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Infinitesimal structure of log canonical thresholds

Jihao Liu, Fanjun Meng, and Lingyao Xie

Abstract. We show that log canonical thresholds of fixed dimension are standardized. More precisely, we show that any sequence of log canonical thresholds in fixed dimension d accumulates either (i) in a way which is similar to how standard and hyperstandard sets accumulate, or (ii) to log canonical thresholds in dimension $\leq d-2$. This provides an accurate description on the infinitesimal structure of the set of log canonical thresholds. We also discuss similar behaviors of minimal log discrepancies, canonical thresholds, and K-semistable thresholds.

1. Introduction

We work over the field of complex numbers $\mathbb{C}$. For any set $\Gamma \subset \mathbb{R}$, we let $\partial\Gamma$ be the set of accumulation points of Γ and $\bar{\Gamma} := \Gamma \cup \partial\Gamma$ the closure of Γ. We let $\partial^0\Gamma := \bar{\Gamma}$, and denote the set of k-th order accumulation points of Γ by $\partial^k\Gamma$ for any $k > 0$. It is clear that $\partial^k\Gamma = \partial^k\bar{\Gamma}$ for any non-negative integer k.

Log canonical thresholds. The log canonical threshold (lct for short) is a fundamental invariant in algebraic geometry. It originates from analysis which measures the integrability of a holomorphic function. In birational geometry, the log canonical threshold measures the complexity of the singularities of a triple $(X, B; D)$ where (X, B) is a pair and D is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor.

Definition 1.1. Let (X, B) be a pair and $D \geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We define

$$\operatorname{lct}(X, B; D) := \sup\{t \geq 0 \mid (X, B + tD) \text{ is log canonical (lc)}\}$$

to be the lct of D with respect to (X, B). The set of lcts in dimension d is defined as

$$\operatorname{lct}(d) := \left\{ \operatorname{lct}(X, B; D) \;\middle|\; \begin{array}{l} \dim X = d,\ (X, B) \text{ is lc, } B \text{ is an effective Weil divisor,} \\ \text{and } D \text{ is an effective } \mathbb{Q}\text{-Cartier Weil divisor} \end{array} \right\}.$$

It is well known that the set of log canonical thresholds of fixed dimension satisfies the ascending chain condition (ACC) [10, Theorem 1.1] and their accumulation points are log canonical thresholds from lower dimension [10, Theorem 1.11]. The purpose of

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this paper is to discuss how lcts approach their accumulation points. More precisely, we show that the sets of lcts of fixed dimension are *standardized sets*. Roughly speaking, this says that the infinitesimal behavior of the sets of lcts is similar to the behavior of standard and hyperstandard sets, especially near their first order accumulation points. We first give definitions of standardized sets.

Definition 1.2 (Standardized sets). Let $\Gamma \subset \mathbb{R}$ be a set and γ_0 a real number. We say that Γ is *standardized near* γ_0 if there exist a positive real number ε, a positive integer m, and real numbers $b_1, \dots, b_m$, such that

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{ \gamma_0 + \frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \le i \le m \right\}.$$

We say that Γ is

(1) *weakly standardized* if Γ is standardized near any $\gamma_0 \in \bar{\Gamma} \backslash \partial^2 \Gamma$, and

(2) *standardized* if $\partial^k \Gamma$ is weakly standardized for any non-negative integer k and $\partial^l \Gamma = \emptyset$ for some positive integer l.

By Lemma 2.16 below, Γ is weakly standardized (resp. standardized) if and only if $\partial^0 \Gamma = \bar{\Gamma}$ is weakly standardized (resp. standardized).

Roughly speaking, a standardized set Γ has a filtration

$$\bar{\Gamma} = \partial^0 \Gamma \supset \partial \Gamma \supset \partial^2 \Gamma \cdots \supset \partial^l \Gamma = \emptyset$$

which consists of weakly standardized sets. Note that when $\Gamma \subset [0, 1]$ and 1 is the only possible accumulation point of Γ, a standardized set is always a subset of a hyperstandard set [25, Section 3.2]. This is the reason why we adopt the word "standardized".

The main theorem of our paper is as follows.

Theorem 1.3 (Main theorem). *For any positive integer d, the set* $\mathrm{lct}(d)$ *is standardized.*

We remark that Theorem 1.3 also holds for pairs which allow more complicated boundary coefficients. See Theorem 4.2 for more details.

To obtain a better understanding of Theorem 1.3, we provide several examples below.

Example 1.4. Let $d, a_1, \dots, a_d$ be fixed positive integers. For any positive integer n, we consider the "diagonal" polynomial $f_n = x_1^{a_1} + x_2^{a_2} + \cdots + x_d^{a_d} + x_{d+1}^n$ and the divisor $S_n := (f_n = 0)$ on $X := \mathbb{C}^{d+1}$. Suppose that $\sum_{i=1}^d \frac{1}{a_i} < 1$ and $n \gg 0$, then it is well known that

$$\gamma_n := \mathrm{lct}(X, 0; S_n) = \min \left\{ 1, \sum_{i=1}^d \frac{1}{a_i} + \frac{1}{n} \right\} = \sum_{i=1}^d \frac{1}{a_i} + \frac{1}{n}.$$

Let $\gamma_0 := \sum_{i=1}^d \frac{1}{a_i}$. Then γ_0 is the accumulation point of $\{\gamma_n\}_{n=1}^{+\infty}$. Moreover, γ_n approaches γ_0 in a "standardized way" as

$$\gamma_n = \gamma_0 + \frac{1}{n}.$$

In other words, $\{\gamma_n\}_{n=1}^{+\infty}$ is standardized near γ_0, hence $\{\gamma_n\}_{n=1}^{+\infty}$ is a standardized set. In fact, it is not hard to check that the set of lcts of "diagonal" polynomials of dimension $d+1$,

$$\left\{\sum_{i=1}^{d+1}\frac{1}{c_i}\;\middle|\; c_1,\ldots,c_{d+1}\in\mathbb{N}^+\right\}\cap[0,1],$$

is a standardized set.

Example 1.5. It is known that the set of lcts on $\mathbb{C}^2$ is

$$\mathcal{HT}_2=\left\{\frac{c_1+c_2}{c_1c_2+a_1c_2+a_2c_1}\;\middle|\;\begin{array}{l}a_1+c_1\geq\max\{2,a_2\},\ a_2+c_2\geq\max\{2,a_1\},\\ a_1,a_2,c_1,c_2\in\mathbb{N}\end{array}\right\},$$

and the set of lcts on $\mathbb{C}^1$ is

$$\mathcal{HT}_1=\operatorname{lct}(1)=\left\{\frac{1}{k}\;\middle|\;k\in\mathbb{N}^+\right\}\cup\{0\}$$

(cf. [16, (15.5)]). We have

$$\partial\mathcal{HT}_2=\mathcal{HT}_1\backslash\{1\}=\left\{\frac{1}{k}\;\middle|\;k\in\mathbb{N}^+,\ k\geq 2\right\}\cup\{0\}$$

and $\partial^2\mathcal{HT}_2=\{0\}$ (cf. [16, Theoerem 7]). It is not hard to see that, for any $k\in\mathbb{N}^+$ and any sequence

$$\{\gamma_n\}_{n=1}^{+\infty}\subset\mathcal{HT}_2$$

such that

$$0<\frac{1}{k}=\gamma_0:=\lim_{n\to+\infty}\gamma_n,$$

possibly by passing to a subsequence and switching c_1 and c_2, we have

$$\gamma_n=\frac{c_{1,n}+c_2}{c_{1,n}c_2+a_1c_2+a_2c_{1,n}}$$

for some fixed a_1,a_2,c_2 and strictly increasing sequence of integers $c_{1,n}$, such that $a_2+c_2=k$ and $a_1\leq k$. Therefore, γ_n approaches γ_0 in a "standardized way" as

$$\gamma_n=\gamma_0+\frac{c_2(k-a_1)}{k}\cdot\frac{1}{kc_{1,n}+a_1c_2}\in\left\{\gamma_0+\frac{c_2(k-a_1)}{m}\;\middle|\;m\in\mathbb{N}^+\right\}.$$

It is not hard to deduce that $\mathcal{HT}_2$ is standardized near any $\gamma_0\in\partial\mathcal{HT}_2\backslash\partial^2\mathcal{HT}_2$.

On the other hand, it is clear that the values in $\mathcal{HT}_2$ may not approach 0 in a standardized way, hence $\mathcal{HT}_2$ is not standardized near 0. Nevertheless, 0 is a second order accumulation point of $\mathcal{HT}_2$, hence $\mathcal{HT}_2$ is still a weakly standardized set. Moreover, since 0 is an accumulation point of $\mathcal{HT}_1$ and $\mathcal{HT}_1$ is standardized near 0, we know that $\mathcal{HT}_2$ is standardized.

Theorem 1.3 could be potentially applied to the study on Han's uniform boundedness conjecture of minimal log discrepancies (cf. [11, Conjecture 7.2]), especially on its weaker version for fixed germs (cf. [21, Conjecture 1.1]), as the accumulation points of lcts naturally appear in the study of these conjectures.

We also expect Theorem 1.3 to be useful when estimating the precise values of log canonical thresholds in high dimension, especially the 1-gap of lc thresholds.

Nevertheless, with Theorem 1.3 settled, it will be interesting to ask whether other invariants in birational geometry behave similarly, such as the minimal log discrepancy and the canonical threshold. We will confirm that the sets of these invariants are standardized in some special cases in the following.

Minimal log discrepancies. The minimal log discrepancy (mld for short) is another fundamental invariant in algebraic geometry, which measures how singular a variety is. The smaller the mld is, the worse the singularity is.

Definition 1.6. Let $(X \ni x)$ be an lc singularity. We define

$$\mathrm{mld}(X \ni x) := \min\left\{a(E, X) \mid E \text{ is over } X \ni x\right\}$$

to be the mld of $(X \ni x)$. The set of mlds for varieties of dimension d is defined as

$$\mathrm{mld}(d) := \left\{\mathrm{mld}(X \ni x) \mid \dim X = d,\ X \ni x \text{ is lc}\right\}.$$

Similar to the sets of log canonical thresholds, the sets of minimal log discrepancies of fixed dimension are also conjectured to satisfy the ACC [27, Problem 5], and its accumulation points are expected to come from lower dimension (cf. [12, Version 1, Remark 1.2]). Unfortunately, the ACC conjecture for mlds is open in dimension ≥ 3, hence it will be difficult to show the standardized behavior of this invariant. Nevertheless, we are able to prove that the sets of mlds are standardized in some special cases.

Theorem 1.7. *The following sets of minimal log discrepancies are standardized.*

(1) *The sets* $\mathrm{mld}(1)$ *and* $\mathrm{mld}(2)$ *are standardized.*

(2) *The set* $\{\mathrm{mld}(X) \mid \dim X = 3,\ X \textit{ is canonical}\}$ *is standardized.*

(3) *For any positive integer* d *and positive real number* ε*,*

$$\left\{\mathrm{mld}(X \ni x) \mid \dim X = d,\ X \ni x \textit{ has an } \varepsilon\textit{-plt blow up}\right\}$$

is standardized. In particular,

$$\left\{\mathrm{mld}(X \ni x) \mid \dim X = d,\ X \ni x \textit{ is exceptional}\right\}$$

is standardized.

We remark that Theorem 1.7 also holds for lc pairs whose boundary coefficients belong to a finite set. See Section 3 for more details.

We also remark that Theorem 1.7 (3) is actually important in the proof of Theorem 1.3. Note that the standardized behavior of mlds is very important due to the following example, which shows that a conjectural standardized behavior is already very helpful in the study of the ACC conjecture for mlds. In fact, the standardized behavior of lcts and mlds was observed when the first author examined the following example in [19].

Example 1.8. A recent work of the first author and Luo shows that $\frac{5}{6}$ is the second largest accumulation point of global mlds in dimension 3 [19, Theorem 1.3]. An important ingredient of the proof is [19, Theorem 3.5], which essentially uses the conjectural standardized behavior of mld(3) in $(\frac{5}{6}, 1)$. The idea is as follows.

For any real number $a \in (\frac{5}{6}, 1)$, we associate infinite equations to a such that $a \in \operatorname{mld}(3)$ (almost) only if these equations have a common solution (cf. [19, Definition 3.4]). Denote the set of these equations by $\mathcal{E}(a)$. For each fixed a, the equations in $\mathcal{E}(a)$ are computable. As there are infinitely many equations, in practice, it is not hard to verify that the equations in $\mathcal{E}(a)$ do not have any common solution by checking finitely many of them, but it is difficult to verify that the equations in $\mathcal{E}(a)$ have a common solution. Moreover, since there are uncountably many real numbers a in $(\frac{5}{6}, 1)$, we cannot consider all $\mathcal{E}(a)$ at the same time. To resolve these issues, a key idea is to decompose $(\frac{5}{6}, 1)$ as a disjoint union of subsets

$$\left(\frac{5}{6}, 1\right) = \bigcup_n \Gamma_n \cup \widetilde{\Gamma}$$

such that

(1) $\widetilde{\Gamma}$ satisfies the ACC and only accumulates to $\frac{5}{6}$ (hence these values will not influence the proof of [19, Theorem 3.5]), and

(2) for any fixed n, Γ_n is an open interval, and the equations in $\bigcap_{a\in\Gamma_n} \mathcal{E}(a)$ do not have any common solution. This implies that $\Gamma_n \cap \operatorname{mld}(3) = \emptyset$.

The difficulty is that we need to guess what $\widetilde{\Gamma}$ is. Since the equations in $\mathcal{E}(a)$ have a common solution (almost) whenever $a \in \operatorname{mld}(3) \cap (\frac{5}{6}, 1)$ and $\operatorname{mld}(3) \cap (\frac{5}{6}, 1)$ is known to be an infinite set, we need to find a regular pattern of the values in $\widetilde{\Gamma}$.

