Exact Asymptotics for Discrete Noiseless Channels

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***Abstract*—Analytic combinatorics in several variables (ACSV) is a powerful tool for deriving the asymptotic behavior of combinatorial quantities by analyzing multivariate generating functions. We use ACSV to derive the first-order sub-exponential asymptotics of sequences generated by a discrete noiseless channel under an average cost constraint. As a by-product of the analysis, we obtain a new proof of the equivalence of the combinatorial and probabilistic definitions of the cost-constrained capacity.**

I. Introduction

Since their introduction in Part I of Shannon's landmark 1948 paper, *A Mathematical Theory of Communication* [1], discrete noiseless channels have been a focus of research by information and coding theorists. They have also found practical use in the design of transmission codes for digital communication systems and recording codes for data storage systems [2].

In this paper, we consider discrete noiseless channels under an *average cost constraint*. Such a constraint can arise from limitations on the transmission power in an optical fiber [3], the recording voltage in a non-volatile memory [4]–[6], or the synthesis time per nucleotide in a DNA-based storage system [7].

A. Background

Constrained channels with cost. The labeled, directed graph G in Fig. 1 represents an exemplary discrete noiseless channel describing the synthesis of DNA strands using the alternating synthesis sequence $\mathsf{ACGT\ ACGT}\dots$ (see [7]).

The channel graph generates sequences of symbols over the alphabet $\Sigma = \{\mathsf{A}, \mathsf{C}, \mathsf{G}, \mathsf{T}\}$ by following paths through the directed graph and reading off the symbols $\sigma(e) \in \Sigma$ labeling the edges e in a path. Each edge e also has an associated positive cost $\tau(e) \in \mathbb{N}$, denoting the synthesis time of the edge label $\sigma(e)$. The edge labels and costs are shown in the figure as $\sigma(e)|\tau(e)$. The cost is assumed to be additive, so the cost of a sequence generated by a path in the graph is the sum of its edge costs.

Discrete noiseless channels in which all edges have unit cost are well studied [2]. Our interest is in the more general setting of varying edge costs, as in Fig. 1.

Cost-constrained capacity. Shannon introduced the concept of *(combinatorial) capacity* of a discrete noiseless channel as the asymptotic growth rate of the number of sequences (of *variable* length) as a function of the sequence cost. In the case of Fig. 1, this represents the maximum rate at which

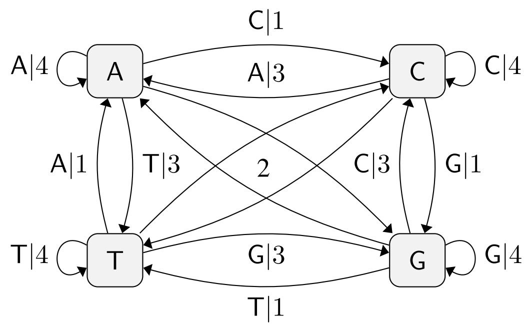

Fig. 1: Channel graph for DNA synthesis using the alternating sequence $\mathsf{ACGT\ ACGT}\ \dots$.

information can be encoded into the synthesized DNA strands per unit of synthesis time. Under the assumption of integer edge costs, Shannon analyzed a system of difference equations and derived the now-classical result that the capacity is equal to logarithm of the largest root of a determinental equation associated with the channel. For a channel represented by a graph G, we denote this capacity as C_G.

Khandekar et al. [8] extended Shannon's result to non-integer symbol costs under a mild assumption about the density of sequence costs, and expressed the capacity C_G in terms of the radius of convergence of a series that can be interpreted as a generating function for the sequence $N(t)$ representing the number of sequences with cost equal to t. Their results were extended to a more general class of channels by Böcherer et al. [9], who expressed the capacity in terms of a singularity of a complex generating function $F(x)$ for $N(t)$. They interpreted this as a generalization of results from analytic combinatorics in a single variable [10], a connection that was established by Böcherer [11], who used it to analyze the sub-exponential asymptotics of $N(t)$. Khandekar et al. [8] also clarified and extended Shannon's proof of the equivalence of the combinatorial capacity and a probabilistic capacity defined as the maximum entropy rate of a Markov process generating the sequences of the channel. This relationship was further addressed in the setting of more general channels in [12], [13].

