EXISTENCE OF GOOD MINIMAL MODELS FOR KÄHLER VARIETIES OF MAXIMAL ALBANESE DIMENSION

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Abstract. In this short article we show that if (X, B) is a compact Kähler klt pair of maximal Albanese dimension, then it has a good minimal model, i.e. there is a bimeromorphic contraction $\phi : X \dashrightarrow X'$ such that $K_{X'} + B'$ is semi-ample.

1. Introduction

The main result of this article is the following

Theorem 1.1. *Let (X, B) be a compact Kähler klt pair of maximal Albanese dimension. Then (X, B) has a good minimal model.*

This generalizes the main result of [Fuj15] from the projective case to the Kähler case. The main idea is to observe that replacing X by an appropriate resolution, then the Albanese morphism $X \to A$ is projective and so by [DHP22] and [Fuj22a] we may run the relative MMP over A. Thus we may assume that $K_X + B$ is nef over A. If X is projective and $K_X + B$ is not nef, then by the cone theorem, X must contain a $K_X + B$ negative rational curve C. Since A contains no rational curves, then C is vertical over A, contradicting the fact that $K_X + B$ is nef over A [Fuj15]. Unluckily, the cone theorem is not known for Kähler varieties and so we pursue an different argument. It would be interesting to find an alternative proof based on the arguments of [CH20].

2. Preliminaries

An *analytic variety* or simply a *variety* is a reduced irreducible complex space. Let X be a compact Kähler manifold and $\mathrm{Alb}(X)$ is the *Albanese torus* (not necessarily an Abelian variety). Then by $a : X \to \mathrm{Alb}(X)$ we will denote the *Albanese morphism*. This morphism can also be characterized via the following universal property: $a : X \to \mathrm{Alb}(X)$ is the Albanese morphism if for every morphism $b : X \to T$ to a complex torus T there is a unique morphism $\phi : \mathrm{Alb}(X) \to T$ such that $b = \phi \circ a$.
The Albanese dimension of X is defined as $\dim a(X)$. We say that X has maximal Albanese dimension if $\dim a(X) = \dim X$ or equivalently, the Albanese morphism $a : X \to \mathrm{Alb}(X)$ is *generically finite* onto its image. For the

definition of *singular* Kähler space see [HP16] or [DH20].
A compact analytic variety X is said to be in *Fujiki's class* $\mathcal{C}$ if X is bimeromorphic to a compact Kähler manifold Y. In particular, there is a resolution of singularities $f : Y \to X$ such that Y is a compact Kähler manifold.

Definition 2.1. Let X be a compact analytic variety in Fujiki's class $\mathcal{C}$. Assume that X has rational singularities. Choose a resolution of singularities $\mu : Y \to X$ such that Y is a Kähler manifold and let $a_Y : Y \to \mathrm{Alb}(Y)$ be the Albanese morphism of Y. Then from the proof of [Kaw85, Lemma 8.1] it follows that $a_Y \circ \mu^{-1} : X \dashrightarrow \mathrm{Alb}(Y)$ extends to a unique morphism $a : X \to \mathrm{Alb}(X) := \mathrm{Alb}(Y)$. We call this morphism the Albanese morphism of X. Observe that $a : X \to \mathrm{Alb}(X)$ satisfies the universal property stated above. The Albanese dimension of X is defined as above. Note that if X is a compact analytic variety with rational singularities, bimeromorphic to a complex torus A, then $A \cong \mathrm{Alb}(X)$ and $X \to A$ is a bimeromorphic morphism.

The following result is well known, however, for a lack of proper reference and convenience of the readers we give a complete proof here.

Lemma 2.2. *Let A be a complex torus and $X \subset A$ is an analytic subvariety. Then for any resolution of singularities $\mu : Y \to X$, $H^0(Y, \omega_Y) \neq \{0\}$.*

Proof. Let $\mu : Y \to X$ be a resolution of singularities of X. Choose a point $p \in X \setminus (X_{\mathrm{sing}} \cup \mu(\mathrm{Ex}(\mu)))$ and let $(x_1, x_2, \ldots, x_n)$ be a local coordinate on A near p, where $n = \dim A$. Since $p \in X$ is a smooth point, X is local complete intersection near p. Thus we may assume that $X = \mathrm{Zero}(x_{d+1}, \ldots, x_n)$ near p, where $d = \dim X$. Then $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_d$ is a local section of Ω^d_A near p whose restriction to X is a non-zero local section of Ω^d_X near $p \in X$. Since Ω^d_A is a trivial vector bundle, there is a non-zero global section $\Theta \in H^0(A, \Omega^d_A)$ which restricts to $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_d$ near p, and hence $\mu^*\Theta|_X$ is a non-zero global section of $\Omega^d_Y := \omega_Y$. In particular, $H^0(Y, \omega_Y) \neq \{0\}$. □

