{2} \left[\sqrt{-1} \Lambda_{\omega_t} F_{H_t}, h \right].$$

The motivation for this operator is that it is close to the linearization of the Hermitian-Yang-Mills equation $\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_H = 0$ at the approximate solution H_t , the difference being the use of g_t instead of $g_{\text{FLY},t}$. We start by proving uniform Schauder estimates independent of t on the deformations X_t .

Proposition 6.7. *Let $\beta \leq 0$. There exists $C > 1$ and $a \in (0, 1)$ such that for all $t \in \mathbb{C}^*$ and $h \in \Gamma(\text{End } T^{1,0}X_t)$, we can estimate*

$$\|h\|_{C_{\beta}^{2,a}(X_t)} \leq C(\|h\|_{C_{\beta}^0(X_t)} + \|L_t h\|_{C_{\beta-2}^{0,a}(X_t)})$$

where the weighted Hölder norms are defined in Section 5.2 using (g_t, H_t) .

Proof. On $X_t \cap \{r > 1\}$, the geometry is uniform in t and the estimate holds by the usual Schauder estimates. Let $\hat{x} \in X_t \cap \{r(x) \leq 1\}$. We denote the scale of this point by the constant $\hat{r} := r(\hat{x})$. We will work in holomorphic cylindrical coordinates $\{w^i\}$ given in Lemma 2.11 on the set

$$U_{\hat{r}} = \{(1/4)\hat{r} < r < 4\hat{r}\}.$$

By Lemma 2.11 and Lemma 5.1, the operator $\hat{r}^2 L_t$ is uniformly elliptic with uniform derivative estimates in coordinates $\{w^i\}$. The standard Schauder estimates applied to each matrix entry $\hat{r}^{-\beta} h^i_j$ imply

$$\|\hat{r}^{-\beta} h\|_{C_{\text{geuc}}^{2,\alpha}(B_{1/2})} \leq C(\|\hat{r}^{-\beta} h\|_{C_{\text{geuc}}^0(B_1)} + \|\hat{r}^2 L_t(\hat{r}^{-\beta} h)\|_{C_{\text{geuc}}^{\alpha}(B_1)})$$

with usual (non-scaled) norms

$$\|u\|_{C_{\text{geuc}}^k(B_1)} = \sum_{i=0}^k \sup_{B_1} |D^i u|, \quad [u]_{C_{\text{geuc}}^{\alpha}(B_1)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}.$$

As observed in (5.6), since $\hat{r}^{-2} g_t$ is uniformly and smoothly equivalent to g_{geuc} in coordinates $\{w^i\}$, the weighted Hölder norms are equivalent to these local Euclidean norms, and we have

$$\|h\|_{C_{\beta}^{2,a}(U_{\hat{r}})} \leq C(\|h\|_{C_{\beta}^0(U_{\hat{r}})} + \|L_t h\|_{C_{\beta-2}^a(U_{\hat{r}})}).$$

The norm $\|\cdot\|_{C_\beta^{2,\alpha}(U_\rho)}$ involves connection terms from ∇^{H_t} , but these are bounded in coordinates $\{w^i\}$ by Lemma 5.3. By Lemma 5.2, these local estimates on sets U_ρ imply the global bound. $\square$

The next step is to improve this estimate for endomorphisms orthogonal to the identity. For a related argument used in a gluing construction of Kähler-Einstein metrics on nodal surfaces, see [79].

Proposition 6.8. *Let $\beta \in (-2, 0)$. There exists $C > 1$ and $\alpha \in (0, 1)$ with the following property. Let $t \in \mathbb{C}$ with $0 < |t| < 1$ be arbitrary. We can estimate*

$$\|h\|_{C_\beta^{2,\alpha}(X_t)} \leq C \|L_t h\|_{C_{\beta-2}^\alpha(X_t)},$$

for all $h \in \Gamma(\text{End } T^{1,0}X_t)$ satisfying $h^{\dagger_{H_t}} = h$ and $\int_{X_t} (\text{Tr } h) d\text{vol}_{g_{\text{FLY},t}} = 0$.

Proof. Suppose there exists a sequence of $t_i \rightarrow 0$ such that

$$\|h_i\|_{C_\beta^{2,\alpha}} \geq M_i \|L_{t_i} h_i\|_{C_{\beta-2}^\alpha}$$

with $M_i \rightarrow \infty$ and h_i defined on X_{t_i} . Replacing h_i with $h_i / \|h_i\|_{C_\beta^{2,\alpha}}$, we have a sequence with

$$\|h_i\|_{C_\beta^{2,\alpha}(X_{t_i})} = 1, \quad \|L_{t_i} h_i\|_{C_{\beta-2}^\alpha(X_{t_i})} \rightarrow 0.$$

Let $K \subset \underline{X}$ be a compact set on the central fiber disjoint from the singular points. For all t small enough, we have a sequence $\Phi_{t_i}^* h_i$ of tensors defined on K . Since $C^{-1}r(x) \leq r(\Phi_t(x)) \leq Cr(x)$, we have a uniform bound on

$$\|\Phi_{t_i}^* h_i\|_{C_\beta^{2,\alpha}(K, \Phi_{t_i}^* g_i, \Phi_{t_i}^* H_i)} \leq C.$$

By Lemma 2.16, $\Phi_{t_i}^* g_i \rightarrow g_0$ smoothly uniformly on compact sets, and the definition of H_t implies that $\Phi_{t_i}^* H_{t_i} \rightarrow H_0$ smoothly uniformly on compact sets. We can thus extract a limiting tensor $h_0 \in C_{loc}^{2,\alpha/2}(\underline{X})$ which satisfies the growth estimates

$$|h_0|_{g_0} \leq Cr^\beta, \quad |\nabla_{H_0} h_0|_{g_0} \leq Cr^{\beta-1}.$$

By the construction of Φ_t in Lemma 2.13, the limit h_0 preserves $T^{1,0}\underline{X}$ and is an endomorphism of this bundle which satisfies the identities

$$(g_0)^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_0} h_0 = 0, \quad h_0^\dagger = h_0 \quad (6.8)$$

where $\dagger$ is with respect to H_0 . We now show that

$$\int_{\underline{X}} (\text{Tr } h_0) d\text{vol}_{g_0} = 0. \quad (6.9)$$

For this, we let $\delta > 0$, so that $(\Phi_{t_i})^* h_{t_i}, (\Phi_{t_i})^* g_{t_i}$ converge uniformly on $\{r > \delta\}$. By Lemma 6.6, we also have that $(\Phi_{t_i})^* g_{\text{FLY}, t_i} \rightarrow g_0$ uniformly in the C^0 norm on $\{r > \delta\}$. Therefore

$$\int_{\underline{X} \cap \{r > \delta\}} (\text{Tr } h_0) d\text{vol}_{g_0} = \lim_i \int_{X_{t_i} \cap \{r > \delta\}} (\text{Tr } h_i) d\text{vol}_{g_{\text{FLY}, t_i}}.$$

Since $\int_{X_{t_i}} \text{Tr } h_i = 0$, then

$$\int_{\underline{X}} (\text{Tr } h_0) d\text{vol}_{g_0} = -\lim_{\delta \rightarrow 0} \lim_i \int_{X_{t_i} \cap \{r < \delta\}} (\text{Tr } h_i) d\text{vol}_{g_{\text{FLY}, t_i}}. \quad (6.10)$$

From $d\text{vol}_{g_{\text{FLY}, t}} \leq C d\text{vol}_{g_t}$ (Lemma 6.6) and $|h_i| \leq r^\beta$, we have

$$\begin{aligned} \left| \int_{X_{t_i} \cap \{r < \delta\}} (\text{Tr } h_i) d\text{vol}_{g_{\text{FLY}, t_i}} \right| &\leq C \delta^\beta \int_{X_{t_i} \cap \{r < \delta\}} d\text{vol}_{g_{t_i}} \\ &= C \delta^\beta \int_{X_1 \cap \{r < \delta |t_i|^{-1/3}\}} S_{t^{1/3}}^* d\text{vol}_{g_{t_i}} \\ &= C \delta^\beta \int_{X_1 \cap \{r < \delta |t_i|^{-1/3}\}} |t_i|^2 d\text{vol}_{g_{c0,1}} \end{aligned}$$

where $S_{t^{1/3}} : V_1 \rightarrow V_t$, $S_{t^{1/3}}(z) = t^{1/2}z$ is the scaling action (2.7) which satisfies $S_{t^{1/3}}^* g_{c0,t} = |t|^{2/3} g_{c0,1}$. We have $\text{Vol}_{g_{c0,1}}(\{r < R\}) = O(R^6)$ since $g_{c0,1}$ is asymptotically conical, and so

$$\left| \int_{X_{t_i} \cap \{r < \delta\}} (\text{Tr } h_i) d\text{vol}_{g_{\text{FLY}, t_i}} \right| \leq C \delta^{6+\beta}$$

which together with (6.10) proves (6.9).