At this point, a key observation is that the equations in $\mathcal{E}(a)$ heavily rely on the denominator of a. This is the reason why we conjectured that the denominator of a grows standardly respect to $a - \frac{5}{6}$ when a approaches $\frac{5}{6}$. With this conjecture in mind, an attempt on setting

$$\widetilde{\Gamma} = \left\{\frac{5n+m}{6n+m} \;\middle|\; m, n \in \mathbb{N}^+,\ 1 \le m \le 5\right\} \cup \left\{\frac{12}{13}\right\}$$

is successful. In summary, the conjecture on the standardized behavior of the set of mlds was essentially applied in the proof of [19, Theorem 3.5], in a way that we directly "guess the set of mlds out". Note that $\widetilde{\Gamma}$ is standardized as the only accumulation point of $\widetilde{\Gamma}$ is $\frac{5}{6}$ and

$$\frac{5n+m}{6n+m} = \frac{5}{6} + \frac{m}{36n+6m} \in \left\{\frac{5}{6} + \frac{m}{l} \;\middle|\; l, m \in \mathbb{N}^+,\ 1 \le m \le 5\right\}.$$

Similar strategies in [19] can be applied to further studies on mld(3), especially for those values that are $\geq \frac{1}{2}$ as $\frac{1}{2}$ is the conjectured largest second order accumulation point of threefold mlds. See [14, 20, 26] for related results.

Canonical thresholds. The canonical threshold is another important invariant in birational geometry. In particular, canonical thresholds in dimension 3 are deeply related to Sarkisov links in dimension 3 (cf. [9, 24]). It is known that, in dimension ≤ 3, the set of canonical thresholds satisfies the ACC [7, 8, 11], and its accumulation points come from lower dimension [7, 11]. We show that the set of canonical thresholds is also standardized in dimension ≤ 3, which actually follows from the proofs in [7, 11].

Theorem 1.9. *The set of canonical thresholds* $\mathrm{ct}(d)$ *in dimension* d *is standardized when* $d \leq 3$.

Further discussions. Yuchen Liu informed us that an invariant in K-stability and wall-crossing theory, the *K-semistable threshold (walls)*, may also behave in a standardized way.

Example 1.10 ([2, Theorem 5.16]). The list of K-moduli walls of $\overline{\mathfrak{M}}_c^K$ (the K-moduli stack which parametrizes K-polystable log Fano pairs (X, cD) admitting a $\mathbb{Q}$-Gorenstein smoothing to $(\mathbb{P}^3, cS)$ where S is a quartic surface) is

$$\left\{1 - \frac{4}{n} \;\middle|\; n \in \{6, 8, 10, 12, 13, 14, 16, 18, 22\}\right\},$$

which is a subset of a hyperstandard set (Definition 2.1). Although this is a very specific example and the value of the walls are finite, using a hyperstandard set to describe the thresholds is natural in this case, and it is possible that larger classes of K-semistable thresholds also behave in a standardized way.

2. Preliminaries

We adopt the standard notation and definitions in [5, 17] and will freely use them.

2.1. Sets

Definition 2.1. Let $\Gamma \subset \mathbb{R}$ be a set. We say that

(1) Γ satisfies the *descending chain condition* (DCC) if any decreasing sequence in Γ stabilizes,

(2) Γ satisfies the *ascending chain condition* (ACC) if any increasing sequence in Γ stabilizes,

(3) Γ is *the standard set* if $\Gamma = \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$, and

(4) ([25, Section 3.2], [3, Section 2.2]) Γ is a *hyperstandard set* if there exists a finite set $\Gamma_0 \subset \mathbb{R}_{\geq 0}$ such that $0, 1 \in \Gamma_0$ and $\Gamma = \{1 - \frac{\gamma}{n} \mid n \in \mathbb{N}^+, \gamma \in \Gamma_0\} \cap [0, 1]$.

2.2. Pairs and singularities

Definition 2.2 (Pairs, cf. [6, Definition 3.2]). A *pair* $(X/Z \ni z, B)$ consists of a contraction $\pi : X \to Z$, a (not necessarily closed) point $z \in Z$, and an $\mathbb{R}$-divisor $B \geq 0$ on X, such that $K_X + B$ is $\mathbb{R}$-Cartier over a neighborhood of z. If π is the identity map and $z = x$, then we may use $(X \ni x, B)$ instead of $(X/Z \ni z, B)$. In addition, if $B = 0$, then we use $X \ni x$ instead of $(X \ni x, 0)$. If $(X \ni x, B)$ is a pair for any codimension ≥ 1 point $x \in X$, then we call (X, B) a pair. A pair $(X \ni x, B)$ is called a *germ* if x is a closed point.

Definition 2.3 (Singularities of pairs). Let $(X \ni x, B)$ be a pair and E a prime divisor over X such that $x \in \operatorname{center}_X E$. Let $f : Y \to X$ be a log resolution of (X, B) such that $\operatorname{center}_Y E$ is a divisor, and suppose that $K_Y + B_Y = f^*(K_X + B)$ over a neighborhood of x. We define $a(E, X, B) := 1 - \operatorname{mult}_E B_Y$ to be the *log discrepancy* of E with respect to (X, B).

For any prime divisor E over X, we say that E is *over* $X \ni x$ if $\operatorname{center}_X E = \bar{x}$. We define

$$\operatorname{mld}(X \ni x, B) := \inf\{a(E, X, B) \mid E \text{ is over } X \ni x\}$$

to be the *minimal log discrepancy (mld)* of $(X \ni x, B)$. We define

$$\operatorname{mld}(X, B) := \inf\{a(E, X, B) \mid E \text{ is exceptional over } X\}.$$

We define

$$\operatorname{tmld}(X, B) := \inf\{a(E, X, B) \mid E \text{ is over } X\}$$

to be the *total minimal log discrepancy (tmld)* of (X, B).

Let ε be a non-negative real number. We say that $(X \ni x, B)$ is *lc* (resp. *klt, ε-lc, ε-klt*) if $\operatorname{mld}(X \ni x, B) \geq 0$ (resp. $> 0, \geq \varepsilon, > \varepsilon$). We say that (X, B) is *lc* (resp. *klt, ε-lc, ε-klt*) if $\operatorname{tmld}(X, B) \geq 0$ (resp. $> 0, \geq \varepsilon, > \varepsilon$).

We say that (X, B) is *canonical* (resp. *terminal*, *plt*, *ε-plt*) if $\operatorname{mld}(X, B) \geq 1$ (resp. > 1, $> 0, > \varepsilon$).

Definition 2.4. Let a be a non-negative real number, $(X \ni x, B)$ (resp. (X, B)) an lc pair, and $D \geq 0$ an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. We define

$$a\text{-}\operatorname{lct}(X \ni x, B; D) := \sup\{-\infty, t \mid t \geq 0,\ (X \ni x, B + tD) \text{ is } a\text{-lc}\}$$
$$\big(\text{resp. } a\text{-}\operatorname{lct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0,\ (X, B + tD) \text{ is } a\text{-lc}\}\big)$$

to be the *a-lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define

$$\operatorname{ct}(X \ni x, B; D) := \sup\{-\infty, t \mid t \geq 0,\ (X \ni x, B + tD) \text{ is 1-lc}\}$$
$$\big(\text{resp. } \operatorname{ct}(X, B; D) := \sup\{-\infty, t \mid t \geq 0,\ (X, B + tD) \text{ is canonical}\}\big)$$

to be the *canonical threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)). We define $\operatorname{lct}(X \ni x, B; D) := 0\text{-}\operatorname{lct}(X \ni x, B; D)$ (resp. $\operatorname{lct}(X, B; D) := 0\text{-}\operatorname{lct}(X, B; D)$) to be the *lc threshold* of D with respect to $(X \ni x, B)$ (resp. (X, B)).

Definition 2.5. Assume that X is a normal variety and B is an $\mathbb{R}$-divisor on X. We write $B \in \Gamma$ if the coefficients of B belong to Γ. For any positive integer d, we define

$$\begin{aligned} \operatorname{mld}(d,\Gamma) &:= \left\{\operatorname{mld}(X \ni x, B) \mid \dim X = d,\ (X \ni x, B) \text{ is lc},\ B \in \Gamma\right\}, \\ \operatorname{lct}(d,\Gamma) &:= \left\{\operatorname{lct}(X, B; D) \mid \dim X = d,\ (X, B) \text{ is lc},\ B \in \Gamma,\ D \in \mathbb{N}^+\right\}, \\ \operatorname{ct}(d,\Gamma) &:= \left\{\operatorname{ct}(X, B; D) \mid \dim X = d,\ (X, B) \text{ is canonical},\ B \in \Gamma,\ D \in \mathbb{N}^+\right\}. \end{aligned}$$

We let $\operatorname{mld}(0,\Gamma) = \operatorname{lct}(0,\Gamma) = \operatorname{ct}(0,\Gamma) := \{0\}$. For any non-negative integer d, we let $\operatorname{mld}(d) := \operatorname{mld}(d, \{0\})$, $\operatorname{lct}(d) := \operatorname{lct}(d, \{0, 1\})$, and $\operatorname{ct}(d) := \operatorname{ct}(d, \{0, 1\})$.

2.3. Complements

Definition 2.6. Let n be a positive integer, $\Gamma_0 \subset (0, 1]$ a finite set, and $(X/Z \ni z, B)$ and $(X/Z \ni z, B^+)$ two pairs. We say that $(X/Z \ni z, B^+)$ is an $\mathbb{R}$-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is lc,
- $B^+ \geq B$, and
- $K_X + B^+ \sim_{\mathbb{R}} 0$ over a neighborhood of z.

We say that $(X/Z \ni z, B^+)$ is an n-*complement* of $(X/Z \ni z, B)$ if

- $(X/Z \ni z, B^+)$ is lc,
- $nB^+ \geq \lfloor (n+1)\{B\} \rfloor + n\lfloor B \rfloor$, and
- $n(K_X + B^+) \sim 0$ over a neighborhood of z.

We say that $(X/Z \ni z, B)$ is $\mathbb{R}$-complementary if $(X/Z \ni z, B)$ has an $\mathbb{R}$-complement. We say that $(X/Z \ni z, B^+)$ is a *monotonic* n-*complement* of $(X/Z \ni z, B)$ if $(X/Z \ni z, B^+)$ is an n-complement of $(X/Z \ni z, B)$ and $B^+ \geq B$.

We say that $(X/Z \ni z, B^+)$ is an (n, Γ_0)-*decomposable* $\mathbb{R}$-*complement* of $(X/Z \ni z, B)$ if there exist a positive integer k, $a_1, \ldots, a_k \in \Gamma_0$, and $\mathbb{Q}$-divisors $B_1^+, \ldots, B_k^+$ on X, such that

- $\sum_{i=1}^k a_i = 1$ and $\sum_{i=1}^k a_i B_i^+ = B^+$,
- $(X/Z \ni z, B^+)$ is an $\mathbb{R}$-complement of $(X/Z \ni z, B)$, and
- $(X/Z \ni z, B_i^+)$ is an n-complement of itself for each i.

Theorem 2.7 ([12, Theorem 1.10]). *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Then there exist a positive integer n and a finite set $\Gamma_0 \subset (0, 1]$ depending only on d and Γ satisfying the following.*

Assume that $(X/Z \ni z, B)$ is a pair of dimension d and $B \in \Gamma$, such that X is of Fano type over Z and $(X/Z \ni z, B)$ is $\mathbb{R}$-complementary. Then $(X/Z \ni z, B)$ has an (n, Γ_0)-decomposable $\mathbb{R}$-complement. Moreover, if $\overline{\Gamma} \subset \mathbb{Q}$, then $(X/Z \ni z, B)$ has a monotonic n-complement.

2.4. Plt blow-ups

Definition 2.8. Let $(X \ni x, B)$ be a klt germ and ε a positive real number. A *plt* (resp. *ε-plt*) *blow-up* of $(X \ni x, B)$ is a divisorial contraction $f : Y \to X$ with a prime exceptional divisor E over $X \ni x$, such that $(Y/X \ni x, f_*^{-1}B + E)$ is plt (resp. ε-plt) and $-E$ is ample over X.

Lemma 2.9 ([28, Section 3.1], [23, Proposition 2.9], [18, Theorem 1.5], [29, Lemma 1]). *Assume that* $(X \ni x, B)$ *is a klt germ such that* $\dim X \geq 2$. *Then there exists a plt blow-up of* $(X \ni x, B)$.

Definition 2.10. Let $(X \ni x, B)$ be an lc germ. We say that $(X \ni x, B)$ is *exceptional* if for any $\mathbb{R}$-divisor $G \geq 0$ on X such that $(X \ni x, B + G)$ is lc, there exists at most one lc place of $(X \ni x, B + G)$.

2.5. Special sets

Definition 2.11. Let $\Gamma \subset [0, 1]$ be a set, d a positive integer, and c a positive real number. We define

$$\Gamma_+ := \left(\{0\} \cup \left\{\sum_{i=1}^{n} \gamma_i \mid \gamma_1, \ldots, \gamma_n \in \Gamma\right\}\right) \cap [0, 1],$$

$$D(\Gamma) := \left\{\frac{m-1+\gamma}{m} \,\middle|\, m \in \mathbb{N}^+,\ \gamma \in \Gamma_+\right\},$$

$$D(\Gamma, c) := \left\{\frac{m-1+\gamma+kc}{m} \,\middle|\, m, k \in \mathbb{N}^+,\ \gamma \in \Gamma_+\right\} \cap [0, 1],$$

$$\mathfrak{N}(d, \Gamma, c) := \left\{(X, B) \,\middle|\, \begin{array}{l} (X, B) \text{ is projective lc},\ K_X + B \equiv 0,\ \dim X = d, \\ B = L + C,\ L \in D(\Gamma),\ 0 \neq C \in D(\Gamma, c) \end{array}\right\},$$

$$\mathfrak{R}(d, \Gamma, c) := \left\{(X, B) \,\middle|\, \begin{array}{l} (X, B) \in \mathfrak{N}(n, \Gamma, c),\ (X, B) \text{ is } \mathbb{Q}\text{-factorial klt}, \\ \rho(X) = 1,\ 1 \leq n \leq d \end{array}\right\},$$

$$N(d, \Gamma) := \{c \mid c \in [0, 1],\ \mathfrak{N}(d, \Gamma, c) \neq \emptyset\},$$

$$K(d, \Gamma) := \{c \mid c \in [0, 1],\ \mathfrak{R}(d, \Gamma, c) \neq \emptyset\}.$$

We define

$$N(0, \Gamma) = K(0, \Gamma) := \left\{\frac{1-\gamma}{n} \,\middle|\, \gamma \in \Gamma_+,\ n \in \mathbb{N}^+\right\} \cup \{0\},$$

$$N(-1, \Gamma) = K(-1, \Gamma) := \{0\}.$$

The following results in [10] are used in the proof of the main theorem.

