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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

On the fixed part of pluricanonical systems for surfaces

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Abstract

We show that $|mK_X|$ defines a birational map and has no fixed part for some bounded positive integer m for any $\frac{1}{2}$-lc surface X such that K_X is big and nef. For every positive integer $n \geq 3$, we construct a sequence of projective surfaces $X_{n,i}$, such that $K_{X_{n,i}}$ is ample, $\mathrm{mld}(X_{n,i}) > \frac{1}{n}$ for every i, $\lim_{i\to+\infty} \mathrm{mld}(X_{n,i}) = \frac{1}{n}$, and for any positive integer m, there exists i such that $|mK_{X_{n,i}}|$ has nonzero fixed part. These results answer the surface case of a question of Xu.

KEYWORDS
effective birationality, minimal model program, singularity, surface, Xu's conjecture

MSC (2020)
14E30, 14B05

1 | INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

Pluricanonical systems are central objects in the study of birational geometry. More precisely, given a normal projective variety X such that K_X is effective, we would like to study the behavior of the linear systems $|mK_X|$ for any positive integer m.

It is well known that for any sufficiently divisible $m \gg 0$, the rational map given by $|mK_X|$ is birationally equivalent to the Iitaka fibration of K_X. In 2014, Hacon–McKernan–Xu proved that for any lc projective variety X of general type and of fixed dimension, there exists a uniform positive integer m such that $|mK_X|$ defines a birational map [8, Theorem 1.3] (see also [7, 15, 16]). In other words, $|mK_X|$ defines a birational morphism $X \setminus \mathrm{Bs}(|mK_X|) \to \mathbf{P}(|mK_X|)$ for some uniform positive integer m, where $\mathrm{Bs}(|mK_X|)$ is the base locus of $|mK_X|$.

It is then natural to ask whether the behavior $|mK_X|$ can be described more accurately. Since we already have a birational morphism $X \setminus \mathrm{Bs}(|mK_X|) \to \mathbf{P}(|mK_X|)$ for some uniform positive integer m, one would like to focus on the asymptotic behavior of $\mathrm{Bs}(|mK_X|)$. As the very first step, we have the following question proposed by Prof. C. Xu to the first author in 2018:

Question 1 (Xu). Assume that X is a klt projective variety of fixed dimension such that K_X is big and nef. When will we have a uniform positive integer m, such that $|mK_X|$ defines a birational map and does not have a fixed part?

Note that it is natural to assume K_X to be nef as we can always run an MMP with scaling and reach a minimal model for varieties of general type (cf. [3, Corollary 1.4.2]).

Question 1 naturally arises as a combination of [8, Theorem 1.3] and the effective base-point-freeness theorem [9, 1.1 Theorem]. Note that when the Cartier index is bounded, $|mK_X|$ not only defines a birational map but is also base-point-free for some uniform positive integer m. The interesting cases of Question 1 appear when the Cartier index of K_X is unbounded, in which case, the uniform base-point-freeness cannot be guaranteed.

Question 1 is trivial in dimension 1 but remained widely open in dimension ≥ 2. In this paper, we study Question 1 when $\dim X = 2$. The main theorem of this paper is the following:

Theorem 1.1. *There exists a uniform positive integer m satisfying the following. Assume that X is a $\frac{1}{2}$-lc projective surface and K_X is big and nef. Then, $|mK_X|$ defines a birational map and does not have a fixed part.*

The following theorem is a complementary statement for Theorem 1.1, which shows that if the Cartier index of K_X is not bounded and X is not $\frac{1}{2}$-lc, then Theorem 1.1 is not expected to hold.

Theorem 1.2. *For any integer $n \geq 3$, there exists a sequence of projective surfaces $\{X_i\}_{i=1}^{+\infty}$, such that*

1. $\mathrm{mld}(X_i) > \frac{1}{n}$ *for each i and* $\lim_{i\to+\infty} \mathrm{mld}(X_i) = \frac{1}{n}$,
2. K_{X_i} *is ample, and*
3. *if m_i is the minimal positive integer such that $|m_iK_{X_i}|$ defines a birational map and has no fixed part, then* $\lim_{i\to+\infty} m_i = +\infty$.

Note that the assumptions on $\mathrm{mld}(X)$ in Theorem 1.1 and Theorem 1.2 are natural assumptions: We are only interested in varieties such that the Cartier index of K_X is not bounded, and if we consider a family of singularities $\{(X \ni x)\}$ such that the index of K_X is unbounded, then $\{\mathrm{mld}(X \ni x)\}$ is an infinite set (cf. [4, Proposition 7.4]) and the accumulation points of $\{\mathrm{mld}(X \ni x)\}$ belong to $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{\geq 2}\}$ (cf. [1, Corollary 3.4]). The $\frac{1}{2}$ accumulation point case is resolved by Theorem 1.1 and the remaining cases are resolved by Theorem 1.2.

It is also interesting to ask whether Question 1 has a positive answer for canonical or terminal threefolds in dimension 3, as 1 is the largest accumulation points of $\mathrm{mld}(X \ni x)$ in dimension 3 (cf. [13, Appendix, Theorem]). We will not address this question in this paper, but we will provide a related example (cf. Theorem 5.7).

2 | PRELIMINARIES

We adopt the standard notation and definitions in [11], and will freely use them.

Definition 2.1 (Pairs and singularities). A pair (X, B) consists of a normal quasi-projective variety X and an $\mathbb{R}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Moreover, if the coefficients of B are ≤ 1, then B is called a boundary of X.

Let E be a prime divisor on X and D an $\mathbb{R}$-divisor on X. We define $\mathrm{mult}_E D$ to be the multiplicity of E along D. Let $\phi : W \to X$ be any log resolution of (X, B) and let

$$K_W + B_W := \phi^*(K_X + B).$$

The *log discrepancy* of a prime divisor D on W with respect to (X, B) is $1 - \mathrm{mult}_D B_W$ and it is denoted by $a(D, X, B)$. For any positive real number ϵ, we say that (X, B) is lc (resp. klt, ϵ-lc, ϵ-klt) if $a(D, X, B) \geq 0$ (resp. $> 0, \geq \epsilon, > \epsilon$) for every log resolution $\phi : W \to X$ as above and every prime divisor D on W. We say that X is lc (resp. klt, ϵ-lc, ϵ-klt) if $(X, 0)$ is lc (resp. klt, ϵ-lc, ϵ-klt).

A germ $(X \ni x, B)$ consists of a pair (X, B) and a closed point $x \in X$. $(X \ni x, B)$ is called an lc (resp. a klt, an ϵ-lc) germ if (X, B) is lc (resp. klt, ϵ-lc) near x. $(X \ni x, B)$ is called ϵ-lc at x if $a(D, X, B) \geq \epsilon$ for any prime divisor D over $X \ni x$ (i.e., $\mathrm{center}_X D = x$).

Definition 2.2. Let $\mathcal{I}$ be a set of real numbers. We say that $\mathcal{I}$ satisfies the *descending chain condition* (DCC) if any decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_k \geq \cdots$ in $\mathcal{I}$ stabilizes. We say that $\mathcal{I}$ satisfies the *ascending chain condition* (ACC) if any increasing sequence in $\mathcal{I}$ stabilizes.

Definition 2.3 (Minimal log discrepancies). Let (X, B) be a pair and $x \in X$ a closed point. The *minimal log discrepancy* of (X, B) is defined as

$$\mathrm{mld}(X, B) := \inf\{a(E, X, B) \mid E \text{ is an exceptional prime divisor over } X\}.$$

The *minimal log discrepancy* of $(X \ni x, B)$ is defined as

$$\mathrm{mld}(X \ni x, B) := \inf\{a(E, X, B) \mid E \text{ is a prime divisor over } X \ni x\}.$$

If X is $\mathbb{Q}$-Gorenstein, we define $\mathrm{mld}(X) := \mathrm{mld}(X, 0)$. If X is $\mathbb{Q}$-Gorenstein near x, we define $\mathrm{mld}(X \ni x) := \mathrm{mld}(X \ni x, 0)$. For any positive integer d, we define

$$\mathrm{mld}(d) := \{\mathrm{mld}(X \ni x) \mid (X \ni x, 0) \text{ is lc, } \dim X = d\}.$$

Definition 2.4. Let X be a normal projective variety and D an $\mathbb{R}$-divisor on X. We define

$$|D| := \{D' \mid 0 \le D' \sim \lfloor D \rfloor\}.$$

For any $\mathbb{R}$-divisor D such that $|D| \neq \emptyset$, the *base locus* of D is

$$\mathrm{Bs}(D) := \cap_{D' \sim D} \operatorname{Supp} D',$$

the *fixed part* of D is the unique $\mathbb{R}$-divisor $F \geq 0$, such that

(1) for any $D' \in |D|$, $D' \geq F$, and
(2) $\mathrm{Bs}(|D - F|)$ does not contain any divisor,

and the *movable part* of D is $D - F$. We also say that F is the fixed part of $|D|$.

We denote by $\rho(X)$ the Picard number of X.

Definition 2.5. A *surface* is a variety of dimension 2. A *rational surface* is a projective surface that is birational to $\mathbb{P}^2$. For ever nonnegative integer k, the *Hirzebruch surface* $\mathbb{F}_k$ is $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$.

Definition 2.6. Let n be a nonnegative integer, and $C = \cup_{i=1}^n C_i$ a collection of proper curves on a smooth surface U. The *determinant* of C is defined as $\det(C) := \det(\{-(C_i \cdot C_j)\}_{1 \le i,j \le n})$ if $C \neq \emptyset$, and we define $\det(\emptyset) = 1$. We define the dual graph $\mathcal{DG}(C)$ of C as follows.

1. The vertices $v_i = v_i(C_i)$ of $\mathcal{DG}(C)$ correspond to the curves C_i.
2. For each i, v_i is labeled by the integer $e_i := -(C_i^2)$. e_i is called the *weight* of v_i.
3. For $i \neq j$,the vertices v_i and v_j are connected by $C_i \cdot C_j$ edges.

The *determinant* of $\mathcal{DG}(C)$ is defined as $\det(C)$. For any birational morphism $f : Y \to X$ between normal surfaces, let $E = \cup_{i=1}^n E_i$ be the reduced exceptional divisor for some nonnegative integer n. We define $\mathcal{DG}(f) := \mathcal{DG}(E)$. If f is the minimal resolution of X (resp. the minimal resolution of $(X \ni x, 0)$ for some closed point $x \in X$), we define $\mathcal{DG}(X) := \mathcal{DG}(f)$ (resp. $\mathcal{DG}(X \ni x) := \mathcal{DG}(f)$).

Theorem 2.7 (cf. [1, Theorem 3.2, Corollary 3.4], [14]). $\mathrm{mld}(2)$ *satisfies the ACC, and the set of accumulation points of* $\mathrm{mld}(2)$ *is* $\{\frac{1}{n} \mid n \geq 2\} \cup \{0\}$.

Proposition 2.8 (cf. [4, Proposition A.5]). *Let* $\mathcal{I}_0 \subset [0, 1]$ *be a finite set. Then, there exists a positive integer* I *depending only on* $\mathcal{I}_0$ *satisfying the following. Assume that* $(X \ni x, 0)$ *is an lc surface germ such that* $\mathrm{mld}(X \ni x) \in \mathcal{I}_0$. *Then,* IK_X *is Cartier near* x.

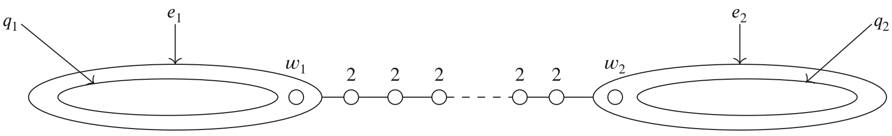


FIGURE 1 .

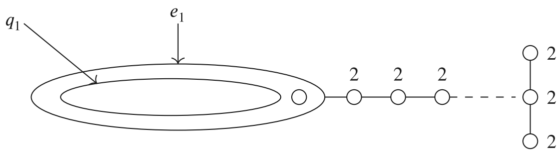


FIGURE 2 .