Next, (6.8) implies that the identity

$$\Delta_{g_0} |h_0|_{H_0}^2 = 2|\bar{\nabla} h_0|_{H_0, g_0}^2$$

holds pointwise away from the nodes. Let η_δ be a cutoff function such that $\eta_\delta \equiv 0$ on $\{r < \frac{\delta}{2}\}$, $\eta_\delta \equiv 1$ on $\{r > \delta\}$ and $|\Delta_g \eta_\delta| \leq C\delta^{-2}$. Then

$$2 \int_{U_\delta} |\bar{\nabla} h_0|_{H_0, g_0}^2 d\text{vol}_{g_0} \leq \int_{\underline{X}} \eta_\delta \Delta_{g_0} |h_0|_{H_0}^2 d\text{vol}_{g_0}.$$

Recall that g_0 is balanced on $\underline{X}$ and so we can integrate by parts.

$$2 \int_{U_\delta} |\bar{\nabla} h_0|_{H_0, g_0}^2 d\text{vol}_{g_0} \leq C \delta^{-2} \int_{\{\frac{\delta}{2} < r < \delta\}} |h_0|_{H_0}^2 d\text{vol}_{g_{c0,0}}.$$

Since $h_0 \in C_\beta^1$ and $d\text{vol}_{g_{co,0}} = r^5 dr d\text{vol}_{g_L}$, then

$$2 \int_{U_\delta} |\bar{\nabla} h_0|_{H_0, g_0}^2 d\text{vol}_{g_0} \leq C \delta^{-2+2\beta+6}$$

and we conclude that

$$\limsup_{\delta \rightarrow 0} \int_{U_\delta} |\bar{\nabla} h_0|_{H_0}^2 d\text{vol}_{g_0} = 0$$

for $\beta \in (-2, 0)$. Therefore $\bar{\partial} h_0 = 0$ on $\underline{X}$, and h_0 is a holomorphic endomorphism. By Hartogs's theorem, h_0 extends across the holomorphic curves to the small resolution X . Since $T^{1,0}X$ is stable with respect to ω_{CY} , it must be the case that $h_0 = cI$ is a multiple of the identity. We showed in (6.9) that the integral of $\text{Tr } h_0$ is zero, so we conclude that

$$h_0 \equiv 0.$$

The goal now is to obtain a contradiction to this by using that $\|h_i\|_{C_\beta^{2,\alpha}} = 1$ along the sequence $t_i \rightarrow 0$. The uniform Schauder estimates in Proposition 6.7 imply

$$1 \leq C(\|h_i\|_{C_\beta^0} + M_i^{-1}),$$

and hence

$$|r^{-\beta} h_i|(z_i) \geq C^{-1}$$

for a sequence $z_i \in X_{t_i}$.

- Case 1: Suppose $\liminf r(z_i) > 0$. It then follows that after taking a subsequence, we have $z_i \rightarrow z_0 \in \underline{X}$ with $r(z_0) > 0$. Then

$$|h_0(z_0)| \geq C^{-1} r(z_0)^\beta > 0$$

which contradicts $h_0 \equiv 0$.

- Case 2: Suppose $r(z_i) \rightarrow 0$. In this case, we can assume that all points z_i are in the region of X_t which can be identified with a subset of $V_t = \{\sum z_i^2 = t\}$ and where $g_t = g_{co,t}$. Define the function $u_i : V_{t_i} \cap \{\|z\|^2 \leq 1\} \rightarrow \mathbb{R}$ given by

$$u_i = |h_i|_{H_{t_i}}^2.$$

The sequence $\{u_i\}$ satisfies the uniform growth estimate

$$\|u_i\|_{C_{2\beta}^{2,\alpha}(g_t)} \leq C, \tag{6.11}$$

which written in full is

$$|u| + r^{-1}|\nabla u|_{g_{co,t}} + r^{-2}|\nabla^2_{g_{co,t}} u|_{g_{co,t}} + [\nabla^2_{g_{co,t}} u]_{C^{0,a}_{\beta-2-a}} \leq Cr^{2\beta}.$$

This definition of $\|u\|_{C^{2,\alpha}_{2\beta}(g_t)}$ is slightly different than the weighted Hölder norms used previously for h_t , since we use ∇ with respect to g_t (rather than H_t). These estimates for u_i follow from $\|h_i\|_{C^{2,\alpha}_{\beta}(g_t, H_t)} \leq 1$ and (5.3), which allow us to uniformly convert norms in H_t to norms in g_t .

Direct computation gives the identity

$$\Delta_{g_i} u_i = 2\operatorname{Re} \langle L_i h_i, h_i \rangle_{H_i} + 2|\bar{\nabla} h_i|_{H_i, g_i}^2.$$

We are assuming $\|L_{t_i} h_i\|_{C^{\alpha}_{\beta-2}} \leq \varepsilon_i$ with $\varepsilon_i \rightarrow 0$, hence

$$\Delta_{g_i} u_i \geq -C\varepsilon_i r^{2\beta-2}. \quad (6.12)$$

We will rescale the functions u_i to take a limit. For ease of notation, we write $\lambda_i = r(z_i)$. We will use the scaling map $S_{\lambda_i} : V_{t_i \lambda_i^{-3}} \rightarrow V_{t_i}$ from (2.7) given by $S(x) = \lambda_i^{3/2} x$. We rescale and pullback u_i via S_{λ_i} to obtain a function

$$\tilde{u}_i : V_{t_i \lambda_i^{-3}} \cap \{\|x\|^2 \leq \lambda_i^{-3}\} \rightarrow \mathbb{R}$$

defined by

$$\tilde{u}_i(x) = \lambda_i^{-2\beta} u(\lambda_i^{3/2} x).$$

The rescaling is setup so that the estimates for u_i imply estimates for $\tilde{u}_i$. For example, we have

$$\Delta_{g_{t_i \lambda_i^{-3}}} \tilde{u}_i \geq -C\varepsilon_i r^{2\beta-2}. \quad (6.13)$$

Indeed, pulling back the Laplacian gives

$$S_{\lambda_i}^* (\Delta_{g_{t_i}} u_i) = \lambda_i^{2\beta} \Delta_{S^* g_{t_i}} \tilde{u}_i = \lambda_i^{2\beta} \lambda_i^{-2} \Delta_{g_{t_i \lambda_i^{-3}}} \tilde{u}_i$$

by using the rescaling relation $S_{\lambda}^* g_t = \lambda^2 g_{t\lambda^{-3}}$. Using $r(\lambda^{3/2} x) = \lambda r(x)$, we obtain (6.13) from (6.12). Similar computations show that

$$|\tilde{u}_i| + r^{-1}|\nabla \tilde{u}_i|_{g_{co, t_i \lambda_i^{-3}}} + r^{-2}|\nabla^2 \tilde{u}_i|_{g_{co, t_i \lambda_i^{-3}}} + [\nabla^2 \tilde{u}_i]_{C^{0,a}_{\beta-2-a}} \leq Cr^{2\beta}. \quad (6.14)$$

Recall that the points z_i satisfy $|h_i|(z_i) \geq C^{-1}\lambda_i^{\beta}$. Then the points $x_i = \lambda_i^{-3/2} z_i$ satisfy

$$\tilde{u}_i(x_i) \geq C^{-1}.$$

The sequence $\{t_i \lambda_i^{-3}\}$ lies in $[0, 1]$, since $\|z\|^2 \geq |t|$ implies $r^3(z_i) \geq |t_i|$. After taking a subsequence, we have convergence $t_i \lambda_i^{-3} \rightarrow \kappa$ for $\kappa \in [0, 1]$.

- Case 2a: $t_i \lambda_i^{-3} \rightarrow \kappa > 0$. The points $x_i = \lambda_i^{-3/2} z_i$ satisfy $\|x_i\| = 1$. We may assume

$$x_i \rightarrow x_\infty \in V_\kappa, \quad \|x_\infty\| = 1.$$

We can take a limit of $\{\tilde{u}_i\}$ on compact sets and obtain a $C_{loc}^{2,\alpha}$ limiting function $u_\infty \geq 0$ on V_κ satisfying $u_\infty \leq C r^{2\beta}$ and $\Delta_{g_\kappa} u_\infty \geq 0$. By the maximum principle,

$$\sup_{r \leq R} u_\infty \leq \sup_{r=R} u_\infty \leq R^{2\beta}.$$

Letting $R \rightarrow \infty$, we obtain that $u_\infty \equiv 0$ since $\beta < 0$. This contradicts $u_\infty(x_\infty) > 0$.

- Case 2b: $t_i \lambda_i^{-3} \rightarrow 0$. In this case, the points $x_i = \lambda_i^{-3/2} z_i$ converge after a subsequence to

$$x_i \rightarrow x_\infty \in V_0, \quad \|x_\infty\| = 1.$$

Let $v_i : \{t_i |\lambda_i^{-3} < \|x\|^2 < \frac{1}{2} \lambda_i^{-3}\} \cap V_0 \rightarrow \mathbb{R}$ with $v_i = \Phi_{t_i \lambda_i^{-3}}^* \tilde{u}_i$ be the corresponding sequence of functions on the cone V_0 (recall Φ_t is defined in (2.9)). Since $\|x\|^2 \leq \|\Phi(x)\|^2 \leq 2\|x\|^2$, we have the growth estimate

$$v_i \leq C r^{2\beta}.$$

By pulling-back (6.14), on compact sets K we have the estimate

$$\|v_i\|_{C^{2,\alpha}(K, \Phi_{t_i \lambda_i^{-3}}^*)} \leq C(K), \quad \Delta_{\Phi_{t_i \lambda_i^{-3}}^*} v_i \geq -\varepsilon_i C(K).$$

Corollary 2.9 implies $\Phi_{t_i \lambda_i^{-3}}^* g_{t_i \lambda_i^{-3}} \rightarrow g_{c0,0}$ uniformly on K . Taking a limit of $\{v_i\}$ on compact sets produces a $C_{loc}^{2,\alpha}$ limiting function $v_\infty \geq 0$ on the cone V_0 satisfying

$$\Delta_{g_{c0,0}} v_\infty \geq 0, \quad v_\infty \leq C r^{2\beta}$$

for $\beta \in (-2, 0)$. Lemma 6.9 below implies that $v_\infty \equiv 0$ which contradicts $v_\infty(x_\infty) > 0$. $\square$