Theorem 2.12 ([10, Lemmas 11.2 and 11.4, Proposition 11.5, Theorem 1.11]). *Let d be a non-negative integer and* $\Gamma \subset [0, 1]$ *a set. Then*

(1) $\operatorname{lct}(d, \Gamma) \subset \operatorname{lct}(d + 1, \Gamma)$ *and* $N(d - 1, \Gamma) \subset N(d, \Gamma)$,

(2) $N(d, \Gamma \cup \{1\}) = K(d, \Gamma)$,

(3) *if* $\Gamma = \Gamma_+$, *then* $\mathrm{lct}(d, \Gamma) = N(d-1, \Gamma)$, *and*

(4) *if* $1 \in \Gamma$, $\Gamma = \Gamma_+$, *and* $\partial\Gamma \subset \{1\}$, *then* $\partial\,\mathrm{lct}(d+1, \Gamma) = \mathrm{lct}(d, \Gamma)\backslash\{1\}$.

2.6. Basic properties of standardized sets

The behavior of standardized sets is generally similar to the behavior of DCC sets and ACC sets. However, there are still some differences.

Example 2.13. It is clear that any subset of a DCC (resp. ACC) set is still DCC (resp. ACC). However, a subset of a standardized set may no longer be standardized. Consider the sets $\Gamma_1 := \{\frac{n}{n^2+1} \mid n \in \mathbb{N}^+\}$ and $\Gamma_2 = \Gamma_1 \cup \{\frac{1}{n} + \frac{1}{m} \mid n, m \in \mathbb{N}^+\}$. Then $\Gamma_1 \subset \Gamma_2$. It is not hard to check that Γ_2 is a standardized set but Γ_1 is not. This is because although Γ_1 and Γ_2 are both not standardized near 0, $0 \notin \partial^2\Gamma_1$, but $0 \in \partial^2\Gamma_2$.

We summarize the following properties on standardized sets below which we will use in this paper.

Lemma 2.14. *Let* Γ *be a set of real numbers and* γ_0 *a real number. Then:*

(1) *If* $\gamma_0 \notin \partial\Gamma$, *then* Γ *is standardized near* γ_0.

(2) *If* $\gamma_0 \in \partial^2\Gamma$, *then* Γ *is not standardized near* γ_0.

(3) *For any real number* a, Γ *is standardized near* γ_0 *if and only if* $\{\gamma + a \mid \gamma \in \Gamma\}$ *is standardized near* $a + \gamma_0$.

(4) *For any non-zero number* c, Γ *is standardized near* γ_0 *if and only if* $\{c\gamma \mid \gamma \in \Gamma\}$ *is standardized near* $c\gamma_0$.

(5) *Suppose that* $\Gamma = \bigcup_{i=1}^k \Gamma_i$. *Then* Γ *is standardized near* γ_0 *if and only if each* Γ_i *is standardized near* γ_0. *In particular,* Γ *is standardized near* γ_0 *if and only if any subset of* Γ *is standardized near* γ_0.

(6) Γ *is standardized near* γ_0 *if and only if* $\Gamma \cap (\gamma_0 - \varepsilon_0, \gamma_0 + \varepsilon_0)$ *is standardized near* γ_0 *for some positive real number* ε_0.

(7) *If* $\{\gamma - \gamma_0 \mid \gamma \in \Gamma\} \subset \mathbb{Q}$, *then* Γ *is standardized near* γ_0 *if and only if there exists a positive integer* I *and a positive real number* ε, *such that*

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{I}{n} \,\middle|\, n \in \mathbb{Z}\backslash\{0\}\right\} \cup \{\gamma_0\}.$$

(8) Γ *is standardized near* γ_0 *if and only if* $\overline{\Gamma}$ *is standardized near* γ_0.

Proof. For any $\gamma_0 \notin \partial\Gamma$, we may pick a positive real number ε such that

$$(\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \cap \Gamma = \{\gamma_0\} \text{ or } \emptyset.$$

This implies (1).

Suppose that Γ is standardized near γ_0 for some $\gamma_0 \in \partial^2\Gamma$. Then there exist a positive real number ε, a positive integer m, and real numbers $b_1, \dots, b_m$, such that

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \le i \le m\right\}.$$

Thus the only accumulation point of $\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$ is γ_0, hence $\gamma_0 \notin \partial^2\Gamma$, a contradiction. This implies (2).

(3) (4) (6) are obvious.

We prove (5). If Γ is standardized near γ_0, then there exist a positive real number ε, a positive integer m, and non-zero real numbers $b_1, \dots, b_m$, such that

$$\Gamma_i \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{b_j}{n} \;\middle|\; j, n \in \mathbb{N}^+,\ 1 \le j \le m\right\}.$$

for each $1 \le i \le k$, hence Γ_i is standardized near γ_0 for each i. If Γ_i is standardized near γ_0 for each i, then there exist positive integers $m_1, \dots, m_k$, a finite set of real numbers $\{b_{i,j}\}_{1 \le i \le k,\ 1 \le j \le m_i}$, and real numbers $\varepsilon_1, \dots, \varepsilon_k$, such that

$$\Gamma_i \cap (\gamma_0 - \varepsilon_i, \gamma_0 + \varepsilon_i) \subset \left\{\gamma_0 + \frac{b_{i,j}}{n} \;\middle|\; j, n \in \mathbb{N}^+,\ 1 \le j \le m_i\right\}$$

for each i. Let $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_k\}$, then

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{b_{i,j}}{n} \;\middle|\; i, j, n \in \mathbb{N}^+,\ 1 \le i \le k,\ 1 \le j \le m_i\right\},$$

hence Γ is standardized.

We prove (7). The if part is obvious. Suppose that Γ is standardized near γ_0, then there exist a positive real number ε, a positive integer m, and non-zero real numbers $b_1, \dots, b_m$, such that

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \le i \le m\right\}.$$

Since $\{\gamma - \gamma_0 \mid \gamma \in \Gamma\} \subset \mathbb{Q}$, we may assume that $b_i \in \mathbb{Q}$ for each i. We may let I be a common denominator of the elements in $\left\{\frac{1}{b_i} \mid b_i \neq 0\right\}$.

The if part of (8) follows from (5). Suppose that Γ is standardized near γ_0, then there exist a positive real number ε, a positive integer m, and non-zero real numbers $b_1, \dots, b_m$, such that

$$\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \left\{\gamma_0 + \frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \le i \le m\right\}.$$

Thus

$$\bar{\Gamma} \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon) \subset \overline{\Gamma \cap (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)} \subset \left\{\gamma_0 + \frac{b_i}{m} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \le i \le n\right\} \cup \{\gamma_0\},$$

hence $\bar{\Gamma}$ is standardized near γ_0. ∎

Lemma 2.15. *Let Γ be a set of real numbers and γ_0 a real number. Suppose that for any sequence $\{\gamma_i\}_{i=1}^{+\infty} \subset \Gamma$ such that $\lim_{i\to+\infty} \gamma_i = \gamma_0$, there exists an infinite subsequence of $\{\gamma_i\}_{i=1}^{+\infty}$ which is standardized near γ_0. Then Γ is standardized near γ_0.*

Proof. For any non-zero real number γ, we let $[\gamma]$ be its $\mathbb{Q}$-class under multiplication: $[\gamma] = [\gamma']$ if and only if $\gamma = s\gamma'$ for some $s \in \mathbb{Q}^{\times}$. By Lemma 2.14 (3), possibly by replacing γ_0 with 0 and Γ with $\{\gamma - \gamma_0 \mid \gamma \in \Gamma\}$, we may assume that $\gamma_0 = 0$. For any positive real number ε, we consider $\Gamma_{\mathbb{Q},\varepsilon} := \{[\gamma] \mid \gamma \in (\Gamma \cap (-\varepsilon, \varepsilon))\backslash\{0\}\}$.

Suppose that $\Gamma_{\mathbb{Q},\varepsilon}$ is an infinite set for any positive real number ε. Then we may pick a sequence $\{\gamma_i\}_{i=1}^{+\infty} \subset \Gamma$ such that $[\gamma_i] \neq [\gamma_j]$ for any $i \neq j$ and $\lim_{i\to+\infty} \gamma_i = 0$. By assumption, there exists a strictly increasing sequence of integers $\{r_i\}_{i=1}^{+\infty}$ such that $\{\gamma_{r_i}\}_{i=1}^{+\infty}$ is standardized near 0, so there exist a positive integer m and real numbers $b_1, \dots, b_m$, such that

$$\{\gamma_{r_i}\}_{i=1}^{+\infty} \subset \left\{\frac{b_j}{n} \;\middle|\; j, n \in \mathbb{N}^+,\ 1 \le j \le m\right\}.$$

This is not possible as the $\mathbb{Q}$-classes of $\{\frac{b_j}{n} \mid j, n \in \mathbb{N}^+,\ 1 \le j \le m\}$ are finite. Therefore, $\Gamma_{\mathbb{Q},\varepsilon_0}$ is a finite set for some positive real number ε_0. By Lemma 2.14 (6), we may replace Γ with $\Gamma \cap (-\varepsilon_0, \varepsilon_0)$ and write $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$ for some positive integer k, such that $[\gamma_i] \neq [\gamma_j]$ for any $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ and any $i \neq j$, and $[\alpha] = [\beta]$ for any $\alpha, \beta \in \Gamma_i$ and any i. By Lemma 2.14 (5), we only need to show that Γ_i is standardized near 0 for any i. Therefore, we may assume that $k = 1$. In particular, there exists a non-zero real number c and a set $\Gamma' \subset \mathbb{Q}$ such that $\Gamma = \{c\gamma' \mid \gamma' \in \Gamma'\}$. By Lemma 2.14 (4), possibly by replacing Γ with Γ', we may assume that $\Gamma \subset \mathbb{Q}$.

Suppose that Γ is not standardized near 0. By Lemma 2.14 (7), there exists a sequence

$$\left\{\frac{p_i}{q_i}\right\}_{i=1}^{+\infty} \subset \Gamma$$

such that $\gcd(p_i, q_i) = 1$, $\lim_{i\to+\infty} |p_i| = +\infty$, and $\lim_{i\to+\infty} \frac{p_i}{q_i} = 0$. It is clear that no infinite subsequence of $\{\frac{p_i}{q_i}\}_{i=1}^{+\infty}$ is standardized near 0, a contradiction. ∎

Lemma 2.16. *Let Γ, Γ' be two sets of real numbers. Then:*

(1) *Γ is weakly standardized (resp. standardized) if and only if $\overline{\Gamma}$ is weakly standardized (resp. standardized).*

(2) *If Γ and Γ' are weakly standardized (resp. standardized), then $\Gamma \cup \Gamma'$ is weakly standardized (resp. standardized).*

Proof. Since $\overline{\Gamma}\backslash\partial^2\Gamma = \overline{\Gamma}\backslash\partial^2\overline{\Gamma} = \overline{\overline{\Gamma}}\backslash\partial^2\overline{\Gamma}$ and $\partial^k\Gamma = \partial^k\overline{\Gamma}$ for any non-negative integer k, (1) follows from Lemma 2.14 (8).

Since $\partial^k(\Gamma \cup \Gamma') = \partial^k\Gamma \cup \partial^k\Gamma'$ for any non-negative integer k, (2) follows from Lemma 2.14 (5). ∎

We summarize some additional properties of standardized sets in the following lemma. The lemma is interesting in its own right. However, we do not need this lemma in the rest of this paper.

Lemma 2.17. *Let Γ, Γ' be two sets of real numbers.*

(1) *Γ is weakly standardized if and only if Γ is standardized near any $\gamma_0 \in \partial\Gamma \backslash \partial^2\Gamma$.*

(2) *If Γ, Γ' are standardized and both satisfy the DCC (resp. ACC), then*

$$\Gamma'' := \{\gamma + \gamma' \mid \gamma \in \Gamma,\ \gamma' \in \Gamma'\}$$

is standardized.

(3) *If Γ' is a finite set, then Γ is standardized if and only if $\Gamma \cup \Gamma'$ is standardized.*

(4) *If Γ' is an interval, Γ is DCC or ACC, and Γ is standardized, then $\Gamma \cap \Gamma'$ is standardized.*

(5) *If $\Gamma \subset [0, 1]$ satisfies the DCC and is standardized, then Γ_+ and $D(\Gamma)$ are standardized.*

(6) *If $\Gamma \subset [0, +\infty)$ satisfies the ACC and is standardized, then $\{\frac{\gamma}{n} \mid \gamma \in \Gamma, n \in \mathbb{N}^+\}$ is standardized.*

Proof. (1) It follows from Lemma 2.14 (1).