Corollary 2.3. *Let X be a compact analytic variety in Fujiki's class $\mathcal{C}$ with canonical singularities. If X has maximal Albanese dimension, then $\kappa(X) \geq 0$.*

Proof. First note that if $f : W \to X$ is a proper bimeromorphic morphism, then $\kappa(X) \geq 0$ if and only if $\kappa(W) \geq 0$, since X has canonical singularities. Now let $a : X \to \mathrm{Alb}(X)$ be the Albanese morphism, $Y := a(X)$, and $\pi : Z \to Y$ is a resolution of singularities of Y. Then $\kappa(Z) \geq 0$ by Lemma 2.2. Note that there is a generically finite meromorphic map $\phi : X \dashrightarrow Z$; resolving the graph of ϕ we may assume that X is smooth and $\phi : X \to Z$ is a morphism. Then $K_X = \phi^* K_Z + E$, where $E \geq 0$ is an effective divisor. Therefore $\kappa(X) \geq 0$, since $\kappa(Z) \geq 0$. □

2.1. **Fourier-Mukai transform.** Let T be a complex torus of dimension g and $\hat{T} = \mathrm{Pic}^0(T)$ its dual torus. Let $p_T : T \times \hat{T} \to T$ and $p_{\hat{T}} : T \times \hat{T} \to \hat{T}$ be the projections, and $\mathcal{P}$ the normalized Poincaré line bundle on $T \times \hat{T}$ so that $\mathcal{P}|_{T\times\{0\}} \cong \mathcal{O}_T$ and $\mathcal{P}|_{\{0\}\times\hat{T}} \cong \mathcal{O}_{\hat{T}}$. Let $\hat{S}$ be the functor from the category of $\mathcal{O}_T$-sheaves to the category of $\mathcal{O}_{\hat{T}}$-sheaves, defined by

$$\hat{S}(\mathscr{F}) := p_{\hat{T},*}(p_T^*\mathscr{F} \otimes \mathcal{P}),$$

where $\mathscr{F}$ is a sheaf of $\mathcal{O}_T$-modules. Similarly, S is a functor from the category of $\mathcal{O}_{\hat{T}}$-sheaves to the category of $\mathcal{O}_T$-sheaves, defined as

$$S(\mathscr{G}) := p_{T,*}(p_{\hat{T}}^*\mathscr{G} \otimes \mathcal{P}),$$

where $\mathscr{G}$ is a sheaf of $\mathcal{O}_{\hat{T}}$-modules.
The corresponding derived functors are

$$\mathbf{R}\hat{S}(\cdot) := \mathbf{R}p_{\hat{T},*}(p_T^*(\cdot) \otimes \mathcal{P}) \text{ and } \mathbf{R}S(\cdot) := \mathbf{R}p_{T,*}(p_{\hat{T}}^*(\cdot) \otimes \mathcal{P}).$$

Recall the following fundamental result of Mukai [Muk81, Theorem 2.2, and (3.8)], [PPS17, Theorem 13.1]

Theorem 2.4. *With notations and hypothesis as above, there are isomorphisms of functors (on the bounded derived category of coherent sheaves)*

$$\mathbf{R}\hat{S} \circ \mathbf{R}S \cong (-1)_{\hat{T}}^*[-g], \qquad \mathbf{R}S \circ \mathbf{R}\hat{S} \cong (-1)_T^*[-g],$$
$$\mathbf{\Delta}_T \circ \mathbf{R}S = ((-1_T)^* \circ \mathbf{R}S \circ \mathbf{\Delta}_{\hat{T}})[-g].$$

Recall that $\mathbf{\Delta}_T(\cdot) := \mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_T)[g]$ is the dualizing functor.

Definition 2.5. Let A be a complex torus. For $a \in A$, let $t_a : A \to A$ be the usual translation morphism defined by a. A vector bundle $\mathcal{E}$ on A is called *homogeneous*, if $t_a^*\mathcal{E} \cong \mathcal{E}$ for all $a \in A$.

