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Abstract

Suppose $\mathcal{H}$ is an admissible Heegaard diagram for a balanced sutured manifold (M,γ). We prove that the number of generators of the associated sutured Heegaard Floer complex is an upper bound on the dimension of the sutured instanton homology $SHI(M,\gamma)$. It follows, in particular, that strong L-spaces are instanton L-spaces.

1. Introduction

Let (M,γ) be a balanced sutured manifold. Kronheimer and Mrowka conjectured [KM10] that its sutured instanton homology is isomorphic to its sutured Heegaard Floer homology,

$$SHI(M,\gamma) \cong SFH(M,\gamma) \otimes \mathbb{C}. \tag{1.1}$$

Proving this remains a major open problem. In particular, it would imply isomorphisms

$$I^{\#}(Y) \cong \widehat{HF}(Y) \otimes \mathbb{C},$$

$$KHI(Y,K) \cong \widehat{HFK}(Y,K) \otimes \mathbb{C},$$

between the invariants of closed 3-manifolds and knots in the instanton and Heegaard Floer settings.

There has been a flood of recent work proving these isomorphisms for various families of closed 3-manifolds and knots; see [BS21, ABDS22, LPCS20, LY22, GL19]. In this paper, we initiate a systematic approach to the general isomorphism (1.1). Before stating our main result, let us establish some notation.

Given a sutured Heegaard diagram

$$\mathcal{H} = \big(\Sigma, \alpha = \{\alpha_1, \dots, \alpha_k\}, \beta = \{\beta_1, \dots, \beta_k\}\big)$$

for a balanced sutured manifold (M,γ), let

$$\mathbb{T}_\alpha := \alpha_1 \times \cdots \times \alpha_k \quad \text{and} \quad \mathbb{T}_\beta := \beta_1 \times \cdots \times \beta_k \subset \mathrm{Sym}^k(\Sigma)$$

denote the usual tori in the k-fold symmetric product of Σ, and let

$$\mathfrak{S}(\mathcal{H}) := \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^k(\Sigma).$$

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If $\mathcal{H}$ is *admissible*, then $\mathfrak{S}(\mathcal{H})$ is the set of generators for the sutured Heegaard Floer complex $SFC(\mathcal{H})$ as defined by Juhász in [Juh06].[1] Our main theorem is the following.

Theorem 1.1. *If $\mathcal{H}$ is an admissible sutured Heegaard diagram for (M,γ), then*

$$\dim_{\mathbb{C}} SHI(M,\gamma) \leq |\mathfrak{S}(\mathcal{H})|.$$

Remark 1.2. Theorem 1.1 does not hold without the assumption that $\mathcal{H}$ is admissible; see Remark 3.6.

Remark 1.3. Our proof of Theorem 1.1 also works for sutured monopole homology (SHM) in place of SHI. Of course, the SHM version of our main result follows from the isomorphism

$$SHM(M,\gamma) \cong SFH(M,\gamma),$$

which is a consequence of the equivalence between monopole and Heegaard Floer homology, see [Lek13]. Still, it may be of value to know that one can prove the inequality

$$\dim_{\mathbb{Z}} SHM(M,\gamma) \leq |\mathfrak{S}(\mathcal{H})|$$

without going through the proof of this equivalence.

Given a balanced sutured manifold (M,γ), we define the *simultaneous trajectory number* $\mathcal{T}(M,\gamma)$ to be the minimum of $|\mathfrak{S}(\mathcal{H})|$ over all admissible sutured Heegaard diagrams $\mathcal{H}$ for (M,γ). This is the generalization to balanced sutured manifolds of a notion originally defined for rational homology 3-spheres by Ozsváth and Szabó in [OS04a]. It admits a purely Morse-theoretic interpretation when $H_1(M,\partial M;\mathbb{Q}) = 0$, and is a measure of the topological complexity of (M,γ); for example, $\mathcal{T}(M,\gamma) = 1$ if and only if (M,γ) is a product sutured manifold.[2] Further, it is clear from the definition that

$$\mathrm{rk}_{\mathbb{Z}} SFH(M,\gamma) \leq \mathcal{T}(M,\gamma).$$

We have the following immediate corollary of Theorem 1.1.