(2) Since Γ and Γ' both satisfy the DCC (resp. ACC), we have

$$\partial^k\Gamma'' = \bigcup_{i=0}^{k}\{\gamma + \gamma' \mid \gamma \in \partial^i\Gamma,\ \gamma' \in \partial^{k-i}\Gamma'\}.$$

Since Γ and Γ' are standardized, $\partial^l\Gamma = \emptyset$ and $\partial^{l'}\Gamma' = \emptyset$ for some positive integers l, l'. Thus $\partial^{l+l'}\Gamma'' = \emptyset$. By induction on $l + l'$ and Lemma 2.16 (2), we only need to show that Γ'' is weakly standardized. By Lemma 2.15 and (1), we only need to show that for any $\gamma''_0 \in \partial\Gamma''\backslash\partial^2\Gamma''$ and any sequence $\{\gamma''_i\}_{i=1}^{+\infty}$ such that $\lim_{i\to+\infty}\gamma''_i = \gamma''_0$, a subsequence of $\{\gamma''_i\}_{i=1}^{+\infty}$ is standardized near γ''_0. We may write $\gamma''_i = \gamma_i + \gamma'_i$ where $\gamma_i \in \Gamma$ and $\gamma'_i \in \Gamma'$. Possibly by passing to a subsequence, we may assume that γ_i, γ'_i are increasing (resp. decreasing), $\lim_{i\to+\infty}\gamma_i = \gamma_0$, and $\lim_{i\to+\infty}\gamma'_i = \gamma'_0$. If $\{\gamma_i\}_{i=1}^{+\infty}$ and $\{\gamma'_i\}_{i=1}^{+\infty}$ both have strictly increasing (resp. strictly decreasing) subsequences, then possibly by passing to subsequences, we may assume that $\{\gamma_i\}_{i=1}^{+\infty}$ and $\{\gamma'_i\}_{i=1}^{+\infty}$ are strictly increasing (resp. strictly decreasing). Since

$$\gamma''_0 = \lim_{j\to+\infty}(\gamma_0 + \gamma'_j) = \lim_{j\to+\infty}\lim_{i\to+\infty}(\gamma_i + \gamma'_j),$$

$\gamma''_0 \in \partial^2\Gamma''$, a contradiction. Thus possibly by passing to a subsequence and switching Γ, Γ', we may assume that $\gamma_i = \gamma_0$ for each i. By Lemma 2.14 (3), $\{\gamma''_i\}_{i=1}^{+\infty}$ is standardized near γ''_0, and we get (2).

(3) We have $\partial^k(\Gamma \cup \Gamma') = \partial^k\Gamma$ for any positive integer k. For any real number γ_0 and non-negative integer k, since Γ' is a finite set, by Lemma 2.14 (5), $\partial^k\Gamma$ is standardized near γ_0 if and only if $\partial^k(\Gamma \cup \Gamma')$ is standardized near γ_0. This implies (3).

(4) By Lemma 2.14 (4), possibly by replacing Γ with $\{-\gamma \mid \gamma \in \Gamma\}$ and Γ' with $\{-\gamma' \mid \gamma' \in \Gamma'\}$, we may assume that Γ is DCC. Let $a := \inf\Gamma' \in \{-\infty\} \cup \mathbb{R}$ and $c := \sup\Gamma' \in \{+\infty\} \cup \mathbb{R}$. Since Γ satisfies the DCC, a is not an accumulation point of $\Gamma \cap \Gamma'$. Thus $\partial^k(\Gamma \cap \Gamma') = (\partial^k\Gamma \cap \Gamma')\backslash\{a, c\}$ or $(\partial^k\Gamma \cap \Gamma'\backslash\{a\}) \cup \{c\}$ for any positive integer k.

By (3) and induction on the minimal non-negative integer l such that $\partial^l\Gamma=\emptyset$, we only need to show that $\Gamma\cap\Gamma'$ is weakly standardized. By (1) and Lemma 2.14 (6), we only need to show that $\Gamma\cap\Gamma'$ is standardized near c when $c<+\infty$ and $c\in\partial(\Gamma\cap\Gamma')\backslash\partial^2(\Gamma\cap\Gamma')$. Since Γ satisfies the DCC, $c\in\partial(\Gamma\cap\Gamma')\backslash\partial^2(\Gamma\cap\Gamma')$ if and only if $c\in\partial\Gamma\backslash\partial^2\Gamma$, hence Γ is standardized near c when $c\in\partial(\Gamma\cap\Gamma')\backslash\partial^2(\Gamma\cap\Gamma')$. Statement (4) follows from Lemma 2.14 (5).

(5) Suppose that $\Gamma\subset[0,1]$ satisfies the DCC and is standardized. First we show that Γ_+ is standardized. We let $\Gamma_1:=\Gamma$ and let $\Gamma_k:=\{\gamma+\tilde{\gamma}\mid\gamma\in\Gamma,\ \tilde{\gamma}\in\Gamma_{k-1}\}$ for any integer $k\geq 2$. Since Γ satisfies the DCC, we may let $\bar{\gamma}:=\min\{1,\gamma\in\Gamma\mid\gamma>0\}$. By (2), Γ_k is standardized for any positive integer k. Since $\Gamma_+=(\Gamma_{\lfloor\frac{1}{\bar{\gamma}}\rfloor}\cup\{0\})\cap[0,1]$, by (3) (4), Γ_+ is standardized.

Now we show that $D(\Gamma)$ is standardized. We may replace Γ with Γ_+ and suppose that $\Gamma=\Gamma_+$. Then $(\partial^k\Gamma)_+=\partial^k\Gamma\cup\{0\}$ for any non-negative integer k. We have

$$D(\Gamma)=\left\{\frac{m-1+\gamma}{m}\;\middle|\; m\in\mathbb{N}^+,\ \gamma\in\Gamma\right\}.$$

Let k_0 be the minimal positive integer such that $\partial^{k_0}\Gamma\neq\emptyset$. By induction, we have

$$\partial^k D(\Gamma)=\{1\}\cup\left\{\frac{m-1+\gamma}{m}\;\middle|\; m\in\mathbb{N}^+,\ \gamma\in\partial^k\Gamma\right\}$$

for any $1\leq k\leq k_0$, $\partial^{k_0+1}D(\Gamma)=\{1\}$, and $\partial^k D(\Gamma)=\emptyset$ for any $k\geq k_0+2$. By (3) and induction on k_0, we only need to show that $D(\Gamma)$ is weakly standardized. There are two cases.

Case 1. The set Γ is a finite set. Then 1 is the only accumulation point of $D(\Gamma)$, and it is clear that $D(\Gamma)$ is standardized near 1. By (1), $D(\Gamma)$ is weakly standardized.

Case 2. The set Γ is not a finite set. Then $1\in\partial^2 D(\Gamma)$. For any $c\in[0,1)$ and any sequence $\{c_i\}_{i=1}^{+\infty}\subset D(\Gamma)$ such that $\lim_{i\to+\infty}c_i=c$, possibly by passing to a subsequence we have

$$c_i=\frac{m-1+\gamma_i}{m}$$

such that m is a constant and $\gamma_i\in\Gamma$ for each i. If $c\notin\partial^2 D(\Gamma)$, then by Lemma 2.14 (3) (4), $\{c_i\}_{i=1}^{+\infty}$ is standardized near c. By Lemma 2.15, $D(\Gamma)$ is standardized near c, hence $D(\Gamma)$ is weakly standardized.

(6) Since Γ satisfies the ACC,

$$\Gamma\subset[0,M]$$

for some positive integer M. By Lemma 2.14 (4), possibly by replacing Γ with $\{\frac{\gamma}{M}\mid\gamma\in\Gamma\}$, we may assume that $M=1$. Let $\Gamma':=\{1-\gamma\mid\gamma\in\Gamma\}$, then $\Gamma'\subset[0,1]$ satisfies the DCC and is standardized. By the same argument as in (5) and Lemma 2.14 (3) (4),

$$\left\{\frac{\gamma}{n}\;\middle|\; n\in\mathbb{N}^+,\ \gamma\in\Gamma\right\}=\left\{1-\frac{m-1+\gamma'}{m}\;\middle|\; m\in\mathbb{N}^+,\ \gamma'\in\Gamma'\right\}$$

is standardized. ∎

3. Standardization of (some) log discrepancies

Theorem 3.1. *Let* $\Gamma \subset [0,1]$ *be a finite set. Then* $\mathrm{mld}(1,\Gamma)$ *and* $\mathrm{mld}(2,\Gamma)$ *are standardized.*

Proof. Since $\mathrm{mld}(1,\Gamma)=\{1, 1-\gamma \mid \gamma\in\Gamma\}$ is a finite set, $\mathrm{mld}(1,\Gamma)$ is standardized.

We first show that $\mathrm{mld}(2,\Gamma)$ is standardized near any $a_0>0$. Fix $a_0>0$ and let $\{a_i\}_{i=1}^{+\infty}\subset \mathrm{mld}(2,\Gamma)$ be any sequence such that $\lim_{i\to+\infty} a_i=a_0$. Let $(X_i\ni x_i, B_i)$ be a surface singularity such that $B_i\in\Gamma$ and $\mathrm{mld}(X_i\ni x_i,B_i)=a_i$.

Claim 3.2. *Possibly by passing to a subsequence, we have*

$$a_i=\frac{\alpha A_i+\beta}{A_i+\delta},$$

where $\alpha,\delta\geq 0$ *and* $\beta>0$ *are constants such that* $\delta\in\mathbb{Q}$*, and* $A_i\in\mathbb{N}^+$.

Proof. This essentially follows from [6, Lemma A.2] (see also [1, Lemma 3.3]), but since the argument of [6, Lemma A.2] is very long, we provide a short proof here.

Let $\varepsilon:=\min\{a_0,\Gamma_{>0}\}$. If the possibilities of the dual graphs of the minimal resolution of $X_i\in x_i$ is finite, then there are only finitely many possibilities of $a_i=\mathrm{mld}(X_i\in x_i,B_i)$, which is not possible. Therefore, by [6, Lemma A.6], possibly by passing to a subsequence, we may assume that

$$a_i=\mathrm{mld}(X_i\ni x_i,B_i)=\mathrm{pld}(X_i\ni x_i,B_i).$$

Possibly by passing to a subsequence, we may assume that $(X_i\ni x_i,B_i)$ is of one of the types as in [6, Lemma A.2 (1)] ($\mathfrak{F}_{\varepsilon,\Gamma}$), [6, Lemma A.2 (2)] ($\mathfrak{C}_{\varepsilon,\Gamma}$), or [6, Lemma A.2 (3)] ($\mathfrak{T}_{\varepsilon,\Gamma}$). If $(X_i\ni x_i,B_i)$ is of type $\mathfrak{F}_{\varepsilon,\Gamma}$ for each i, then there are finitely many possibilities of the dual graph of the minimal resolution of $X_i\ni x_i$, which is again not possible. If $(X_i\ni x_i,B_i)$ is of type $\mathfrak{T}_{\varepsilon,\Gamma}$ for each i, then [6, Lemma A.2 (3)] implies that

$$a_i=\mathrm{pld}(X_i\ni x_i,B_i)=\frac{\alpha_i}{m_i-q_i},$$

where $q_i<m_i\leq\lfloor\frac{2}{\varepsilon}\rfloor^{\lfloor\frac{2}{\varepsilon}\rfloor}$, q_i and m_i are integers, and α_i belongs to a finite set as it is constructed as in [6, p. 35, line 17] and Γ is a finite set. In this case, $\frac{\alpha_i}{m_i-q_i}$ belongs to a finite set, which is not possible. Therefore, we may assume that $(X_i\ni x_i,B_i)$ is of type $\mathfrak{C}_{\varepsilon,\Gamma}$ for each i. [6, Lemma A.2 (2)] implies that

$$a_i=\mathrm{pld}(X_i\ni x_i,B_i)=\frac{\left(A_i+\frac{m_{2,i}}{m_{2,i}-q_{2,i}}\right)\frac{\alpha_{1,i}}{m_{1,i}-q_{1,i}}+\frac{q_{1,i}}{m_{1,i}-q_{1,i}}\cdot\frac{\alpha_{2,i}}{m_{2,i}-q_{2,i}}}{A_i+\frac{q_{1,i}}{m_{1,i}-q_{1,i}}+\frac{m_{2,i}}{m_{2,i}-q_{2,i}}},$$

where $q_{1,i}<m_{1,i}\leq\lfloor\frac{2}{\varepsilon}\rfloor^{\lfloor\frac{2}{\varepsilon}\rfloor}$ and $q_{2,i}<m_{2,i}\leq\lfloor\frac{2}{\varepsilon}\rfloor^{\lfloor\frac{2}{\varepsilon}\rfloor}$, $q_{1,i},q_{2,i},m_{1,i},m_{2,i}$ are integers, and A_i is a non-negative integers. Therefore, possibly passing to a subsequence, we may

assume that $q_{1,i}, q_{2,i}, m_{1,i}, m_{2,i}$ are constants, and $A_i > 0$. We may let

$$\alpha := \frac{\alpha_{1,i}}{m_{1,i} - q_{1,i}}, \quad \beta := \frac{m_{2,i}}{m_{2,i} - q_{2,i}} \cdot \frac{\alpha_{1,i}}{m_{1,i} - q_{1,i}} + \frac{q_{1,i}}{m_{1,i} - q_{1,i}} \cdot \frac{\alpha_{2,i}}{m_{2,i} - q_{2,i}},$$

and

$$\delta := \frac{q_{1,i}}{m_{1,i} - q_{1,i}} + \frac{m_{2,i}}{m_{2,i} - q_{2,i}}. \qquad \blacksquare$$

Proof of Theorem 3.1 continued. Let α, β, δ and A_i be as in Claim 3.2. Then $\alpha = a_0$, and

$$a_i = a_0 + \frac{\beta - a_0\delta}{A_i + \delta}.$$

Therefore, $\{a_i\}_{i=1}^{+\infty}$ is standardized near a_0. By Lemma 2.15, $\operatorname{mld}(2, \Gamma)$ is standardized near a_0.