Remark 2.6. Let A be a complex torus, $\hat{A}$ the dual torus and $\dim A = \dim \hat{A} = g$. Then from the proof of [Muk81, Example 3.2] it follows that $R^g\hat{S}$ gives an equivalence of categories

$$\mathbf{H}_A := \{\text{Homogeneous vector bundles on } A\}, \qquad \text{and}$$
$$\mathbf{C}_{\hat{A}}^f := \{\text{Coherent sheaves on } \hat{A} \text{ supported at finitely many points}\}.$$

Note that in [Muk81] the results are all stated for abelian varieties, however, we observe that in the proof of [Muk81, Example 3.2] the main arguments follow from Theorem 2.4 and the isomorphisms in [Muk81, (3.1), page 158], both of which hold over complex tori. In particular, [Muk81, Example 3.2] holds for complex tori.

We will need the following result on the rational singularity of (log) canonical models of klt pairs.

Proposition 2.7. *Let (X, B) be a klt pair, where X is a compact analytic variety in Fujiki's class $\mathcal{C}$. Assume that the Kodaira dimension $\kappa(X, K_X+B) \geq 0$. Then $R(X, K_X + B) := \oplus_{m\geq 0} H^0(X, m(K_X + B))$ is a finitely generated $\mathbb{C}$-algebra and*

$$\bar{Z} = \operatorname{Proj} R(X, K_X + B)$$

has rational singularities.

Proof. The finite generation of $R(X, K_X + B)$ follows from [DHP22, Theorem 1.3] and [Fuj15, Theorem 5.1]. Let $f : X \dashrightarrow Z$ be the Iitaka fibration of $K_X + B$. Resolving Z, f and X, we may assume that X is a compact Kähler manifold, B has SNC support, Z is a smooth projective variety and f is a morphism. Then from the proof of [Fuj15, Theorem 5.1] it follows that there is a smooth projective variety Z' which is birational to Z and an effective $\mathbb{Q}$-divisor $B_{Z'} \geq 0$ such that $(Z', B_{Z'})$ is klt, $K_{Z'} + B_{Z'}$ is big and the following holds

$$R(X, K_X + B)^{(d)} \cong R(Z', K_{Z'} + B_{Z'})^{(d')},$$

where the superscripts d and d' represent the corresponding d and d'-Veronese subrings.
Thus $\bar{Z} = \operatorname{Proj} R(X, K_X + B) \cong \operatorname{Proj} R(Z', K_{Z'} + B_{Z'})$ is the log-canonical model of $(Z', B_{Z'})$. If $(Z'', B_{Z''})$ is a minimal model of $(Z', B_{Z'})$ as in [BCHM10, Theorem 1.2(2)], then by the base-point free theorem, there is a birational morphism $\phi : Z'' \to \bar{Z}$ such that $K_{Z''} + B_{Z''} = \phi^*(K_{\bar{Z}} + B_{\bar{Z}})$, where $B_{\bar{Z}} := \phi_* B_{Z''} \geq 0$. Thus $(\bar{Z}, B_{\bar{Z}})$ is a klt pair, and hence $\bar{Z}$ has rational singularities. □

3. Main Theorem

In this section we will prove our main theorem. We begin with some preparation.

Definition 3.1. Let X be a smooth compact analytic variety. Then the m-th *plurigenera* of X is defined as

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \omega_X^m).$$

The next result is one of our main tools in the proof of the main theorem, it is also of independent interest. It follows immediately from the main results of [PPS17].

Theorem 3.2. *Let X be a compact Kähler variety with terminal singularities. Assume that X has maximal Albanese dimension and $\kappa(X) = 0$. Then X is bimeromorphic to a torus. Additionally, if K_X is also nef, then X is isomorphic to a torus.*

Remark 3.3. Note that the above result holds if we simply assume that X is in Fujiki's class $\mathcal{C}$. Indeed, if $X' \to X$ is a resolution of singularities such that X' is Kähler, then $\kappa(X') = 0$ and so $X' \to \mathrm{Alb}(X')$ is bimeromorphic, and hence so is $X \to \mathrm{Alb}(X')$. Note also that if X is a complex manifold of maximal Albanese dimension, then X is automatically in Fujiki's class $\mathcal{C}$. To see this, consider the Stein factorization $X' \to Y \to A$. Then $Y \to A$ is finite and so Y is also Kähler (see [Var89, Prop. 1.3.1(v) and (vi), page 24]). Let $X' \to X$ be a resolution of singularities such that $X' \to Y$ is projective, then X' is Kähler and so X is in Fujiki's class $\mathcal{C}$.