In this paper, we consider generalizations of these results to discrete noiseless channels **subject to an average cost constraint**. In the context of Fig. 1, this corresponds to a constraint on the average synthesis time per nucleotide.

B. Overview of Results

For a channel graph G with **integer** edge costs, we consider $N(t, \lfloor \alpha t \rfloor)$, $\alpha > 0$, the number of sequences with cost at most t and length $\lfloor \alpha t \rfloor$, i.e., with average cost per symbol at most $\frac{1}{\alpha}$ (asymptotically in t). We study the singularities of a *bivariate* complex generating function $F(x, y)$ for $N(t, \lfloor \alpha t \rfloor)$ and use methods of analytic combinatorics in several variables (ACSV) [14], [15] to determine precise first-order sub-exponential asymptotics of $N(t, \lfloor \alpha t \rfloor)$. Defining the *cost-constrained (combinatorial) capacity* $C_G(\alpha)$ as the asymptotic growth rate of $N(t, \lfloor \alpha t \rfloor)$ as a function of t, we obtain an expression for $C_G(\alpha)$ in terms of a particular singularity of $F(x, y)$ determined by α. As a by-product of the analysis, we obtain a new proof of the equivalence of the combinatorial and probabilistic definitions of cost-constrained capacity [16]–[18].

Our analysis integrates results across three disparate areas: (a) spectral properties of channel graphs, (b) singularities of bivariate generating functions of channel graphs, and (c) asymptotic expansions via ACSV.

Formal definitions pertaining to channel graphs and channel capacity are provided in Section II. Our main results on spectral porperties of channel graphs, singularities of bivariate generating functions, and precise asymptotics and channel capacity are stated in Section III. Due to space limitations, proofs are omitted and will appear in a longer version (see [19]).

II. Preliminaries

A. Channel Graphs

Consider a labeled, directed graph $G = (V, E, \sigma, \tau)$ that has vertices V and directed edges E. The edges are labeled with $\sigma : E \mapsto \Sigma$ and have weights $\tau : E \mapsto \mathbb{N}$. A path $\boldsymbol{p} = (e_1, \ldots, e_n)$ of length n is a connected sequence of edges $e_1, \ldots, e_n \in E$. It generates a word $\sigma(\boldsymbol{p}) = (\sigma(e_1), \ldots, \sigma(e_n)) \in \Sigma^n$ and has cost $\tau(\boldsymbol{p}) = \tau(e_1) + \cdots + \tau(e_n)$. We call G a channel graph.

Definition II.1 (Connected and deterministic graphs)**.** *A directed graph is* strongly connected *if there is a path connecting any two vertices.*

A labeled, directed graph G is deterministic *if for all vertices $v \in V$ the labels of all edges $e \in E$ that emanate from v are distinct.*

When the context is clear, we often refer to a labeled, directed graph $G = (V, E, \sigma, \tau)$ simply as G. The following graph property plays a key role in our results. It generalizes a concept introduced in [4].

Definition II.2 (Cost diversity and period)**.** *A strongly connected graph G is* cost-uniform *if for each pair of vertices v_i, v_j and each length m, the costs of all length-m paths $\boldsymbol{p}$ from v_i to v_j are the same. If G is not cost-uniform, then we say that G is* cost-diverse.

For cost-diverse graphs, we further say that G has cost-period $c \in \mathbb{N}$ *if for each pair of vertices v_i, v_j and each length m the costs $\tau(\boldsymbol{p})$ of all length-m paths $\boldsymbol{p}$ connecting v_i and v_j are congruent modulo c.*

We also recall that, for a strongly connected graph G, we say G has period d *if for each pair of vertices v_i, v_j, the lengths of all paths $\boldsymbol{p}$ connecting v_i and v_j are congruent modulo d.*

Fig. 2 below shows a graph with cost-period 2.

Definition II.3 (Coboundary condition)**.** *A graph G satisfies the c-*periodic coboundary condition *if there exists a function $B : V \to \mathbb{R}$ and a constant $b \in \mathbb{Q}$ such that if $e \in E$ is an edge from vertex v_i to vertex v_j then the edge cost satisfies*

$$\tau(e) \equiv b + B(v_j) - B(v_i) \pmod{c}.$$

A graph satisfies the coboundary condition *if the congruence above holds without the modulo operation.*

We associate to G a *cost-enumerator matrix*, which is defined as follows.