By [6, Lemma A.2] (see also [1, Lemma 3.3]), we have

$$\left\{0, \frac{1}{n} \,\middle|\, n \in \mathbb{N}^+\right\} \subset \partial \operatorname{mld}(2, \Gamma) \subset \left\{0, \frac{1-\gamma}{n} \,\middle|\, n \in \mathbb{N}^+,\ \gamma \in \Gamma_+\right\}$$

and

$$\partial^2 \operatorname{mld}(2, \Gamma) = \{0\}.$$

Since Γ is a finite set, Γ_+ is a finite set, hence $\{0, \frac{1-\gamma}{n} \mid \gamma \in \Gamma_+,\ n \in \mathbb{N}^+\}$ is standardized near 0. By Lemma 2.14 (5), $\partial \operatorname{mld}(2, \Gamma)$ is standardized near 0. Since $\overline{\operatorname{mld}(2, \Gamma)} \subset [0, +\infty)$, $\operatorname{mld}(2, \Gamma)$ is standardized. $\blacksquare$

Theorem 3.3. *Let $\Gamma \subset [0, 1]$ be a finite set. Then*

$$\Gamma' := \left\{\operatorname{mld}(X, B) \mid \dim X = 3,\ B \in \Gamma\right\} \cap [1, +\infty)$$

is standardized and its only accumulation point is 1.

Proof. By [22, Corollary 1.5], 1 is the only accumulation point of Γ', so we only need to show that Γ' is standardized near 1. By Lemma 2.15, we only need to show that for any sequence of pairs $\{(X_i, B_i)\}_{i=1}^{+\infty}$ such that $\dim X_i = 3$, $B_i \in \Gamma$, and $\operatorname{mld}(X_i, B_i) \geq 1$, $\{\operatorname{mld}(X_i, B_i)\}_{i=1}^{+\infty}$ has a subsequence which is standardized near 1. Possibly by passing to a subsequence and replacing each (X_i, B_i) with a $\mathbb{Q}$-factorialization, we may assume that each X_i is $\mathbb{Q}$-factorial. By [11, Theorem 6.12], possibly by passing to a subsequence, we may find a positive integer l depending only on Γ, and prime divisors E_i that are exceptional over X_i, such that $a(E_i, X_i, B_i) = \operatorname{mld}(X_i, B_i) > 1$ and $a(E_i, X_i, 0) \leq 1 + \frac{l}{I_i}$, where I_i is the Cartier index of K_{X_i} near the generic point x_i of $\operatorname{center}_{X_i} E_i$. By [15, Corollary 5.2], for any prime divisor D on X_i, $I_i D$ is Cartier near x_i. Since Γ is a finite set, possibly by passing to a subsequence, we may assume that $a(E_i, X_i, B_i) = 1 + \frac{\gamma}{I_i}$ where $\gamma \in (0, l]$ is a constant. It is clear that $\{a(E_i, X_i, B_i)\}_{i=1}^{+\infty}$ is standardized near 1 and the theorem follows. $\blacksquare$

Lemma 3.4. *Let d be a positive integer, ε and c two positive real numbers, and $\Gamma \subset [0,1]$ and $\Gamma' \subset [0,+\infty) \cap \mathbb{Q}$ two finite sets. Then there exist a finite set $\Gamma_1 \subset (0,+\infty)$ and a finite set $\Gamma_2 \subset [0,+\infty) \cap \mathbb{Q}$ depending only on $d, \varepsilon, c, \Gamma$, and Γ' which satisfy the following.*

Assume that $(X \ni x, B)$ is an lc pair of dimension d, such that

(1) $B = \Delta + sS$ such that $\Delta \in \Gamma$, $S \in \Gamma'$, and

(2) $(X \ni x, \Delta + cS)$ has an ε-plt blow-up $f : Y \to X$ which extracts a prime divisor E.

Then $a(E, X, B) = \frac{\alpha-(s-c)\beta}{n}$, where $\alpha \in \Gamma_1, \beta \in \Gamma_2$, and $n \in \mathbb{N}^+$.

Proof. Possibly by replacing c, s and Γ', we may assume that S is a Weil divisor. By cutting X by general hyperplane sections and applying induction on dimension, we may assume that x is a closed point. Let Δ_Y, B_Y, and S_Y be the strict transforms of Δ, B and S on Y respectively, and $a := a(E, X, B)$.

$$K_Y + B_Y + (1-a)E = f^*(K_X + B).$$

Since $(X \ni x, B)$ is lc, $a \geq 0$. Let

$$K_E + B_E := (K_Y + B_Y + E)|_E \quad \text{and} \quad K_E + B'_E := (K_Y + \Delta_Y + cS_Y + E)|_E.$$

Since $a \geq 0$ and $-E$ is ample$/X$, $-(K_E + B_E)$ is nef. Since f is an ε-plt blow-up of $(X \ni x, \Delta + cS)$, (E, B'_E) is an ε-klt log Fano pair. By [4, Theorem 1.1], E belongs to a bounded family. Thus there exist a positive integer M depending only on d and ε, and a very ample divisor H on E, such that

$$-K_E \cdot H^{d-2} \leq M.$$

By adjunction (cf. [12, Theorem 3.10]), we may write

$$B_E = \sum_D \frac{m_D - 1 + \gamma_D + sk_D}{m_D} D \quad \text{and} \quad B'_E = \sum_D \frac{m_D - 1 + \gamma_D + ck_D}{m_D} D,$$

where the sums are taken over all prime divisors D on E, m_D are positive integers, k_D are non-negative integers, and $\gamma_D \in \Gamma_+$. Since (E, B'_E) is ε-klt and $\Gamma \subset [0,1]$ is a finite set, γ_D belongs to a finite set of non-negative real numbers, m_D belongs to a finite set of positive integers, and k_D belongs to a finite set of non-negative integers.

Since $0 < -(K_E + B'_E) \cdot H^{d-2} \leq M$ and $D \cdot H^{d-2}$ is a positive integer for each D, $-(K_E + B_E) \cdot H^{d-2}$ is of the form $\alpha' - (s-c)\beta'$, where $\alpha' = -(K_E + B'_E) \cdot H^{d-2}$ belongs to a finite set of positive real numbers and $\beta' := (\sum_D \frac{k_D}{m_D} D) \cdot H^{d-2}$ belongs to a finite set of non-negative rational numbers.

Let $H_1, \ldots, H_{d-2}$ be general elements in $|H|$, $C := E \cap H_1 \cap H_2 \cdots \cap H_{d-2}$, and $r := \lfloor \frac{1}{\varepsilon} \rfloor !$. Since $(Y/X \ni x, B_Y + E)$ is ε-plt, by [12, Theorem 3.10], $rE|_E$ is a $\mathbb{Q}$-Cartier Weil divisor. In particular, $-E \cdot C$ belongs to the discrete set $\frac{1}{r}\mathbb{N}^+$. Since

$$(K_Y + B_Y + (1-a)E) \cdot C = 0,$$

we have

$$a = \frac{-(K_Y + B_Y + E) \cdot C}{-E \cdot C} = \frac{-(K_E + B_E) \cdot H^{d-2}}{-E \cdot C}.$$

Thus $a = \frac{r(\alpha' - (s-c)\beta')}{n}$, where $n \in \mathbb{N}^+$. We may let $\alpha := r\alpha'$ and $\beta := r\beta'$. ■

Lemma 3.5. *Let d be a positive integer, ε a positive real number, and $\Gamma \subset [0,1]$ a finite set. Then*

$$\left\{ a(E, X, B) \;\middle|\; \begin{array}{l} (X \ni x, B) \text{ has an } \varepsilon\text{-plt blow-up } f : Y \to X \\ \text{which extracts } E, \ \dim X = d, B \in \Gamma \end{array} \right\}$$

is standardized and its only possible accumulation point is 0.

Proof. It follows from Lemma 3.4 by letting $c = s = 1$ and $S = 0$. ■

Theorem 3.6. *Let d be a positive integer, ε a positive real number, and $\Gamma \subset [0,1]$ a finite set. Then*

$$\Gamma_1(d, \varepsilon, \Gamma) := \left\{ \operatorname{mld}(X \ni x, B) \mid \dim X = d, \ (X \ni x, B) \text{ has an } \varepsilon\text{-plt blow up} \right\}$$

is standardized and its only possible accumulation point is 0. *In particular,*

$$\Gamma_2(d, \Gamma) := \left\{ \operatorname{mld}(X \ni x, B) \mid \dim X = d, \ (X \ni x, B) \text{ is exceptional} \right\}$$

is standardized and its only possible accumulation point is 0.

Proof. By [12, Theorems 1.2, 1.3], the only possible accumulation point of $\Gamma_1(d, \varepsilon, \Gamma)$ and $\Gamma_2(d, \Gamma)$ is 0. By Lemma 2.14 (1), we only need to show that $\Gamma_1(d, \varepsilon, \Gamma)$ and $\Gamma_2(d, \Gamma)$ are standardized near 0.

By [12, Lemma 3.22], there exists a positive real number ε' depending only on d and Γ, such that for any exceptional pair $(X \ni x, B)$ of dimension d with $B \in \Gamma$, $(X \ni x, B)$ has an ε'-plt blow-up. In particular, $\Gamma_2(d, \Gamma) \subset \Gamma_1(d, \varepsilon', \Gamma)$.

By [12, Theorem 1.3] and Lemma 3.5,

$$\Gamma_1(d, \varepsilon, \Gamma) \cap [0, \varepsilon] \quad \text{and} \quad \Gamma_1(d, \varepsilon', \Gamma) \cap [0, \varepsilon']$$

are standardized near 0. By Lemma 2.14 (6), $\Gamma_1(d, \varepsilon, \Gamma)$ and $\Gamma_1(d, \varepsilon', \Gamma)$ are standardized near 0. By Lemma 2.14 (5), $\Gamma_2(d, \Gamma)$ is standardized near 0, and we are done. ■

Proof of Theorem 1.7. It follows from Theorems 3.1, 3.3, and 3.6 when $\Gamma = \{0\}$. ■

4. Standardization of log canonical thresholds

Lemma 4.1. *Let d be a positive integer, c a non-negative real number, and $\Gamma \subset [0,1]$ a set such that $1 \in \Gamma_+$. Suppose that $(X, B) \in \mathfrak{N}(d, \Gamma, c)$ is a pair such that (X, B) is not klt. Then $c \in N(d-1, \Gamma)$.*

Proof. Possibly by replacing (X, B) with a dlt modification, we may assume that $\lfloor B \rfloor \neq 0$ and X is $\mathbb{Q}$-factorial. We may assume that $c \neq 0$. We may write $B = L + C$ such that $L \in D(\Gamma)$ and $0 \neq C \in D(\Gamma, c)$. If $\lfloor C \rfloor \neq 0$, then

$$\frac{m - 1 + \gamma + kc}{m} = 1$$

for some $m, k \in \mathbb{N}^+$ and $\gamma \in \Gamma_+$. By Theorem 2.12 (1), $c = \frac{1-\gamma}{k} \in N(0, \Gamma) \subset N(d-1, \Gamma)$. Thus we may assume that $\lfloor C \rfloor = 0$.

If $\lfloor L \rfloor = 0$, then

$$\frac{m_1 - 1 + \gamma_1}{m_1} + \frac{m_2 - 1 + \gamma_2 + kc}{m_2} = 1$$

for some $m_1, m_2, k \in \mathbb{N}^+$ and $\gamma_1, \gamma_2 \in \Gamma_+$. Since $c \neq 0$, either $m_1 = 1$ or $m_2 = 1$. If $m_1 = 1$, then

$$c = \frac{1 - \gamma_2 - m_2\gamma_1}{k} \in N(0, \Gamma) \subset N(d-1, \Gamma),$$

and if $m_2 = 1$, then

$$c = \frac{1 - \gamma_1 - m_1\gamma_2}{m_1 k} \in N(0, \Gamma) \subset N(d-1, \Gamma).$$

Thus we may assume that $\lfloor L \rfloor \neq 0$. We let T be an irreducible component of $\lfloor L \rfloor$. We run a $(K_X + L)$-MMP $\phi : X \dashrightarrow X'$ which terminates with a Mori fiber space $X' \to Z$. Then this MMP is C-positive, hence C is not contracted by this MMP.

If T is contracted by ϕ, then there exists a step of the MMP $\psi : X'' \to X'''$ which is a divisorial contraction and contracts the strict transform of T on X''. Let B'', L'', C'', T'' be the strict transforms of B, L, C, T on X'' respectively. Since ψ is C''-positive, T'' intersects C''. Since (X, L) is dlt, (X'', L'') is dlt, hence T'' is normal. Let

$$K_{T''} + B_{T''} := (K_{X''} + B'')|_{T''},$$

then $(T'', B_{T''}) \in \mathfrak{N}(d-1, \Gamma, c)$. Thus $c \in N(d-1, \Gamma)$. Therefore, we may assume that T is not contracted by ϕ.

We let B', L', C', T' be the strict transforms of B, L, C, T on X' respectively. Note that T' is normal as (X', L') is dlt. Since ϕ is C-positive, C' dominates Z.

If $\dim Z > 0$, then we let F be a very general fiber of $X' \to Z$, and let

$$K_F + B_F := (K_{X'} + B')|_F,$$

then $(F, B_F) \in \mathfrak{N}(d - \dim Z, \Gamma, c)$. Thus $c \in N(d - \dim Z, \Gamma) \subset N(d-1, \Gamma)$. Thus we may assume that $\dim Z = 0$ and $\rho(X') = 1$.

If $d \geq 2$, then T' intersects C'. Let

$$K_{T'} + B_{T'} := (K_{X'} + B')|_{T'},$$

then $(T', B_{T'}) \in \mathfrak{N}(d-1, \Gamma, c)$. Thus $c \in N(d-1, \Gamma)$ and we are done.