Lemma 6.9. *Let V_0 be a Riemannian cone of dimension $n > 2$ with metric $g = dr^2 + r^2 g_L$. Let u be a C^2 function satisfying $\Delta_g u \geq 0$ and $u \geq 0$. Suppose there exists $M > 0$ such that $u \leq M r^{-\delta}$ where $\delta \in (0, n-2)$. Then $u \equiv 0$.*

Proof. This is a standard PDE result (e.g. [46]), but we give the proof for completeness. Recall that the real Laplacian is

$$\Delta_g u = \partial_r \partial_r u + (n-1) r^{-1} \partial_r u + r^{-2} \Delta_{g_L} u.$$

Let $B_R(0) = \{x \in V_0 : r(x) < R\}$. We start by noting that for any $\varphi \in C^2(\partial B_R(0), \mathbb{R})$, there exists $h \in C^2(B_R(0), \mathbb{R})$ such that $h|_{\partial B_R(0)} = \varphi$ and

$$\Delta_g h = 0, \quad \sup_{B_R(0)} |h| \leq \sup_{\partial B_R(0)} |\varphi|.$$

To obtain such a harmonic function h , we start by expanding $\varphi|_{\partial B_R(0)} = \sum_{\lambda \in \text{Spec}(\Delta_{g_L})} c_\lambda \psi_\lambda$, where ψ_λ are an L^2 orthogonal basis of eigenfunctions of Δ_{g_L} on the link $L = \{r = 1\}$, with eigenvalue convention $\Delta_L \psi_\lambda = -\lambda \psi_\lambda$. We then let

$$h = \sum_{\lambda \in \text{Spec}(\Delta_{g_L})} c_\lambda \left(\frac{r}{R}\right)^{a(\lambda)} \psi_\lambda$$

where $a(\lambda) = \frac{1}{2}(-(n-2) + \sqrt{(n-2)^2 + 4\lambda}) > 0$. Direct computation gives $\Delta_g h = 0$, and by the maximum principle

$$\sup_{B_R \setminus B_\varepsilon} |h| \leq \max \left\{ \sup_{\partial B_R} |\varphi|, \sup_{\partial B_\varepsilon} |h| \right\}.$$

for any $\varepsilon > 0$. As $\varepsilon \rightarrow 0$, we see that $\sup_{\partial B_\varepsilon} |h|$ selects the $\lambda = 0$ mode $c_0 \psi_0$. This is a constant equal to $\frac{1}{\text{Vol}(B_R)} \int_{\partial B_R} \varphi$, which is bounded by $\sup_{\partial B_R} |\varphi|$.

We now prove that the subharmonic function u given in the lemma is constant. Let $R > 1$. Let h_R be the harmonic function mentioned above with $h_R|_{\partial B_R} = u$. Then

$$|h_R| \leq \sup_{\partial B_R} |u| \leq MR^{-\delta}. \quad (6.15)$$

Let $0 < \varepsilon < 1$, and consider

$$v = u - h_R - 2M\varepsilon^{n-2-\delta} r^{-(n-2)}$$

defined on $B_R \setminus B_\varepsilon$. Since $\Delta_g r^{-(n-2)} = 0$ we have that $\Delta_g v \geq 0$, and

$$v|_{\partial B_\varepsilon} \leq |u| + |h_R| - 2M\varepsilon^{-\delta} \leq M\varepsilon^{-\delta} + MR^{-\delta} - 2M\varepsilon^{-\delta} \leq 0.$$

We also have

$$v|_{\partial B_R} = -\frac{2M\varepsilon^{n-2-\delta}}{R^{n-2}} < 0.$$

By the maximum principle, $v \leq 0$ in $B_R \setminus B_\varepsilon$. We now fix $x \in B_R \setminus B_\varepsilon$. Then $v(x) \leq 0$ implies

$$u(x) \leq h_R(x) + 2M \frac{\varepsilon^{n-2-\delta}}{r^{n-2}(x)}.$$

This holds true for all $0 < \varepsilon < 1$, hence taking $\varepsilon \rightarrow 0$ we obtain

$$u(x) \leq h_R(x).$$

By (6.15), we conclude

$$u(x) \leq MR^{-\delta}.$$

We now take $R \rightarrow \infty$ to conclude $u \leq 0$. Since $u \geq 0$ by assumption, we obtain that $u \equiv 0$. $\square$

6.3 | Inverting the linearized operator

Let

$$W_t = \left\{ u \in \Gamma(\text{End } T^{1,0}X_t) : u^\dagger = u, \int_{X_t} (\text{Tr } u) d\text{vol}_{g_{\text{FLY},t}} = 0 \right\} \quad (6.16)$$

where $\dagger$ is the adjoint with respect to H_t . When linearizing the equation $\Lambda_{\omega_{\text{FLY},t}} F_H = 0$ at the approximate solution H_t we obtain an operator $\mathcal{L}$ that acts on endomorphisms by

$$\mathcal{L}u = -(g_{\text{FLY},t})^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_t} u - \frac{1}{2} \left[\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_{H_t}, u \right].$$

This operator $\mathcal{L}$ involves $g_{\text{FLY},t}$ rather than g_t , so it is a perturbation of the operator L in Proposition 6.8.

We note that $\mathcal{L} : W_t \rightarrow W_t$. Indeed, since $u^\dagger = u$, $(\sqrt{-1} \Lambda F)^\dagger = \sqrt{-1} \Lambda F$ and $(g^{j\bar{k}} \nabla_j \nabla_{\bar{k}} u)^\dagger = g^{j\bar{k}} \nabla_{\bar{k}} \nabla_j u$, we have

$$(\mathcal{L}u)^\dagger = -(g_{\text{FLY},t})^{j\bar{k}} \nabla_j^{H_t} \partial_{\bar{k}} u + \frac{1}{2} \left[\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_{H_t}, u \right].$$

The commutator identity for $[\nabla_j, \nabla_{\bar{k}}]$ now shows that $(\mathcal{L}u)^\dagger = \mathcal{L}u$. Next, since $\omega_{\text{FLY},t}$ is balanced, we do have

$$\int_{X_t} (\Delta_{g_{\text{FLY},t}} \text{Tr } u) (\omega_{\text{FLY},t})^3 = 0.$$

This verifies that $\mathcal{L} : W_t \rightarrow W_t$ preserves the subspace W_t .

Having obtained the estimate $\|h\| \leq C \|Lh\|$ from Proposition 6.8, we can invert $\mathcal{L}$ with a bound on the inverse.

Lemma 6.10. *Let $\mathcal{L} : C_{\beta}^{2,\alpha}(W_t) \rightarrow C_{\beta-2}^{0,\alpha}(W_t)$ for $\alpha \in (0, 1)$ and $\beta \in (-2, 0)$. There exists $|t_0| > 0$ such that for all $0 < |t| \leq |t_0|$, then $\mathcal{L}$ is invertible and the operator norm*

$$\|\mathcal{L}^{-1}\| \leq C \quad (6.17)$$

is bounded independent of t .

Proof. We start by discussing the uniform estimate. Let L be the operator that was estimated in Proposition 6.8. For $u \in C_{\beta}^{2,\alpha}(W_t)$, we have

$$\begin{aligned} \|u\|_{C_{\beta}^{2,\alpha}} &\leq C \|Lu\|_{C_{\beta-2}^{0,\alpha}} \\ &\leq C \left\| (g_{\text{FLY}})^{j\bar{k}} - g_t^{j\bar{k}} \right\| \partial_{\bar{k}} \nabla_j^{H_t} u \|_{C_{\beta-2}^{0,\alpha}} \end{aligned}$$

$$\begin{aligned}
& + C \|g_{\text{FLY}}^{-1} - g_t^{-1}\|_{C_0^{0,\alpha}} \|F_{H_t}\|_{C_{-2}^{0,\alpha}} \|u\|_{C_\beta^{0,\alpha}} + C \|\mathcal{L}u\|_{C_{\beta-2}^{0,\alpha}} \\
& \leq C |t|^{2/3} \|u\|_{C_\beta^{2,\alpha}} + C |t|^{2/3} \|F_{H_t}\|_{C_{-2}^{0,\alpha}} \|u\|_{C_\beta^{0,\alpha}} + C \|\mathcal{L}u\|_{C_{\beta-2}^{0,\alpha}}
\end{aligned}$$

by Lemma 6.6. By Lemma 5.3, we have $r^2|F_{H_t}|_{g_t} + r^3|\nabla_{g_t} F_{H_t}|_{g_t} \leq C$ and hence

$$\|F_{H_t}\|_{C_{-2}^{0,\alpha}} \leq C \quad (6.18)$$

uniformly in t . We conclude that for t small enough, then

$$\|u\|_{C_\beta^{2,\alpha}} \leq C \|\mathcal{L}u\|_{C_{\beta-2}^{0,\alpha}}, \quad u \in W_t. \quad (6.19)$$

The proof of the lemma now follows from standard elliptic PDE theory on the smooth compact manifold X_t . We will use the space of sections denoted by

$$\mathcal{H} = \left\{ u \in \Gamma(\text{End } T^{1,0}X_t) : u^\dagger = u \right\}$$

with L^2 inner product

$$\langle s, h \rangle_{L^2} = \int_X \langle s, h \rangle_{H_t} d\text{vol}_{g_{\text{FLY},t}}, \quad s, h \in \mathcal{H}.$$

Then we can orthogonally decompose $\mathcal{H} = W \oplus \mathbb{C}I$. We consider $P : C^{2,\alpha}(\mathcal{H}) \rightarrow C^{0,\alpha}(\mathcal{H})$ with $Pu = \mathcal{L}u$. From (6.19), we see that $\mathbb{C}I = \ker P$. Furthermore, by the balanced condition of $\omega_{\text{FLY},t}$ we see that

$$\text{im } P \subseteq W.$$

Therefore $(\ker P^\dagger)^\perp \subseteq W$ and so $\mathbb{C}I \subseteq \ker P^\dagger$. We will show $\mathbb{C}I = \ker P^\dagger$.