Corollary 1.4. *If (M,γ) is a balanced sutured manifold, then*

$$\dim_{\mathbb{C}} SHI(M,\gamma) \leq \mathcal{T}(M,\gamma).$$

For the following corollaries, we recall the natural sutured manifolds associated to closed 3-manifolds and knots therein. Given a closed 3-manifold Y, let $(Y(1),\delta)$ denote the sutured manifold obtained by removing a 3-ball from Y, where δ is a simple closed curve on $\partial Y(1) \cong S^2$. Similarly, given a knot $K \subset Y$, let $(Y(K), m \cup -m)$ be the sutured manifold obtained by removing a tubular neighborhood of K, where m and $-m$ are oppositely oriented meridional curves on $\partial Y(K) \cong T^2$. The framed instanton and Heegaard Floer homologies of a closed 3-manifold Y are given by

$$I^{\#}(Y) \cong SHI(Y(1),\delta),$$

$$\widehat{HF}(Y) \cong SFH(Y(1),\delta).$$

Likewise the instanton and Heegaard knot Floer homologies of a knot $K \subset Y$ are given by

$$KHI(Y,K) \cong SHI(Y(K), m \cup -m),$$

$$\widehat{HFK}(Y,K) \cong SFH(Y(K), m \cup -m).$$

[1] See § 3.2 for the definition of *admissible*; every $\mathcal{H}$ is admissible when $H_1(M,\partial M;\mathbb{Q}) = 0$.

[2] This is a fun exercise which we have not seen written down before: it is a generalization of the well-known fact that S^3 is the only closed 3-manifold with simultaneous trajectory number one [OS04a].

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Recall that the ranks of the Heegaard Floer homology and framed instanton homology of a rational homology 3-sphere Y are each bounded below by $|H_1(Y)|$. An *L-space*, respectively *instanton L-space*, is a rational homology 3-sphere which achieves these lower bounds

$$\mathrm{rk}_{\mathbb{Z}}\widehat{HF}(Y) = |H_1(Y)|,$$
$$\dim_{\mathbb{C}} I^{\#}(Y) = |H_1(Y)|,$$

respectively. A *strong L-space*, as defined by Levine and Lewallen in [LL12], is a rational homology 3-sphere Y which satisfies the stronger condition[3]

$$\mathcal{T}(Y(1), \delta) = |H_1(Y)|.$$

Indeed, this condition implies that Y is an L-space, because

$$|H_1(Y)| \leq \mathrm{rk}_{\mathbb{Z}}\widehat{HF}(Y) \leq \mathcal{T}(Y(1), \delta).$$

The following is then an immediate corollary of Theorem 1.1.

Corollary 1.5. *If Y is a strong L-space, then it is an instanton L-space.*

More generally, we say that a sutured Heegaard diagram $\mathcal{H}$ for a sutured manifold (M, γ) is *strong* if $\mathcal{H}$ is admissible and the sutured Floer complex $SFC(\mathcal{H})$ has trivial differential. We then have the following.

Corollary 1.6. *If (M, γ) is a balanced sutured manifold which has a strong sutured Heegaard diagram, then*

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leq rk_{\mathbb{Z}} SFH(M, \gamma).$$

For example, when K is a $(1, 1)$-knot in a lens space $L(p, q)$, the sutured manifold

$$SHI((L(p, q))(K), m \cup -m)$$

has a strong Heegaard diagram. We thus reproduce the following result of Li and Ye [LY22].

Corollary 1.7. *If $K \subset L(p, q)$ is a $(1, 1)$-knot, then*

$$\dim_{\mathbb{C}} KHI(L(p, q), K) \leq rk_{\mathbb{Z}}\widehat{HFK}(L(p, q), K).$$

1.1 On the proof

Given a vertical tangle T in a balanced sutured manifold (M, γ), one forms an associated sutured manifold (M_T, γ_T) by removing a neighborhood of T from M, and adding meridians of the components of T to γ; see § 2 for more details. Li and Ye proved the following dimension inequality in [LY22, Proposition 3.14].