If $d = 1$, then we have

$$\sum_{j=1}^{l_1} \frac{m_j - 1 + \gamma_j}{m_j} + \sum_{j=1}^{l_2} \frac{n_j - 1 + \gamma_j' + k_j c}{n_j} = 1$$

for some $l_1 \in \mathbb{N}$, $l_2, m_j, n_j, k_j \in \mathbb{N}^+$, and $\gamma_j, \gamma_j' \in \Gamma_+$. Since $c > 0$, possibly by reordering indices, either $m_j = 1$ for every j and $n_j = 1$ for every $j \geq 2$, or $m_j = 1$ for every $j \geq 2$ and $n_j = 1$ for every j. Thus either

$$c = \frac{1 - n_1(\sum_{j=1}^{l_1} \gamma_j + \sum_{j=2}^{l_2} \gamma_j') - \gamma_1'}{k_1 + n_1 \sum_{j=2}^{l_2} k_j} \in N(0, \Gamma) \subset N(d-1, \Gamma)$$

or

$$c = \frac{1 - m_1(\sum_{j=2}^{l_1} \gamma_j + \sum_{j=1}^{l_2} \gamma_j') - \gamma_1}{m_1 \sum_{j=1}^{l_2} k_j} \in N(0, \Gamma) \subset N(d-1, \Gamma)$$

and we are done. ∎

Theorem 4.2. *Let d be a non-negative integer and $\Gamma \subset [0,1]$ a set, such that $1 \in \Gamma$, $\Gamma = \Gamma_+$, 1 is the only possible accumulation point of Γ, and Γ is standardized. Then* $\operatorname{lct}(d, \Gamma)$ *is standardized.*

Proof. The proof consists of eight steps.

Step 1. In this step, we reduce our theorem to the case when $d \geq 2$ and show that we only need to prove that $\operatorname{lct}(d, \Gamma)$ is standardized near any

$$c \in \operatorname{lct}(d-1, \Gamma) \backslash \big(\operatorname{lct}(d-2, \Gamma) \cup \{1\}\big).$$

Since 1 is the only possible accumulation point of Γ and Γ is standardized, there exists a positive integer m and non-negative real numbers $b_1, \ldots, b_m$, such that

$$\Gamma \subset \left\{1 - \frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \leq i \leq m\right\}.$$

In particular, Γ satisfies the DCC. By [10, Theorem 1.1], $\operatorname{lct}(d, \Gamma)$ satisfies the ACC for any non-negative integer d.

If $d = 0$, then the theorem follows from the definition. If $d = 1$, then

$$\operatorname{lct}(d, \Gamma) = \left\{\frac{1-\gamma}{k} \;\middle|\; k \in \mathbb{N}^+,\ \gamma \in \Gamma\right\} \subset \left\{\frac{b_i}{nk} \;\middle|\; i, n, k \in \mathbb{N}^+,\ 1 \leq i \leq m\right\}$$
$$\subset \left\{\frac{b_i}{n} \;\middle|\; i, n \in \mathbb{N}^+,\ 1 \leq i \leq m\right\}.$$

Thus the only possible accumulation point of $\operatorname{lct}(d, \Gamma)$ is 0, and by Lemma 2.14 (5), $\operatorname{lct}(d, \Gamma)$ is standardized near 0. Thus $\operatorname{lct}(d, \Gamma)$ is standardized and we are done.

Therefore, we may assume that $d \geq 2$. By induction on dimension and Theorem 2.12 (4), we only need to show that $\mathrm{lct}(d,\Gamma)$ is weakly standardized, that is, for any

$$c \in \partial\,\mathrm{lct}(d,\Gamma)\backslash\partial^2\,\mathrm{lct}(d,\Gamma) = \mathrm{lct}(d-1,\Gamma)\backslash\big(\mathrm{lct}(d-2,\Gamma)\cup\{1\}\big),$$

$\mathrm{lct}(d,\Gamma)$ is standardized near c.

Step 2. For any $c \in \mathrm{lct}(d-1,\Gamma)\backslash(\mathrm{lct}(d-2,\Gamma)\cup\{1\})$ and $c_i \in \mathrm{lct}(d,\Gamma)$ such that

$$\lim_{i\to+\infty} c_i = c,$$

we construct pairs $(X_i, B_i) \in \mathfrak{R}(d,\Gamma,c_i)$ in this step.

For $c \in \mathrm{lct}(d-1,\Gamma)\backslash(\mathrm{lct}(d-2,\Gamma)\cup\{1\})$, we define

$$\varepsilon_c := \sup\big\{t \mid 0 \leq t \leq 1,\ (c, c+t)\cap \mathrm{lct}(d-1,\Gamma) = \emptyset\big\}.$$

Since $c \notin \partial\,\mathrm{lct}(d-1,\Gamma)$, $\varepsilon_c > 0$. By Theorem 2.12 (4), c is the only accumulation point of $(c, c+\varepsilon_c)\cap\mathrm{lct}(d,\Gamma)$. Since $\mathrm{lct}(d,\Gamma)$ satisfies the ACC, by Lemma 2.14 (6), we only need to show that $(c, c+\varepsilon_c)\cap\mathrm{lct}(d,\Gamma)$ is standardized near c. By Lemma 2.15, we only need to show that for any sequence $\{c_i\}_{i=1}^{+\infty} \subset (c, c+\varepsilon_c)\cap\mathrm{lct}(d,\Gamma)$ such that $\lim_{i\to+\infty} c_i = c$, a subsequence of $\{c_i\}_{i=1}^{+\infty}$ is standardized near c. Possibly by passing to a subsequence, we may assume that c_i is strictly decreasing.

In the following, we will fix $c \in \mathrm{lct}(d-1,\Gamma)\backslash(\mathrm{lct}(d-2,\Gamma)\cup\{1\})$ and a sequence

$$\{c_i\}_{i=1}^{+\infty} \subset (c, c+\varepsilon_c)\cap\mathrm{lct}(d,\Gamma)$$

such that $\lim_{i\to+\infty} c_i = c$. In particular,

$$c_i \notin \mathrm{lct}(d-1,\Gamma)$$

for each i. By Theorem 2.12 (2) (3), $c_i \in K(d-1,\Gamma)\backslash K(d-2,\Gamma)$ for each i and $c \in K(d-2,\Gamma)\backslash K(d-3,\Gamma)$. Since $d \geq 2$ and $c \notin \mathrm{lct}(d-2,\Gamma)$, $c \neq 0$. Thus there exists a sequence of pairs $(X_i, B_i = L_i + C_i)$, such that

(1) $\dim X_i = d-1$ and $\rho(X_i) = 1$,

(2) $K_{X_i} + B_i \equiv 0$ and (X_i, B_i) is $\mathbb{Q}$-factorial klt, and

(3) $L_i \in D(\Gamma)$ and $0 \neq C_i \in D(\Gamma, c_i)$.

In particular, for each i, we may write

$$L_i = \sum_j \frac{m_{i,j} - 1 + \gamma_{i,j}}{m_{i,j}} L_{i,j} \quad\text{and}\quad C_i = \sum_j \frac{n_{i,j} - 1 + \gamma'_{i,j} + k_{i,j}c_i}{n_{i,j}} C_{i,j},$$

such that $m_{i,j}, n_{i,j}, k_{i,j} \in \mathbb{N}^+$, $\gamma_{i,j}, \gamma'_{i,j} \in \Gamma$, and $L_{i,j}, C_{i,j}$ are prime divisors. We write $C_i = R_i + c_i S_i$ where

$$R_i := \sum_j \frac{n_{i,j} - 1 + \gamma'_{i,j}}{n_{i,j}} C_{i,j} \quad\text{and}\quad S_i := \sum_j \frac{k_{i,j}}{n_{i,j}} C_{i,j}.$$

We let $a_i := \mathrm{tmld}(X_i, L_i + R_i + cS_i)$. Possibly by passing to a subsequence, we may assume that a_i is increasing or decreasing, and let $a := \lim_{i\to+\infty} a_i$.

Step 3. In this step, we show that $a = 0$.

Suppose that a is a positive real number. Then there exists a positive real number ε such that $(X_i, L_i + R_i + cS_i)$ is ε-lc for each i, hence X_i belongs to a bounded family by [4, Theorem 1.1]. Thus there exist a positive integer M which does not depend on i, and very ample divisors H_i on X_i, such that $-K_{X_i} \cdot H_i^{d-2} \le M$ for each i.

Since $(X_i, L_i + R_i + cS_i)$ is ε-lc for each i,

$$\frac{m_{i,j} - 1 + \gamma_{i,j}}{m_{i,j}} \le 1 - \varepsilon \quad \text{and} \quad \frac{n_{i,j} - 1 + \gamma'_{i,j} + k_{i,j}c}{n_{i,j}} \le 1 - \varepsilon \quad \text{for any } i, j.$$

Since 1 is the only possible accumulation point of Γ and $c > 0$, $m_{i,j}, \gamma_{i,j}, n_{i,j}, \gamma'_{i,j}, k_{i,j}$ belong to a finite set. Thus $L_i \cdot H_i^{d-2} \ge 0$, $R_i \cdot H_i^{d-2} \ge 0$, and $S_i \cdot H_i^{d-2} > 0$ belong to discrete sets. Since $K_{X_i} + B_i \equiv 0$, we have

$$(K_{X_i} + L_i + R_i + c_i S_i) \cdot H_i^{d-2} = 0.$$

Thus $c_i = \frac{p_i}{q_i}$, where $p_i = -(K_{X_i} + L_i + R_i) \cdot H_i^{d-2}$ belongs to a finite set of positive real numbers, and $q_i = S_i \cdot H_i^{d-2}$ belongs to a discrete set of positive real numbers. Thus the only possible accumulation point of $\{c_i\}_{i=1}^{+\infty}$ is 0, which is not possible as $c \neq 0$.

Thus $a = 0$. Let $a'_i := \operatorname{tmld}(X_i, B_i)$. Since $a = 0$ and $0 < a'_i \le a_i$, $\lim_{i\to+\infty} a'_i = 0$. Possibly by passing to a subsequence, we may assume that a'_i is strictly decreasing.

Step 4. In this step, we find a positive integer N, a finite set $\Gamma_0 \subset (0, 1]$, a positive real number ε_0, and divisors T_i over X_i. We then construct (N, Γ_0)-decomposable $\mathbb{R}$-complements $(X_i, L_i + R_i + cS_i + G_i)$ and Mori fiber spaces $(X'_i, B'_i) \to Z_i$, and reduce our theorem to the case when $\dim Z_i = 0$ and $\rho(X'_i) = 1$.

By Theorem 2.7, there exist a positive integer N and a finite set $\Gamma_0 \subset (0, 1]$ depending only on d, Γ and c, such that for any $\mathbb{R}$-complementary pair $(X/Z \ni z, B)$ where X is of Fano type over Z, $\dim X = d - 1$, and $B \in D(\Gamma \cup \{c\})$, $(X/Z \ni z, B)$ has an (N, Γ_0)-decomposable $\mathbb{R}$-complement. We let $\varepsilon_0 := \min\{\frac{\gamma_0}{2N} \mid \gamma_0 \in \Gamma_0\} > 0$. Since $0 = a = \lim_{i\to+\infty} a_i$, possibly by passing to a subsequence, we may assume that $a_i \le \min\{\varepsilon_0, 1\}$ for each i. We let T_i be a prime divisor over X_i such that $a(T_i, X_i, B_i) = \operatorname{tmld}(X_i, B_i) = a'_i$. We let $(X_i, L_i + R_i + cS_i + G_i)$ be an (N, Γ_0)-decomposable $\mathbb{R}$-complement of $(X_i, L_i + R_i + cS_i)$. By our construction, $a(T_i, X_i, L_i + R_i + cS_i + G_i) = 0$.

We construct a pair (X'_i, B'_i) and a Mori fiber space $X'_i \to Z_i$ in the following way:

- If T_i is on X_i, we let $Z_i := \{pt\}$, $(X'_i, B'_i) = (X_i, B_i)$, $L'_i := L_i - L_i \wedge T_i$, $L'_{i,j} := L_{i,j}$, $C'_i := C_i - C_i \wedge T_i$, $C'_{i,j} := C_{i,j}$, $R'_i := R_i - R_i \wedge T_i$, $S'_i := S_i - (\operatorname{mult}_{T_i} S_i)T_i$, and $G'_i := G_i - G_i \wedge T_i$. We let $T'_i := T_i$.
- If T_i is exceptional over X_i, we let $f_i : Y_i \to X_i$ be a birational contraction which only extracts T_i, and let B_{Y_i}, L_{Y_i}, $L_{Y_i,j}$, C_{Y_i}, $C_{Y_i,j}$, R_{Y_i}, S_{Y_i}, G_{Y_i} be the strict transforms of $B_i, L_i, L_{i,j}, C_i, C_{i,j}, R_i, S_i, G_i$ on Y_i respectively. We run a $(K_{Y_i} + B_{Y_i})$-MMP, which terminates with a Mori fiber space $X'_i \to Z_i$. We let L'_i, $L'_{i,j}$, C'_i, $C'_{i,j}$, R'_i, S'_i, G'_i, T'_i be the strict transforms of $L_{Y_i}, L_{Y_i,j}, C_{Y_i}, C_{Y_i,j}, R_{Y_i}, S_{Y_i}, G_{Y_i}, T_i$ on X'_i respectively, and let $B'_i := (1 - a'_i)T'_i + L'_i + C'_i$.

By our construction, $T_i' \neq 0$, T_i' dominates Z_i, $B_i' = (1-a_i')T_i' + L_i' + C_i'$, and $C_i' = R_i' + c_i S_i'$. Moreover, since

$$K_{X_i} + L_i + R_i + cS_i + G_i \equiv 0 \quad \text{and} \quad a(T_i, X_i, L_i + R_i + cS_i + G_i) = 0,$$

$(X_i, L_i + R_i + cS_i + G_i)$ and $(X_i', T_i' + L_i' + R_i' + cS_i' + G_i')$ are crepant. Since

$$K_{X_i} + L_i + R_i + c_i S_i \equiv 0 \quad \text{and} \quad a(T_i, X_i, L_i + R_i + c_i S_i) = a_i',$$

$(X_i', (1-a_i')T_i' + L_i' + R_i' + c_i S_i')$ and (X_i, B_i) are crepant. Thus $K_{X_i'} + (1-a_i')T_i' + L_i' + R_i' + c_i S_i' \equiv 0$ and $(X_i', (1-a_i')T_i' + L_i' + R_i' + c_i S_i')$ is klt. Since $a_i' > 0$ and $\lim_{i\to+\infty} a_i' = 0$, by [10, Theorem 1.5], possibly by passing to a subsequence, we may assume that $S_i' \neq 0$.