An integration by parts argument using the balanced property shows that the operator $\Delta = (g_{\text{FLY},t})^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_t}$ is L^2 self-adjoint and so has degree zero. Thus P also has degree zero. Therefore $\dim \ker P^\dagger = 1$ and $\mathbb{C}I = \ker P^\dagger$. It follows that $\text{im } P = W$. Hence $\mathcal{L} : W \rightarrow W$ is invertible, and the bound on the inverse in weighted norms is (6.19). $\square$

6.4 | Fixed point theorem

In this section, we perturb the glued metric H_t to a solution of the Hermitian-Yang-Mills equation $\sqrt{-1}\Lambda_{\omega_{\text{FLY},t}} F_H = 0$ on the smoothing X_t via a fixed point theorem. The general approach of constructing an approximate solution and deforming it to a true solution to solve equations in differential geometry goes back to Taubes [85] and is now widely used; in this section we will follow the notation used in [84].

Our space of deformations will be the space W_t defined in (6.16). We introduce the operator

$$\mathcal{F} : W_t \rightarrow W_t$$

given

$$\mathcal{F}(u) = e^{u/2} \left(\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_u \right) e^{-u/2}, \quad (6.20)$$

where F_u is the curvature of the metric $H_{t,u} = H_t e^u$.

We note that $\mathcal{F} : W_t \rightarrow W_t$. Indeed, for $u \in W_t$, we have $\int \text{Tr } \mathcal{F}(u) = 0$ since

$$\int_{X_t} c_1(T^{1,0} X_t) \wedge \omega_{\text{FLY},t}^2 = 0$$

as X_t has trivial canonical bundle. The adjoint action of $e^{u/2}$ in (6.20) is added to ensure that $\mathcal{F}(u)^\dagger = \mathcal{F}(u)$. Indeed, $\sqrt{-1} \Lambda_{F_{H_{t,u}}}$ is self-adjoint with respect to the metric $H_{t,u}$. In coordinates where $H_t = I$ and e^u is diagonal, it is straight-forward to check that

$$\left(e^{u/2} \left(\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_u \right) e^{-u/2} \right)^\dagger = e^{u/2} \left(\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_u \right) e^{-u/2}$$

where $\dagger$ is the adjoint with respect to H_t .

A well-known computation gives the linearization of $\mathcal{F}_u$.

$$\delta(\mathcal{F}_u) = \bar{\partial} \partial_{H_{t,u}} (e^{-u} \delta e^u).$$

The linearization of $\mathcal{F}$ is then

$$\begin{aligned} (\delta \mathcal{F})|_u(\delta u) &= -e^{u/2} \left[(g_{\text{FLY},t})^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_{t,u}} (e^{-u} \delta e^u) \right] e^{-u/2} \\ &\quad + (\delta e^{u/2}) \left(\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_u \right) e^{-u/2} \\ &\quad + (e^{u/2}) \left(\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_u \right) \delta e^{-u/2}. \end{aligned}$$

Let $\mathcal{L}$ be the linearization $(\delta \mathcal{F})|_0$ at $u = 0$. That is, $\mathcal{L} : W_t \rightarrow W_t$ with

$$\mathcal{L} w = -(g_{\text{FLY},t})^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_t} w - \frac{1}{2} \left[\sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_{H_t}, w \right].$$

We previously inverted $\mathcal{L}$ and gave a bound on the inverse $\mathcal{L}^{-1}$ uniform in t . We can write

$$\mathcal{F}(u) = \mathcal{F}(0) + \mathcal{L}(u) + \mathcal{Q}(u)$$

where by definition

$$\mathcal{Q}(u) = \mathcal{F}(u) - \mathcal{F}(0) - \mathcal{L}(u). \quad (6.21)$$

We define

$$\mathcal{N} : C_\beta^{2,\alpha}(W) \rightarrow C_\beta^{2,\alpha}(W)$$

given by

$$\mathcal{N}(u) = \mathcal{L}^{-1}(-\mathcal{F}(0) - \mathcal{Q}(u)).$$

To solve $\mathcal{F}(u) = 0$, it is equivalent to find a fixed point

$$\mathcal{N}(u) = u.$$

Proposition 6.11. *Let $a \in (0, 1)$. There exists $\varepsilon > 0$ and $\beta \in (-1, 0)$ with the following property. Let*

$$\mathcal{V}_t = \left\{ u \in C_{\beta}^{2,a}(W(X_t)) : \|u\|_{C_{\beta}^{2,a}} < |t|^{(2/3)|\beta|} \right\}.$$

Then for all $0 < |t| < \varepsilon$, the mapping $\mathcal{N}$ preserves $\mathcal{V}_t$ and satisfies

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{C_{\beta}^{2,a}} \leq \frac{1}{2} \|u - v\|_{C_{\beta}^{2,a}} \quad (6.22)$$

for all $u, v \in \mathcal{V}_t$.

Proof. We start by assuming (6.22) and prove that $\mathcal{N}$ preserves $\mathcal{V}_t$. We have

$$\mathcal{F}(0) = \sqrt{-1} \Lambda_{\omega_{\text{FLY},t}} F_{H_t}$$

and we can estimate

$$\|\mathcal{F}(0)\|_{C_{-2}^{0,a}} \leq \|\Lambda_{\omega_t} F_{H_t}\|_{C_{-2}^{0,a}} + \|\omega_t^{-1} - \omega_{\text{FLY},t}^{-1}\|_{C_0^{0,a}} \|F_{H_t}\|_{C_{-2}^{0,a}}.$$

Proposition 5.3, Lemma 6.6 and (6.18) imply

$$\|\mathcal{F}(0)\|_{C_{-2}^{0,a}} \leq C(|t|^{2/3} + |t|^{|\alpha\lambda|/3}).$$

The contribution $|t|^{|\alpha\lambda|/3}$ is the slowest rate. For any $\beta \in (-1, 0)$, we therefore have

$$\|\mathcal{F}(0)\|_{C_{\beta-2}^{0,a}} \leq C|t|^{|\alpha\lambda|/3}.$$

Since $\|\mathcal{L}^{-1}\| \leq C$ by (6.17), it follows that

$$\|\mathcal{N}(0)\|_{C_{\beta}^{2,a}} \leq C|t|^{|\alpha\lambda|/3},$$

and hence (6.22) implies that for $u \in \mathcal{V}_t$ then

$$\|\mathcal{N}(u)\| \leq \|\mathcal{N}(u) - \mathcal{N}(0)\| + \|\mathcal{N}(0)\| \leq \frac{1}{2} |t|^{(2/3)|\beta|} + C|t|^{|\alpha\lambda|/3} < |t|^{(2/3)|\beta|}$$

for $\beta = -\frac{1}{4}|\alpha\lambda|$ and t small enough.

Thus it remains to prove (6.22). By definition

$$\mathcal{N}(u) - \mathcal{N}(v) = \mathcal{L}^{-1}(Q(v) - Q(u)).$$

Since $\|\mathcal{L}^{-1}\| \leq C$ by (6.17), we have

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{C_{\beta}^{2,a}} \leq C \|Q(u) - Q(v)\|_{C_{\beta-2}^{0,a}}. \quad (6.23)$$

So we need to estimate $Q(u) - Q(v)$. One way to write this is

$$Q(u) - Q(v) = \int_0^1 \frac{d}{ds} Q(w_s) ds$$

where $w_s = su + (1-s)v$. By the definition (6.21) of Q , its variation is

$$\frac{d}{ds} Q(w_s) = (\delta \mathcal{F})|_{w_s}(u - v) - \mathcal{L}(u - v).$$

We claim that the approximate linearized operator $\mathcal{L}$ is close enough to the actual linearization $\delta \mathcal{F}$, so that for all $w, s \in \mathcal{U}$ then

$$\|(\delta \mathcal{F})|_w(s) - \mathcal{L}(s)\|_{C_{\beta-2}^{0,a}} \leq C \|w\|_{C_0^{2,a}} \|s\|_{C_{\beta}^{2,a}}. \quad (6.24)$$

Assuming this, we conclude

$$\|Q(u) - Q(v)\|_{C_{\beta-2}^{0,a}} \leq C(\|u\|_{C_0^{2,a}} + \|v\|_{C_0^{2,a}})\|u - v\|_{C_{\beta}^{2,a}}.$$

By (6.23), we see that

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{C_{\beta}^{2,a}} \leq C(\|u\|_{C_0^{2,a}} + \|v\|_{C_0^{2,a}})\|u - v\|_{C_{\beta}^{2,a}}.$$

If $u \in \mathcal{U}$, then $\|u\|_{C_{\beta}^{2,a}} < |t|^{\frac{2}{3}|\beta|}$, and since $r^3 \geq |t|$, we have

$$\|u\|_{C_0^{2,a}} \leq \frac{1}{\min_{X_t} r^{|\beta|}} \|u\|_{C_{\beta}^{2,a}} \leq \frac{1}{|t|^{|\beta|/3}} |t|^{(2/3)|\beta|} \leq \varepsilon^{|\beta|/3}.$$

Thus if ε is small enough, then $\mathcal{N}$ is a contraction map. The proof is complete given (6.24), which is proved in the following lemma. $\square$