Lemma 2.9. *Let ϵ be a positive real number and $(X \ni x, 0)$ an ϵ-lc (resp. ϵ-klt) surface germ. Then, for any vertex v of $\mathcal{DG}(X \ni x)$, the weight of v is $\leq \frac{2}{\epsilon}$ (resp. $< \frac{2}{\epsilon}$).*

Proof. [1, Corollary 2.19] proves the ϵ-lc case and the ϵ-klt case immediately follow. □

Lemma 2.10 (cf. [1, Lemma 3.3], [4, Lemma A.1]). *Let ϵ be a positive real number. Then, there exists a finite set $\mathcal{G} = \mathcal{G}(\epsilon)$ of dual graphs and a finite set $\mathcal{I}_0 = \mathcal{I}_0(\epsilon)$ of positive integers, such that for any ϵ-lc germ $(X \ni x, 0)$, one of the following holds:*

1. *$\mathcal{DG}(X \ni x, 0) \in \mathcal{G}$.*
2. *$\mathcal{DG}(X \ni x, 0)$ is of the type as in Figure 1. Here, $e_1 = e_1(X \ni x), q_1 = q_1(X \ni x)$ and $e_2 = e_2(X \ni x), q_2 = q_2(X \ni x)$ are the determinants of the subdual graphs, such that $e_1, e_2, q_1, q_2 \in \mathcal{I}_0$, and*

$$\min\left\{\frac{1}{e_1 - q_1}, \frac{1}{e_2 - q_2}\right\} \geq \epsilon.$$

   Moreover, we may assume that
   (a) either $e_1 = w_1 = 2$ and $q_1 = 1$, or $w_1 > 2$; and
   (b) either $e_2 = w_2 = 2$ and $q_2 = 1$, or $w_2 > 2$.
3. *$\mathcal{DG}(X \ni x, 0)$ is of the type as in Figure 2. Here, $e_1 = e_1(X \ni x)$ and $q_1 = q_1(X \ni x)$ are the determinants of the subdual graphs, such that $e_1, q_1 \in \mathcal{I}_0$, and*

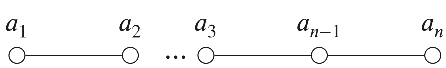
$$\mathrm{mld}(X \ni x) = \frac{1}{e_1 - q_1} \geq \epsilon.$$

We remark that each oval in Figures 1 and 2 corresponds to a subdual graph, which is a chain, as shown in [1, Lemma 3.3, 2] and [4, Appendix, Notation].

Proof. The statement on the structure of the dual graphs are explained both in [1, Lemma 3.3] and in [4, Lemma A.1]. By taking the coefficient set $\Gamma = \{0\}$, the inequality $\min\{\frac{1}{e_1-q_1}, \frac{1}{e_2-q_2}\} \geq \epsilon$ in (2) follows from the moreover part of [4, Lemma A.1(2)], and the inequality $\frac{1}{e_1-q_1} \geq \epsilon$ follows from the moreover part of [4, Lemma A.1(3)].

For the moreover part of (2), note that if $w_1 \leq 2$, then we may add the vertex corresponding to w_1 to the 2-chains and repeat this process unless this vertex is the tail of the chain. This implies (2.a), and (2.b) is similar to (2.a). □

Lemma 2.11 [10, 3.1.11]. *Let $(X \ni x, 0)$ be a klt surface germ such that $\mathcal{DG}(X \ni x, 0)$ is a chain. Then, $X \ni x$ is a cyclic quotient singularity. Moreover, if the dual graph of $X \ni x$ is*



then $X \ni x$ is a cyclic quotient singularity of form $\frac{1}{r}(1, a)$, such that $\frac{r}{a} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\dots a_n}}}$ and $\gcd(r, a) = 1$.

Lemma 2.12 (cf. [12, Lemma 2.11], [2, Theorem 1]). *Let $X \ni x$ be a cyclic quotient singularity of form $\frac{1}{r}(1, a)$ such that* $\gcd(r, a) = 1$. *Then,*

$$\operatorname{mld}(X \ni x) = \min\left\{\frac{k}{r} + \left\{\frac{ka}{r}\right\} \mid 1 \le k \le r-1, k \in \mathbb{N}^+\right\}.$$

Lemma 2.13. *Let $(X \ni x, 0)$ and $(Y \ni y, 0)$ be two klt surface germs such that $\mathcal{DG}(X \ni x, 0)$ is a subgraph of $\mathcal{DG}(Y \ni y, 0)$. Then,* $\operatorname{mld}(X \ni x) \ge \operatorname{mld}(Y \ni y, 0)$.

Proof. Let $f : W \to Y$ be a partial resolution, which extracts all divisors corresponding to vertices contained in $\mathcal{DG}(Y \ni y, 0) \backslash \mathcal{DG}(X \ni x, 0)$. Then, $(X \ni x) \cong (W \ni w)$ for some $w \in W$. Since $f^*K_Y = K_W + B_W$ for some $B_W \ge 0$, we have

$$\operatorname{mld}(Y \ni y, 0) \le \operatorname{mld}(W \ni w, B_W) \le \operatorname{mld}(W \ni w, 0) = \operatorname{mld}(X \ni x, 0). \qquad \square$$

Lemma 2.14. *Let $(X \ni x, 0)$ be a $\frac{2}{5}$-klt surface singularity. Then, either $(X \ni x) \cong \frac{1}{7}(1, 2)$, or $(X \ni x) \cong \frac{1}{4}(1, 1)$, or the weight of any vertex of $\mathcal{DG}(X \ni x)$ is ≤ 3.*

Proof. By Lemma 2.9, the weight of any vertex of $\mathcal{DG}(X \ni x)$ is ≤ 4. By [11, Theorem 4.7], $\mathcal{DG}(X \ni x, 0)$ is connected and contains no cycle. We may assume that $\mathcal{DG}(X \ni x)$ contains a vertex of weight 4. We have the following cases.

Case 1. $\mathcal{DG}(X \ni x, 0)$ only contains one point. Then, $(X \ni x) \cong \frac{1}{4}(1, 1)$ and we are done.

Case 2. $\mathcal{DG}(X \ni x, 0)$ contains the subgraph $\mathcal{G}_n$:

for some $n \ge 3$. By Lemma 2.11, the singularity corresponding to the dual graph $\mathcal{G}_n$ is a cyclic quotient singularity of type $\frac{1}{4n-1}(1, 4)$. By Lemma 2.12, when $n \ge 4$,

$$\operatorname{mld}\left(\frac{1}{4n-1}(1, 4)\right) \le \frac{5}{4n-1} \le \frac{1}{3} < \frac{2}{5},$$

and when $n = 3$,

$$\operatorname{mld}\left(\frac{1}{4n-1}(1, 4)\right) = \operatorname{mld}\left(\frac{1}{11}(1, 4)\right) = \frac{4}{11} < \frac{2}{5}.$$

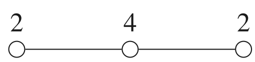
We get a contradiction to Lemma 2.13.

Case 3. $\mathcal{DG}(X \ni x, 0)$ contains the subgraph $\mathcal{G}_2$:

but does not contain the subgraph $\mathcal{G}_n$ as in Case 1.2 for any $n \ge 3$. We have the following cases.

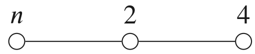
Case 3.1. $\mathcal{DG}(X \ni x, 0) = \mathcal{G}_2$. By Lemma 2.11, $(X \ni x)$ is a cyclic quotient singularity of type $\frac{1}{7}(1, 2)$ and we are done.

Case 3.2. $\mathcal{DG}(X \ni x, 0)$ contains a subgraph $\mathcal{H}$:



By Lemma 2.11, the singularity corresponds to $\mathcal{H}$, which is a cyclic quotient singularity of type $\frac{1}{12}(1, 7)$. Since $\operatorname{mld}\left(\frac{1}{12}(1, 7)\right) = \frac{1}{3} < \frac{2}{5}$, we get a contradiction to Lemma 2.13.

Case 3.3. $\mathcal{DG}(X \ni x, 0)$ contains a subgraph $\mathcal{L}_n$:



for some integer $n \geq 2$. By Lemma 2.11, singularity corresponds to $\mathcal{L}_n$, which is a cyclic quotient singularity of type $\frac{1}{7n-4}(1,7)$. By Lemma 2.12, when $n \geq 4$,

$$\mathrm{mld}\left(\frac{1}{7n-4}(1,7)\right) \leq \frac{8}{7n-4} \leq \frac{1}{3} < \frac{2}{5}.$$

When $n = 3$,

$$\mathrm{mld}\left(\frac{1}{7n-4}(1,7)\right) = \mathrm{mld}\left(\frac{1}{17}(1,7)\right) = \mathrm{mld}\left(\frac{1}{17}(5,1)\right) = \frac{6}{17} < \frac{2}{5}.$$

When $n = 2$,

$$\mathrm{mld}\left(\frac{1}{7n-4}(1,7)\right) = \mathrm{mld}\left(\frac{1}{10}(1,7)\right) = \mathrm{mld}\left(\frac{1}{10}(3,1)\right) = \frac{2}{5}.$$

We get a contradiction to Lemma 2.13. □

Lemma 2.15. *Let $(X \ni x)$ be a $\frac{2}{5}$-klt surface germ and $f : Y \to X$ the minimal resolution of $X \ni x$. Suppose that*

$$K_Y + \sum_{i=1}^{n} a_i E_i = f^* K_X,$$

where $E_1, \dots, E_n$ are the prime exceptional divisors of f. Then, $K_Y \cdot \sum_{i=1}^n a_i E_i \leq n$.

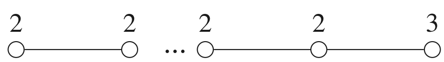
Proof. By Lemma 2.14, there are three cases.

Case 1. $E_i^2 \geq -3$ for each i. Since $(K_Y + E_i) \cdot E_i = -2$ for each i, $K_Y \cdot E_i \leq 1$ for each i. Since $a_i < 1$ for each i, the lemma follows.

Case 2. $n = 1$ and $E_1^2 = -4$. Then, $K_Y \cdot E_1 = 2$ and $(K_Y + a_1 E_1) \cdot E_1 = 0$, hence $a_1 = \frac{1}{2}$. We have $K_Y \cdot \sum_{i=1}^n a_i E_i = \frac{1}{2} K_Y \cdot E_1 = 1 = n$.

Case 3. $n = 2$, and possibly reordering indices, $E_1^2 = -2$ and $E_2^2 = -4$. Then, $K_Y \cdot E_1 = 0$, $K_Y \cdot E_2 = 2$, $(K_Y + a_1 E_1 + a_2 E_2) \cdot E_1 = 0$, and $(K_Y + a_1 E_1 + a_2 E_2) \cdot E_2 = 0$. Thus, $a_1 = \frac{2}{7}$ and $a_2 = \frac{4}{7}$, hence $K_Y \cdot \sum_{i=1}^n a_i E_i = \frac{8}{7} < 2 = n$. □

Lemma 2.16. *Let $X \ni x$ be a cyclic quotient singularity of type $\frac{1}{2k+1}(1,k)$ for some positive integer k. Then, $\mathcal{DG}(X \ni x)$ is the following graph, where there are $k-1$ "2" in the graph.*



Moreover, let $E_1, \dots, E_k$ be the divisors corresponding to the vertices of $\mathcal{DG}(X \ni x)$, such that $E_i^2 = -2$, when $1 \leq i \leq k-1$, $E_k^2 = -3$, and $E_i \cdot E_j \neq 0$ if and only if $|i-j| \leq 1$. Then, $a(E_i, X, 0) = \frac{2k+1-i}{2k+1}$ for each i.

Proof. It is clear that the cyclic quotient singularity is uniquely determined by its dual graph. Since $\frac{2k+1}{k} = 3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{\dots 2}}}$ where there are $k-1$ "2" in the fraction, the first part of the lemma follows from Lemma 2.11. For the remaining part of the lemma, let $a_i := 1 - a(E_i, X, 0)$ for each i. Since $K_Y \cdot E_i = -2 - E_i^2$ for each i and $(K_Y + \sum_{i=1}^k a_i E_i) \cdot E_i = 0$ for each i, when $k = 1$, $a(E_1, x, 0) = \frac{2}{3}$ and we are done, and when $k \geq 2$, we have

(1) $2a_1 = a_2$,
(2) $2a_i = a_{i-1} + a_{i+1}$ for any $2 \leq i \leq k-1$, and
(3) $3a_k = a_{k-1} + 1$.