Theorem 1.8. *If T is a vertical tangle in (M, γ) such that $[T_i] = 0$ in $H_1(M, \partial M; \mathbb{Q})$ for each component T_i of T, then*

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leq \dim_{\mathbb{C}} SHI(M_T, \gamma_T).$$

To prove Theorem 1.1, we first establish the same inequality under the weaker assumption that T (rather than each of its components) is rationally nullhomologous, in § 2.

Theorem 1.9. *If T is a vertical tangle in (M, γ) such that $[T] = 0$ in $H_1(M, \partial M; \mathbb{Q})$, then*

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leq \dim_{\mathbb{C}} SHI(M_T, \gamma_T).$$

[3] Strong L-spaces are also of interest because Levine and Lewallen were able to show that their fundamental groups are not left-orderable, as predicted by the L-space Conjecture.

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Next, given an admissible sutured Heegaard diagram $\mathcal{H}$ for (M,γ), we construct a vertical tangle $T \subset (M,\gamma)$ with $[T]=0$ in $H_1(M,\partial M;\mathbb{Q})$, such that

$$\dim_{\mathbb{C}} SHI(M_T,\gamma_T) = |\mathfrak{S}(\mathcal{H})|.$$

This is the content of § 3. Theorem 1.1 then follows from Theorem 1.9.

Remark 1.10. Theorem 1.8 suffices to prove Theorem 1.1 in the case where $H_1(M,\partial M;\mathbb{Q})=0$, but we need the stronger Theorem 1.9 in general.

1.2 Organization

In § 2, we prove the inequality in Theorem 1.9. We then use this in § 3 to prove our main result, Theorem 1.1. Finally, in § 4, we discuss further directions, some of which are in progress. In particular, we discuss the possibility of using the proof of Theorem 1.1 to construct a grading on *SHI* by homotopy classes of 2-plane fields, and the prospects for upgrading Theorem 1.1 to a proof of the isomorphism (1.1).

2. A dimension inequality

A *vertical tangle* in a balanced sutured manifold (M,γ) is a properly embedded 1-manifold

$$T = T_1 \cup \cdots \cup T_n$$

in M, with boundary in $R(\gamma)$, whose components T_i satisfy

$$\partial T_i \cap R_+(\gamma) \neq \emptyset \neq \partial T_i \cap R_-(\gamma),$$

and are oriented from $R_+(\gamma)$ to $R_-(\gamma)$ (in particular, T has no closed components). One forms an associated balanced sutured manifold (M_T,γ_T) by removing tubular neighborhoods of the components T_i, and adding positively-oriented meridians of these components to the suture γ, as in [LY22, § 3] and depicted in Figure 7. In this section, we prove Theorem 1.9, which states that

$$\dim_{\mathbb{C}} SHI(M,\gamma) \leq \dim_{\mathbb{C}} SHI(M_T,\gamma_T)$$

when $[T]=0$ in $H_1(M,\partial M;\mathbb{Q})$. The rough idea is to turn T into a related tangle T' whose components are rationally nullhomologous, and apply Theorem 1.8.

Proof of Theorem 1.9. Let $T_1,\dots,T_n$ be the components of T. For $i=1,\dots,n$, let

$$\partial T_i = q_i - p_i,$$

where $p_i \in R_+(\gamma)$ and $q_i \in R_-(\gamma)$. We may assume that γ is connected, because we can achieve this by adding contact 1-handles to (M,γ), an operation which does not change $SHI(M,\gamma)$ (equivalently, *SHI* is invariant under product disk decomposition [KM10, BS16]). Then we can find a sequence of pairwise disjoint arcs

$$\xi_1,\dots,\xi_n \subset \partial M$$

such that, for each $i=1,\dots,n$, we have:

- $\partial\xi_i = p_{i+1} - q_i$ (where $p_{n+1} := p_1$); and
- ξ_i intersects γ in exactly one point.

For every $i \in \{2,\dots,n\}$, choose an arc $t_i \subset \partial M$ in a neighborhood of the unique intersection point $\xi_{i-1} \cap \gamma$, as depicted in Figures 1 and 2. Push the interior of t_i into the interior of M to

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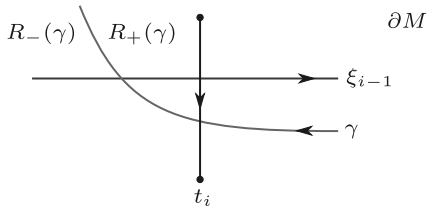

FIGURE 1. The point of view is from the interior of M, looking at ∂M.