Lemma 6.12. *There exists $C > 1$ such that for all $t \in \mathbb{C}^*$ and $u, s \in W_t$ with $\|u\|_{C_0^{2,a}(X_t)} \leq 1$, we can estimate*

$$\|(\delta \mathcal{F})|_u(s) - \mathcal{L}(s)\|_{C_{\beta-2}^{0,a}(X_t)} \leq C \|u\|_{C_0^{2,a}(X_t)} \|s\|_{C_{\beta}^{2,a}(X_t)}.$$

Proof. To simplify notation, for this estimate we will write $g = g_{\text{FLY},t}$, $H_0 = H_t$ and $H_u = H_{t,u}$. The linearization was computed in (6.21), and the difference is explicitly given by

$$\begin{aligned} & (\delta F)|_u(s) - \mathcal{L}(s) \\ &= [[\delta \exp]|_u(s/2)] \left(\sqrt{-1} \Lambda_\omega F_{H_u} \right) e^{-u/2} - \frac{1}{2} s \left(\sqrt{-1} \Lambda_\omega F_{H_0} \right) \\ &+ (e^{u/2}) \left(\sqrt{-1} \Lambda_\omega F_{H_u} \right) [[\delta \exp]|_u(-s/2)] + \frac{1}{2} \left(\sqrt{-1} \Lambda_\omega F_{H_0} \right) s \\ &- e^{u/2} \left[g^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_u} e^{-u} [\delta \exp]|_u(s) \right] e^{-u/2} + g^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_0} s \\ &= \text{(I)} + \text{(II)} + \text{(III)}, \end{aligned}$$

where (I), (II), (III) denotes the terms on each line and the derivative of the matrix exponential is given by the formula

$$[\delta \exp]|_u(\delta u) = \int_0^1 e^{\lambda u} (\delta u) e^{(1-\lambda)u} d\lambda.$$

- Terms (I) and (II). These two terms are estimated in the same way, so we only give the estimate for (I). We start by writing

$$\begin{aligned} \text{(I)} &= [\delta \exp]|_u(s/2) (\sqrt{-1} \Lambda_\omega F_{H_u}) e^{-u/2} - \frac{1}{2} s (\sqrt{-1} \Lambda_\omega F_{H_0}) \\ &= \int_0^1 \frac{d}{d\gamma} \left[[\delta \exp]|_{\gamma u}(s/2) (\sqrt{-1} \Lambda_\omega F_{H_{\gamma u}}) e^{-(\gamma/2)u} \right] d\gamma. \end{aligned} \quad (6.25)$$

There are three terms depending on where $\frac{d}{d\gamma}$ lands. The first is

$$(\text{Ia})_\gamma = \left[\frac{d}{d\gamma} [\delta \exp]|_{\gamma u}(s/2) \right] (\sqrt{-1} \Lambda_\omega F_{H_{\gamma u}}) e^{-(\gamma/2)u}.$$

This can be estimated as

$$\|(\text{Ia})_\gamma\|_{C_{\beta-2}^{0,a}} \leq C \|F_{H_{\gamma u}}\|_{C_{-2}^{0,a}} \|u\|_{C_0^{0,a}} \|s\|_{C_\beta^{0,a}}$$

Next, we consider

$$(\text{Ib})_\gamma = [\delta \exp]|_{\gamma u}(s/2) \left[\frac{d}{d\gamma} (\sqrt{-1} \Lambda_\omega F_{H_{\gamma u}}) \right] e^{-(\gamma/2)u}.$$

Since $\frac{d}{d\gamma} F_{H_{\gamma u}} = \bar{\partial} \partial_{H_{\gamma u}} u$, we have

$$\|(\text{Ib})_{\gamma}\|_{C_{\beta-2}^{0,a}} \leq C \|\bar{\partial} \nabla_{H_{\gamma u}} u\|_{C_{-2}^{0,a}} \|s\|_{C_{\beta}^{0,a}}.$$

The other term is

$$(\text{Ic})_{\gamma} = [\delta \exp]_{\gamma u}(s/2)(\sqrt{-1} \Lambda_{\omega} F_{H_{\gamma u}}) \left[\frac{d}{d\gamma} e^{-(\gamma/2)u} \right]$$

and can be estimated by

$$\|(\text{Ic})_{\gamma}\|_{C_{\beta-2}^{0,a}} \leq C \|F_{H_{\gamma u}}\|_{C_{-2}^{0,a}} \|u\|_{C_0^{0,a}} \|s\|_{C_{\beta}^{0,a}}.$$

Altogether, using $\|u\|_{C_0^{2,a}} \leq 1$, the formula for the difference of connections $A_{H_{\gamma u}} - A_{H_0} = e^{-\gamma u} \nabla_{H_0} e^{\gamma u}$ which gives a formula for $F_{H_{\gamma u}}$, and $\|F_{H_0}\|_{C_{-2}^{0,a}} \leq C$, we have

$$\|(\text{I})\|_{C_{\beta-2}^{0,a}} \leq C \|u\|_{C_0^{2,a}} \|s\|_{C_{\beta}^{0,a}}$$

- Term (III). We write

$$(\text{III}) = - \int_0^1 \frac{d}{d\gamma} \left[e^{(\gamma/2)u} \left[g^{j\bar{k}} \partial_{\bar{k}} \nabla_j^{H_{\gamma u}} e^{-\gamma u} [\delta \exp]_{\gamma u}(s) \right] e^{-(\gamma/2)u} \right] d\gamma.$$

Using $\|u\|_{C_0^{2,a}} \leq 1$, from here we can derive the estimate

$$\|(\text{III})\|_{C_{\beta-2}^{0,a}} \leq C \|u\|_{C_0^{2,a}} \|s\|_{C_{\beta}^{2,a}}.$$

To do this, the covariant derivative $\nabla_{H_{\gamma u}}$ can be converted to ∇_{H_0} via $\nabla_{H_{\gamma u}} - \nabla_{H_0} = e^{-\gamma u} \nabla_{H_0} e^{\gamma u}$ and we can use the variational formula $\frac{d}{d\gamma} \nabla_{H_{\gamma u}} = \nabla_{H_{\gamma u}} u$. $\square$

By Proposition 6.11 and the Banach fixed point theorem, there exists $\check{u} \in C_{\beta}^{2,a}(W(X_t))$ with $\|\check{u}\|_{C_{\beta}^{2,a}} < |t|^{(2/3)\beta}$ such that $\mathcal{N}(\check{u}) = \check{u}$, meaning that $\Lambda_{\omega_{\text{FLY},t}} F_{\check{u}} = 0$ where F_u is the curvature of $H_t e^u$. This proves the existence of a pair solving

$$d\omega_{\text{FLY},t}^2 = 0, \quad F_{\check{H}_t} \wedge \omega_{\text{FLY},t}^2 = 0$$

on X_t . To complete the proof of the main theorem, we describe the behavior of $(g_{\text{FLY},t}, \check{H}_t)$ near the vanishing cycles.

Proof (Theorem 6.1). The local estimate (6.1) follows from Lemma 6.6 since $g_t = c_i g_{co,t}$ in a fixed neighborhood of the vanishing cycles. The Hermitian-Yang-Mills metric is given by $\dot{H}_t = H_t e^{\check{u}}$ where $\check{u}$ is the fixed-point solving $\mathcal{N}(\check{u}) = \check{u}$ and satisfying $\|\check{u}\|_{C_{\beta}^{2,a}} \leq |t|^{(2/3)|\beta|}$. Near the nodes,

by the gluing construction (see (5.2)) we have $\check{H}_t = d_i g_{co,t} e^{\check{u}}$ in the region

$$\mathcal{R}_\lambda = \left\{ |t| \leq \|z\|^2 \leq |t|^{\frac{3}{3+\lambda}} \right\},$$

which implies

$$|\check{H}_t - d_i g_{co,t}|_{g_{co,t}} \leq C r^{-|\beta|} |t|^{(2/3)|\beta|} \leq C |t|^{|\beta|/3}$$

since $r^3 \geq |t|$. We obtain similar estimates for $\nabla^k \check{H}_t$ for $k = 1$ and $k = 2$. For the higher order estimates, we write the equation $g^{j\bar{k}}(F_{\check{H}})_{j\bar{k}} = 0$ in holomorphic cylindrical coordinates as

$$r^2(g_{\text{FLY},t})^{j\bar{k}} \partial_j \partial_{\bar{k}} \check{H}_t = r^2(g_{\text{FLY},t})^{j\bar{k}} \partial_j \check{H}_t \check{H}_t^{-1} \partial_{\bar{k}} \check{H}_t.$$

Note that since $g_t = r^2 O(I)$ (see Lemma 2.11), by (6.1) we also have $g_{\text{FLY},t} = r^2 O(1)$ for small t . Schauder estimates and a bootstrap argument imply

$$|\partial^\ell \check{H}_{\bar{k}j}|_{g_{\text{euc}}} \leq C_\ell |t|^{|\beta|/3}.$$

Converting g_{euc} in holomorphic cylindrical coordinates to $g_{co,t}$ gives the stated estimate with $\lambda = |\beta|/3$. $\square$