Proof of Theorem 3.2. Since X is terminal, it has rational singularities, and thus by Definition 2.1 the Albanese morphism $a : X \to \mathrm{Alb}(X)$ exists. Let $\pi : \tilde{X} \to X$ be a resolution of singularities of X. Then $a \circ \pi : \tilde{X} \to \mathrm{Alb}(X)$ is the Albanese morphism of $\tilde{X}$. Moreover, since X has terminal singularities, $\kappa(\tilde{X}) = \kappa(X) = 0$. Thus replacing X by $\tilde{X}$, we may assume that X is a compact Kähler manifold. Let $d = \dim X$ and pick a general element $\Theta \in H^0(\Omega^d_A)$, where $A = \mathrm{Alb}(X)$. Then $0 \neq a^*\Theta \in H^0(\Omega^d_X)$ and so $P_1(X) > 0$. It follows that $P_k(X) = h^0(X, \omega^k_X) > 0$ for all $k > 0$. Since $\kappa(X) = 0$, we have $P_1(X) = P_2(X) = 1$. Thus by [PPS17, Theorem 19.1], $X \to A$ is surjective, and hence $\dim X = \dim A = h^{1,0}(X)$. Thus by [PPS17, Theorem B], X is bimeromorphic to a complex torus and so $a : X \to A$ is (surjective and) bimeromorphic.

Assume now that X has terminal singularities and K_X is nef. Let $a : X \to A$ be the Albanese morphism. By what we have seen above, this morphism is bimeromorphic.

Thus $K_X \equiv a^*K_A + E \equiv E$, where $E \geq 0$ is an effective Cartier divisor such that $\mathrm{Supp}(E) = \mathrm{Ex}(a)$ (since A is smooth). By the negativity lemma $E = 0$, and hence a is an isomorphism. □

Corollary 3.4. *Let (X, B) be a compact Kähler klt pair. Assume that X has maximal Albanese dimension and $\kappa(X, K_X + B) = 0$. Then X is bimeromorphic to a torus. Additionally, if $K_X + B \sim_{\mathbb{Q}} 0$, then X is isomorphic to a torus.*

Proof. Passing to a terminalization by running an appropriate MMP over X (using [DHP22, Theorem 1.4]) we may assume that (X, B) has $\mathbb{Q}$-factorial terminal singularities. Now since $\kappa(X) \geq 0$ by Corollary 2.3, $\kappa(X, K_X + B) = 0$ implies that $\kappa(X, K_X) = 0$. Thus by Theorem 3.2, $a : X \to A := \mathrm{Alb}(X)$ is a surjective bimeromorphic morphism. Now assume that $K_X + B$ is nef. Then $K_X + B = a^*K_A + E + B \sim_{\mathbb{Q}} B + E$, where $E \geq 0$ is an effective Cartier divisor such that $\mathrm{Supp}(E) = \mathrm{Ex}(a)$, since A is smooth. Thus $(B + E) \sim_{\mathbb{Q}} 0$, as $K_X + B \sim_{\mathbb{Q}} 0$, and hence $B = E = 0$ (as X is Kähler). In particular, $a : X \to A$ is an isomorphism. ☐

Now we are ready to prove our main theorem.

Proof of Theorem 1.1. Let $a : X \to A$ be the Albanese morphism. Since X has maximal Albanese dimension, a is generically finite over its image $a(X)$. By the relative Chow lemma (see [Hir75, Corollary 2] and [DH20, Theorem 2.12]) there is a log resolution $\mu : X' \to X$ of (X, B) such that the Albanese morphism $a' = a \circ \mu : X' \to A$ is projective. Let $K_{X'} + B' = \mu^*(K_X + B) + F$, where $F \geq 0$ such that $\mathrm{Supp}(F) = \mathrm{Ex}(\mu)$, and (X', B') has klt singularities. Note that if (X', B') has a good minimal model $\psi : X' \dashrightarrow X^m$, then ψ contracts every component of F and the induced bimeromorphic map $X \dashrightarrow X^m$ is a good minimal model of (X, B) (see [HX13, Lemmas 2.5 and 2.4] and their proofs). Thus, we may replace (X, B) by (X', B') and assume that (X, B) is a log smooth pair and $X \to A$ is a projective morphism. From Corollary 2.3 it follows that $\kappa(X) \geq 0$. In particular, $\kappa(X, K_X + B) \geq 0$. Now we split the proof into two parts. In Step 1 we deal with the $\kappa(X, K_X + B) = 0$ case, and the remaining cases are dealt with in Step 2.

Step 1. Suppose that $\kappa(X, K_X + B) = 0$. Then by Theorem 3.2, the Albanese morphism $a : X \to A := \mathrm{Alb}(X)$ is bimeromorphic. Let D be an irreducible component of the unique effective divisor $G \in |m(K_X + B)|$ for $m > 0$ sufficiently divisible. We make the following claim.

Claim 3.5. D is a-exceptional; in particular, G is a-exceptional.