Thus, $a_i = ia_1$ for each i, and we have

$$3ka_1 = 3a_k = a_{k-1} + 1 = (k-1)a_1 + 1,$$

hence $a_1 = \frac{1}{2k+1}$ and $a_i = \frac{i}{2k+1}$ for each i. The lemma follows. □

3 | GLOBAL GEOMETRY OF SMOOTH SURFACES

3.1 | Some elementary lemmas

Lemma 3.1. *Let X be a smooth projective surface, D a pseudo-effective $\mathbb{R}$-divisor on X, and C an irreducible curve on X. If $D \cdot C < 0$, then $C^2 < 0$.*

Proof. Let $D = P + N$ be the Zariski decomposition of D such that P is the positive part and N is the negative part. Since $D \cdot C < 0$ and P is nef, $N \cdot C < 0$. Since $N \geq 0$, $C \subset \operatorname{Supp} N$ and $C^2 < 0$. □

Lemma 3.2. *Let X be a smooth projective surface such that K_X is pseudo-effective. Let C be an irreducible curve on X such that $K_X \cdot C < 0$. Then, $C^2 = K_X \cdot C = -1$. In particular, C is a smooth rational curve.*

Proof. By Lemma 3.1, $C^2 < 0$. Since X is smooth, $K_X \cdot C \leq -1$ and $C^2 \leq -1$. Thus, $(K_X + C) \cdot C \leq -2$, which implies that $(K_X + C) \cdot C = -2$, $C^2 = K_X \cdot C = -1$, and C is a smooth rational curve. □

Lemma 3.3. *Let X be a smooth projective surface such that K_X is pseudo-effective, and C a smooth rational curve on X. Then, $C^2 \leq -1$.*

Proof. If not, then $C^2 \geq 0$. Since $(K_X + C) \cdot C = -2$, $K_X \cdot C \leq -2 < 0$. Since K_X is pseudo-effective, $C^2 < 0$, a contradiction. □

Lemma 3.4. *Let X be a smooth projective surface, C an irreducible curve on X, $f : Y \to X$ a blow-up of a closed point, E the exceptional divisor of f, and C_Y the strict transform of C on Y. If $C_Y \cdot E \leq 1$ and C_Y is a smooth rational curve, then C is a smooth rational curve.*

Proof. Since X is smooth, Y is smooth. Thus, $C_Y \cdot E \in \{0, 1\}$. If $C_Y \cdot E = 0$, then f is an isomorphism near a neighborhood of C_Y and hence C is a smooth rational curve. If $C_Y \cdot E = 1$, then $K_X \cdot C = K_Y \cdot C_Y - 1$ and $C^2 = C_Y^2 + 1$, and hence $(K_X + C) \cdot C = (K_Y + C_Y) \cdot C_Y = -2$. Thus, C is a smooth rational curve. □

Lemma 3.5. *Let X be a smooth projective surface such that K_X is pseudo-effective, and E_1, E_2 two different smooth rational curves on X such that $E_1^2 = E_2^2 = -1$. Then, $E_1 \cdot E_2 = 0$.*

Proof. Assume that $E_1 \cdot E_2 \neq 0$, then $E_1 \cdot E_2 = n \geq 1$ for some positive integer n. Let $f : X \to Y$ be the contraction of E_1 and $E_{2,Y} := f_* E_2$. Then, $E_{2,Y}^2 = -1 + n^2 \geq 0$ and $K_Y \cdot E_{2,Y} = -1 - n < 0$. Since K_X is pseudo-effective, K_Y is pseudo-effective, which contradicts Lemma 3.1. □

Lemma 3.6. *Let X be a smooth projective surface such that K_X is pseudo-effective, and E_1, E_2, E_3 three different smooth rational curves on X. If $E_1^2 = E_3^2 = -2$ and $E_2^2 = -1$, then either $E_1 \cdot E_2 = 0$ or $E_2 \cdot E_3 = 0$.*

Proof. Assume that $E_1 \cdot E_2 = n_1 > 0$ and $E_2 \cdot E_3 = n_3 > 0$ for some positive integers n_1 and n_3. Let $f : X \to Y$ be the contraction of E_2. Then, Y is smooth and K_Y is pseudo-effective. Let $E_{1,Y} := f_* E_1$, and $E_{3,Y} := f_* E_3$. Then, $E_{1,Y}^2 = -2 +$

n_1^2, $E_{3,Y}^2 = -2 + n_3^2$, $K_Y \cdot E_{1,Y} = -n_1$, and $K_Y \cdot E_{3,Y} = -n_3$. Thus, by Lemma 3.1, $n_1 = n_3 = 1$, which implies that $E_{1,Y}^2 = E_{3,Y}^2 = -1$ and $E_{1,Y} \cdot E_{3,Y} > 0$. By Lemma 3.4, $E_{1,Y}$ and $E_{3,Y}$ are smooth rational curves, which contradicts Lemma 3.5. □

Lemma 3.7. *Let X be a smooth rational surface. Then, $K_X^2 = 10 - \rho(X)$.*

Proof. We may run a K_X-MMP $f : X := X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$ such that either $X_n = \mathbb{F}_k$ for some nonnegative integer k or $X_n = \mathbb{P}^2$. For any $i \in \{0, 1, 2, \dots, n-1\}$, we have $K_{X_i}^2 = K_{X_{i+1}}^2 - 1$ and $\rho(X_i) = \rho(X_{i+1}) + 1$. Thus, $K_X^2 + \rho(X) = K_{X_n}^2 + \rho(X_n)$. If $X_n = \mathbb{F}_k$ for some nonnegative integer k, then $K_{X_n}^2 + \rho(X_n) = 8 + 2 = 10$. If $X_n = \mathbb{P}^2$, then $K_{X_n}^2 + \rho(X_n) = 9 + 1 = 10$. Thus, $K_X^2 = 10 - \rho(X)$. □

3.2 | Zariski decomposition

Lemma 3.8. *Let X be a smooth projective surface, and $D, \tilde{D}$ two $\mathbb{Q}$-divisors on X, such that $D \geq \tilde{D}$ and $\tilde{D}$ is nef. Let $D = P + N$ be the Zariski decomposition of D, where P is the positive part and N is the negative part. Then, $P \geq \tilde{D}$.*

Proof. Assume that $N = \sum_{i=1}^n a_i C_i$ and $D - \tilde{D} = \sum_{i=1}^n b_i C_i + D_0$, where n is a nonnegative integer, C_i are distinct irreducible curves, $D_0 \geq 0$, and for each i, $a_i > 0$, $b_i \geq 0$, and $C_i \not\subset \operatorname{Supp} D_0$. Then, for every $j \in \{1, 2, \dots, n\}$,

$$\sum_{i=1}^n a_i (C_i \cdot C_j) = N \cdot C_j = D \cdot C_j = \tilde{D} \cdot C_j + (D - \tilde{D}) \cdot C_j$$

$$\geq (D - \tilde{D}) \cdot C_j = \sum_{i=1}^n b_i (C_i \cdot C_j) + D_0 \cdot C_j \geq \sum_{i=1}^n b_i (C_i \cdot C_j),$$

which implies that $\sum_{i=1}^n (a_i - b_i)(C_i \cdot C_j) \geq 0$ for every j. Since the intersection matrix $\{(C_i \cdot C_j)\}_{1 \leq i,j \leq n}$ is negative definite, $a_i \leq b_i$ for each i. Thus, $D - \tilde{D} \geq N$, hence $P \geq \tilde{D}$. □

Lemma 3.9. *Let X be a smooth projective surface, D a big Weil divisor on X, $\tilde{D}$ an nef Weil divisor on X, and E a Weil divisor on X, such that*

(1) $D = P + N$ is the Zariski decomposition of D, where P is the positive part and $N \geq 0$ is the negative part,
(2) $E = D - \tilde{D} \geq 0$, and
(3) $|D|$ defines a birational map.

Then, there exist a big Weil divisor D_1 on X and a Weil divisor E_1 on X, such that

1. *$D_1 = \lfloor P \rfloor$,*
2. *$E \geq E_1 = D_1 - \tilde{D} \geq 0$,*
3. *$|D_1|$ defines a birational map, and*
4. *either $N = 0$ and $D = P$, or there exists at least one irreducible component F of $\operatorname{Supp} E$ such that $\operatorname{mult}_F(E - E_1) \geq 1$.*

Proof. We let $D_1 := \lfloor P \rfloor$, then (1) holds. Let $E_1 := D_1 - \tilde{D}$. Since $\tilde{D}$ is nef and $D \geq \tilde{D}$, by Lemma 3.8, $P \geq \tilde{D}$. Thus, $P - \tilde{D} \geq 0$, and hence

$$E_1 = D_1 - \tilde{D} = \lfloor P \rfloor - \tilde{D} = \lfloor P - \tilde{D} \rfloor \geq 0.$$

Since

$$E - E_1 = D - D_1 = P + N - \lfloor P \rfloor = \{P\} + N \geq 0,$$

we deduce (2). Since $|D_1| = |\lfloor P \rfloor| = |P| \cong |D|$,$|D_1|$ defines a birational map, hence (3). Finally, if $E - E_1 \neq 0$, then we are done; otherwise, $E - E_1 = 0$, hence $\{P\} + N = 0$. Thus, $N = 0$, which implies that $D = P$, hence (4). □

Proposition 3.10. *Let X be a smooth projective surface, D a big Weil divisor on X, and $\tilde{D}$ an nef Weil divisor on X, such that*

(1) $D = P + N$ is the Zariski decomposition of D, where P is the positive part and $N \geq 0$ is the negative part,
(2) $D - \tilde{D} \geq 0$, and
(3) $|D|$ defines a birational map.

Then, there exists a Weil divisor D' on X, such that

1. $D \geq D' \geq \tilde{D}$,
2. D' defines a birational map, and
3. D' is big and nef.

Proof. Let $D_0 := D, P_0 := P, N_0 := N$, and $E_0 := D - \tilde{D}$, and let r_0 be the sum of all the coefficients of E_0. Then, r_0 is a nonnegative integer.

For any nonnegative integer k, assume that there exist big Weil divisors $D_1, \dots, D_k$ on X, Weil divisors $E_1, \dots, E_k$ on X, and nonnegative integers $r_1, \dots, r_k$, such that for every $i \in \{0, 1, \dots, k\}$,

(1) $D_i = P_i + N_i$ is the Zariski decomposition of D_i, where P_i is the positive part and $N_i \geq 0$ is the negative part;
(2) $E_0 \geq E_i = D_i - \tilde{D} \geq 0$;
(3) $|D_i|$ defines a birational map;
(4) r_k is the sum of all the coefficients of the components of E_i such that $0 \leq r_k \leq r_0 - k$; and
(5) if $i \geq 1$, then $D_i = \lfloor P_{i-1} \rfloor$.

It is clear that these assumptions hold when $k = 0$. By Lemma 3.9, there are two cases:

Case 1. $N_k = 0$ and $D_k = P_k$. In this case, by our assumptions,

(1) $D_k - \tilde{D} \geq 0$, hence $D_k \geq \tilde{D}$;
(2) $E_0 \geq D_k - \tilde{D}$, hence $D \geq D_k$;
(3) D_k is big and defines a birational map; and
(4) $D_k = P_k$ is nef.

Thus, we may let $D' := D_k$.

Case 2. There exists a big Weil divisor D_{k+1} on X, a Weil divisor E_{k+1} on X, and a nonnegative integer r_{k+1}, such that

(1) $D_{k+1} = \lfloor P_k \rfloor$,
(2) $E_0 \geq E_{k+1} = D_{k+1} - \tilde{D} \geq 0$,
(3) $|D_{k+1}|$ defines a birational map, and
(4) $0 \leq r_{k+1} \leq r_k - 1$.

In this case, we may replace k with $k + 1$ and apply induction on k. Since $0 \leq r_k \leq r_0 - k$, we have $k \leq r_0$. Thus, this process must terminate and we are done. □

3.3 | Effective birationality and existence of special nef $\mathbb{Q}$-divisors

Lemma 3.11. *Let X be a klt projective surface such that K_X is big and nef, $f : Y \to X$ the minimal resolution of X, and $E_1, \dots, E_n$ the prime f-exceptional divisors. Assume that $K_Y + \sum_{i=1}^n a_i E_i = f^* K_X$. Then, for any positive integer m, if there exist integers $r_1, \dots, r_n$, such that*

1. $0 \leq r_i \leq \lfloor m a_i \rfloor$, and
2. $K_Y + \sum_{i=1}^n \frac{r_i}{m} E_i$ is big and nef,

then $|192 m K_X|$ does not have a fixed part.

Proof. Let $\Delta := \sum_{i=1}^{n} \frac{r_i}{m} E_i$ and $L := m(K_Y + \Delta)$. Then, L is big and nef and Cartier. In particular, $2L - (K_Y + \Delta) \sim_{\mathbb{Q}} \left(2 - \frac{1}{m}\right) L$ is big and nef. By [6, Theorem 1.1, Remark 1.2] (see also [9, 1.1 Theorem]), $192L$ is base-point-free, which implies that the fixed part of $192m(K_Y + \sum_{i=1}^{n} a_i E_i)$ is supported on $\cup_{i=1}^{n} E_i$. Thus, $|192mK_X|$ does not have a fixed part. □

Theorem 3.12 (cf. [8, Theorem 1.3]). *There exists a uniform positive integer m_1, such that for any lc surface X such that K_X is big, $|m_1 K_X|$ defines a birational map.*

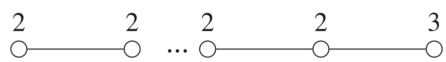
4 | $\frac{1}{3}$-KLT SURFACES

In this section, we will prove Theorem 1.1. The structure of this section is as follows. In Section 4.1, we give a detailed classification of $\frac{1}{3}$-klt surface singularities. In Section 4.2, we consider the intersection numbers of the form $K_X \cdot C$ where X is $\frac{1}{3}$-klt, K_X is big and nef, and C is a curve satisfying special properties. For some lemmas and propositions, we need to restrict ourselves to $\frac{2}{5}$-klt surfaces. With a good description of these intersection numbers and with the help of the results on Zariski decomposition in Section 3, in Section 4.3, we will construct special nef $\mathbb{Q}$-divisors on the minimal resolution of $\frac{2}{5}$-klt surfaces. We will prove our main theorem in Section 4.4.