Remark 6.13. To simultaneously solve the Hermitian-Yang-Mills and the conformally balanced equations

$$d(\|\Omega_t\|_{\check{\omega}_t} \check{\omega}_t^2) = 0, \quad F_{\check{H}_t} \wedge \check{\omega}_t^2 = 0,$$

we can set

$$\check{\omega}_t = \|\Omega_t\|_{\omega_{\text{FLY},t}}^{-2} \omega_{\text{FLY},t}$$

so that $\|\Omega_t\|_{\check{\omega}_t} \check{\omega}_t^2 = \omega_{\text{FLY},t}^2$. Here Ω_t is a holomorphic volume form on X_t , whose existence through conifold transitions is proved in [32], normalized by $\int_{X_t} \sqrt{-1} \Omega_t \wedge \bar{\Omega}_t = 1$. It is straightforward to show that this conformally balanced metric $\check{g}_t$ associated to $\check{\omega}_t$ satisfies a decay estimate near the vanishing cycles. Namely, near any ODP $p_i \in \underline{X}$ there is a constant c_i such that we have the estimates

$$|\nabla_{g_{\text{FLY},t}}^k (\check{g}_t - c_i g_{\text{FLY},t})|_{g_{\text{FLY},t}} \leq C_k r^{\frac{3}{2}-k},$$

for all $k \in \mathbb{Z}_{\geq 0}$. Note that, unlike Lemma 6.6, one can no longer expect decay in $|t|$ for fixed r . This is due to the fact that, near the ODP singularities, the holomorphic volume form Ω_t will not necessarily converge to a constant multiple of the natural equivariant holomorphic volume form on the conifold. Nevertheless, these estimates still imply that there is a constant $d_i > 0$ so that at a suitable scale the pair $(\check{g}_t, \check{H}_t)$ converges smoothly to the pair $(g_{co,0}, d_i g_{co,0})$ as $|t| \rightarrow 0$. In particular, at suitably small scales, the pair $(\check{g}_t, \check{H}_t)$ converges to a solution of the Strominger system near the ODP singularities as $t \rightarrow 0$.

Corollary 6.14. *For all $|t| \ll 1$ sufficiently small, the tangent bundle $T^{1,0}X_t$ is stable.*

Proof. We showed that the bundle $T^{1,0}X_t$ admits a Hermitian-Yang-Mills metric. This implies that the bundle is polystable with respect to the Fu-Li-Yau balanced metric. To show $T^{1,0}X_t$ is stable, we must show that it cannot split holomorphically as a product $T^{1,0}X_t = E_t \oplus F_t$.

Suppose there exists a sequence $t_i \rightarrow 0$ such that $T^{1,0}X_{t_i}$ splits. Let h_t be the holomorphic endomorphism of $T^{1,0}X_t$ which is the identity on E_t and zero on F_t . Note that h_t satisfies

$$\bar{\partial}h_t = 0, \quad \text{Tr } h_t = \text{rk } E_t, \quad \det h_t = 0.$$

By the compactness of holomorphic functions, we can take a subsequential limit on compact sets to obtain a limit $h_0 \in \Gamma(\text{End}(T^{1,0}X_0))$ such that $\bar{\partial}h_0 = 0$ and h_0 is non-zero and not a multiple of the identity. By Hartogs's theorem, h_0 extends to the small resolution X . Since $T^{1,0}X$ is stable, it does not admit holomorphic endomorphisms other than multiples of the identity. This is a contradiction.

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APPENDIX A: THE FU-LI-YAU GLUING CONSTRUCTION

In this appendix we will explain the gluing result of Fu-Li-Yau, which establishes Propositions 2.16 and 2.18. Before beginning, we recall the notation. We are primarily interested in the small resolution of the conifold, given by $p : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1$. Let h_{FS} denote the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}^1}(-1)$. Let

$$\pi : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow V_0$$

be the map contracting $\mathbb{P}^1$. The pull-back radial function of the conical Calabi-Yau metric on V_0 is given by

$$r^2 = \left(|u|_{h_{FS}}^2 + |v|_{h_{FS}}^2 \right)^{2/3}$$

where (u, v) are fiber coordinates on $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. The holomorphic Reeb field on V_0 induces a holomorphic $\mathbb{C}^*$ action given by

$$S_\lambda(x, u, v) = (x, \lambda^{3/2}u, \lambda^{3/2}v).$$

where $x \in \mathbb{P}^1$. Unless otherwise specified, we will consider the restriction to $\lambda \in \mathbb{R}_{>0} \subset \mathbb{C}^*$. Note that we have

$$S_\lambda^* r^2 = \lambda^2 r^2.$$

We define a smooth Kähler metric on $\mathcal{O}(-1)^{\oplus 2}$ by

$$\omega_{sm} = \sqrt{-1} \partial \bar{\partial} r^3 + p^* \omega_{FS}.$$

We will consider also the cone metric

$$\omega_{co,0} = \frac{3}{2} \sqrt{-1} \partial \bar{\partial} r^2.$$

It will be important for us to compare $\omega_{co,0}^2$ and ω_{sm}^2 .

Lemma A.1. *There is a uniform constant $C > 0$ such that following estimates hold*

(i) *If $0 < r < 1$, then we have*

$$r^2 C^{-1} \omega_{co,0}^2 \leq \omega_{sm}^2 \leq C r^{-1} \omega_{co,0}^2$$

(ii) *If $0 < r < 1$ then*

$$C^{-1} r^2 \omega_{co,0}^2 \leq \omega_{sm} \wedge \omega_{co,0} \leq r^{-1} \omega_{co,0}^2$$

Proof. First, since $\omega_{co,0}$ and ω_{sm} define smooth Kähler metrics on the compact set $\{1 \leq r \leq 2\}$ we can fix a constant A such that

$$A^{-1} \omega_{co,0} \leq \omega_{sm} \leq A \omega_{co,0}.$$

Consider

$$S_\lambda : \{1 \leq r \leq 2\} \rightarrow \{\lambda \leq r \leq 2\lambda\}.$$

From the homogeneity of r we have

$$S_\lambda^* \omega_{co,0}^2 = \lambda^4 \omega_{co,0}^2.$$

On the other hand,

$$S_\lambda^* \omega_{sm} = \lambda^3 \sqrt{-1} \partial \bar{\partial} r^3 + p^* \omega_{FS},$$

and so

$$S_\lambda^* \omega_{sm}^2 = \lambda^6 (\sqrt{-1} \partial \bar{\partial} r^3)^2 + 2\lambda^3 \sqrt{-1} \partial \bar{\partial} r^3 \wedge p^* \omega_{FS}$$

using that $p^* \omega_{FS}^2 = 0$. If $\lambda \leq 1$ then we get

$$\lambda^6 \omega_{sm}^2 \leq S_\lambda^* \omega_{sm}^2 \leq \lambda^3 \omega_{sm}^2$$

From the definition of A we have

$$S_\lambda^* \omega_{sm}^2 \leq \lambda^3 \omega_{sm}^2 \leq \lambda^3 A^2 \omega_{co,0}^2 \leq \lambda^{-1} A^2 S_\lambda^* \omega_{co,0}^2$$

and similarly

$$S_\lambda^* \omega_{sm}^2 \geq \lambda^6 \omega_{sm}^2 \geq \lambda^2 A^{-2} S_\lambda^* \omega_{co,0}^2.$$

Since $\lambda \leq r \leq 2\lambda$, proves (i). The proof of (ii) is similar. □

Consider the four form

$$\Omega := \sqrt{-1} \partial \bar{\partial} \left(\chi(r^2) \sqrt{-1} \partial \bar{\partial} r^2 \right)$$

where $\chi(\cdot)$ is some smooth function to be determined. We compute

$$\Omega = \chi''(r^2) \sqrt{-1} \partial r^2 \wedge \bar{\partial} r^2 \wedge \sqrt{-1} \partial \bar{\partial} r^2 + \chi'(r^2) \omega_{co,0}^2.$$

To understand the first term write

$$\sqrt{-1} \partial \bar{\partial} r^2 = 4 \sqrt{-1} \partial r \wedge \bar{\partial} r + r^2 \sqrt{-1} \partial \bar{\partial} \log r^2.$$

from which it follows that

$$\left(\sqrt{-1} \partial \bar{\partial} r^2 \right)^2 = 8 r^2 \sqrt{-1} \partial r \wedge \bar{\partial} r \wedge \sqrt{-1} \partial \bar{\partial} \log r^2 + \left(r^2 \sqrt{-1} \partial \bar{\partial} \log r^2 \right)^2$$

and also

$$\sqrt{-1} \partial r \wedge \bar{\partial} r \wedge \sqrt{-1} \partial \bar{\partial} r^2 = r^2 \sqrt{-1} \partial r \wedge \bar{\partial} r \wedge \sqrt{-1} \partial \bar{\partial} \log r^2.$$

Since $\sqrt{-1} \partial \bar{\partial} \log r^2 \geq 0$ we conclude

Lemma A.2. *The following estimate holds everywhere on $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$*

$$\sqrt{-1} \partial r^2 \wedge \bar{\partial} r^2 \wedge \sqrt{-1} \partial \bar{\partial} r^2 \leq \frac{r^2}{2} (\sqrt{-1} \partial \bar{\partial} r^2)^2.$$

In particular, whenever $\chi'' < 0$ we have the lower bound

$$\Omega \geq \left(\chi'(r^2) + \frac{r^2}{2} \chi''(r^2) \right) \omega_{co,0}^2$$

Fix $R \gg 1$. Our goal is to find $R \gg 1$, and a constant $C_R > 0$ such that we can glue

$$\Omega_R := C_R^2 S_R^* \sqrt{-1} \partial \bar{\partial} (\chi(r^2)) \sqrt{-1} \partial \bar{\partial} r^2$$

to the Calabi-Yau metric ω_{CY}^2 . This will require carefully choosing χ , and the constant C_R . We are going to assume that

$$\chi(s) = s$$

for $s \in [0, 4]$, so that Ω_R agrees with a rescaling of $\omega_{co,0}^2$ on $\{0 \leq r \leq \frac{2}{R}\}$. More precisely, we have

$$C_R^2 S_R^* \sqrt{-1} \partial \bar{\partial} (\chi(r^2)) \sqrt{-1} \partial \bar{\partial} r^2 = C_R^2 R^4 \omega_{co,0}^2 \quad \{0 < r < 2R^{-1}\}.$$

In the remainder of the appendix we will determine conditions on χ, C_R, R for this to be possible.