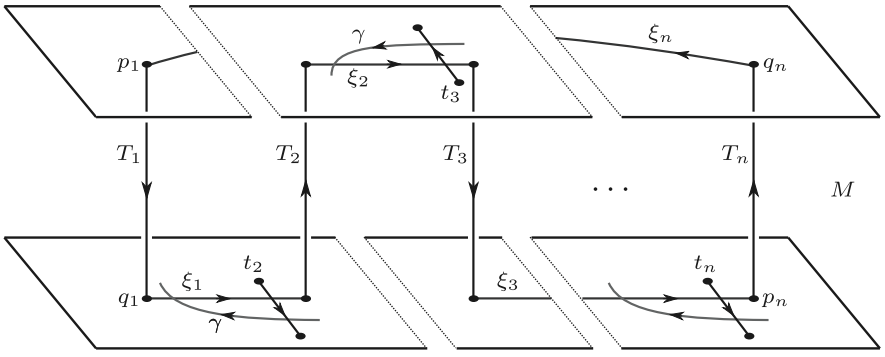

FIGURE 2. The tangle $T = T_1 \cup \cdots \cup T_n$ in M and the arcs ξ_i and t_i in ∂M, together with the suture γ.

turn this arc into a vertical tangle T_i', and let

$$T'' = T_2' \cup \cdots \cup T_n'.$$

Then

$$(M_{T''}, \gamma_{T''}) = (M - N(T''), \gamma \cup \mu_2' \cup \cdots \cup \mu_n'),$$

where μ_i' is a positively oriented meridian of T_i'. Each component T_i' cobounds a disk in M with the arc t_i. These disks then restrict to properly embedded disks

$$D_2, \ldots, D_n \subset M_{T''}$$

with $|D_i \cap \gamma| = 1$ and $|D_i \cap \mu_j'| = \delta_{ij}$, so that

$$|D_i \cap \gamma_{T''}| = 2.$$

Thus, each D_i is a product disk.

Next, consider the arc

$$t_1 = T_1 \cup \xi_1 \cup T_2 \cup \xi_2 \cup \cdots \cup T_{n-1} \cup \xi_{n-1} \cup T_n \subset M_{T''}.$$

Push its interior into the interior of $M_{T''}$ to form a vertical tangle T_1' with $\partial T_1' = q_n - p_1$, as in Figure 3. Let T' be the tangle in M given by

$$T' = T_1' \cup T'' = T_1' \cup \cdots \cup T_n'.$$

We refer to a tangle T' formed in this way as a *mixed tangle for* T. Note that

$$(M_{T'}, \gamma_{T'}) = ((M_{T''})_{T_1'}, (\gamma_{T''})_{T_1'}) = (M - N(T'), \gamma \cup \mu_1' \cup \cdots \cup \mu_n'),$$

where μ_1' is a positively oriented meridian of T_1'. Observe that the disks $D_2, \ldots, D_n \subset M_{T''}$ restrict to properly embedded annuli

$$A_2, \ldots, A_n \subset M_{T'}$$

because T_1' intersects each disk in exactly one point, as shown in Figure 4.

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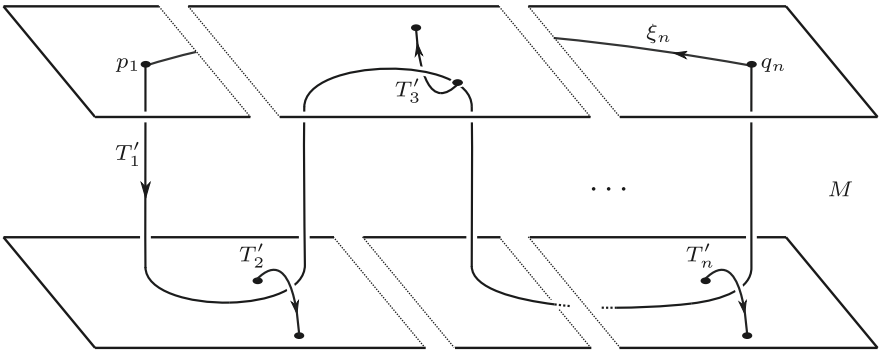

FIGURE 3. The tangle $T' = T_1' \cup \cdots \cup T_n'$ in M.