4.1 | Classification of $(\frac{1}{3} + \epsilon)$-lc singularities

Lemma 4.1. *Let ϵ be a positive real number. Then, there exists a positive integer $n_0 = n_0(\epsilon)$ depending only on ϵ satisfying the following. Assume that $(X \ni x, 0)$ is a $(\frac{1}{3} + \epsilon)$-lc surface germ. Then,*

1. *either $n_0 K_X$ is Cartier near x, or*
2. *$X \ni x$ is a cyclic quotient singularity of type $\frac{1}{2k+1}(1, k)$ for some positive integer $k \geq 10$. In particular, $\mathcal{DG}(X \ni x)$ is the following graph, where there are $k-1$ "2" in the graph.*



Proof. Assume that the lemma does not hold. Then, there exists a sequence of $(\frac{1}{3} + \epsilon)$-lc surface germs $(X_i \ni x_i, 0)$, and a strictly increasing sequence of positive integer n_i, such that

(1) nK_{X_i} is not Cartier near x_i for any positive integer $n \leq n_i$, and
(2) $X_i \ni x_i$ is not a cyclic quotient singularity of type $\frac{1}{2k+1}(1, k)$ for any i and any positive integer k.

We consider the set $\mathcal{A} := \{\mathrm{mld}(X_i \ni x_i)\}_{i=1}^{+\infty}$. Since $\mathrm{mld}(X_i \ni x_i) \geq \frac{1}{3} + \epsilon$, by Theorem 2.7, the only possible accumulation point of $\mathcal{A}$ is $\frac{1}{2}$. If $\mathcal{A}$ is a finite set, it contradicts Proposition 2.8. Thus, possibly passing to a subsequence and replacing $\mathcal{A}$, we may assume that $\mathrm{mld}(X_i \ni x_i)$ is strictly decreasing and $\lim_{i \to +\infty} \mathrm{mld}(X_i \ni x_i) = \frac{1}{2}$.

We let $\mathcal{G} := \mathcal{G}(\frac{1}{3} + \epsilon)$ be the finite set of dual graphs and $\mathcal{I}_0 := \mathcal{I}_0(\frac{1}{3} + \epsilon)$ be the finite set of real numbers as in Lemma 2.10. Then, for any $(X \ni x)$ such that $\mathcal{DG}(X \ni x, 0) \in \mathcal{G}$, $\mathrm{mld}(X \ni x)$ belongs to a finite set. Thus, possibly passing to a subsequence, by Lemma 2.10, we may assume that one of the following holds:

(1) $(X_i \ni x_i)$ satisfies (2) of Lemma 2.10 for each i, and $e_1 = e_1(X_i \ni x_i)$, $q_1 = q_1(X_i \ni x_i)$, $e_2 = e_2(X_i \ni x_i)$, $q_2 = q_2(X_i \ni x_i) \in \mathcal{I}_0$ for each i. Since $\mathcal{I}_0$ is a finite set, possibly passing to a subsequence, we may assume that e_1, e_2, q_1, q_2 are constants for each i.

(2) $(X_i \ni x_i)$ satisfies (3) of Lemma 2.10 for each i, and $e_1 = e_1(X_i \ni x_i), q_1 = q_1(X_i \ni x_i) \in \mathcal{I}_0$ for each i. Since $\mathcal{I}_0$ is a finite set, possibly passing to a subsequence, we may assume that e_1, e_2, q_1, q_2 are constants for each i.

If $(X_i \ni x_i)$ satisfies (3) of Lemma 2.10 for each i, and $e_1 = e_1(X_i \ni x_i), q_1 = q_1(X_i \ni x_i)$ are constants for each i, then by Lemma 2.10(3), $\mathrm{mld}(X_i \ni x_i) = \frac{1}{e_1 - q_1}$ is a constant, a contradiction.

If $(X_i \ni x_i)$ satisfies (2) of Lemma 2.10 for each i, and $e_1 = e_1(X_i \ni x_i), q_1 = q_1(X_i \ni x_i), e_2 = e_2(X_i \ni x_i), q_2 = q_2(X_i \ni x_i)$ are constants for each i, then by Lemma 2.10(2),

$$\min\left\{\frac{1}{e_1 - q_1}, \frac{1}{e_2 - q_2}\right\} \geq \frac{1}{3} + \epsilon.$$

Thus, $e_1 - q_1 \leq 2$ and $e_2 - q_2 \leq 2$. We get a contradiction by enumerating possibilities as follows:

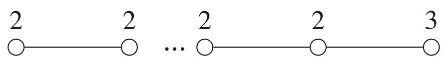
Case 1. $q_1 = 1$. Then $e_1 = 2$ or 3.

Case 1.1 $e_1 = 2$.

Case 1.1.1 $q_2 = 1$. Then $e_2 = 2$ or 3.

Case 1.1.1.1 $e_2 = 2$. In this case, all the weights in $\mathcal{DG}(X_i \ni x_i)$ are 2. Thus, $\mathrm{mld}(X_i \ni x_i) = 1$ for every i, a contradiction.

Case 1.1.1.2 $e_2 = 3$. In this case, for each i, the dual graph of $X_i \ni x_i$ is of the following form



By Lemma 2.11, $X_i \ni x_i$ is a cyclic quotient singularity of type $\frac{1}{2k_i+1}(1, k_i)$ for some positive integer k_i, and $k_i \to +\infty$ when $i \to +\infty$, a contradiction.

Case 1.1.2 $q_2 \geq 2$. In this case, there exist an integer $w_2 \geq 3$ and a nonnegative integer $d_2 < q_2$, such that $e_2 = w_2 q_2 - d_2$. Thus,

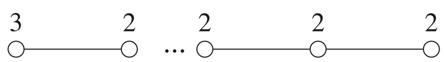
$$2 \geq e_2 - q_2 = (w_2 - 1)q_2 - d_2 \geq (w_2 - 2)q_2 + 1 \geq q_2 + 1 \geq 3,$$

a contradiction.

Case 1.2 $e_1 = 3$.

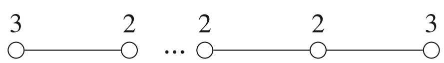
Case 1.2.1 $q_2 = 1$. Then, $e_2 = 2$ or 3.

Case 1.2.1.1 $e_2 = 2$. In this case, for each i, the dual graph of $X_i \ni x_i$ is of the following form



By Lemma 2.11, $X_i \ni x_i$ is a cyclic quotient singularity of type $\frac{1}{2k_i+1}(1, k_i)$ for some positive integer k_i, and $k_i \to +\infty$ when $i \to +\infty$, a contradiction.

Case 1.2.1.2 $e_2 = 3$. In this case, for each i, the dual graph of $X_i \ni x_i$ is of the following form:



By Lemma 2.11, $X_i \ni x_i$ is a cyclic quotient singularity of type $\frac{1}{4k_i+8}(1, 2k_i + 3)$ for some nonnegative integer k_i. By Lemma 2.12, $\mathrm{mld}(X_i \ni x_i) = \frac{1}{2}$, a contradiction.

Case 1.2.2 $q_2 \geq 2$. In this case, exactly the same argument as in Case 1.1.2 holds and we get a contradiction.

Case 2. $q_1 \geq 2$. In this case, there exists an integer $w_1 \geq 3$ and a nonnegative integer $d_1 < q_1$, such that $e_1 = w_1 q_1 - d_1$. Thus,

$$2 \geq e_1 - q_1 = (w_1 - 1)q_1 - d_1 \geq (w_1 - 2)q_1 + 1 \geq q_1 + 1 \geq 3,$$

a contradiction. □

4.2 | Intersection numbers

Lemma 4.2. *Let X be a projective klt surface such that K_X is nef and $f : Y \to X$ the minimal resolution of X. If X is not rational, then K_Y is pseudo-effective.*

Proof. If X is not rational, Y is not rational. If K_Y is not pseudo-effective, then there exists a birational morphism $g : Y \to W$ to a smooth projective surface W and a $\mathbb{P}^1$-fibration $h : W \to R$. Since Y is not a rational surface, $g(R) \geq 0$. Thus, for any exceptional curve F of f, F does not dominate R. Pick a general h-vertical curve Σ and let Σ_Y, Σ_X be the strict transforms of Σ on Y and X, respectively. Then,

$$0 \leq K_X \cdot \Sigma_X = K_Y \cdot \Sigma_Y = K_W \cdot \Sigma = -2,$$

a contradiction. □

Lemma 4.3. *Let X be a $\frac{1}{3}$-klt surface such that K_X is big and nef, C an irreducible curve on X, $x \in C$ a closed point, $f : Y \to X$ the minimal resolution of X, and C_Y the strict transform of C on Y. Assume that*

- *X is not a rational surface,*
- *$K_Y \cdot C_Y < 0$,*
- *$X \ni x$ is a cyclic quotient singularity of type $\frac{1}{2k+1}(1,k)$ for some integer $k \geq 5$, and*
- *$E_1, \dots, E_k$ are prime f-exceptional divisors over $X \ni x$, such that*
  1. *$E_i^2 = -2$ when $1 \leq i \leq k-1$,*
  2. *$E_k^2 = -3$, and*
  3. *$E_i \cdot E_j \neq 0$ if and only if $|i-j| \leq 1$.*

Then,

1. *$C_Y \cdot E_i = 0$ when $1 \leq i \leq k-1$, and*
2. *$C_Y \cdot E_k = 1$.*

Proof. By Lemma 4.2, K_Y is pseudo-effective. By Lemma 3.2, $K_Y \cdot C_Y = -1$ and $C_Y^2 = -1$. Moreover, each E_i is a smooth rational curve. Let $g : Y \to W$ be the contraction of C_Y and $E_{i,W} := g_* E_i$ for each i. Then, W is smooth and K_W is pseudo-effective. □

Claim 4.4. $C_Y \cdot E_j \leq 1$ for every $j \in \{1, 2, \dots, k\}$.

Proof of Claim 4.4. Suppose this is not the case, then there exists an integer $n \geq 2$ and an integer $j \in \{1, 2, \dots, k\}$, such that $C_Y \cdot E_j = n$. We have

$$E_{j,W}^2 = E_j^2 + n^2 \geq -3 + 4 \geq 1$$

and

$$K_W \cdot E_{j,W} = K_Y \cdot E_j - n \leq -1,$$

which contradicts Lemma 3.1 as K_W is pseudo-effective. □

Claim 4.5. $E_{i,W}$ are smooth rational curves for every i.

Proof of Claim 4.5. It immediately follows from Lemma 3.4 and Claim 4.4. □

Claim 4.6. $C_Y \cdot E_j = 0$ for every $j \in \{2, 3, \dots, k-2\}$.

Proof of Claim 4.6. Suppose that the claim does not hold. Then, by Claim 4.4, there exists $j \in \{2, 3, \dots, k-2\}$ such that $C_Y \cdot E_j = 1$. There are three cases:

Case 1. $C_Y \cdot E_{j+1} = 1$. In this case, $E^2_{j,W} = E^2_{j+1,W} = -1$ and $E_{j,W} \cdot E_{j+1,W} = 2$. By Claim 4.5, $E_{j,W}$ and $E_{j+1,W}$ are smooth rational curves. Since K_W is pseudo-effective, this contradicts Lemma 3.5.

Case 2. $C_Y \cdot E_{j-1} = 1$. We get a contradiction by the same arguments as Case 1 except that we replace E_{j+1} with E_{j-1}.

Case 3. $C_Y \cdot E_{j-1} = C_Y \cdot E_{j+1} = 0$. In this case, $E^2_{j,W} = -1$, $E^2_{j-1,W} = E^2_{j+1,W} = -2$, $E_{j,W} \cdot E_{j-1,W} = E_{j,W} \cdot E_{j+1,W} = 1$, which contradicts Lemma 3.6. □

Claim 4.7. $C_Y \cdot E_{k-1} = 0$.

Proof of Claim 4.7. Suppose that the claim does not hold. Then, by Claim 4.4, $C_Y \cdot E_{k-1} = 1$. By Claim 4.6, $C_Y \cdot E_j = 0$ for every $j \in \{2, 3, \dots, k-2\}$. There are two cases:

Case 1. $C_Y \cdot E_k = 1$. In this case, $E^2_{k-1,W} = -1$, $E^2_{k,W} = -2$, $E^2_{k-2,W} = -2$, $E_{k-1,W} \cdot E_{k,W} = 2$, and $E_{k-1,W} \cdot E_{k-2,W} = 1$. This contradicts Lemma 3.6.

Case 2. $C_Y \cdot E_k = 0$. In this case, $E^2_{k-1,W} = -1$, $E^2_{k,W} = -3$, $E^2_{k-2,W} = E^2_{k-3,W} = -2$, and for every $i, j \in \{k-3, k-2, k-1, k\}$, $E_i \cdot E_j = 1$ if $|i-j| = 1$ and $E_i \cdot E_j = 0$ if $|i-j| \geq 2$.