Definition II.4 (Cost-enumerator matrix)**.** *Let $v_1, \ldots, v_{|V|}$ be an arbitrary ordering of the vertices V. Then $\boldsymbol{P}_G(x)$ is the $|V| \times |V|$ matrix with entries*

$$[\boldsymbol{P}_G(x)]_{ij} = \sum_{e \in E : v_i \to v_j} x^{\tau(e)},$$

where $x \in \mathbb{C}$. The spectral radius *of $\boldsymbol{P}_G(x)$ is denoted by $\rho_G(x)$ and defined as the largest absolute value of the eigenvalues of $\boldsymbol{P}_G(x)$.*

For real $x > 0$, the spectral radius $\rho_G(x)$ is equal to the unique real eigenvalue of $\boldsymbol{P}_G(x)$ with the largest magnitude, which we refer to as the *Perron root*. Later we will see that $\rho_G(x)$ plays a central role in the asymptotic behavior of $N(t, \lfloor \alpha t \rfloor)$.

B. Channel Capacity

For a strongly connected, deterministic graph G we are interested in the number of words of bounded cost generated by paths in the graph.

Definition II.5 (Follower set size)**.** *For any vertex $v \in V$ we define $N_{G,v}(t)$ to be the size of the cost-t* follower set *of v, i.e., the set of words that are gene-rated by some path of cost at most t that starts in v.*

Similarly, we define $N_{G,v}(t, n)$ to be the size of the cost-t length-n follower set *of v, i.e., the set of length-n words in the cost-t follower set of v.*

Now we formally define the (combinatorial) capacity of a discrete noiseless channel. The capacity of a strongly connected graph is independent of the starting vertex, and we use this fact implicitly in the definition.

Definition II.6 (Capacity)**.** *The* ***combinatorial (or variable-length) capacity*** *of a discrete noiseless channel G is defined as*

$$C_G = \limsup_{t \to \infty} \frac{\log_2(N_{G,v}(t))}{t}.$$

Similarly, the ***cost-constrained combinatorial (or fixed-length) capacity*** *is defined as*

$$C_G(\alpha) = \limsup_{t\to\infty} \frac{\log_2(N_{G,v}(t, \lfloor \alpha t \rfloor))}{t}.$$

These capacities have counterparts that are defined probabilistically, in terms of stationary Markov chains $\mathcal{P}$ on the channel graph.

Definition II.7 (Probabilistic capacity)**.** *The* ***probabilistic capacity*** *of a discrete noiseless channel G is defined as*

$$C_{G,\text{prob}} = \sup_{\mathcal{P}} \frac{H(\mathcal{P})}{T(\mathcal{P})},$$

where $H(\mathcal{P})$ and $T(\mathcal{P})$ are the entropy rate and average cost, respectively, of $\mathcal{P}$, and the supremum is taken over all stationary Markov chains on G.

Similarly, the ***cost-constrained probabilistic capacity*** *is defined as*

$$C_{G,\text{prob}}(\alpha) = \sup_{\mathcal{P}:T(\mathcal{P})\le \frac{1}{\alpha}} \frac{H(\mathcal{P})}{T(\mathcal{P})},$$

where the supremum is over all stationary Markov chains on G with average cost at most $\frac{1}{\alpha}$.

A concise parametric characterization of $C_{G,\text{prob}}(\alpha)$, found by constrained optimization methods, is stated in [8], [16], [17].

For channels with integer edge costs, Shannon proved the fundamental equivalence $C_G = C_{G,\text{prob}}$. A rigorous proof of this equivalence for non-integer edge costs is given in [8]. The cost-constrained extension, $C_G(\alpha) = C_{G,\text{prob}}(\alpha)$, was proved in [18].