We now consider the Calabi-Yau metric ω_{CY} . Consider the set $U = \{r < 4\} \subset X$. Since U is contractible onto $\mathbb{P}^1$, we can write

$$\omega_{CY} = \lambda p^* \omega_{FS} + \partial \beta + \bar{\partial} \bar{\beta}$$

for some $(1,0)$ form β on U , and some $\lambda > 0$. To simplify notation, let us assume $\lambda = 1$. By solving the $\bar{\partial}$ -equation we can write

$$\omega_{CY} = p^* \omega + \sqrt{-1} \partial \bar{\partial} \varphi$$

where ω is a Kähler form on $\mathbb{P}^1$, with $[\omega] = [\omega_{FS}] \in H^{1,1}(\mathbb{P}^1, \mathbb{R})$ and $\varphi : U \rightarrow \mathbb{R}$ a smooth function with $\varphi|_{\mathbb{P}^1} = 0$; see [35, Lemma 2.4]. Let h_1 denote the degree $\frac{3}{2}$ part of φ under S_λ (recall that S_λ corresponds to scaling with weight $\frac{3}{2}$ along the fibers), so that $|\varphi - h_1| \sim O(|u|^2 + |v|^2)$, or in other words

$$|\varphi - h_1| \leq Cr^3.$$

Let $\sigma(x)$ be a positive cut-off function with $0 \leq \sigma(x) \leq 1$ and $\sigma(x) = 1$ for $x \in [0, 1]$ and $\sigma(x) = 0$ for $x \geq 8$. Define

$$\begin{aligned} \Psi_R &= \omega_{CY}^2 - \sqrt{-1} \partial \bar{\partial} \Gamma_R \\ \Gamma_R &= \sqrt{-1} \partial \bar{\partial} \left(\sigma(R^3 r^3) \left((\varphi - h_1)(2p^* \omega + \sqrt{-1} \partial \bar{\partial} (\varphi + h_1)) + h_1 \sqrt{-1} \partial \bar{\partial} h_1 \right) \right) \end{aligned}$$

where, as before, $R \gg 1$ is a parameter to be determined. From the definition of σ we have

$$\Psi_R = \omega_{CY}^2 \text{ if } \{r > 2R^{-1}\}$$

We claim that

$$\Psi_R = 0 \text{ if } \{r < R^{-1}\}.$$

To see this, note that if x, y, w are commutative variables satisfying $yw = 0$, then

$$(x - y)(2w + (x + y)) = x(2w + x) + xy - 2wy - xy - y^2 = x(2w + x) - y^2.$$

We apply this formula with $x = \sqrt{-1}\partial\bar{\partial}\varphi, y = \sqrt{-1}\partial\bar{\partial}h_1, w = p^*\omega$ and product being the wedge product. We only need to check that $\sqrt{-1}\partial\bar{\partial}h_1 \wedge p^*\omega_{FS} = \sqrt{-1}\partial\bar{\partial}h_1^2 = 0$, but this is clear since $\sqrt{-1}\partial\bar{\partial}h_1$ is linear along the fibers of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1$.

The main task is to find a lower bound for Ψ_R in terms of $\omega_{co,0}^2$ in the transition region $\{R^{-1} < r < 2R^{-1}\}$. To do this we expand

$$\Psi_R = (1 - \sigma)\omega_{CY}^2 - \text{(I)} - \text{(II)} - \text{(III)}$$

where

$$\text{(I)} = 2\text{Re}\left(\left(\sqrt{-1}\sigma'(R^3r^3)R^3\partial r^3 \wedge \bar{\partial}(\varphi - h_1) \wedge (2p^*\omega + \sqrt{-1}\partial\bar{\partial}(\varphi + h_1))\right)\right)$$

$$+ 2\text{Re}\left(\sigma'(R^3r^3)R^3\partial r^3 \wedge \bar{\partial}h_1 \wedge \sqrt{-1}\partial\bar{\partial}h_1\right)$$

$$\text{(II)} = \sigma''(L^3r^3)L^6\sqrt{-1}\partial r^3 \wedge \bar{\partial}r^3 \wedge \left((\varphi - h_1)(2p^*\omega + \sqrt{-1}\partial\bar{\partial}(\varphi + h_1)) + h_1\sqrt{-1}\partial\bar{\partial}h_1\right)$$

$$\text{(III)} = \sigma'(L^3r^3)L^3\sqrt{-1}\partial\bar{\partial}r^3 \wedge \left((\varphi - h_1)(2p^*\omega + \sqrt{-1}\partial\bar{\partial}(\varphi + h_1)) + h_1\sqrt{-1}\partial\bar{\partial}h_1\right).$$

Our goal is to estimate each term from below by $\omega_{co,0}^2$. Each term will be treated differently, depending on whether it is homogeneous or not.

- Term (I). Observe that

$$\partial r^3 \wedge \bar{\partial}(\varphi - h_1) \sim r^3\omega_{sm}$$

To see this recall that, in coordinates we have

$$r^3 = |u|_{h_{FS}}^2 + |v|_{h_{FS}}^2$$

so that, in coordinates where $\partial h_{FS} = 0$, we have

$$\partial r^3 = \bar{u}du + \bar{v}dv.$$

On the other hand, since h_1 is linear along the fibers of $p : \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \mathbb{P}^1$ we have

$$\bar{\partial}(\varphi - h_1) = O(u, v)(d\bar{u} + d\bar{v})$$

and so

$$\partial r^3 \wedge \bar{\partial}(\varphi - h_1) \leq Cr^3\omega_{sm}.$$

Thus, by Lemma A.1, the first term in (I) can be controlled by $r^3 \omega_{sm}^2 \leq r^2 \omega_{co,0}^2$.

To analyze the second term in (I) we observe that

$$S_\lambda^*(\partial r^3 \wedge \bar{\partial} h_1 \wedge \sqrt{-1} \partial \bar{\partial} h_1) = \lambda^6 \partial r^3 \wedge \bar{\partial} h_1 \wedge \sqrt{-1} \partial \bar{\partial} h_1,$$

which, by the homogeneity of $\omega_{co,0}$, implies

$$\partial r^3 \wedge \bar{\partial} h_1 \wedge \sqrt{-1} \partial \bar{\partial} h_1 \leq C r^2 \omega_{co,0}^2.$$

In total, we have

$$(I) \leq C r^2 \omega_{co,0}^2$$

- Term (II). Again, by homogeneity we have

$$0 \leq \sqrt{-1} \partial r^3 \wedge \bar{\partial} r^3 \leq C r^2 \omega_{co,0},$$

while the bound $|\varphi - h_1| \leq C r^3$ yields a bound for the first term in (II)

$$\begin{aligned} \sqrt{-1} \partial r^3 \wedge \bar{\partial} r^3 \wedge \left((\varphi - h_1)(2p^* \omega + \sqrt{-1} \partial \bar{\partial}(\varphi + h_1)) \right) &\leq r^2 \omega_{co,0} \wedge r^3 \omega_{sm} \\ &\leq C r^4 \omega_{co,0}^2 \end{aligned}$$

where we used Lemma A.1, (ii). The second term in (II) can be treated directly by scaling. We have

$$\sqrt{-1} \partial r^3 \wedge \bar{\partial} r^3 \wedge h_1 \sqrt{-1} \partial \bar{\partial} h_1 \leq C r^3 \omega_{co,0}^2.$$

In total, we have

$$(II) \leq C r^3 \omega_{co,0}^2$$

- Term (III) can be treated similarly to term (II). The first term can be estimated as

$$\sqrt{-1} \partial \bar{\partial} r^3 \wedge (\varphi - h_1)(2p^* \omega + \sqrt{-1} \partial \bar{\partial}(\varphi + h_1)) \leq C r^3 \omega_{sm}^2 \leq C r^2 \omega_{co,0}^2,$$

thanks to Lemma A.1 (i) again. The homogeneous term is easily estimated as

$$\sqrt{-1} \partial \bar{\partial} r^3 \wedge h_1 \sqrt{-1} \partial \bar{\partial} h_1 \leq C r^2 \omega_{co,0}^2.$$

In summation, we have proved

Lemma A.3. *There is a constant $C > 0$ so that on the region $\{R^{-1} < r < 2R^{-1}\}$ we have*

$$\Psi_R \geq -C R^3 r^2 \omega_{co,0}^2 \geq -C R \omega_{co,0}^2.$$

At this point we consider

$$\Psi_R + C_0 C_R^2 S_R^* \Omega.$$

In order for this form to be positive we need to show that C_0, C_R can be chosen consistently. To do this we consider the conditions for positivity in four different regions. In the following C will denote a uniform constant which can change from line to line, but depends only on the fixed background data and is, in particular, independent of R, C_0, C_R .

- **Region** $\{0 < r < R^{-1}\}$.

In this region we have $\Psi_R = 0$ and $S_R^* \chi(r^2) = R^2 r^2$, so that

$$\Psi_R + C_0 C_R^2 S_R^* \Omega = \frac{4}{9} C_0 C_R^2 R^4 \omega_{co,0}^2 > 0$$

so this region contributes no restriction.

- **Region** $\{R^{-1} < r < 2R^{-1}\}$.