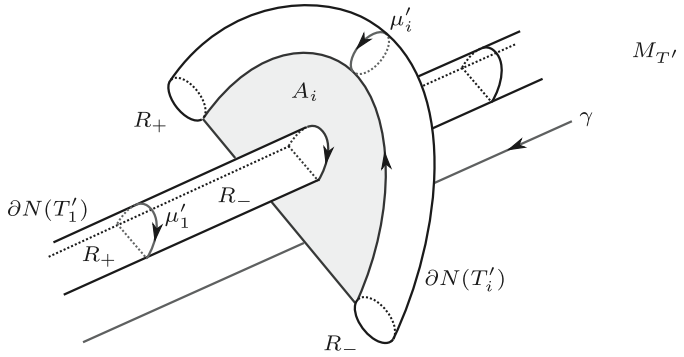

FIGURE 4. The annulus A_i in $M_{T'}$ near the boundaries of the tubular neighborhoods of the components T_1' and T_i', as seen from inside $M_{T'}$.

The endpoints of the arc $\xi_n \subset \partial M_{T''}$ agree with $\partial T_1'$, and $|\xi_n \cap \gamma| = 1$. We can, thus, use ξ_n together with γ and μ_1' to define a sequence of sutures $\Gamma_m \subset \partial M_{T'}$ for $m \in \mathbb{N}$, as in [LY22, § 3.2], which one should regard as 'longitudinal' sutures for T_1'; see Figure 5. By the construction of T_1' and the assumption that T is rationally nullhomologous in $(M, \partial M)$, we have

$$[T_1'] = [T] = 0 \in H_1(M_{T''}, \partial M_{T''}; \mathbb{Q}).$$

Therefore, by [LY22, Lemmas 3.21 and 3.22], we have the following.

LEMMA 2.1. *There is an exact triangle*

$$\begin{array}{ccc} SHI(-M_{T'}, -\Gamma_m) & \longrightarrow & SHI(-M_{T'}, -\Gamma_{m+1}) \\ & {\scriptstyle G_m} \nwarrow \quad \swarrow {\scriptstyle F_{m+1}} & \\ & SHI(-M_{T''}, -\gamma_{T''}), & \end{array}$$

coming from the surgery exact triangle associated to surgeries on the meridian μ_1' of T_1'. Furthermore, $G_m \equiv 0$ for m sufficiently large.

For each $i = 2, \ldots, n$ and every $m \in \mathbb{N}$, we have that

$$|A_i \cap \Gamma_m| = 4.$$

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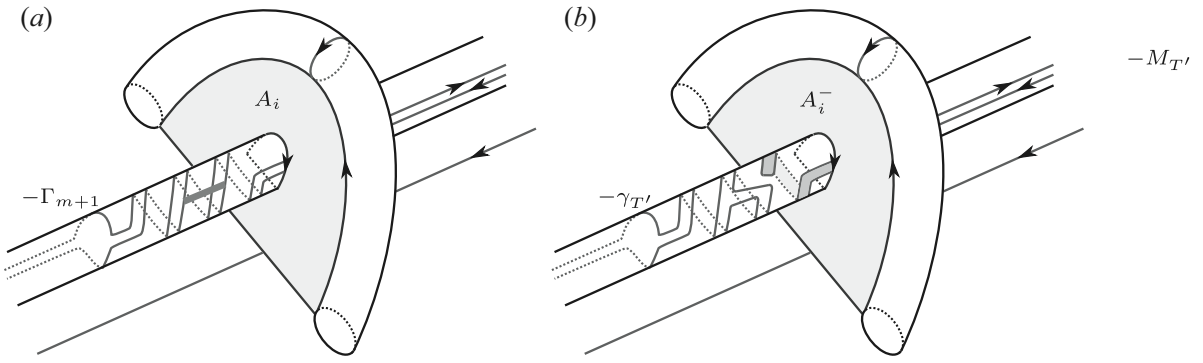

FIGURE 5. (a) The suture $-\Gamma_{m+1}$ and the bypass arc η_- shown in bold. (b) The suture $-\gamma_{T'}$ resulting from the bypass attachment along η_-, and the negatively stabilized annulus $A_i^- \subset (-M_{T'}, -\gamma_{T'})$.