C. ACSV and Singularities

Analytic combinatorics in several variables (ACSV) [14], [15] considers multivariate sequences $N(\boldsymbol{t}) = N(t_1, \dots, t_d)$, $\boldsymbol{t} \in \mathbb{N}^d$, and generating functions

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{t}\in\mathbb{N}^d} N(\boldsymbol{t})\boldsymbol{x}^{\boldsymbol{t}} = \sum_{\boldsymbol{t}\in\mathbb{N}^d} N(t_1, \dots, t_d) x_1^{t_1} \cdots x_d^{t_d}.$$

ACSV resembles univariate analytic combinatorics [10], transferring behavior of a generating function near singularities to an asymptotic expansion of its coefficients. There are many ways for the coefficient vector $\boldsymbol{t}$ to grow to infinity in the multivariate setting. Under mild assumptions, if $\boldsymbol{t}=t\boldsymbol{\alpha}$ with $t\to\infty$ and $\boldsymbol{\alpha}=(\alpha_1, \dots, \alpha_d)\in\mathbb{R}^d_{>0}$ fixed, then the asymptotic behavior of $N(\boldsymbol{t})$ is uniform as $\boldsymbol{\alpha}$ varies in cones of $\mathbb{R}^d$. Thus, ACSV derives asymptotic behavior of *$\boldsymbol{\alpha}$-diagonals* $N(t\boldsymbol{\alpha})$ as functions of $\boldsymbol{\alpha}$.

Fix a direction $\boldsymbol{\alpha} \in \mathbb{R}^d_{>0}$ and coprime polynomials $Q(\boldsymbol{x})$ and $H(\boldsymbol{x})$ such that the multivariate rational function $F(\boldsymbol{x}) = Q(\boldsymbol{x})/H(\boldsymbol{x})$ has a convergent power series expansion $F(\boldsymbol{x}) = \sum_{\boldsymbol{t}\in\mathbb{N}^d} N(\boldsymbol{t})\boldsymbol{x}^{\boldsymbol{t}}$ near the origin. If $H(\boldsymbol{x})$ is not a constant then, when $d \ge 2$, $F(\boldsymbol{x})$ admits an infinite set $\mathcal{V}$ of singularities defined by the vanishing of $H(\boldsymbol{x})$.

Our application of ACSV depends on finding singularities of a bivariate generating function $F_{G,v}(x, y)$ for $N_{G,v}(t, n)$ (specified in Lemma III.3) that have particular properties.

Minimal points are the singularities of $F(\boldsymbol{x})$ on the boundary of its domain of absolute convergence, i.e., $\boldsymbol{x} \in \mathcal{V}$ is minimal if and only if $H(\boldsymbol{x})$ has no other root $\boldsymbol{y}$ with strictly smaller coordinate-wise modulus. A minimal point is *strictly minimal* if no other singularity has the same coordinate-wise modulus, and *finite minimal* if only a finite number of other singularities have the same coordinate-wise modulus.

Critical points are singularities whose explicit definition depends heavily on the geometry of $\mathcal{V}$. The simplest situation is the *square-free smooth case*, when H and all of its partial derivatives do not simultaneously vanish. Then the critical points for direction $\boldsymbol{\alpha}$ are solutions of the system of equations

$$\begin{aligned} H(\boldsymbol{x}) &= \alpha_2 x_1 H_{x_1}(\boldsymbol{x}) - \alpha_1 x_2 H_{x_2}(\boldsymbol{x}) \\ = \cdots &= \alpha_d x_1 H_{x_1}(\boldsymbol{x}) - \alpha_1 x_d H_{x_d}(\boldsymbol{x}) = 0 \end{aligned} \qquad (1)$$

containing d equations in d variables. If $\mathcal{V}$ is locally a manifold near a point $\boldsymbol{x} \in \mathcal{V}$ but H and its partial derivatives simultaneously vanish at $\boldsymbol{x}$, then $\boldsymbol{x}$ is a critical point if and only if (1) holds when H is replaced by the product of its unique irreducible factors. The only other situation we encounter is the *complete intersection case* (see [15, Ch. 9]), when $\mathcal{V}$ is locally the union of d manifolds near $\boldsymbol{x} \in \mathcal{V}$, and any such $\boldsymbol{x}$ is a critical point.

III. Main results

A. Spectral Properties of Channel Graphs

The following lemmas reveal properties of the spectral radius $\rho_G(x)$ of the cost-enumerator matrix $\boldsymbol{P}_G(x)$ of a cost-diverse channel graph G.