Proof. First passing to a higher model of X we may assume that D has SNC support. Consider the short exact sequence

$$0 \to \omega_X \to \omega_X(D) \to \omega_D \to 0.$$

Let $V^0(\omega_D) := \{P \in \mathrm{Pic}^0(A) \mid h^0(D, \omega_D \otimes a^*P) \neq 0\}$. If $\dim V^0(\omega_D) > 0$, then it contains a subvariety $K + P$, where P is torsion in $\mathrm{Pic}^0(A)$ and K is a subtorus of $\mathrm{Pic}^0(A)$ with $\dim K > 0$ (see [PPS17, Corollary 17.1]). Since $a : X \to A$ is surjective and bimeromorphic, we have $H^i(X, a^*Q) = H^i(A, Q) = 0$ for any $\mathcal{O}_A \neq Q \in \mathrm{Pic}^0(A)$; in particular, $H^1(X, \omega_X \otimes a^*Q) = H^{n-1}(X, a^*Q^{-1})^\vee = 0$, where $n = \dim X$. Thus $H^0(X, \omega_X(D) \otimes a^*Q) \to H^0(D, \omega_D \otimes a^*Q)$ is surjective for all $\mathcal{O}_A \neq Q \in \mathrm{Pic}^0(A)$, and so $h^0(X, \omega_X(D) \otimes a^*Q) > 0$ for all $\mathcal{O}_A \neq Q \in P + K$. Since P is torsion, $\ell P = 0$ for some $\ell > 0$. Consider the morphism

$$|K_X + D + P + Q_1| \times \cdots \times |K_X + D + P + Q_\ell| \to |\ell(K_X + D)|, \tag{3.1}$$

where $Q_i \in K$ such that $\sum_{i=1}^{\ell} Q_i = 0$.
Since $\dim K > 0$, for $\ell \geq 2$, the $Q_1, \ldots, Q_\ell$ vary in the subvariety $\mathcal{K} \subset K^{\times \ell}$ defined by the equation $\sum_{i=1}^{\ell} Q_i = 0$. Thus $\dim \mathcal{K} \geq \ell \cdot (\dim K) - 1 \geq \ell - 1 \geq 1$. Therefore $\dim |\ell(K_X + D)| > 0$, i.e. $h^0(X, \ell(K_X + D)) > 1$. Since D is contained in the support of G, we have $(r - \ell)G \geq \ell D$ for some $r > 0$. Then $h^0(X, rm(K_X + B)) \geq h^0(X, \ell(K_X + D)) > 1$, which is a contradiction. Therefore, $\dim V^0(\omega_D) \leq 0$. By [PPS17, Theorem A], $a_*\omega_D$ is a GV sheaf so that $\mathbf{R}\hat{S}\mathbf{\Delta}_A(a_*\omega_D) = \mathbf{R}^0\hat{S}\mathbf{\Delta}_A(a_*\omega_D)$. If $\dim V^0(\omega_D) = 0$, then $\mathbf{R}^0\hat{S}(\mathbf{\Delta}_A(a_*\omega_D))$ is an Artinian sheaf of modules on A, and hence by Theorem 2.4 and Remark 2.6

$$\mathbf{\Delta}_A(a_*\omega_D) = (-1_A)^*\mathbf{R}S(\mathbf{R}\hat{S}\mathbf{\Delta}_A(a_*\omega_D))[g] = (-1_A)^*\mathbf{R}S(\mathbf{R}^0\hat{S}\mathbf{\Delta}_A(a_*\omega_D))[g]$$

is a shift of a homogeneous vector bundle which we denote by $\mathcal{E}$ (see Remark 2.6). But then

$$a_*\omega_D = \mathbf{\Delta}_A(\mathbf{\Delta}_A(a_*\omega_D)) = \mathcal{E}^\vee$$

is also a homogeneous vector bundle and hence its support is either empty or entire A. The latter is clearly impossible, since $\operatorname{Supp}(a_*\omega_D) \neq A$, and hence $V^0(\omega_D) = \emptyset$. Thus by [PPS17, Proposition 13.6(b)], $a_*\omega_D = 0$; in particular D is a-exceptional.