Let $h : W \to Z$ be the contraction of $E_{k-1,W}$ and $E_{i,Z} := h_* E_{i,W}$ for any $i \neq k-1$. Then, Z is smooth and K_Z is pseudo-effective. By Lemma 3.4, $E_{k-3,Z}, E_{k-2,Z}$, and $E_{k,Z}$ are smooth rational curves. Moreover, $E^2_{k-3,Z} = E^2_{k,Z} = -2$, $E^2_{k-2,Z} = -1$, and $E_{k-3,Z} \cdot E_{k-2,Z} = E_{k-2,Z} \cdot E_{k,Z} = 1$. This contradicts Lemma 3.6. □

Claim 4.8. $C_Y \cdot E_1 = 0$.

Proof of Claim 4.8. Suppose that the claim does not hold. Then, by Claim 4.4, $C_Y \cdot E_1 = 1$. By Claim 4.6 and Claim 4.7, $C_Y \cdot E_j = 0$ for every $j \in \{2, \dots, k-1\}$. By Claim 4.4, there are two cases:

Case 1. $C_Y \cdot E_k = 1$. In this case, $E^2_{1,W} = -1$, $E^2_{2,W} = E^2_{k,W} = -2$, $E_{1,W} \cdot E_{2,W} = E_{1,W} \cdot E_{k,W} = 1$, which contradicts Lemma 3.6.

Case 2. $C_Y \cdot E_k = 0$. The are two subcases:

Case 2.1. For any closed point $y \in C$ such that $y \neq x$, X is smooth near y. In this case, let $a := 1 - a(E_1, X, 0) = \frac{1}{2k+1}$. Since K_X is big and nef,

$$0 \leq K_X \cdot C = f^* K_X \cdot C_Y = (K_Y + (1-a)E_1) \cdot C_Y = -1 + (1-a) = -a < 0,$$

a contradiction.

Case 2.2. There exists a closed point $y \in C$ such that $y \neq x$ and X is not smooth near y. Then, there exists a prime divisor F on Y that is over $X \ni y$, such that $C_Y \cap F \neq \emptyset$. Moreover, F is a smooth rational curve. Since X is $\frac{1}{3}$-klt, by Lemma 2.9, $F^2 \geq -5$. Let $F_W := g_* F$.

We have $F_W \cdot E_{i,W} = 0$ for every $i \neq 1$, $E^2_{1,W} = -1$, $E^2_{2,W} = E^2_{3,W} = E^2_{4,W} = -2$, and for every $i, j \in \{1, 2, 3, 4\}$, $E_{i,W} \cdot E_{j,W} = 1$ when $|i-j| = 1$ and $E_{i,W} \cdot E_{j,W} = 0$ when $|i-j| \geq 2$.

There are two subcases:

Case 2.2.1. $C_Y \cdot F = 1$. In this case, by Lemma 3.4, F_W is a smooth rational curve. Moreover, $F^2_W \geq -4$ and $F_W \cdot E_{1,W} = 1$,

Let $h : W \to Z$ be the contraction of $E_{1,W}$, $E_{i,Z} := h_* E_{i,W}$ for each $i \neq 1$, and $F_Z := h_* F_W$. Then, Z is smooth and K_Z is pseudo-effective. By Lemma 3.4, $E_{2,Z}, E_{3,Z}, E_{4,Z}$, and F_Z are smooth rational curves. Moreover, $E^2_{2,Z} = -1$, $E^2_{3,Z} = E^2_{4,Z} = -2$, $E_{2,Z} \cdot E_{3,Z} = E_{3,Z} \cdot E_{4,Z} = F_Z \cdot E_{2,Z} = 1$, $F_{2,Z} \cdot E_{3,Z} = F_{2,Z} \cdot E_{4,Z} = E_{2,Z} \cdot E_{4,Z} = 0$, and $F^2_Z \geq -3$.

Let $p : Z \to T$ be the contraction of $E_{2,Z}$, $E_{i,T} := p_* E_{i,Z}$ for each $i \neq 1, 2$, and $F_T := p_* F_Z$. Then, T is smooth and K_T is pseudo-effective. By Lemma 3.4, $E_{3,T}, E_{4,T}$, and F_T are smooth rational curves. Moreover, $E^2_{3,T} = -1$, $E^2_{4,T} = -2$, $F^2_T \geq -2$, and $E_{3,T} \cdot E_{4,T} = F_T \cdot E_{3,T} = 1$.

By Lemma 3.3, $F^2_T \in \{-1, -2\}$. By Lemma 3.5, $F^2_T = -2$. But this contradicts Lemma 3.6.

Case 2.2.2. $C_Y \cdot F \geq 2$. In this case, we let $b := F^2$ and $c := C_Y \cdot F$. Then, $F^2_W = b + c^2$, $K_W \cdot F_W = K_Y \cdot F - c = -2 - b - c$, and $F_W \cdot E_{1,W} = c$.

Let $h : W \to Z$ be the contraction of $E_{1,W}$ and $F_Z := h_*F_W$. Then, Z is smooth and K_Z is pseudo-effective. Moreover, $F_Z^2 = F_W^2 + c^2 = b + 2c^2$, and $K_Z \cdot F_Z = K_W \cdot F_W - c = -2 - b - 2c$. Since $b \geq -5$ and $c \geq 2$, $F_Z^2 \geq 3 > 0$ and $K_Z \cdot F_Z \leq -1 < 0$, which contradicts Lemma 3.1. □

Proof of Lemma 4.3 continued. By Claim 4.6, Claim 4.7, and Claim 4.8, we get (1). Since $x \in C$, C_Y intersects $\cup_{i=1}^{k} C_i$, which implies that C_Y intersects E_k. Thus, $C_Y \cdot E_k \geq 1$. (2) follows from Claim 4.4. □

Lemma 4.9. *Let X be a rational $\frac{2}{5}$-klt surface such that K_X is big and nef and $k \geq 10$ an integer. Then, X does not contain a cyclic quotient singularity of type $\frac{1}{2k+1}(1,k)$.*

Proof. Assume not. Then, there exists a closed point $x \in X$ such that x is a cyclic quotient singularity of type $\frac{1}{2k+1}(1,k)$. By Lemma 2.16, we may let $f : Y \to X$ be the minimal resolution of X and write

$$K_Y + \sum_{i=1}^{k} \frac{i}{2k+1} E_i + \sum_{i=1}^{s} b_i F_i = f^* K_X$$

where $E_1, \dots, E_k, F_1, \dots, F_s$ are the prime f-exceptional divisors, where

(1) $E_1, \dots, E_k$ are the prime f-exceptional divisors over $X \ni x$ such that $E_i^2 = -2$ when $1 \leq i \leq k-1$ and $E_k^2 = -3$, and
(2) for every $i \in \{1, 2, \dots, s\}$, $\mathrm{center}_X F_i = x_i$ for some closed point $x_i \in X$, such that $x_i \neq x$.

In particular, $K_Y \cdot E_i = 0$ when $i \neq k$ and $K_Y \cdot E_k = 1$. Since X is $\frac{2}{5}$-klt, by Lemma 2.15, $K_Y \cdot \sum_{i=1}^{s} b_i F_i \leq s$. Since f extracts $k + s$ divisors, we have $\rho(Y) \geq 1 + k + s$. Since K_X is big and nef, we have $K_X^2 > 0$, which implies that

$$K_Y^2 = K_X^2 - K_Y \cdot \left(\sum_{i=1}^{k} \frac{i}{2k+1} E_i + \sum_{i=1}^{s} b_i F_i \right) > -\frac{k}{2k+1} - s > -\frac{1}{2} - s.$$

Since X is rational, Y is rational. By Lemma 3.7, $K_Y^2 = 10 - \rho(Y)$. Thus,

$$-\frac{1}{2} - s < K_Y^2 = 10 - \rho(Y) \leq 10 - (1 + k + s) = 9 - k - s,$$

which implies that $k < \frac{19}{2} < 10$, a contradiction. □

Lemma 4.10. *Then, there exists a positive integer n_1, a DCC set $\mathcal{I}$ of nonnegative real numbers, and a positive real number γ_0 satisfying the following. Assume the following:*

- *X is a $\frac{2}{5}$-klt surface such that K_X is big and nef,*
- *C is an irreducible curve on X,*
- *$f : Y \to X$ is the minimal resolution of X,*
- *C_Y is the strict transform of C on Y, and*
- *$K_Y \cdot C_Y < 0$,*

then

1. *$K_X \cdot C \in \mathcal{I}$,*
2. *if $K_X \cdot C = 0$, then $n_1 K_X$ is Cartier near C, and*
3. *if $K_X \cdot C > 0$, then $K_X \cdot C \geq \gamma_0$.*

Proof. By Lemma 4.1, there exists a positive integer $n_0 = n_0(\frac{1}{15})$, such that for any closed point $X \ni x$, either $n_0 K_X$ is Cartier near x, or x is a cyclic quotient singularity of type $\frac{1}{2k+1}(1,k)$ for some positive integer $k \geq 10$. Now we let

$$\mathcal{I} := \left\{ \gamma \mid \gamma \geq 0, \gamma = -1 + \sum_{i=1}^{m} \frac{k_i}{2k_i+1} + \frac{l}{n_0}, m, l, k_1, \dots, k_m \in \mathbb{N} \right\}.$$

Then, $\mathcal{I}$ is a DCC set of nonnegative real numbers. Since $\mathcal{I}$ satisfies the DCC, we may let $\gamma_0 := \min\{1, \gamma \in \mathcal{I} \mid \gamma > 0\}$.

Consider the equation

$$\sum_{i=1}^{m} \frac{k_i}{2k_i+1} + \frac{l}{n_0} = 1,$$

where $m, l, k_1, \dots, k_m \in \mathbb{N}$. Then, there exists a finite set $\mathcal{I}_0 \subset \mathbb{N}$ such that $k_i \in \mathcal{I}_0$ for each i: to see this, note that $\frac{k_i}{2k_i+1}$ belongs to a DCC set of positive real numbers and the sum $\sum_{i=1}^{m} \frac{k_i}{2k_i+1}$ belongs to the finite set $\{\frac{n_0-l}{n_0} \mid 1 \leq l \leq n_0\}$, which implies that $\frac{k_i}{2k_i+1}$ belongs to a finite set, hence k_i belongs to a finite set. We define

$$n_1 := n_0 \prod_{\gamma \in \mathcal{I}_0} (2\gamma + 1).$$

We show that $n_1, \mathcal{I}$, and γ_0 satisfy our requirements. For any curve C as in the assumption, there exists a nonnegative integer s, such that

(1) there are closed points $x_1, \dots, x_s$ on X, such that $x_i \in C$ and x_i is a cyclic quotient singularity of type $\frac{1}{2k_i+1}(1, k_i)$ for some positive integer $k_i \geq 10$ for each i, and
(2) for any closed point $y \notin \{x_1, \dots, x_s\}$, $n_0 K_X$ is Cartier near y.

By Lemma 4.9, we may assume that X is not rational. By Lemma 4.3, we may write

$$K_Y + \sum_{i=1}^{s} \sum_{j=1}^{k_i} a_{i,j} E_{i,j} + \sum_{k=1}^{t} \frac{c_k}{n_0} F_k = f^* K_X,$$

where

(1) $E_{i,j}$ and F_k are distinct prime f-exceptional divisors for every i, j, k,
(2) for any i, j, $\operatorname{center}_X E_{i,j} = x_i$,
(3) k_i, c_k are positive integers,
(4) $a_{i,k_i} = \frac{k_i}{2k_i+1}$ for each i, and
(5) $C_Y \cdot E_{i,u_i} = 1$ and $C_Y \cdot E_{i,j} = 0$ for every $j \neq u_i$.

By Lemma 3.2, $K_Y \cdot C_Y = -1$. Thus,

$$f^* K_X \cdot C_Y = \left(K_Y + \sum_{i=1}^{s} \sum_{j=1}^{k_i} a_{i,j} E_{i,j} + \sum_{k=1}^{t} \frac{c_k}{n_0} F_k \right) \cdot C_Y = -1 + \sum_{i=1}^{s} \frac{k_i}{2k_i+1} + \frac{l}{n_0}$$

for some nonnegative integer l. Moreover, since K_X is big and nef,

$$0 \leq K_X \cdot C = f^* K_X \cdot C_Y.$$

Thus, $K_X \cdot C = f^*K_X \cdot C_Y \in \mathcal{I}$, and we get (1). (3) follows from (1). Moreover, if $K_X \cdot C = 0$, then

$$0 = -1 + \sum_{i=1}^{s} \frac{k_i}{2k_i+1} + \frac{l}{n_0},$$

which implies that $k_i \in \mathcal{I}_0$ for each i. Thus, $n_1 K_X$ is Cartier near C by construction of n_1, and we get (2). □

4.3 | Construction of nef $\mathbb{Q}$-divisors

Proposition 4.11. *There exists a positive integer m_0 satisfying the following. Assume the following:*

(1) X a $\frac{2}{5}$-klt surface such that K_X is big and nef,
(2) $f : Y \to X$ is the minimal resolution of X, and
(3) $K_Y + \sum_{i=1}^{s} a_i E_i = f^*K_X$, where E_i are the prime f-exceptional divisors,

then $m_0 K_Y + \sum_{i=1}^{s} c_i E_i$ is nef for some nonnegative integers $c_1, \dots, c_s$, such that $c_i \le \lfloor m_0 a_i \rfloor$ for each i.