Lemma III.1. *Let G be a strongly connected graph. The following statements are equivalent.*

(a) *The graph G has cost-period c.*
(b) *The graph G satisfies the c-periodic coboundary condition.*
(c) *For all $x\in\mathbb{C}$ and $k\in\mathbb{Z}$, $\rho_G(x e^{2\pi i k/c}) = \rho_G(x)$.*

Further, if G is cost-uniform, then the spectral radius $\rho_G(x)$ is log-log-linear on $x\in\mathbb{R}^+$. If G is cost-diverse then the spectral radius $\rho_G(x)$ is strictly log-log-convex on $x\in\mathbb{R}^+$.

Lemma III.2. *Let G be a strongly connected and cost-diverse graph with largest cost-period c. Then, for any $x\in\mathbb{R}^+$, there are precisely c solutions $\phi_k = 2\pi k/c$, $k\in\{0, 1, \dots, c-1\}$ to the equation $\rho_G(x e^{i\phi}) = \rho_G(x)$, in the interval $0 \le \phi < 2\pi$. For all other ϕ, $\rho_G(x e^{i\phi}) < \rho_G(x)$.*

B. Generating Functions and Singularities

For the channel graph G, we define the bivariate generating function $\boldsymbol{F}_G(x, y)$ for $N_{G,v}(t, n)$, characterized in the following lemma.

Lemma III.3. *Let G be a deterministic graph, and let v be a vertex. Let $(1, \dots, 1)$ denote the all-ones vector of length $|V|$*

and $\boldsymbol{I}$ denote the $|V| \times |V|$ identity matrix. The generating function $F_{G,v}(x,y)$ of $N_{G,v}(t,n)$ is given by the entry of

$$\boldsymbol{F}_G(x,y) = \frac{1}{1-x} \cdot (\boldsymbol{I} - y\boldsymbol{P}_G(x))^{-1}(1,\ldots,1)^{\mathrm{T}}$$

corresponding to the vertex v.

Note that $\boldsymbol{F}_G(x,y) = \boldsymbol{Q}_G(x,y)/H_G(x,y)$ for a polynomial vector $\boldsymbol{Q}_G(x,y)$ and polynomial $H_G(x,y) = (1-x)\det(\boldsymbol{I} - y\boldsymbol{P}_G(x))$. In particular, every coordinate of $\boldsymbol{F}_G(x,y)$ is a rational function with the same denominator. We need to understand the singularities of $\boldsymbol{F}_G(x,y)$, i.e., the values (x,y) where $H_G(x,y) = 0$, to determine the asymptotic behavior of the integer sequence $N_{G,v}(t,n)$ using ACSV.

Using the results in Section III-A, we prove the following proposition, which characterizes properties of the singularities of $\boldsymbol{F}_G(x,y)$ associated with the direction $\boldsymbol{\alpha} = (1,\alpha)$.

Proposition III.4. *Let G be a strongly connected and cost-diverse graph with largest period d and largest cost-period c.*

(a) *The points $\{(x_0, 1/\rho_G(x_0)) : 0 < x_0 < 1\} \cup \{(1, y_0) : y_0 \in \mathbb{C}, |y_0| \leq 1/\rho_G(1)\}$ are minimal singularities of each coordinate of $\boldsymbol{F}_G(x,y)$. All other minimal singularities have the form*

$$\left(x_0 \mathrm{e}^{\mathrm{i}2\pi k/c}, \mathrm{e}^{-2\pi\mathrm{i}(kb/c+j/d)}/\rho_G(x_0)\right) \qquad (2)$$

for some $0 < x_0 \leq 1$, $k \in \{0,1,\ldots,c-1\}$, and $j \in \{0,1,\ldots,d-1\}$, where b is the constant of the c-periodic coboundary condition.

(b) *For all $x_0 \in \mathbb{R}^+$ with $x_0 \neq 1$ and all $k \in \{0,1,\ldots,c-1\}$, $j \in \{0,1,\ldots,d-1\}$, the points in (2) are smooth points of $\boldsymbol{F}_G(x,y)$ and critical if and only if $\alpha x_0 \rho'_G(x_0) = \rho_G(x_0)$. Any point $(1,y_0)$ with $y_0 \in \mathbb{C}$ and $|y_0| < \rho_G(1)$ is not a root of $\det(\boldsymbol{I} - y\boldsymbol{P}_G(x))$ and thus is a smooth point that is never critical.*

(c) *For all $x_0 \in \mathbb{R}^+$ and $k \in \{0,1,\ldots,c-1\}$, $j \in \{0,1,\ldots,d-1\}$, the points in (2) are non-degenerate, meaning that the second derivative of $\phi(\theta) = \log(\lambda_j(\xi_k)/\lambda_j(\xi_k e^{i\theta})) + (i\theta/\alpha)$ is non-zero at $\theta = 0$, where $\xi_k = e^{2\pi i k/c}$.*

We also make use of the following result.