□

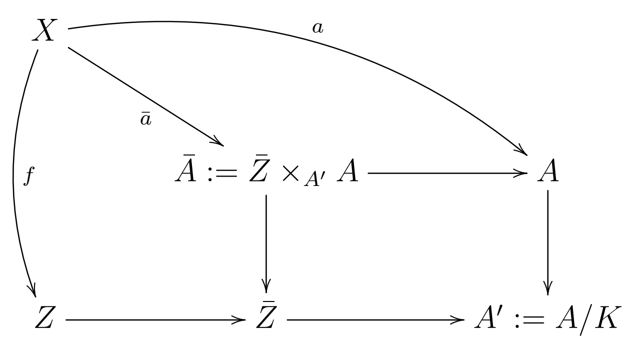
Now by [DHP22, Theorem 1.4] and [Fuj22a, Theorem 1.1] we can run the relative minimal model program over A and hence may assume that $K_X + B$ is nef over A. From our claim above we know that $K_X + B \sim_{\mathbb{Q}} E \geq 0$ for some effective a-exceptional divisor $E \geq 0$. Then by the negativity lemma we have $E = 0$; thus $\mathcal{O}_X(m(K_X + B)) \cong \mathcal{O}_X$ for sufficiently divisible $m > 0$, and hence we have a good minimal model.

Step 2. Suppose now that $\kappa(X, K_X + B) \geq 1$ and let $f : X \dashrightarrow Z$ be the Iitaka fibration. Note that the ring $R(X, K_X + B) := \oplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + B) \rfloor))$ is a finitely generated $\mathbb{C}$-algebra by [DHP22, Theorem 1.3]. Define $\bar{Z} := \operatorname{Proj} R(X, K_X + B)$. Then $Z \dashrightarrow \bar{Z}$ is a birational map of projective varieties. Resolving the graph of $Z \dashrightarrow \bar{Z}$ we may assume that Z is a smooth projective variety and $\nu : Z \to \bar{Z}$ is a birational morphism. Then passing to a resolution of X we may assume that f is a morphism and (X, B) is a log smooth pair. Write $K_F + B_F = (K_X + B)|_F$, where F is a very general fiber of f, so that $\kappa(F, K_F + B_F) = 0$. Note that $a|_F$ is also generically finite (as F is a very general fiber of f) and thus F has maximal Albanese dimension. In particular, (F, B_F) has a good minimal model by Step 1. Let $\psi : F \dashrightarrow F'$ be this minimal model; then $K_{F'} + B_{F'} \sim_{\mathbb{Q}} 0$. Thus by Corollary 3.4, F' is a torus and $B_{F'} = 0$; in particular, $\psi : F \to F'$ is the Albanese morphism. Thus $a|_F : F \to A$ factors through $\psi : F \to F'$; let $\alpha : F' \to A$ be the induced morphism. Let $K := \alpha(F')$; then K is a torus, and α is étale over K, as F' and K are both homogeneous varieties. Now since A contains at most countably many subtori and F is a very general fiber, K is independent of the very general points $z \in Z$, and hence so is F'. Define $A' := A/K$, then A' is again a torus. Since the composite morphism $X \to A'$ contracts F and $\dim F = \dim K$, from the rigidity lemma (see [BS95, Lemma 4.1.13]) and dimension count it follows that there is a meromorphic map $Z \dashrightarrow A'$ generically finite onto its image. Since Z is smooth, we may assume that $Z \to A'$ is a morphism (see [Kaw85, Lemma 8.1]). Similarly, since $\bar{Z}$ has rational singularities by Proposition 2.7, again from [Kaw85, Lemma 8.1] it follows that $\bar{Z} \to A'$ is a morphism.

Now choose an ample $\mathbb{Q}$-divisor $\bar{H}$ on $\bar{Z}$. Let H_X be the pull-back of $\bar{H}$ to X such that $K_X + B \sim_{\mathbb{Q}} H_X + E$ and $|k(K_X + B)| = |kH_X| + kE$ for any sufficiently large and divisible integer $k > 0$, where $E \geq 0$ is effective.

Now let $\bar{A} := \bar{Z} \times_{A'} A$. Observe that there is a unique morphism $\bar{a} : X \to \bar{A}$ determined by the universal property of fiber products. We claim that E is exceptional over $\bar{A}$. If not, then let D be a component of E which is not exceptional over $\bar{A}$. Let $h : X \to \bar{Z}$ be the composite morphism $X \to Z \to \bar{Z}$ and $W := h(D)$. Choose a sufficiently divisible and large positive integer $s > 0$ such that $s\bar{H}$ is very ample, $r(K_X + B)$ is Cartier, $rE \geq D$ and $|r(K_X + B)| = |rH_X| + rE$, where $r = (n + 1)s$ and $n = \dim X$.

(3.2)



Claim 3.6. $|K_D + (n + 1)sH_D| \neq \emptyset$, where $H_D = H_X|_D$.