Proof. Let n_1 and γ_0 be the numbers given by Lemma 4.10, n_0 the number given by Lemma 4.1, $n_2 := \max\{10, n_1, \lceil \frac{1}{\gamma_0} \rceil\}$, and

$$m_0 := n_0 n_1 \prod_{i=1}^{n_2} (2i+1).$$

We show that m_0 satisfies our requirements.

We classify the singularities on X into three classes:

Class 1. Cyclic quotient singularities of type $\frac{1}{2k+1}(1,k)$ where $k \ge n_2$. Let these singularities be $x_1, \dots, x_s$ for some nonnegative integer s. We may assume that x_i is a cyclic quotient singularity of type $\frac{1}{2k_i+1}(1,k_i)$ for some integer $k_i \ge n_2$ for every $1 \le i \le s$.

Class 2. Singularities of type $\frac{1}{2k+1}(1,k)$ where $5 \le k < n_2$. Let these singularities be $x_{s+1}, \dots, x_t$ for some integer $t \ge s$. In particular, by the definition of m_0. $m_0 K_X$ is Cartier near x_i for every $s+1 \le i \le t$.

Class 3. Other singularities. Let these singularities be $x_{t+1}, \dots, x_r$ for some integer $r \ge t$. In particular, by Lemma 4.1, and the definition of m_0, $m_0 K_X$ is Cartier near x_i for every $t+1 \le i \le r$.

Now we may write

$$K_Y + \sum_{i=1}^{s} \sum_{j=1}^{k_i} \frac{j}{2k_i+1} E_{i,j} + \frac{1}{m_0} F = f^*K_X,$$

where

(1) for every $1 \le i \le s$ and $1 \le j \le k_i$, $\mathrm{center}_X E_{i,j} = x_i$;
(2) for every $1 \le i \le s$ and $1 \le j \le k_i - 1$, $E_{i,j}^2 = -2$;
(3) for every $1 \le i \le s$, $E_{i,k_i}^2 = -3$;
(4) $F \ge 0$ is a f-exceptional Weil divisor, such that $x_i \notin \mathrm{center}_X F$ for every $1 \le i \le s$.

We show that we may take

$$\sum_{i=1}^{l} c_i E_i := \sum_{i=1}^{s} \sum_{j=k_i-n_2+1}^{k_i} \frac{m_0(j-(k_i-n_2))}{2n_2+1} E_{i,j} + F.$$

Indeed, by our constructions, $0 \leq c_i \leq \lfloor m_0 a_i \rfloor$ for each i, and we only left to check that $(m_0K_Y + \sum_{i=1}^{l} c_iE_i) \cdot C_Y \geq 0$ for any irreducible curve C_Y on Y. We have the following cases:

Case 1. K_Y is not pseudo-effective. In this case, by Lemma 4.2, X is rational. By Lemma 4.9, $s = 0$. Thus, $\sum_{i=1}^{l} c_iE_i = F$ and

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right) = m_0f^*K_X$$

is nef. Thus, $(m_0K_Y + \sum_{i=1}^{l} c_iE_i) \cdot C_Y \geq 0$ for any irreducible curve C_Y on Y.

Case 2. K_Y is pseudo-effective.

Case 2.1. C_Y is not exceptional over X. Let $C := f_*C_Y$.

Case 2.1.1. $K_Y \cdot C_Y \geq 0$. In this case, $E_{i,j} \cdot C_Y \geq 0$ and $F \cdot C_Y \geq 0$, hence $(m_0K_Y + \sum_{i=1}^{l} c_iE_i) \cdot C_Y \geq 0$.

Case 2.1.2. $K_Y \cdot C_Y < 0$. By Lemma 3.2, $K_Y \cdot C_Y = C_Y^2 = -1$. By Lemma 4.3, $C_Y \cdot E_{i,j} = 0$ for every i and every $j \leq k_i - 1$, and $C_Y \cdot E_{i,k_i} \in \{0, 1\}$ for every i. By Lemma 4.10, there are two possibilities.

Case 2.1.2.1. n_1K_X is Cartier near C. In this case, since $n_2 \geq n_1$, we have $2k_i + 1 \geq 2n_2 + 1 > n_1$ for every i. Since the Cartier index of K_X near x_i is $2k_i + 1$ and n_1K_X is Cartier near C, C does not pass through x_i. Thus, C_Y does not intersect $E_{i,j}$ for any i, j, and hence

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right) \cdot C_Y = (m_0K_Y + F) \cdot C_Y$$

$$= \left(m_0K_Y + \sum_{i=1}^{s}\sum_{j=1}^{k_i} \frac{m_0 j}{2k_i + 1}E_{i,j} + F\right) \cdot C_Y = m_0f^*K_X \cdot C_Y \geq 0.$$

Case 2.1.2.2. $K_X \cdot C \geq \gamma_0$. Possibly reordering indices, we may assume that there exists an integer $t \in \{0, 1, 2, \dots, s\}$, such that $C_Y \cdot E_{i,k_i} = 1$ when $1 \leq i \leq t$ and $C_Y \cdot E_{i,k_i} = 0$ when $t + 1 \leq i \leq s$. There are two cases:

Case 2.1.2.2.1. $t \leq 2$. In this case, since $n_2 \geq \frac{1}{\gamma_0}$, $\gamma_0 > \frac{1}{2n_2+1}$. Thus,

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right) \cdot C_Y = m_0f^*K_X \cdot C - \sum_{i=1}^{t}\left(\frac{m_0k_i}{2k_i + 1} - \frac{m_0n_2}{2n_2 + 1}\right)$$

$$\geq m_0\gamma_0 - m_0\sum_{i=1}^{t}\left(\frac{1}{2} - \frac{n_2}{2n_2 + 1}\right) \geq m_0\gamma_0 - \frac{m_0}{2n_2 + 1} > 0.$$

Case 2.1.2.2.2. $t \geq 3$. In this case, we have

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right) \cdot C_Y \geq m_0K_Y \cdot C_Y + \sum_{i=1}^{t} \frac{m_0n_2}{2n_2 + 1} = m_0\left(-1 + \sum_{i=1}^{t} \frac{n_2}{2n_2 + 1}\right)$$

$$\geq m_0\left(-1 + \frac{3n_2}{2n_2 + 1}\right) = \frac{m_0(n_2 - 1)}{2n_2 + 1} > 0.$$

Case 2.2. C_Y is exceptional over X. Then, $C \subset \operatorname{Supp}\left(\cup_{i=1}^{s} \cup_{j=1}^{k_i} E_{i,j}\right) \cup \operatorname{Supp} F$.

Case 2.2.1 $C_Y \subset \operatorname{Supp} F$. In this case, $C_Y \cdot E_{i,j} = 0$ for every i, j, and hence

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right) \cdot C_Y = (m_0K_Y + F) \cdot C_Y$$

$$= \left(m_0K_Y + \sum_{i=1}^{s}\sum_{j=1}^{k_i} \frac{m_0 j}{2k_i + 1}E_{i,j} + F\right) \cdot C_Y = m_0f^*K_X \cdot C_Y = 0.$$

Case 2.2.2 $C_Y \subset \mathrm{Supp}\left(\cup_{i=1}^{s}\cup_{j=1}^{k_i} E_{i,j}\right)$. We may assume that $C_Y = E_{i,j_0}$ for some i and some $1 \le j_0 \le k_i$. In this case,

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right)\cdot C_Y = \left(m_0K_Y + \sum_{j=k_i-n_2+1}^{k_i} \frac{m_0(j-(k_i-n_2))}{2n_2+1}E_{i,j}\right)\cdot C_Y.$$

There are four possibilities:

Case 2.2.2.1 $j_0 = k_i$. In this case, $\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right)\cdot C_Y = m_0\left(1 + \frac{n_2-1}{2n_2+1} - \frac{3n_2}{2n_2+1}\right) = 0.$

Case 2.2.2.2 $k_i - n_2 + 1 \le j_0 \le k_i - 1$. In this case,

$$\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right)\cdot C_Y = m_0\left(0 + \frac{j_0-1-(k_i-n_2)}{2n_2+1} - \frac{2(j_0-(k_i-n_2))}{2n_2+1} + \frac{j_0+1-(k_i-n_2)}{2n_2+1}\right) = 0.$$

Case 2.2.2.3 $j_0 = k_i - n_2$. In this case, $\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right)\cdot C_Y = \frac{m_0}{2n_2+1} > 0.$

Case 2.2.2.4 $1 \le j_0 \le k_i - n_2$. In this case, $\left(m_0K_Y + \sum_{i=1}^{l} c_iE_i\right)\cdot C_Y = m_0K_Y\cdot C_Y = 0.$ □

Proposition 4.12. *There exists a uniform positive integer m_2 satisfying the following. Assume that*

1. *X a $\frac{2}{5}$-klt surface such that K_X is big and nef,*
2. *$f : Y \to X$ is the minimal resolution of X, and*
3. *$K_Y + \sum_{i=1}^{s} a_iE_i = f^*K_X$, where E_i are the prime f-exceptional divisors,*

then $m_2K_Y + \sum_{i=1}^{l} r_iE_i$ is big and nef for some nonnegative integers $r_1,\dots,r_l$, such that $r_i \le \lfloor m_2a_i\rfloor$ for each i.

Proof. By Proposition 4.11, there exist a positive integer $m_0 = m_0$ which does not depend on X, and non-negative integers $c_1,\dots,c_l$, such that $m_0K_Y + \sum_{i=1}^{l} c_iE_i$ is nef and $c_i \le \lfloor m_0a_i\rfloor$ for each i. By Theorem 3.12, there exists a uniform positive integer m_1 such that $|m_1K_X|$ defines a birational map. Let $m_2 := m_0m_1$. Then, $|m_2K_X|$ defines a birational map, and hence

$$\left|m_2K_Y + \sum_{i=1}^{l}\lfloor m_2a_i\rfloor E_i\right| = \left|m_2K_Y + \sum_{i=1}^{l} m_2a_iE_i\right| = |f^*(m_2K_X)|$$

defines a birational map.

Let $D := m_2K_Y + \sum_{i=1}^{l}\lfloor m_2a_i\rfloor E_i$ and $\tilde{D} := m_2K_Y + \sum_{i=1}^{l} m_1c_iE_i$. Since $c_i \le \lfloor m_0a_i\rfloor$,

$$m_1c_i \le m_1\lfloor m_0a_i\rfloor \le \lfloor m_1m_0a_i\rfloor = \lfloor m_2a_i\rfloor.$$

Thus, $D \ge \tilde{D}$. By Proposition 3.10, there exists a Weil divisor D' on X, such that $D \ge D' \ge \tilde{D}$ and D' is big and nef. In particular, we may write $D' = m_2K_Y + \sum_{i=1}^{l} r_iE_i$ for some integers $r_1,\dots,r_l$ such that $0 \le c_i \le r_i \le \lfloor m_2a_i\rfloor$ for each i. m_2 and $r_1,\dots,r_l$ satisfy our requirements. □

4.4 | Proof of the main theorem

Proof of Theorem 1.1. Let $f : Y \to X$ be the minimal resolution of X such that $K_Y + \sum_{i=1}^{n} a_iE_i = f^*K_X$, where $E_1,\dots,E_n$ are the prime exceptional divisors of f. By Proposition 4.12, there exists a uniform positive integer m_2, such that $K_Y + \sum_{i=1}^{n}\frac{r_i}{m_2}E_i$ is big and nef for some integers $r_1,\dots,r_n$ such that $0 \le r_i \le \lfloor m_2a_i\rfloor$ for each i. By Lemma 3.11, $|192m_2K_X|$ defines a birational map and we may let $m := 192m_2$. □

5 | EXAMPLES

In this section, we will provide two theorems where we construct some interesting examples. The first one is Theorem 5.3 (= Theorem 1.2), which shows that the $\frac{1}{2}$-lc assumption in Theorem 1.1 is necessary. The second one is Theorem 5.7. It shows that, even if we only have a very strong control on $\mathrm{mld}(X)$ (i.e., when X is a terminal threefold), "$|mK_X|$ has no fixed part" is the best we may expect, as we cannot expect $|mK_X|$ to be free in codimension 2 for any bounded m.