Lemma III.5. *Let G be a strongly connected and cost-diverse graph. Then, the critical point equation $\alpha x \rho'_G(x) = \rho_G(x)$ has a positive real solution x_0 if and only if*

$$\lim_{x\to\infty} \frac{\rho_G(x)}{x\rho'_G(x)} < \alpha < \lim_{x\to 0^+} \frac{\rho_G(x)}{x\rho'_G(x)}.$$

This solution, if it exists, is unique among all positive real x. If $\alpha > \rho_G(1)/\rho'_G(1)$ then $x_0 < 1$, and $x_0 > 1$ otherwise.

C. Asymptotics and Capacity via ACSV

Our main theorem describes the exact asymptotics of the number of bounded-cost followers in a graph $G = (V, E, \sigma, \tau)$. It follows by an application of ACSV, namely [15, Theorem 5.1] and [15, Prop. 9.1 and Thm. 9.1], using the results established in Section III-B. In the statement of the theorem, we mean by largest period d and largest cost-period c the largest integers such that the graph G has period d and cost-period c, respectively. We also associate to G two critical values of α, defined as $\alpha_G^{\mathsf{lo}} \triangleq \rho_G(1)/\rho'_G(1)$ and $\alpha_G^{\mathsf{up}} \triangleq \lim_{x\to 0^+} \rho_G(x)/(x\rho'_G(x))$.

Theorem III.6. *Let G be a strongly connected, deterministic, and cost-diverse graph with largest period d and largest cost-period c. Denote by b and $B(v_j)$ the quantities from the c-periodic coboundary condition. For all α with $0 < \alpha < \alpha_G^{\mathsf{lo}}$, $\alpha t \in \mathbb{N}$ and for any $v \in V$, $N_{G,v}(t,\alpha t)$ has the asymptotic expansion*

$$N_{G,v}(t,\alpha t) = \sum_{j=0}^{d-1} (\lambda_j(1))^{\alpha t} [\boldsymbol{u}_j^{\mathrm{T}}(1)\boldsymbol{v}_j(1)\boldsymbol{1}^{\mathrm{T}}]_v + O\big(\tau^t\big),$$

where $0 < \tau < (\rho_G(1))^\alpha$ and $\boldsymbol{u}_j(x), \boldsymbol{v}_j(x)$, with $\boldsymbol{v}_j(x)\boldsymbol{u}_j^{\mathrm{T}}(x) = 1$, are the right and left eigenvectors of $\boldsymbol{P}_G(x)$, corresponding to the eigenvalues $\lambda_j(x) = \rho_G(x)\mathrm{e}^{2\pi\mathrm{i}j/d}$. For $\alpha_G^{\mathsf{lo}} < \alpha < \alpha_G^{\mathsf{up}}$, $\alpha t \in \mathbb{N}$,

$$N_{G,v}(t,\alpha t) = \sum_{k=0}^{c-1}\sum_{j=0}^{d-1} \left(\frac{(\mathrm{e}^{2\pi\mathrm{i}bk/c}\lambda_j(x_0))^\alpha}{x_0\mathrm{e}^{2\pi\mathrm{i}k/c}}\right)^t \frac{t^{-1/2}}{\sqrt{2\pi\alpha J(x_0)}} \left(\frac{[\boldsymbol{D}_k^{-1}\boldsymbol{u}_j^{\mathrm{T}}(x_0)\boldsymbol{v}_j(x_0)\boldsymbol{D}_k\boldsymbol{1}^{\mathrm{T}}]_v}{(1-x_0\mathrm{e}^{2\pi\mathrm{i}k/c})} + O\left(\frac{1}{t}\right)\right),$$

where $J(\mathrm{e}^s) = \frac{\partial^2}{\partial s^2}\log\rho_G(\mathrm{e}^s)$, x_0 is the unique positive solution to $\alpha x \rho'_G(x) = \rho_G(x)$, and the $\boldsymbol{D}_k$ are the diagonal matrices with $[\boldsymbol{D}_k]_{jj} = \mathrm{e}^{2\pi\mathrm{i}kB(v_j)/c}$. For all $\alpha > \alpha_G^{\mathsf{up}}$, $N_{G,v}(t,\alpha t)$ is eventually 0.