Suppose that $\dim Z_i > 0$ for infinitely many i. Possibly by passing to a subsequence, we may assume that $\dim Z_i > 0$ for each i and $\dim Z_i = \dim Z_j$ for each i and j. Let F_i be a general fiber of $X_i' \to Z_i$, and $B_{F_i} := B_i'|_{F_i}$, $L_{F_i} := L_i'|_{F_i}$, $C_{F_i} := C_i'|_{F_i}$, $R_{F_i} := R_i'|_{F_i}$, $S_{F_i} := S_i'|_{F_i}$, and $T_{F_i} := T_i'|_{F_i}$. Then

$$(F_i, B_{F_i} = (1-a_i')T_{F_i} + L_{F_i} + R_{F_i} + c_i S_{F_i})$$

is klt and $K_{F_i} + B_{F_i} \equiv 0$. Since T_i' dominates Z_i, $T_{F_i} \neq 0$. Since $a_i' > 0$ and $\lim_{i\to+\infty} a_i' = 0$, by [10, Theorem 1.5], possibly by passing to a subsequence, we may assume that $S_{F_i} \neq 0$. Since $\dim F_i \leq \dim X_i' - 1 = d-2$, by [10, Proposition 11.7], $c \in N(d-3, \Gamma)$. By Theorem 2.12 (3), $c \in \mathrm{lct}(d-2, \Gamma)$, a contradiction. Thus possibly by passing to a subsequence, we may assume that $\dim Z_i = 0$ for each i. In particular, $\rho(X_i') = 1$ for each i.

Step 5. We reduce our theorem to the case when $G_i' = 0$ and

$$c_i' := \mathrm{lct}(X_i', T_i' + L_i' + R_i'; S_i') \geq c_i$$

in this step.

Since $(X_i', T_i' + L_i' + R_i' + cS_i' + G_i')$ is lc, $(X_i', T_i' + L_i' + R_i' + cS_i')$ is lc. We consider $c_i' := \mathrm{lct}(X_i', T_i' + L_i' + R_i'; S_i')$, then $c_i' \geq c$. Possibly by passing to a subsequence, we may assume that either $c_i' \in [c, c_i)$ for each i or $c_i' \geq c_i$ for each i. Suppose that $c_i' \in [c, c_i)$ for each i. Since $(X_i', (1-a_i')T_i' + L_i' + R_i' + c_i S_i')$ is klt, all lc centers of $(X_i', T_i' + L_i' + R_i' + c_i' S_i')$ are contained in T_i', and there exists an lc center of $(X_i', T_i' + L_i' + R_i' + c_i' S_i')$ which is contained in $T_i' \cap \operatorname{Supp} S_i'$. In particular, there exist general hyperplane sections $H_{i,1}, \dots, H_{i,l_i} \subset X_i'$ for some integer $l_i \geq 0$ and $U_i := X_i' \cap (\bigcap_{j=1}^{l_i} H_{i,j})$, such that

- $(U_i, \Delta_i := T_{U_i} + L_{U_i} + R_{U_i} + c_i' S_{U_i})$ is lc, where $T_{U_i} := T_i'|_{U_i}$, $L_{U_i} := L_i'|_{U_i}$, $R_{U_i} := R_i'|_{U_i}$, and $S_{U_i} := S_i'|_{U_i}$,
- all lc centers of (U_i, Δ_i) are contained in T_{U_i},
- there exists an lc center of (U_i, Δ_i) which is contained in $T_{U_i} \cap \operatorname{Supp} S_{U_i}$, and
- all lc centers of (U_i, Δ_i) which are contained in $T_{U_i} \cap \operatorname{Supp} S_{U_i}$ have dimension 0.

Possibly by passing to a subsequence, we may assume that $l_i = l_0$ is a constant. Let $g_i : W_i \to U_i$ be a dlt modification of (U_i, Δ_i), and S_{W_i} the strict transform of S_{U_i} on W_i. Note that $g_i^* S_{U_i} = S_{W_i} + F_i$ for some $F_i \geq 0$ such that $F_i \subset \operatorname{Exc}(g_i)$. We show that $\operatorname{Supp} S_{W_i} \cap F_i \neq \emptyset$. Suppose that $\operatorname{Supp} S_{W_i} \cap F_i = \emptyset$, then by the negativity lemma, $F_i = 0$, hence g_i is the identity morphism near $\operatorname{Supp} S_{U_i}$. Thus (U_i, Δ_i) is dlt near $\operatorname{Supp} S_{U_i}$. Since $c_i' < c_i < 1$, $\lfloor \Delta_i \rfloor = T_{U_i}$, so (U_i, Δ_i) is plt near $\operatorname{Supp} S_{U_i}$. However, this is not possible since there exists an lc center of (U_i, Δ_i) which is contained in $T_{U_i} \cap \operatorname{Supp} S_{U_i}$.

Thus we can pick a g_i-exceptional prime divisor E_i such that $E_i \cap \operatorname{Supp} S_{W_i} \neq \emptyset$ and $E_i \subset F_i$. Since E_i is an lc place of (U_i, Δ_i), $\operatorname{center}_{U_i} E_i$ is contained in T_{U_i}. Thus $V_i := \operatorname{center}_{U_i} E_i$ is contained in $T_{U_i} \cap \operatorname{Supp} S_{U_i}$, so V_i is a point.

We denote the sum of all g_i-exceptional prime divisors by E_{g_i}. We let

$$B_{W_i} := (g_i^{-1})_* \Delta_i + E_{g_i} \quad \text{and} \quad K_{E_i} + B_{E_i} := (K_{W_i} + B_{W_i})|_{E_i}.$$

Then $K_{E_i} + B_{E_i} \sim_{\mathbb{R}} 0$ as V_i is a point. Since $\dim E_i = d - 2 - l_0$ and $E_i \cap \operatorname{Supp} S_{W_i} \neq \emptyset$,

$$(E_i, B_{E_i}) \in \mathfrak{N}(d - 2 - l_0, \Gamma, c_i').$$

Since $V_i \in T_{U_i}$, (E_i, B_{E_i}) is not klt. By Lemma 4.1, $c_i' \in N(d-3-l_0, \Gamma) \subset N(d-3, \Gamma)$. By Theorem 2.12 (1) (3) (4), $c \in N(d-3, \Gamma)$, which is not possible.

Thus $c_i' \geq c_i$ for each i. Since $\rho(X_i') = 1$, we may let c_i'' be the unique real number such that $K_{X_i'} + T_i' + L_i' + R_i' + c_i'' S_i' \equiv 0$. Since $K_{X_i'} + (1 - a_i') T_i' + L_i' + R_i' + c_i S_i' \equiv 0$ and $K_{X_i'} + T_i' + L_i' + R_i' + c S_i' + G_i' \equiv 0$, $c \leq c_i'' < c_i \leq c_i'$. By Lemma 4.1, $c_i'' \in N(d-2, \Gamma)$. Since $c \notin N(d-3, \Gamma)$, by Theorem 2.12 (4), possibly by passing to a subsequence, we may assume that $c_i'' = c$ for each i. In particular, $G_i' = 0$ for each i.

Step 6. In this step, we reduce our theorem to the case when $(X_i', T_i' + L_i' + R_i' + c S_i')$ is plt for each i.

Suppose that $(X_i', T_i' + L_i' + R_i' + c S_i')$ is not plt for infinitely many i. Possibly by passing to a subsequence, we may assume that $(X_i', T_i' + L_i' + R_i' + c S_i')$ is not plt for each i. Since $(X_i', (1 - a_i') T_i' + L_i' + R_i' + c S_i')$ is klt, $(X_i', T_i' + L_i' + R_i' + c S_i')$ is not plt near T_i'. Since $\operatorname{lct}(X_i', T_i' + L_i' + R_i'; S_i') = c_i' \geq c_i > c$, $(X_i', T_i' + L_i' + R_i' + c S_i')$ is plt near the generic point of each irreducible component of $T_i' \cap S_i'$. Moreover, since $(X_i', (1 - a_i') T_i' + L_i' + R_i' + c_i S_i')$ is klt, any lc center of $(X_i', T_i' + L_i' + R_i' + c S_i')$ is contained in T_i'. Thus we may take a dlt modification

$$g_i : W_i \to X_i'$$

of $(X_i', T_i' + L_i' + R_i' + c S_i')$ which is an isomorphism near the generic point of each irreducible component of $T_i' \cap S_i'$. We denote the sum of all g_i-exceptional prime divisors by E_{g_i}. We let T_{W_i} and S_{W_i} be the strict transforms of T_i' and S_i' on W_i respectively. Since $T_i' \cap S_i' \neq \emptyset$ and g_i is an isomorphism near the generic point of each irreducible component of $T_i' \cap S_i'$, $T_{W_i} \cap S_{W_i} \neq \emptyset$. Let $B_{W_i} := (g_i^{-1})_*(T_i' + L_i' + R_i' + c S_i') + E_{g_i}$. Since $K_{X_i'} + T_i' + L_i' + R_i' + c S_i' \equiv 0$, $K_{W_i} + B_{W_i} \equiv 0$. Let

$$K_{T_{W_i}} + B_{T_{W_i}} := (K_{W_i} + B_{W_i})|_{T_{W_i}},$$

then

$$(T_{W_i}, B_{T_{W_i}}) \in \mathfrak{N}(d-2, \Gamma, c)$$

and $(T_{W_i}, B_{T_{W_i}})$ is not klt since $(X_i', T_i' + L_i' + R_i' + cS_i')$ is not plt near T_i'. By Lemma 4.1, $c \in N(d-3, \Gamma)$, which is not possible.

Thus possibly by passing to a subsequence, we may assume that

$$(X_i', T_i' + L_i' + R_i' + cS_i')$$

is plt for each i. Since $G_i' = 0$, $(X_i, L_i + R_i + cS_i + G_i)$ and $(X_i', T_i' + L_i' + R_i' + cS_i')$ are crepant. Since $(X_i, L_i + R_i + cS_i + G_i)$ is an (N, Γ_0)-decomposable $\mathbb{R}$-complement of $(X_i, L_i + R_i + cS_i)$, $a(E_i, X_i, L_i + R_i + cS_i + G_i) \geq 2\varepsilon_0$ for any prime divisor $E_i \neq T_i$ over X_i.

Step 7. We prove the case when T_i is on X_i for each i in this step.

Suppose that T_i is on X_i for each i. Then $X_i = X_i'$ is ε_0-klt, hence X_i belongs to a bounded family by [4, Theorem 1.1]. Moreover, since 1 is the only possible accumulation point of Γ, the coefficients of L_i', R_i', S_i' belong to a finite set and $1 - a_i' \in D(\Gamma \cup \{c_i\})$. In particular, there exist a positive integer M which does not depend on i, and very ample divisors H_i on X_i, such that $-K_{X_i} \cdot H_i^{d-2} \leq M$. Since

$$(K_{X_i} + T_i + L_i' + R_i' + cS_i') \cdot H_i^{d-2} = 0$$

and $c > 0$, $T_i \cdot H_i^{d-2} > 0$ and $S_i' \cdot H_i^{d-2}$ belong to a finite set. Since $\rho(X_i) = 1$, possibly by passing to a subsequence, we may assume that there exists a positive rational number $\lambda = \frac{p}{q}$, where p and q are coprime positive integers, such that $T_i \equiv \lambda S_i'$ for each i. Since

$$K_{X_i} + (1 - a_i')T_i + L_i' + R_i' + c_i S_i' \equiv 0 \equiv K_{X_i} + T_i + L_i' + R_i' + cS_i',$$

we have

$$a_i' \lambda S_i' \equiv a_i' T_i \equiv (c_i - c) S_i',$$

hence $c_i - c = a_i' \lambda$. Since $1 - a_i' \in D(\Gamma \cup \{c_i\})$,

$$0 < a_i' = \frac{1 - \gamma_i - k_i c_i}{m_i}$$

for some $\gamma_i \in \Gamma \backslash \{1\}$, $k_i \in \mathbb{N}$, and $m_i \in \mathbb{N}^+$. Since $c_i > c > 0$, possibly by passing to a subsequence, we may assume that $k_i = k$ is a constant. Since

$$\Gamma \subset \left\{1 - \frac{b_j}{n} \;\middle|\; j, n \in \mathbb{N}^+,\ 1 \leq j \leq m\right\},$$

possibly by passing to a subsequence, we have $a_i' = \frac{\frac{b_j}{n_i} - kc_i}{m_i}$ for $n_i \in \mathbb{N}^+$ and some fixed j. Thus

$$c_i = c + \frac{\frac{\lambda b_j}{n_i} - \lambda k c}{m_i + \lambda k}.$$

If $k = 0$, then $c_i = c + \frac{\lambda b_j}{n_i m_i} \in \{c + \frac{\lambda b_j}{n} \mid n \in \mathbb{N}^+\}$, hence $\{c_i\}_{i=1}^{+\infty}$ is standardized near c and we are done. If $k > 0$, then since $c_i > c$, possibly by passing to a subsequence, we may assume that $n_i = n_0$ is a constant, hence

$$c_i \in \left\{c + \frac{q\left(\frac{\lambda b_j}{n_0} - \lambda k c\right)}{n} \;\middle|\; n \in \mathbb{N}^+\right\},$$

so $\{c_i\}_{i=1}^{+\infty}$ is standardized near c and we are done.

Step 8. We conclude the proof in this step. By Step 7, possibly by passing to a subsequence, we may assume that T_i is exceptional over X_i for each i. Then $f_i : Y_i \to X_i$ is the birational contraction which only extracts T_i, and f_i is an ε_0-plt blow-up of $(X_i \ni x_i, L_i + R_i + cS_i)$, where x_i is the generic point of $\mathrm{center}_{X_i} T_i$. Moreover, the coefficients of L_i, R_i belong to a finite set, and the coefficients of S_i belong to a finite rational set. By Lemma 3.4, possibly by passing to a subsequence, there exist a positive real number α and a non-negative rational number β such that

$$a'_i = \frac{\alpha - (c_i - c)\beta}{n_i}$$

for some positive integer n_i.