Lemma 5.1. *Let X be an lc projective surface such that K_X is big and nef, $f : Y \to X$ the minimal resolution of X, and $E_1, \dots, E_n$ the prime f-exceptional divisors of X. Assume that $K_Y + \sum_{i=1}^n a_iE_i = f^*K_X$, where $a_i := 1 - a(E_i, X, 0)$. Let m be a positive integer and $c_1, \dots, c_n$ nonnegative integers, such that*

- $0 \le c_1, \dots, c_n \le \lfloor ma_i \rfloor$,
- $|mK_Y + \sum_{i=1}^n c_iE_i| \neq \emptyset$,
- *the fixed part of* $|mK_Y + \sum_{i=1}^n c_iE_i|$ *is supported on* $\cup_{i=1}^n E_i$, *and*
- $mK_Y + \sum_{i=1}^n c_iE_i$ *is big but not nef,*

then there exist nonnegative integers $c'_1, \dots, c'_n$, such that

1. $0 \le c'_i \le c_i$ *for each* i,
2. *there exists* $j \in \{1, 2, \dots, n\}$ *such that* $c'_j < c_j$,
3. $|mK_Y + \sum_{i=1}^n c_iE_i| \neq \emptyset$, *and*
4. *the fixed part of* $|mK_Y + \sum_{i=1}^n c'_iE_i|$ *is supported on* $\cup_{i=1}^n E_i$,

Proof. Since $0 \le c_1, \dots, c_n \le \lfloor ma_i \rfloor$, $\left(Y, \sum_{i=1}^n \frac{c_i}{m}E_i\right)$ is lc. Thus, we may run a $\left(K_Y + \sum_{i=1}^n \frac{c_i}{m}E_i\right)$-MMP $h : Y \to W$. Since the fixed part of $|mK_Y + \sum_{i=1}^n c_iE_i|$ is supported on $\cup_{i=1}^n E_i$, h only contracts divisors supported on $\cup_{i=1}^n E_i$. Let $B := h_*\left(K_Y + \sum_{i=1}^n \frac{c_i}{m}E_i\right)$, then we have

$$K_Y + \sum_{i=1}^n \frac{c_i}{m}E_i = h^*(K_W + B) + \sum_{i=1}^n b_iE_i,$$

where $b_i \ge 0$ are real numbers. Moreover, since $mK_Y + \sum_{i=1}^n c_iE_i$ is big but not nef, $h \neq \mathrm{id}_Y$. Thus, there exists $j \in \{1, 2, \dots, n\}$ such that $b_j > 0$. We have

$$mh^*(K_W + B) = mK_Y + \sum_{i=1}^n (c_i - mb_i)E_i.$$

Since f is the minimal resolution of X, $E_i^2 \le -2$ for every i. Thus, h is the minimal resolution of W, which implies that $c_i - mb_i \ge 0$ for every i. Let $c'_i := \lfloor c_i - mb_i \rfloor$ for every i. Then, (1)(2) hold. Since

$$\left|mK_Y + \sum_{i=1}^n c_iE_i\right| \cong |m(K_W + B)| \cong |mh^*(K_W + B)| \cong \left|mK_Y + \sum_{i=1}^n c'_iE_i\right|,$$

(3)(4) hold. □

Theorem 5.2. *Let X be an lc projective surface such that K_X is big and nef, $f : Y \to X$ the minimal resolution of X, and $E_1, \dots, E_n$ the prime f-exceptional divisors of X. Assume that $K_Y + \sum_{i=1}^n a_iE_i = f^*K_X$, where $a_i := 1 - a(E_i, X, 0)$. Then, for any positive integer m, if $|mK_X|$ defines a birational map and does not have fixed part, then there exist positive integers $r_1, \dots, r_n$, such that*

1. $0 \le r_i \le \lfloor ma_i \rfloor$, *and*
2. $K_Y + \sum_{i=1}^n \frac{r_i}{m}E_i$ *is big and nef.*

Proof. The fixed part of

$$|f^*(mK_X)| = \left|mK_Y + \sum_{i=1}^n ma_i E_i\right| = \left|mK_Y + \sum_{i=1}^n \lfloor ma_i \rfloor E_i\right|$$

is supported on $\cup_{i=1}^n E_i$. Since $|mK_X|$ defines a birational map, $|mK_Y + \sum_{i=1}^n \lfloor ma_i \rfloor E_i|$ defines a birational map. In particular, $mK_Y + \sum_{i=1}^n \lfloor ma_i \rfloor E_i$ is big.

We inductively define integers c_i^j for every $i \in \{1, 2, \ldots, n\}$ for nonnegative integers j in the following way: Let $c_i^0 := \lfloor ma_i \rfloor$ for every i. If $K_Y + \sum_{i=1}^n \frac{c_i^j}{m} E_i$ is big and nef, then we let $r_i := c_i^j$ for every i and we are done. Otherwise, by Lemma 5.1, there exist integers c_i^{j+1} for every i, such that $0 \le c_i^{j+1} \le c_i^j$, $c_k^{j+1} < c_k^j$ for some $k \in \{1, 2, \ldots, n\}$, $\left|mK_Y + \sum_{i=1}^n \lfloor c_i^{j+1} \rfloor E_i\right| \neq \emptyset$, and the fixed part of $\left|mK_Y + \sum_{i=1}^n \lfloor c_i^{j+1} \rfloor E_i\right|$ is supported on $\cup_{i=1}^n E_i$. This process must terminate after finitely many steps, and we get the desired r_i for every i. □

Theorem 5.3 (= Theorem 1.2). *There exist normal projective surfaces* $\{X_{n,k}\}_{n\ge 4, k\ge 2}$, *such that*

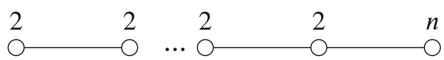
1. $|mK_{X_{n,k}}| \neq \emptyset$ *and has a nonzero fixed part for any positive integers* m, n, *and* $k \ge m$,
2. $K_{X_{n,k}}$ *is ample for every* n, k, *and*
3. $\lim_{k\to+\infty} \mathrm{mld}(X_{n,k}) = \frac{1}{n-1}$ *for any* n.

Proof. **Step 1**. In this step, we construct $X_{n,k}$ for every $n \ge 4$ and $k \ge 2$.

For any positive integer $n \ge 4$ and positive integer $k \ge 2$, we let $Y_{n,k}$ be a general hypersurface of degree $d_{n,k} := 2k(n-2)^2(2k(n-1)-1)$ in the weighted projective space $P_{n,k} := \mathbb{P}(1, 1, 2k(n-2), 2k(n-2)(n-1)+1)$. Since $2k(n-2) \mid d_{n,k}$ and

$$d_{n,k} - 1 = (2k(n-2)(n-1)+1) \cdot (2k(n-2)-1),$$

$Y_{n,k}$ is well formed and has a unique singularity $o_{n,k}$, which is a cyclic quotient singularity of type $\frac{1}{2k(n-2)(n-1)+1}(1, 2k(n-2))$. The dual graph of this cyclic quotient singularity is the following:



where there are $2k(n-2)-1$ "2" in the chain. Let $E_1 = E_1(n,k), \ldots, E_{2k(n-2)} = E_{2k(n-2)}(n,k)$ be the curves in this dual graph in order, that is,

(1) $E_i^2 = -2$ when $i \in \{1, 2, \ldots, 2k(n-2)-1\}$,
(2) $E_{2k(n-2)}^2 = -n$, and
(3) $E_i \cdot E_j \neq 0$ if and only if $|i-j| \le 1$.

Let $h_{n,k} : Z_{n,k} \to Y_{n,k}$ be the minimal resolution, then we have

$$K_{Z_{n,k}} + \sum_{i=1}^{2k(n-2)} \frac{i(n-2)}{2k(n-1)(n-2)+1} E_i = h_{n,k}^* K_{Y_{n,k}}.$$

Now let $g_{n,k} : W_{n,k} \to Z_{n,k}$ be the blow-up of $E_{k(n-1)} \cap E_{k(n-1)+1}$ and $C_{n,k,W}$ the exceptional divisor of $g_{n,k}$. Let $E_{i,W} = E_{i,W}(n,k)$ be the strict transform of E_i on $W_{n,k}$ for each i. Then,

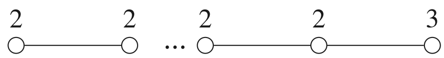
$$K_{W_{n,k}} + \sum_{i=1}^{2k(n-2)} \frac{i(n-2)}{2k(n-1)(n-2)+1} E_{i,W} + \frac{n-3}{2k(n-1)(n-2)+1} C_{n,k,W}$$
$$= g_{n,k}^* \left(K_{Z_{n,k}} + \sum_{i=1}^{2k(n-2)} \frac{i(n-2)}{2k(n-1)(n-2)+1} E_i \right) = (h_{n,k} \circ g_{n,k})^* K_{Y_{n,k}}.$$

Now, we run a $\left(K_{W_{n,k}} + \sum_{i=1}^{2k(n-2)} E_{i,W} + \frac{n-3}{2k(n-1)(n-2)+1} C_{n,k,W}\right)$-MMP over $Y_{n,k}$, which induces a birational contraction $f_{n,k} : W_{n,k} \to X_{n,k}$. Then, $f_{n,k}$ contracts precisely $E_{1,W}, \dots, E_{2k(n-2),W}$. We let $C_{n,k}$ be the pushforward of $C_{n,k,W}$ on $X_{n,k}$ and $p_{n,k} : X_{n,k} \to Y_{n,k}$ the induced contraction.

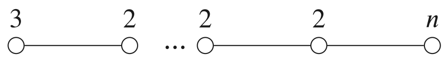
Step 2. In this step, we show the following:

Claim 5.4. For any positive integers m, $n \geq 4$ and $k \geq m$, if $|mK_{X_{n,k}}| \neq \emptyset$ and $K_{X_{n,k}}$ is big, then $|mK_{X_{n,k}}|$ has nonzero fixed part.

Proof. We let $o_1 = o_1(n,k) := (f_{n,k})_*\left(\cup_{i=1}^{k(n-1)} E_{i,W}\right)$ and $o_2 = o_2(n,k) := (f_{n,k})_*\left(\cup_{i=k(n-1)+1}^{2k(n-2)} E_{i,W}\right)$. Then, o_1 is a cyclic quotient singularity of type $\frac{1}{2k(n-1)+1}(1, k(n-1))$ with dual graph



where there are $k(n-1)-1$ "2" in the chain, and o_2 is a cyclic quotient singularity of type $\frac{1}{(n-3)(2k(n-1)-1)}(1, 2k(n-3)-1)$ with dual graph



where there are $k(n-3)-2$ "2" in the chain. Then,

$$K_{X_{n,k}} + \frac{n-3}{2k(n-1)(n-2)+1} C_{n,k} = p_{n,k}^* K_{Y_{n,k}},$$

$$K_{W_{n,k}} + \sum_{i=1}^{2k(n-2)} \frac{i(n-2)}{2k(n-1)(n-2)+1} E_{i,W} + \frac{n-3}{2k(n-1)(n-2)+1} C_{n,k,W}$$
$$= f_{n,k}^*\left(K_{X_{n,k}} + \frac{n-3}{2k(n-1)(n-2)+1} C_{n,k}\right),$$

and

$$K_{W_{n,k}} + \sum_{i=1}^{k(n-1)} \frac{i}{2k(n-1)+1} E_{i,W} + \sum_{i=k(n-1)+1}^{2k(n-2)} \frac{i-1}{2k(n-1)-1} E_{i,W} = f_{n,k}^* K_{X_{n,k}}.$$

We have

$$f_{n,k}^* K_{X_{n,k}} \cdot C_{n,k,W} = -1 + \frac{k(n-1)}{2k(n-1)+1} + \frac{k(n-1)}{2k(n-1)-1} = \frac{1}{4k^2(n-1)^2-1} < \frac{1}{35k^2} < \frac{5}{12k}.$$

Now for any positive even number $m = 2l$, any $n \geq 4$ and any $k \geq l$, we have

$$\frac{\left\{m \cdot \frac{k(n-1)}{2k(n-1)+1}\right\}}{m} = \frac{\left\{\frac{2lk(n-1)}{2k(n-1)+1}\right\}}{2l} = \frac{\left\{-\frac{l}{2k(n-1)+1}\right\}}{2l} \geq \frac{5}{12l} \geq \frac{5}{12k}.$$

Thus, for any positive even number $m = 2l$, any $n \geq 4$ and any $k \geq l$, $K_{W_{n,k}} + \sum_{i=1}^{2k(n-2)} \frac{c_i}{m} E_{i,W}$ is not nef for any integers $c_1, \dots, c_{2k(n-2)}$ such that 1

(1) $0 \leq c_i \leq \lfloor \frac{mi}{2k(n-1)+1} \rfloor$ when $1 \leq i \leq k(n-1)$, and
(2) $0 \leq c_i \leq \lfloor \frac{m(i-1)}{2k(n-1)-1} \rfloor$ when $k(n-1)+1 \leq i \leq 2k(n-2)$.