Proof. Let $D_i = G_1 \cap \ldots \cap G_i$ be the intersection of general divisors $G_1, \ldots, G_m \in |sH_D|$, where $0 \leq i \leq m := \dim W$ and $D_0 := D$. Let $M := K_D + (n+1)sH_D$, then we have the short exact sequences

$$0 \to \mathcal{O}_{D_i}(M - G_{i+1}) \to \mathcal{O}_{D_i}(M) \to \mathcal{O}_{D_{i+1}}(M) \to 0.$$

Recall that $h : X \to \bar{Z}$ is the given morphism; let $h_i := h|_{D_i}$. Then

$$\begin{aligned}(M - G_{i+1})|_{D_i} &\sim (K_D + nsH_D)|_{D_i} \\ &\sim \Big(K_D + \sum_{j=1}^{i} G_j + (n-i)sH_D\Big)\Big|_{D_i} \\ &\sim K_{D_i} + (n-i)sH_{D_i} \\ &\sim K_{D_i} + h_i^*(n-i)s\bar{H},\end{aligned}$$

where $H_{D_i} := H_X|_{D_i}$. By [Fuj22b, Theorem 3.1(i)] the only associated subvarieties of

$$R^1 h_{i,*}\mathcal{O}_{D_i}(M - G_{i+1}) = R^1 h_{i,*}\mathcal{O}_{D_i}(K_{D_i}) \otimes \mathcal{O}_{\bar{Z}}((n-i)s\bar{H})$$

are $W_i := h(D_i) \subset \bar{Z}$, i.e. $R^1 h_{i,*}\mathcal{O}_{D_i}(M - G_{i+1})$ is a torsion free sheaf on W_i.

Therefore, the induced homomorphism $h_{i,*}\mathcal{O}_{D_{i+1}}(M) \to R^1 h_{i,*}\mathcal{O}_{D_i}(M - G_{i+1})$ is zero and we have the following exact sequence

$$0 \longrightarrow h_{i,*}\mathcal{O}_{D_i}(M - G_{i+1}) \longrightarrow h_{i,*}\mathcal{O}_{D_i}(M) \longrightarrow h_{i,*}\mathcal{O}_{D_{i+1}}(M) \longrightarrow 0.$$

By [Fuj22b, Theorem 3.1(ii)] we have

$$H^1(h_{i,*}\mathcal{O}_{D_i}(M - G_{i+1})) = H^1(h_{i,*}\mathcal{O}_{D_i}(K_{D_i}) \otimes \mathcal{O}_{\bar{Z}}((n-i)s\bar{H})) = 0,$$

and thus we have the following surjections

(3.3)

$$H^0(\mathcal{O}_D(M)) \to H^0(\mathcal{O}_{D_1}(M_{D_1})) \to \cdots \to H^0(\mathcal{O}_{D_m}(M_{D_m})) \to H^0(\mathcal{O}_G(M|_G)),$$

where G is a connected (and hence irreducible, as D_m is smooth) component of D_m. Note that G is a general fiber of $D \to W$, since H_D is a pullback from W and $m = \dim W$.

Let $w := h(G) \in W \subset \bar{Z}$. Then $G \to \bar{G} := \bar{a}(G)$ is generically finite (as so is $D \to \bar{a}(D)$ by our assumption), and $\bar{G} \to a(G)$ is an isomorphism, since $\bar{A}_w \to K \subset A$ is an isomorphism, as $\bar{A}_w = (A \times_{A'} \bar{Z})_w = A \times_{A'} \{w\} \cong K$. In particular, G has maximal Albanese dimension, and hence $h^0(G, K_G) > 0$ by Lemma 2.2. Now since $M|_G \sim K_G$, from the surjections in (3.3) it follows that $|M| = |K_D + (n+1)sH_D| \neq \emptyset$, and hence the claim follows.

□

Now consider the short exact sequence

$$0 \to \omega_X(L) \to \omega_X(L + D) \to \omega_D(L) \to 0,$$

where $L = rH_X$. Then by [Fuj22b, Theorem 3.1(i)], $R^1 h_*\omega_X(L) = R^1 h_*\omega_X \otimes \mathcal{O}_{\bar{Z}}(r\bar{H})$ is torsion free, and hence $h_*\omega_X(L + D) \to h_*\omega_D(L)$ is surjective.