Since $(X'_i, T'_i + L'_i + R'_i + cS'_i)$ is plt and $(X'_i, T'_i + L'_i + R'_i + cS'_i)$ is an (N, Γ_0)-decomposable $\mathbb{R}$-complement of itself, X'_i is ε_0-klt. Thus X'_i belongs to a bounded family by [4, Theorem 1.1], and there exist a positive integer M and very ample divisors H_i on X'_i such that $-K_{X'_i} \cdot H_i^{d-2} \leq M$. Since

$$(K_{X'_i} + T'_i + L'_i + R'_i + cS'_i) \cdot H_i^{d-2} = 0$$

and $c > 0$, $T'_i \cdot H_i^{d-2} > 0$ and $S'_i \cdot H_i^{d-2}$ belong to a finite set. Since $\rho(X'_i) = 1$, possibly by passing to a subsequence, we may assume that there exists a positive rational number λ such that $T'_i \equiv \lambda S'_i$ for each i. Since

$$K_{X'_i} + (1 - a'_i)T'_i + L'_i + R'_i + c_i S'_i \equiv 0 \equiv K_{X'_i} + T'_i + L'_i + R'_i + cS'_i,$$

we have

$$a'_i \lambda S'_i \equiv a'_i T'_i \equiv (c_i - c)S'_i,$$

hence $c_i - c = a'_i \lambda$. Thus

$$c_i - c = \frac{\alpha\lambda}{n_i + \beta\lambda}.$$

Let μ be a positive integer such that $\mu\beta\lambda \in \mathbb{N}$, then

$$c_i = c + \frac{\mu\alpha\lambda}{\mu n_i + \mu\beta\lambda} \in \left\{c + \frac{\mu\alpha\lambda}{n} \;\middle|\; n \in \mathbb{N}^+\right\}.$$

Thus $\{c_i\}_{i=1}^{+\infty}$ is standardized near c, and we are done. ∎

Proof of Theorem 1.3. It follows from Theorem 4.2. ∎

5. Standardization of threefold canonical thresholds

Proof of Theorem 1.9. We only need to show that $\mathrm{ct}(3)$ is standardized since $\mathrm{ct}(1) = \{0\}$ and $\mathrm{ct}(2) = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$ by [13, Lemma 2.17]. By [11, Theorem 1.8] and [7, Theorem 1.1], we know

$$\partial\, \mathrm{ct}(3) = \left\{\frac{1}{k} \;\middle|\; k \in \mathbb{N}^+,\ k \geq 2\right\} \cup \{0\}.$$

Thus we only need to show that $\mathrm{ct}(3)$ is standardized near $\frac{1}{k}$ for any integer $k \geq 2$. By Lemma 2.15, we only need to show that for any sequence $\{c_i\}_{i=1}^{+\infty} \subset \mathrm{ct}(3)$ such that $\lim_{i\to+\infty} c_i = \frac{1}{k}$, a subsequence of $\{c_i\}_{i=1}^{+\infty}$ is standardized near $\frac{1}{k}$. In the following, we fix k and $\{c_i\}_{i=1}^{+\infty} \subset \mathrm{ct}(3)$.

Possibly by passing to a subsequence, we may assume that $c_i \in (\frac{1}{k}, \frac{1}{k-1})$ for each i, c_i is strictly decreasing, and $c_i = \mathrm{ct}(X_i \ni x_i, 0; B_i)$, where $X_i \ni x_i$ is a threefold terminal singularity and $B_i \geq 0$ is Weil divisor on X_i. Possibly by replacing X_i with a $\mathbb{Q}$-factorialization, we may assume that X_i is $\mathbb{Q}$-factorial for each i. By [11, Theorem 4.8], 0 is the only accumulation point of canonical thresholds whose ambient variety is neither smooth, nor of cA-type or cA/n-type. Since $\lim_{i\to+\infty} c_i = \frac{1}{k} > 0$, possibly by passing to a subsequence, we may assume that $X_i \ni x_i$ is either smooth, or a cA-type singularity, or a cA/n_i-type singularity for some positive integer n_i. By [7, Propositions 2.1, 2.2], we may assume that $X_i \ni x_i$ is a cA/n_i-type singularity for some positive integer n_i. By [8, Lemma 5.10], $n_i \leq 3k$, so possibly by passing to a subsequence, we may assume that $n = n_i$ is a constant.

By [7, Claims 2.4, 2.5, and 2.6] and [8, Lemma 5.2], we may assume that

$$c_i = \mathrm{ct}(X_i, 0; B_i) = \frac{a_i}{m_i},$$

and there exist positive integers d_i, non-negative integers $l_{2,i}, l_{3,i}, r_{1,i}, r_{2,i}$, such that

- $r_{1,i} + r_{2,i} = a_i d_i n$ and $r_{1,i} \leq r_{2,i}$ [7, Proof of Lemma 2.3, line 8],
- if $a_i \nmid m_i$, then $m_i \geq \frac{r_{1,i} r_{2,i}}{d_i n^2}$ [8, Lemma 5.2],
- if $a_i \geq 6k^2$, then $d_i n \leq 4k$ [7, Claim 2.4],
- $\max\{l_{2,i}, l_{3,i}\} < k$ and either $l_{2,i} > 0$ or $l_{3,i} > 0$ [7, Claim 2.5],
- $d_i n l_{2,i} + l_{3,i} \geq k$ [7, Claim 2.6], and
- $\frac{a_i}{m_i} \leq \frac{a_i - n}{(r_{2,i} - d_i n^2) l_{2,i} + (a_i - n) l_{3,i}}$ [7, Claim 2.6].

Possibly by passing to a subsequence, we may assume that $l_2 := l_{2,i}$ and $l_3 := l_{3,i}$ are constant integers. Since $\lim_{i\to+\infty} c_i = \frac{1}{k}$, $\lim_{i\to+\infty} a_i = \lim_{i\to+\infty} m_i = +\infty$. Therefore, possibly by passing to a subsequence, we have $d_i n \leq 4k$. Possibly by passing to a subsequence, we may assume that $d := d_i$ is a constant. Since $\frac{a_i}{m_i} \in (\frac{1}{k}, \frac{1}{k-1})$, $a_i \nmid m_i$. Thus $m_i \geq \frac{r_{1,i} r_{2,i}}{d n^2}$, so

$$\frac{1}{k} < \frac{a_i}{m_i} = \frac{a_i d n^2}{r_{1,i} r_{2,i}} = \frac{n}{r_{1,i}} + \frac{n}{r_{2,i}} \leq \frac{2n}{r_{1,i}},$$

so $r_{1,i} < 2kn$. Possibly by passing to a sequence, we may assume that $r_1 = r_{1,i}$ is a constant. Therefore,

$$\frac{a_i}{m_i} \le \frac{a_i - n}{(r_2 - dn^2)l_2 + (a_i - n)l_3} = \frac{a_i - n}{(a_i dn - r_1 - dn^2)l_2 + (a_i - n)l_3} = \frac{1 - \frac{n}{a_i}}{dnl_2 + l_3 - \frac{(r_1 + dn^2)l_2 + nl_3}{a_i}}.$$

Since $\lim_{i\to+\infty} \frac{a_i}{m_i} = \frac{1}{k}$, $dnl_2 + l_3 = k$. Let $I := (r_1 + dn^2)l_2 + nl_3$, then

$$\frac{1}{k} < \frac{a_i}{m_i} \le \frac{1 - \frac{n}{a_i}}{k - \frac{I}{a_i}} = \frac{a_i - n}{ka_i - I},$$

so

$$ka_i > m_i \ge ka_i - \frac{a_i}{a_i - n}(I - kn).$$

Since $\lim_{i\to+\infty} a_i = +\infty$, possibly by passing to a subsequence, we may assume that there exists a positive integer I' such that $m_i = ka_i - I'$ for each i. Thus

$$c_i = \frac{a_i}{m_i} = \frac{a_i}{ka_i - I'} = \frac{1}{k} + \frac{I'}{k(ka_i - I')} \in \left\{ \frac{1}{k} + \frac{I'}{m} \,\middle|\, m \in \mathbb{N}^+ \right\},$$

so $\{c_i\}_{i=1}^{+\infty}$ is standardized near $\frac{1}{k}$, and we are done. ∎

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References

[1] V. Alexeev, Two two-dimensional terminations. *Duke Math. J.* **69** (1993), no. 3, 527–545 Zbl 0791.14006 MR 1208810

[2] K. Ascher, K. DeVleming, and Y. Liu, K-stability and birational models of moduli of quartic K3 surfaces. *Invent. Math.* **232** (2023), no. 2, 471–552 Zbl 07676256 MR 4574660

[3] C. Birkar, Anti-pluricanonical systems on Fano varieties. *Ann. of Math. (2)* **190** (2019), no. 2, 345–463 Zbl 1470.14078 MR 3997127

[4] C. Birkar, Singularities of linear systems and boundedness of Fano varieties. *Ann. of Math. (2)* **193** (2021), no. 2, 347–405 Zbl 1469.14085 MR 4224714

[5] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468 Zbl 1210.14019 MR 2601039

[6] G. Chen and J. Han, Boundedness of (ϵ, n)-complements for surfaces. *Adv. Math.* **383** (2021), article no. 107703 Zbl 1473.14029 MR 4236289

[7] J.-J. Chen, Accumulation points on 3-fold canonical thresholds. 2022, arXiv:2202.06230v1

[8] J.-J. Chen, On threefold canonical thresholds. *Adv. Math.* **404** (2022), article no. 108447 Zbl 1491.14027 MR 4423806

[9] A. Corti, Factoring birational maps of threefolds after Sarkisov. *J. Algebraic Geom.* **4** (1995), no. 2, 223–254 Zbl 0866.14007 MR 1311348

[10] C. D. Hacon, J. McKernan, and C. Xu, ACC for log canonical thresholds. *Ann. of Math. (2)* **180** (2014), no. 2, 523–571 Zbl 1320.14023 MR 3224718

[11] J. Han, J. Liu, and Y. Luo, ACC for minimal log discrepancies of terminal threefolds. 2022, arXiv:2202.05287v2

[12] J. Han, J. Liu, and V. V. Shokurov, ACC for minimal log discrepancies of exceptional singularities. [v1] 2019, [v2] 2020, arXiv:1903.04338v2

[13] J. Han and Y. Luo, On boundedness of divisors computing minimal log discrepancies for surfaces. *J. Inst. Math. Jussieu* **22** (2023), no. 6, 2907–2930 Zbl 07750915 MR 4653762

[14] C. Jiang, A gap theorem for minimal log discrepancies of noncanonical singularities in dimension three. *J. Algebraic Geom.* **30** (2021), no. 4, 759–800 Zbl 1509.14035 MR 4372404

[15] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Ann. of Math. (2)* **127** (1988), no. 1, 93–163 Zbl 0651.14005 MR 924674

[16] J. Kollár, Which powers of holomorphic functions are integrable? 2008, arXiv:0805.0756v1

[17] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*. Cambridge Tracts in Math. 134, Cambridge University Press, Cambridge, 1998 Zbl 0926.14003 MR 1658959

[18] S. A. Kudryavtsev, Pure log terminal blow-ups. *Math. Notes* **69** (2001), no. 6, 814–819 Zbl 1015.14007 MR 1861570

[19] J. Liu and Y. Luo, Second largest accumulation point of minimal log discrepancies of threefolds. 2022, arXiv:2207.04610v1

[20] J. Liu and L. Xiao, An optimal gap of minimal log discrepancies of threefold non-canonical singularities. *J. Pure Appl. Algebra* **225** (2021), no. 9, article no. 106674 Zbl 1485.14029 MR 4200812

[21] M. Mustaţă and Y. Nakamura, A boundedness conjecture for minimal log discrepancies on a fixed germ. In *Local and global methods in algebraic geometry*, pp. 287–306, Contemp. Math. 712, American Mathematical Society, Providence, RI, 2018 Zbl 1397.14008 MR 3832408

[22] Y. Nakamura, On minimal log discrepancies on varieties with fixed Gorenstein index. *Michigan Math. J.* **65** (2016), no. 1, 165–187 Zbl 1400.14046 MR 3466821

[23] Y. G. Prokhorov, Blow-ups of canonical singularities. In *Algebra (Moscow, 1998)*, pp. 301–317, de Gruyter, Berlin, 2000 Zbl 1003.14005 MR 1754677

[24] Y. G. Prokhorov, The rationality problem for conic bundles. *Uspekhi Mat. Nauk* **73** (2018), no. 3(441), 3–88; translation in *Russian Math. Surveys* **73** (2018), 375–456 Zbl 1400.14040 MR 3807895

[25] Y. G. Prokhorov and V. V. Shokurov, Towards the second main theorem on complements. *J. Algebraic Geom.* **18** (2009), no. 1, 151–199 Zbl 1159.14020 MR 2448282

[26] M. Reid, Young person's guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, pp. 345–414, Proc. Sympos. Pure Math. 46, American Mathematical Society, Providence, RI, 1987 Zbl 0634.14003 MR 927963

[27] V. V. Shokurov, Problems about Fano varieties. In *Birational geometry of algebraic varieties: Open problems. The XXIIIrd International Symposium*, pp. 30–32, Division of Mathematics, The Taniguchi Foundation, 1988

[28] V. V. Shokurov, 3-fold log models. *J. Math. Sci.* **81** (1996), no. 3, 2667–2699
Zbl 0873.14014 MR 1420223

[29] C. Xu, Finiteness of algebraic fundamental groups. *Compos. Math.* **150** (2014), no. 3, 409–414
Zbl 1291.14057 MR 3187625

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Jihao Liu
Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208, USA; jliu@northwestern.edu

Fanjun Meng
Department of Mathematics, Johns Hopkins University, 3400 N. Charles Street, Baltimore, MD 21218, USA; fmeng3@jhu.edu

Lingyao Xie
Department of Mathematics, The University of Utah, 155 South 1400 East, Salt Lake City, UT 84112, USA; lingyao@math.utah.edu