For any integer $n \geq 4$, any positive integer m such that $|mK_{X_{n,k}}| \neq \emptyset$, and any integer $k \geq m$,

(1) if $K_{X_{n,k}}$ is not nef, then $|mK_{X_{n,k}}|$ has nonzero fixed part, and
(2) if $K_{X_{n,k}}$ is nef, then by Theorem 5.2, $|mK_{X_{n,k}}|$ has nonzero fixed part.

Step 3. In this step, we show that $K_{X_{n,k}}$ is ample.

Claim 5.5. For any integers $n \geq 4$ and $k \geq 2$, $K_{Y_{n,k}}$ is ample, $|K_{Y_{n,k}}|$ defines a birational map, and $|K_{Y_{n,k}}|$ and has no fixed part. In particular, $|K_{Y_{n,k}}|$ defines a birational map.

Proof. Let $d'_{n,k} := d_{n,k} - \deg(-K_{P_{n,k}})$. Then,

$$d'_{n,k} - (2k(n-2)(n-1)+1) = -4 + 2k(n-2)(n-1)(2k(n-2)-3) \geq 116 > 0.$$

Thus, $K_{Y_{n,k}}$ is ample and $|K_{Y_{n,k}}|$ defines a birational map. In particular, $|K_{Y_{n,k}}| \neq \emptyset$.

Let x, y, z, w be the coordinates of $P_{n,k}$ and since $d'_{n,k} = d_{n,k} - (1+1+2k(n-2)+(2k(n-2)(n-1)+1))$. Let $A := \left(x^{d'_{n,k}} = 0\right)$ and $B := \left(y^{d'_{n,k}} = 0\right)$. Then, $A|_{Y_{n,k}} \in |K_{Y_{n,k}}|$ and $B|_{Y_{n,k}} \in |K_{Y_{n,k}}|$. We only need to show that $A|_{Y_{n,k}} \neq B|_{Y_{n,k}}$. This is the same as saying that $Y_{n,k}$ does not contain the line $(x = y = 0)$ in $P_{n,k}$. Suppose that $Y_{n,k}$ is defined by the homogeneous weighted polynomial $q_{n,k}(x,y,z,w)$. Since $Y_{n,k}$ is general, $z^{(n-2)(2k(n-1)-1)} \in q_{n,k}(x,y,z,w)$. Thus, $Y_{n,k}$ does not contain the line $x = y = 0$ and we are done. □

Claim 5.6. For any integers $n \geq 4$ and $k \geq 2$, $K_{X_{n,k}}$ is ample.

Proof. For any n, k, by Claim 5.5, the fixed part of $|p^*_{n,k}K_{Y_{n,k}}|$ is supported on $C_{n,k}$. Since

$$p^*_{n,k}K_{Y_{n,k}} = K_{X_{n,k}} + \frac{(n-3)}{2k(n-1)(n-2)+1}C_{n,k},$$

there exists a nonnegative integer $r = r_{n,k}$ such that $|K_{X_{n,k}} - r_{n,k}C_{n,k}|$ defines a birational map and has no fixed part. In particular, $K_{X_{n,k}} - r_{n,k}C_{n,k}$ is big and nef. If $r_{n,k} = 0$, then $|K_{X_{n,k}}| \neq \emptyset$ and has no fixed part, which contradicts Claim 5.4. Thus, $r_{n,k} > 0$.

Since $K_{X_{n,k}} + \frac{(n-3)}{2k(n-1)(n-2)+1}C_{n,k}$ is nef and big, $\left(K_{X_{n,k}} + \frac{(n-3)}{2k(n-1)(n-2)+1}C_{n,k}\right) \cdot C_{n,k} = 0$. Since $K_{X_{n,k}} - r_{n,k}C_{n,k}$ is nef and big, $K_{X_{n,k}}$ is nef and big and $K_{X_{n,k}} \cdot C_{n,k} > 0$. In particular, $K^2_{X_{n,k}} > 0$.

For any irreducible curve $D_{n,k}$ on $X_{n,k}$ such that $D_{n,k} \neq C_{n,k}$, if $D_{n,k} \cdot C_{n,k} > 0$, we have that

$$K_{X_{n,k}} \cdot D_{n,k} = (K_{X_{n,k}} - r_{n,k}C_{n,k}) \cdot D_{n,k} + r_{n,k}C_{n,k} \cdot D_{n,k} > 0,$$

and if $D_{n,k} \cdot C_{n,k} = 0$, then

$$K_{X_{n,k}} \cdot D_{n,k} = K_{Y_{n,k}} \cdot (p_{n,k})_* D_{n,k} > 0.$$

Thus, $K_{X_{n,k}}$ is ample. □

Step 4. Claim 5.4 and Claim 5.6 imply (1)(2). Since

$$\mathrm{mld}(X_{n,k}) = \mathrm{mld}(X_{n,k} \ni o_2(n,k)) = \frac{2k}{2k(n-1)-1} = \frac{1}{n-1-\frac{1}{2k}},$$

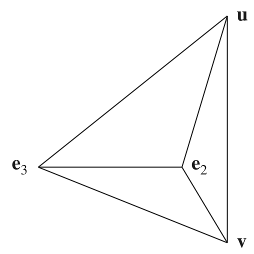
we have $\lim_{k\to+\infty} \mathrm{mld}(X_{n,k}) = \frac{1}{n-1}$ for any $n \geq 4$, which implies (3). □

Theorem 5.7. *For any positive integer m_0, there exists a terminal threefold X such that K_X is ample but $|m_0K_X|$ is not free in codimension 2.*

Proof. **Step 1**. We start with a local construction by using the language of toric varieties.

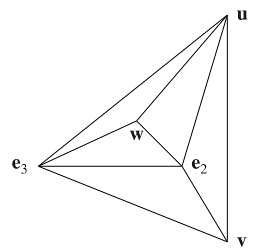
Let $N = \mathbb{Z}^3$, $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$, $\mathbf{w} = (1,1,0)$. Let $\mathbf{u} = (m,1,-b)$ and $\mathbf{v} = (-n,2,1)$, where m, n, b are positive integers such that $nb = m+1$ and $2 \nmid n$. Then, all these vectors above are primitive in N.

Let Σ_1 be the fan determined by the single maximal Cone($\mathbf{e}_3, \mathbf{u}, \mathbf{v}$). Let X_{Σ_1} be the corresponding toric variety. Then, X_{Σ_1} is affine and the cyclic quotient singularity is of the form $\frac{1}{2m+n}(-1, 2, 2b+1)$. Notice that X_{Σ_1} has an isolated singularity.



Let $\Sigma_2 = \Sigma_1^*(\mathbf{e}_2)$ be the star subdivision of Σ_1 at $\mathbf{e}_2$ as above (see [5, Chapter 11]) and X_{Σ_2} be the corresponding toric variety, then $g : X_{\Sigma_2} \to X_{\Sigma_1}$ is a birational morphism, which is an isomorphism outside the unique torus-invariant point $P \in X_{\Sigma_1}$. Since Cone($\mathbf{u}, \mathbf{e}_2, \mathbf{v}$) is smooth, X_{Σ_2} has only two isolated singularities, which are of type $\frac{1}{m}(1, -1, b)$ and $\frac{1}{n}(1, -1, 2)$. In particular, X_{Σ_2} is terminal. We use $D_{\mathbf{e}_2}$, $D_{\mathbf{v}}$, $D_{\mathbf{u}}$, $D_{\mathbf{e}_3}$ to denote the corresponding torus-invariant divisors. We can see that $D_{\mathbf{e}_2}$ is the only exceptional divisor. Let R denote the proper curve in X_{Σ_2} that corresponds to Cone($\mathbf{e}_2, \mathbf{e}_3$) $\in \Sigma_2$. Then, $R \subset D_{\mathbf{e}_2}$. By [5, Proposition 6.4.4], $D_{\mathbf{u}} \cdot R = \frac{1}{m}$, $D_{\mathbf{v}} \cdot R = \frac{1}{n}$, $D_{\mathbf{e}_3} \cdot R = \frac{b}{m} - \frac{1}{n} > 0$, and $D_{\mathbf{e}_2} \cdot R = -(\frac{2}{n} + \frac{1}{m})$. Therefore,

$$0 < \frac{2}{n} - \frac{b}{m} = K_{X_{\Sigma_2}} \cdot R < \frac{1}{n}$$



Let $\Sigma_3 = \Sigma_2^*(\mathbf{w})$ be the star subdivision of Σ_2 at $\mathbf{w}$ as above and X_{Σ_3} be the corresponding toric variety, then $f : X_{\Sigma_3} \to X_{\Sigma_2}$ is a birational morphism, which is an isomorphism outside the torus-invariant point $Q \in X_{\Sigma_2}$ that corresponds to the maximal Cone($\mathbf{w}, \mathbf{e}_2, \mathbf{e}_3$) $\in \Sigma_2$. We use $D'_{\mathbf{e}_2}$, $D'_{\mathbf{v}}$, $D'_{\mathbf{u}}$, $D'_{\mathbf{e}_3}$, $D'_{\mathbf{w}}$ to denote the corresponding torus-invariant divisors. Notice that $D'_{\mathbf{w}}$ is the only exceptional divisor of f and $D'_{\mathbf{e}_2}$, $D'_{\mathbf{v}}$, $D'_{\mathbf{u}}$, $D'_{\mathbf{e}_3}$ are the birational transforms of $D_{\mathbf{e}_2}$, $D_{\mathbf{v}}$, $D_{\mathbf{u}}$, $D_{\mathbf{e}_3}$ on X_{Σ_3}. Let R' denote the birational transform of R on X_3, then R' corresponds to Cone($\mathbf{e}_2, \mathbf{e}_3$) $\in \Sigma_3$.

Since $\mathbf{w} = \frac{1}{m}\mathbf{u} + \frac{b}{m}\mathbf{e}_3 + \frac{m-1}{m}\mathbf{e}_2$, we have $K_{X_{\Sigma_3}} = f^*K_{X_{\Sigma_2}} + \left(\frac{1}{m} + \frac{b}{m} + \frac{m-1}{m} - 1\right) D'_{\mathbf{w}}$, hence

$$f^*K_{X_{\Sigma_2}} = K_{X_{\Sigma_3}} - \frac{b}{m} D'_{\mathbf{w}}.$$

By [5, Lemma 6.4.2], $D'_{\mathbf{w}} \cdot R' = 1$. Thus, for any positive integer k,

$$\lfloor k f^* K_{X_{\Sigma_2}} \rfloor \cdot R' = \left(\frac{2}{n} - \frac{b}{m}\right) k - \left\{\frac{m-kb}{m}\right\}.$$

Step 2. Next, we will use covering trick to make the canonical divisor ample.

Choose a projective threefold Z with the isolated quotient singularity of type $\frac{1}{2m+n}(-1, 2, 2b+1)$ at P, after resolving singularities away from P, we may assume that P is the only singular point on Z. By abuse of notation we continue to use $f : Y \to X$ and $g : X \to Z$ to denote the corresponding toric blow-ups defined in Step 1. Let E be the exceptional divisor of g and $R \subset E$ be the proper curve defined in Step 1. Then, $-E$ is g-ample and we have $f^*K_Z - aE = K_X$, where

$$a = \frac{\frac{2}{n} - \frac{b}{m}}{\frac{2}{n} + \frac{1}{m}} > 0.$$

Let L be a sufficiently ample Cartier divisor on Z such that $g^*(L + K_Z) - aE$ is ample on X. We can find an effective $A \sim 2L$ that is smooth and avoids P. Let $h : Z' \to Z$ be the double cover ramified along A. Then, by Hurwitz's Formula, we have $K'_Z = h^*\left(K_Z + \frac{1}{2}A\right)$ and h is étale around P. Let X', Y', E', f', g' be the corresponding base change of h. Then, $K_{X'} = h_X^*\left(K_X + \frac{1}{2}g^*A\right) = h_X^*\left(g^*(\frac{1}{2}A + K_Z) - aE\right)$ is ample, where $h_X : X' = X \times_Z Z' \to X$ is the canonical projection. Since R is a proper curve in one of the components of E' as in Step 1 and R' its birational transform in Y', we have

$$\lfloor m_0 f^* K_{X'} \rfloor \cdot R' = \left(\frac{2}{n} - \frac{b}{m}\right) m_0 - \left\{\frac{m - m_0 b}{m}\right\}$$

for any positive integer m_0 since Z' and X_{Σ_1} are isomorphic around the isolated singular point and the computation is local.

Now we can choose $m, n \gg 0$ such that $2bm_0 < n < m$, then

$$\left(\frac{2}{n} - \frac{b}{m}\right) m_0 - \left\{\frac{m - m_0 b}{m}\right\} < \frac{m_0}{n} - \frac{1}{2} < 0.$$

Therefore, $\lfloor m_0 f^* K_{X'} \rfloor \cdot R' < 0$, which means any effective divisor in $|m_0 f^* K_{X'}|$ contains R'. Since f' is isomorphic over the generic point of R, this implies that any effective divisor in $|m_0 K_{X'}|$ contains R. □

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