Remark III.7. *The inverse of α_G^{lo} is the average cost per edge, as $n\to\infty$, over all paths of length n in G. Equivalently, it is the average cost per edge associated with the unique stationary Markov chain of maximum entropy on G. The inverse of α_G^{up} is the minimum average cost per edge among the cycles in G.*

To the best of our knowledge this is the first sub-exponential approximation of the size of the cost-t length-$\lfloor \alpha t \rfloor$ follower set of a general channel graph G. Notably, the multiplicative term following $t^{-1/2}$ is (asymptotically) independent of t and only depends on α. Further terms in the asymptotic expansion are effectively computable (with each successive term becoming ever-more unwieldy).

Example III.8. *Consider the graph in Fig. 2, representing DNA synthesis using the alternating sequence* $\mathsf{AC}\ \mathsf{AC}\ \ldots$ *over the binary alphabet $\{\mathsf{A},\mathsf{C}\}$. A direct application of Theorem III.6 gives, for $0 < \alpha < \frac{2}{3}$,*

$$N(t,\alpha t) \sim 2^{\alpha t}$$

and, for $\frac{2}{3} < \alpha < 1$,

$$N(t,\alpha t) \sim \left(\frac{2\alpha-1}{1-\alpha}\right)^t \left(\frac{\alpha(1-\alpha)}{(1-2\alpha)^2}\right)^{\alpha t} t^{-1/2} \cdot \gamma(\alpha)$$
$$= 2^{\alpha h(\alpha^{-1}-1)t} t^{-1/2} \cdot \gamma(\alpha),$$

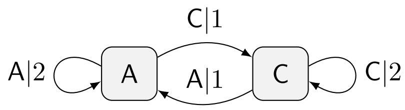

Fig. 2: Channel graph for DNA synthesis using the binary alternating sequence AC AC ….

where $\gamma(\alpha) = \sqrt{(2\alpha-1)\alpha}/(3\alpha-2)\sqrt{2\pi(1-\alpha)}$ *and* $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$.

Our next result, which follows from Theorem III.6, shows that the cost-constrained combinatorial capacity can be expressed as a logarithmic function of the two-dimensional minimal singularity $(x_0, 1/\rho_G(x_0))$ of $F_{G,v}(x,y)$. This generalizes Shannon's classical result which expresses the capacity of a constrained system in terms of the logarithm of a one-dimensional minimal singularity.

Theorem III.9. *Let G be a strongly connected, deterministic, and cost-diverse graph. For all α with $0 \leq \alpha \leq \alpha_G^{\mathsf{lo}}$, we have*

$$C_G(\alpha) = \alpha \log_2 \rho_G(1).$$

For all α with $\alpha_G^{\mathsf{lo}} < \alpha < \alpha_G^{\mathsf{up}}$,

$$C_G(\alpha) = -\log_2 x_0 + \alpha \log_2 \rho_G(x_0), \tag{3}$$

where x_0 is the unique real solution to $\alpha x \rho_G'(x) = \rho_G(x)$ in the interval $0 < x < 1$. For all $\alpha > \alpha_G^{\mathsf{up}}$, we have that $C_G(\alpha) = 0$.

Theorem III.9 improves over previous work [8], [16]–[18] in several ways. None of them recognizes the role of cost-diversity, nor do they address the full domain of the cost-constrained capacity.

Interestingly, our formula (3) for the cost-constrained combinatorial capacity is identical to the formula for the cost-constrained probabilistic capacity in [16], [17] (up to differences in notation). Thus, an immediate corollary of Theorem III.9 is the equivalence between cost-constrained combinatorial and probabilistic capacities.