Again by [Fuj22b, Theorem 3.1(ii)], $H^1(h_*\omega_X(L)) = H^1(h_*\omega_X \otimes \mathcal{O}_{\bar{Z}}(r\bar{H})) = 0$, and so $H^0(\omega_X(L+D)) \to H^0(\omega_D(L))$ is surjective. Since $|K_D + L|_D| \neq 0$ by Claim 3.6, D is not contained in the base locus of $|K_X + L + D|$. Let $0 \le b := \mathrm{mult}_D(B) < 1$ and $e := \mathrm{mult}_D(E) > 0$. Then $\sigma E + B - D \ge 0$ and $\mathrm{mult}_D(\sigma E + B - D) = 0$ for $\sigma = \frac{1-b}{e} > 0$. We may assume that $\sigma \le r$ (as r is sufficiently large and divisible). Adding $rE + B - D$ to a general divisor $G \in |K_X + L + D|$ we get

$$\Gamma := rE + B - D + G \sim_{\mathbb{Q}} (r+1)(K_X + B) \sim_{\mathbb{Q}} (r+1)(H_X + E).$$

Then for any sufficiently divisible $m > 0$ we have

$$\mathrm{mult}_D(m\Gamma) = m(r-\sigma)\mathrm{mult}_D(E) < m(r+1)\mathrm{mult}_D(E),$$

which is a contradiction to the fact that $|k(K_X + B)| = |kH_X| + kE$ for sufficiently divisible $k = m(r+1) > 0$. Thus D is exceptional over $\bar{A}$.

Let $n = \dim X$. We will run a relative $(K_X + B + (2n+3)sH_X)$-MMP over A. Note that since $|(2n+3)sH_X|$ is a base-point free linear system on a smooth compact variety X, by Sard's theorem there is an effective $\mathbb{Q}$-divisor $H' \ge 0$ such that $(2n+3)sH_X \sim_{\mathbb{Q}} H'$ and $(X, B + H')$ has klt singularities. Thus $K_X + B + (2n+3)sH_X \sim_{\mathbb{Q}} K_X + B + H'$ and we can run a $(K_X + B + (2n+3)sH_X)$-MMP over A by [DHP22, Theorem 1.4], and obtain $X \dashrightarrow X'$ so that $K_{X'} + B' + (2n+3)sH_{X'} \sim_{\mathbb{Q}} ((2n+3)s+1)H_{X'} + E'$ is nef over A. Note that if R is a $(K_X + B + (2n+3)sH_X)$-negative extremal ray over A, then it is also $(K_X + B)$-negative and so it is spanned by a rational curve C such that $0 > (K_X + B) \cdot C \ge -2n$ (see [DHP22, Theorem 2.46]). But then C is vertical over $\bar{Z}$, otherwise $(K_X + B + (2n+3)sH_X) \cdot C > 0$, as H_X is the pullback of an ample divisor $\bar{H}$ on $\bar{Z}$, this is a contradiction. Thus it follows that every step of this MMP is also a step of an MMP over $\bar{Z}$, and hence there is an induced morphism $\mu : X' \to \bar{A} := \bar{Z} \times_{A'} A$. It follows that

$$K_{X'} + B' \sim_{\mathbb{Q}} \mu^* H_{\bar{A}} + E' \sim_{\mathbb{Q}, \bar{A}} E' \ge 0,$$

where $H_{\bar{A}}$ is the pullback of the ample divisor $\bar{H}$ by the projection $\bar{A} \to \bar{Z}$. Then E' is nef and exceptional over $\bar{A}$, and hence by the negativity lemma, $E' = 0$. But then $K_{X'} + B' \sim_{\mathbb{Q}} \mu^* H_{\bar{A}}$ and since $H_{\bar{A}}$ is semi-ample, so is $K_{X'} + B'$.

□

Corollary 3.7. *Let (X, B) be a compact Kähler klt pair of maximal Albanese dimension such that $a : X \to A := \mathrm{Alb}(X)$ is a projective morphism. Then we can run a $(K_X + B)$-Minimal Model Program which ends with a good minimal model.*

Proof. Note that since $a : X \to A$ is generically finite over image, $K_X + B$ is relatively big over $a(X)$. Thus by [DHP22, Theorem 1.4] and [Fuj22a, Theorem 1.8], we can run a (K_X+B)-Minimal Model Program over A. Notice that each step of this MMP is also a step of the $(K_X + B)$-MMP. Therefore, we may assume that $K_X + B$ is nef over A and we must check that it is indeed nef on X. Let $(\bar{X}, \bar{B})$ be a good minimal model of (X, B), which exists by Theorem 1.1. By what we have seen, $(\bar{X}, \bar{B})$ is also a minimal model over A. But then $\phi : (X, B) \dashrightarrow (\bar{X}, \bar{B})$ is an isomorphism in codimension 1. If $p : Y \to X$ and $q : Y \to \bar{X}$ is a common resolution, then from the negativity lemma it follows that $p^*(K_X + B) = q^*(K_{\bar{X}} + \bar{B})$. In particular, $p^*(K_X + B)$ is semi-ample, and hence so is $K_X + B$. Thus (X, B) is a good minimal model. □

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