Let us orient each A_i so that the induced orientation on ∂A_i is opposite the orientation of ∂D_i coming from that of T_i', as in Figures 4 and 5. By [GL19], the disks $D_2, \dots, D_n$ induce a $\mathbb{Z}^{n-1}$-grading on

$$SHI(-M_{T''}, -\gamma_{T''}).$$

Similarly, the annuli $A_2, \dots, A_n$ induce a $\mathbb{Z}^{n-1}$-grading on

$$SHI(-M_{T'}, -\Gamma_m)$$

for each $m \in \mathbb{N}$, and we have the following graded version of the triangle in Lemma 2.1.

LEMMA 2.2. *The exact triangle of Lemma 2.1 restricts to the exact triangle*

$$\begin{array}{ccc} SHI(-M_{T'}, -\Gamma_m, (A_2, \dots, A_n), (0, \dots, 0)) & \longrightarrow & SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \dots, A_n), (0, \dots, 0)) \\ & {\scriptstyle G_m} \nwarrow \quad \swarrow {\scriptstyle F_{m+1}} & \\ & SHI(-M_{T''}, -\gamma_{T''}, (D_2, \dots, D_n), (0, \dots, 0)). & \end{array}$$

Proof of Lemma 2.2. We prove that the map F_{m+1} preserves the gradings. The arguments for the other two maps are similar.

Let us first recall the definition of F_{m+1} from [LY22, § 3]. Pick a *closure*

$$(Y_{m+1}, R_{m+1}, \omega_{m+1})$$

for $(-M_{T'}, -\Gamma_{m+1})$ so that each annulus A_i extends to a closed surface $\bar{A}_i \subset Y_{m+1}$, as in [Li21, § 3]. By the construction therein,

$$g(\bar{A}_i) = 2$$

for $i = 2, \dots, n$, because each component of ∂A_i intersects Γ_{m+1} in two points. The sutured instanton homology of $(-M_{T'}, -\Gamma_{m+1})$ is defined as a certain direct summand

$$SHI(-M_{T'}, -\Gamma_{m+1}) = I_*(Y_{m+1}|R_{m+1})_{\omega_{m+1}}$$

of the instanton Floer homology $I_*(Y_{m+1})_{\omega_{m+1}}$ as in [KM10, § 7]. The summand

$$SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \dots, A_n), (i_2, \dots, i_n))$$

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is defined as the simultaneous (generalized) $(2i_2, \dots, 2i_n)$-eigenspace of commuting operators

$$\mu(\bar{A}_2), \dots, \mu(\bar{A}_n) : SHI(-M_{T'}, -\Gamma_{m+1}) \to SHI(-M_{T'}, -\Gamma_{m+1})$$

associated to these surfaces.

The meridian μ_1' of T_1' can be thought of as an embedded circle in Y_{m+1}. Let Y be the manifold obtained from Y_{m+1} via 0-surgery on μ_1', with respect to the framing of μ_1' induced by $\partial M_{T'}$. Since μ_1' is disjoint from R_{m+1} and the $\bar{A}_i$, these surfaces survive in Y. By [BS16, § 3.3], $(Y, R_{m+1}, \omega_{m+1})$ is a closure of $(-M_{T''}, -\gamma_{T''})$. The map F_{m+1} is then induced by the cobordism given by the trace of 0-surgery on μ_1'. Since $\bar{A}_i \subset Y_{m+1}$ is homologous to $\bar{A}_i \subset Y$ in this cobordism, F_{m+1} respects the eigenspaces of $\mu(\bar{A}_i)$. Thus, F_{m+1} maps

$$SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \dots, A_n), (i_2, \dots, i_n))$$