Thanks to Lemma A.3, and the fact that $\chi(s) = s$ for $s \in [0, 4]$ we have

$$\Psi_R \geq -CR\omega_{co,0}^2 \quad C_0 C_R^2 S_R^* \Omega = \frac{4}{9} C_0 C_R^2 R^4 \omega_{co,0}^2.$$

Thus, in order to ensure positivity we need $C_0 > 3C$, $C_R^2 R^4 = R$, so we must take $C_R^2 = R^{-3}$.

- **Region** $\{2R^{-1} < r < 1\}$.

By definition we have

$$\Psi_R = \omega_{CY}^2 \geq C\omega_{sm}^2.$$

On the other hand whenever $\chi'' < 0$, Lemma A.2 gives the estimate

$$C_0 C_R^2 S_R^* \Omega \geq \frac{4}{9} C_0 R (2\chi'(R^2 r^2) + (Rr)^2 \chi''(R^2 r^2)) \omega_{co,0}^2.$$

Now, from Lemma A.1 we have $\omega_{co,0}^2 \leq Cr^{-2} \omega_{sm}^2$ and so we can ensure positivity provided

$$\frac{4}{9} C_0 R r^{-2} (2\chi'(R^2 r^2) + (Rr)^2 \chi''(R^2 r^2)) \geq -C.$$

If we can choose χ so that $\chi' \geq 0$, then when $\chi'' \geq 0$ we have $S_R^* \Omega \geq 0$ so no additional restriction is contributed from this case.

- **Region** $\{1 < r\}$.

In this region we take $\chi(s) = \text{const.}$, and so, by definition

$$\Psi_R + C_0 C_R^2 S_R^* \Omega = \omega_{CY}^2$$

In summary, $\Psi_R + C_0 C_R^2 S_R^* \Omega$ will be positive definite provided we can construct a cut-off function $\chi(s)$ (upon defining $s = R^2 r^2$) with the following properties: for $0 < \varepsilon \ll 1$ given, we have

- $\chi(s) = s$ for $s \in [0, 4]$, and $\chi'(s) \geq 0$,
- $\chi(s)$ is constant for $s \geq R^2$,
- $\chi(s)$ satisfies

$$\frac{1}{s} (2\chi'(s) + s\chi''(s)) \geq -\frac{\varepsilon}{R^3}$$

for $s \in [4, R^2]$

A cut-off function with these properties is constructed in [35, Lemma 2.2], but for the readers convenience we give a slightly different proof here.

Lemma A.4. *For $R \gg 1$ sufficiently large there exists a smooth function $\chi(s)$ with the following properties*

- (i) $\chi(s) = s$ for $s \in [0, 4]$, and $\chi'(s) \geq 0$.
- (ii) $\chi(s)$ is constant for $s > R^2$,
- (iii) there is a uniform constant C' such that

$$\frac{1}{s} (2\chi'(s) + s\chi''(s)) \geq -\frac{C'}{R^4}$$

Proof. Let $v = \chi'$. Then we need to find v such that $v(s) = 1$ for $s \in [0, 4]$, $v(s) = 0$ for $s > R^2$, and

$$\frac{1}{s^2} \frac{d}{ds} (s^2 v) \geq -\frac{C'}{R^4}.$$

Consider

$$v(s) = \begin{cases} 1, & s \in [0, 5] \\ as^{-3} + bs^{-2} + cs^2 + d, & s \in [5, R^2 - 1] \\ 0, & s \geq R^2 - 1. \end{cases}$$

Demanding that $v(s)$ is C^1 gives a system of 4 equations in the unknowns a, b, c, d , whose solutions are given by

$$a = -250 + O(R^{-2}), \quad b = 75 + O(R^{-2}), \quad c = 75R^{-8} + O(R^{-10}), \quad d = -150R^{-4} + O(R^{-6}).$$

On the other hand, if $w(s) = C_\alpha s^\alpha$, then

$$\frac{1}{s^2} \frac{d}{ds} (s^2 w) = (\alpha + 2)C_\alpha s^{\alpha-1} = \begin{cases} \geq 0 & \text{if } \alpha > -2, \text{ and } C_\alpha \geq 0 \\ 0 & \text{if } \alpha = -2 \\ \geq 0 & \text{if } \alpha < -2, \text{ and } C_\alpha \leq 0. \end{cases}$$

Since the terms corresponding to a, b, c fall into one of these cases, it follows that

$$\frac{1}{s^2} \frac{d}{ds} (s^2 v) \geq -\frac{300}{R^4} + O(R^{-6}).$$

It only remains to check that $v(s) \geq 0$ for $s \in [4, R^2]$. To see this, observe that

$$0 = v'(\lambda R^2) \iff \lambda^5 - \lambda = O(R^{-2}).$$

Since the equation $\lambda^5 - \lambda = 0$ has only three real solutions $\lambda = \pm 1, 0$, it follows that for R sufficiently large $v'(s) = 0$ has only two real solutions on $[4, R^2]$. Since these solutions are given by $s = 4$ and $s = R^2 - 1$, it follows that $v'(s) \neq 0$ in $(4, R^2 - 1)$, and hence v does not have an interior minimum in $[4, R^2]$. This immediately implies $v(s) \geq 0$.

Finally, let $\hat{v}$ be the result of convolving v with a positive, symmetric mollifier with sufficiently small support. Then $\hat{v}$ is smooth and has the same properties as v . Integrating $\hat{v}$ yields χ . $\square$

Now let us explain the extension of this construction to the metrics $\omega_{co,a}$ on the small resolution. For this, recall that

$$\omega_{co,a} = \sqrt{-1} \partial \bar{\partial} f_a(\|z\|^2) + 4a^2 \pi^* \omega_{FS}$$

where $f_a(x) = a^2 f_1(\frac{x}{a^3})$ and f_1 satisfies

$$(xf_1')^3 + 6(xf_1')^2 = x^2.$$

Rewriting this equation in terms of the variable z defined by $x = z^{-3/2}$ and applying standard ODE techniques we obtain the following result, whose proof we leave to the reader

Lemma A.5. *For $x \gg 1$, the function $f_1(x)$ has a convergent expansion*

$$f_1(x) = \frac{3}{2}x^{2/3} - 2 \log(x) + \sum_{n=0}^{\infty} c_n x^{-2n/3}.$$

In particular, $f_a(x) \rightarrow \frac{3}{2}x^{2/3}$ smoothly and uniformly on any compact set as $a \rightarrow 0$.

We now describe how to glue $\omega_{co,a}$ to the Calabi-Yau metric ω_{CY} to obtain a balanced metric. We first recall that

$$\Omega_R = C_0 R^{-3} \sqrt{-1} \partial \bar{\partial} (\chi(R^2 r^2) R^2 \sqrt{-1} \partial \bar{\partial} r^2) = C_0 R^{-1} \sqrt{-1} \partial \bar{\partial} (\chi(R^2 r^2) \sqrt{-1} \partial \bar{\partial} r^2)$$

On the other hand, we have

$$\begin{aligned} \omega_{co,a}^2 &= (\sqrt{-1} \partial \bar{\partial} f_a(\|z\|^2))^2 + 8a^2 \sqrt{-1} \partial \bar{\partial} f_a(\|z\|^2) \wedge \pi^* \omega_{FS} \\ &= \sqrt{-1} \partial \bar{\partial} \left(f_a(\|z\|^2) \left(\sqrt{-1} \partial \bar{\partial} f_a(\|z\|^2) + 8a^2 \pi^* \omega_{FS} \right) \right) \end{aligned}$$

This suggests that we define

$$\Omega_{R,a} = C_0 \frac{2R^{-1}}{3} \sqrt{-1} \partial \bar{\partial} \left(\chi \left(\frac{2R^2}{3} f_a(\|z\|^2) \right) \left(\sqrt{-1} \partial \bar{\partial} f_a(\|z\|^2) + 8a^2 \pi^* \omega_{FS} \right) \right)$$

Lemma A.6. *For $0 \leq a \ll 1$ sufficiently small there is an open set U containing the $(-1, -1)$ rational curve such that*

$$\Omega_{R,a} = C_0 R \omega_{co,a}^2$$

Proof. We only need to observe that the formula holds whenever

$$\frac{2R^2}{3} f_a(\|z\|^2) \leq 4.$$

Now since $f_a(\|z\|^2)$ converges uniformly to $\frac{3}{2}r^2$ on compact sets away from the $(-1, -1)$ rational curve, this inequality will hold for a sufficiently small provided $r < 2R^{-1}$. $\square$

Next, recall that the gluing of $\omega_{co,0}$ and ω_{CY} depending on only two estimates.

- The bounds

$$\Psi_R \geq -CR\omega_{co,0}^2, \quad C_0 R^{-3} S_R^* \Omega = \frac{4C_0}{9} R \omega_{co,0}^2$$

in the region $\{R^{-1} < r < 2R^{-1}\}$. Since $\Omega_{R,a}$ converges uniformly to $\omega_{co,0}^2$ on this region, the same bound holds with $\omega_{co,0}$ replaced by $\omega_{co,a}^2$, after possibly changing the constants.

- The bound

$$\omega_{co,0}^2 \leq Cr^{-2} \omega_{sm}^2$$

in the region $\{2R^{-1} < r < 1\}$. Again, from the uniform convergence of $\omega_{co,a}$ to $\omega_{co,0}$ this bound holds, up to possibly increasing C for $\omega_{co,a}^2$ as well, from the uniform convergence.

It follows that the gluing procedure used to glue $\omega_{co,0}^2$ to ω_{CY}^2 carries over in exactly the same way to glue $\omega_{co,a}^2$ to ω_{CY}^2 for $0 < a \ll 1$. Furthermore, from Lemma A.5 we obtain the smooth, uniform convergence of ω_a to ω_0 on compact sets away from the $(-1, -1)$ rational curves.