Corollary III.10. *[18] For any strongly connected and cost-diverse graph G, the cost-constrained combinatorial capacity $C_G(\alpha)$ and the cost-constrained probabilistic capacity $C_{G,\mathrm{prob}}(\alpha)$ are equal.*

Our proof contrasts with that in [18] which used typical sequence arguments, converse inequalities, optimization techniques, and an outer-product relationship between the derivative of $\boldsymbol{P}_G(x)$ and $\rho_G'(x)$.

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References

[1] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, Oct. 1948.
[2] B. H. Marcus, R. M. Roth, and P. H. Siegel, *An Introduction to Coding for Constrained Systems*. Lecture Notes, 2001. [Online]. Available: https://personal.math.ubc.ca/~marcus/Handbook/
[3] I. Djordjevic, W. Ryan, and B. Vasic, *Coding for Optical Channels*. Springer Science & Business Media, 2010.
[4] Y. Liu, "Coding Techniques to Extend the Lifetime of Flash Memories," Ph.D. dissertation, University of California, San Diego, San Diego, CA, USA, 2020.
[5] Y. Liu, P. Huang, A. W. Bergman, and P. H. Siegel, "Rate-constrained shaping codes for structured sources," *IEEE Trans. Inf. Theory*, vol. 66, no. 8, pp. 5261–5281, Aug. 2020.
[6] Y. Liu, Y. Li, P. Huang and P. H. Siegel, "Rate-constrained shaping codes for finite-state channels with cost," in *Proc. Int. Symp. Inf. Theory (ISIT)*. Espoo, Finland, Jun.–Jul. 2022, pp. 1354–1359.
[7] A. Lenz, Y. Liu, C. Rashtchian, P. H. Siegel, A. Wachter-Zeh, and E. Yaakobi, "Coding for efficient DNA synthesis," in *Proc. Int. Symp. Inf. Theory (ISIT)*. Los Angeles, CA, USA, Jun. 2020, pp. 2885–2890.
[8] A. Khandekar, R. J. McEliece, and E. R. Rodemich, "The discrete noiseless channel revisited," in *Coding, Communications, and Broadcasting*. Badlock: Research Studies Series Ltd., UK, 2000, pp. 115–137.
[9] G. Böcherer, V. C. da Rocah Jr., and C. Pimentel, "Capacity of general discrete noiseless channels," *arXiv:0802.2451 [cs.IT]*, Jun. 2010.
[10] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*. Cambridge ; New York: Cambridge University Press, 2009.
[11] G. Böcherer, "Analytic Asymptotics of Discrete Noiseless Channels," Master's thesis, Swiss Federal Institute of Technology Zurich, 2007.
[12] G. Böcherer, V. C. da Rocha Jr., C. Pimentel, and R. Mathar, "Maximum entropy rate of Markov sources for systems with non-regular constraints," *arXiv:0809.1252 [cs.IT]*, Sep. 2008.
[13] G. Böcherer, V. C. da Rocha Jr., C. Pimentel, and R. Mathar, "On the capacity of constrained systems," in *Proc. Int. ITG Conf. Source Channel Coding (SCC)*, 2010.
[14] R. Pemantle and M. C. Wilson, *Analytic Combinatorics in Several Variables*, ser. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2013, no. 140.
[15] S. Melczer, *An Invitation to Analytic Combinatorics: From One to Several Variables*, ser. Texts & Monographs in Symbolic Computation. Springer International Publishing, 2021.
[16] R. J. McEliece and E. R. Rodemich, "A maximum entropy Markov chain," in *Proc. Conf. Inform. Sci. Syst.*, Johns Hopkins University, Mar. 1983, pp. 245–248.
[17] J. Justesen and T. Hohøldt, "Maxentropic Markov chains," *IEEE Trans. Inf. Theory*, vol. 30, no. 4, pp. 665–667, Jul. 1984.
[18] J. B. Soriaga and P. H. Siegel, "On the design of finite-state shaping encoders for partial-response channels," in *Proc. Inf. Theory Appl. Workshop*, San Diego, CA, USA, Feb. 2006.
[19] A. Lenz, S. Melczer, C. Rashtchian, and P. H. Siegel, "Multivariate analytic combinatorics for cost constrained channels and subsequence enumeration," *arXiv:2111.06105 [cs.IT]*, Nov. 2021.