into

$$SHI(-M_{T''}, -\gamma_{T''}, (A_2, \dots, A_n), (i_2, \dots, i_n)).$$

Now, the 0-surgery on μ_1' makes each $\bar{A}_i$ compressible in Y; in particular, each $\bar{A}_i \subset Y$ is homologous to the disjoint union of two tori

$$T_i^1 \cup T_i^2 \subset Y.$$

One of these tori, say T_i^1, is the extension $\bar{D}_i \subset Y$ of $D_i \subset -M_{T''}$ that is used to define the grading on $SHI(-M_{T''}, -\gamma_{T''})$ associated to D_i. Since

$$\bar{A}_i = \bar{D}_i + T_i^2$$

in $H_2(Y)$, the k-eigenspace of $\mu(\bar{A}_i)$ agrees with the k-eigenspace of $\mu(\bar{D}_i)$ for every k, by [BS22, Corollary 2.9]. Thus, we have that

$$\begin{aligned} &SHI(-M_{T''}, -\gamma_{T''}, (A_2, \dots, A_n), (i_2, \dots, i_n)) \\ &\quad = SHI(-M_{T''}, -\gamma_{T''}, (D_2, \dots, D_n), (i_2, \dots, i_n)). \end{aligned}$$

Putting these arguments together, we see that F_{m+1} preserves the $\mathbb{Z}^{n-1}$-gradings as claimed in the lemma. □

Note that decomposing $(-M_{T''}, -\gamma_{T''})$ along $D_2 \cup \dots \cup D_n$ yields $(-M, -\gamma)$. By [Li21, Lemma 4.2], we therefore have

$$SHI(-M_{T''}, -\gamma_{T''}, (D_2, \dots, D_n), (0, \dots, 0)) \cong SHI(-M, -\gamma).$$

Hence, for m sufficiently large, Lemmas 2.1 and 2.2 imply that

$$\begin{aligned} \dim_{\mathbb{C}} SHI(-M, -\gamma) = &\dim_{\mathbb{C}} SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \dots, A_n), (0, \dots, 0)) \\ &- \dim_{\mathbb{C}} SHI(-M_{T'}, -\Gamma_m, (A_2, \dots, A_n), (0, \dots, 0)), \end{aligned} \tag{2.1}$$

since $G_m \equiv 0$.

Next, we consider attaching a bypass to $(-M_{T'}, -\Gamma_{m+1})$ along the arc η_- in Figure 5. By [BS22, § 4], this attachment gives rise to a bypass exact triangle. As discussed in [LY22, § 3], the other two sutures involved in the triangle are $-\Gamma_m$ and $-\gamma_{T'}$. It is straightforward to check that the bypass attachment along η_- creates a *negative stabilization*

$$A_i^- \subset (-M_{T'}, -\gamma_{T'})$$

of A_i, for each $i = 2, \dots, n$, in the sense of [Li21, Definition 3.1]. Hence, as in the proof of [Li21, Proposition 5.5], we have the following graded version of the bypass exact triangle of

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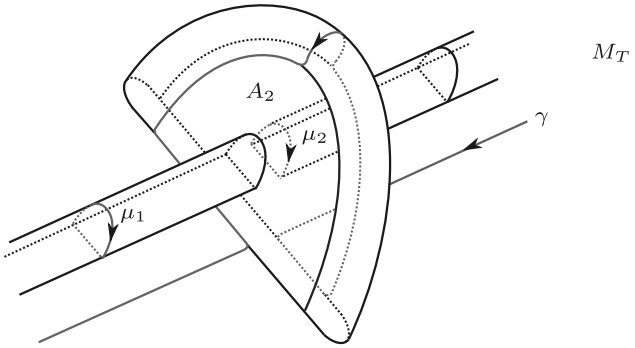

Figure 6. The result of decomposing $(M_{T'}, \gamma_{T'})$ along $A_2 \cup \cdots \cup A_n$ is simply (M_T, γ_T). This is illustrated above in the case $n = 2$.

[BS22, Theorem 1.20]:

$$\begin{array}{ccc} SHI(-M_{T'}, -\Gamma_m, (A_2, \ldots, A_n), (0, \ldots, 0)) & \longrightarrow & SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \ldots, A_n), (0,\ ldots, 0)) \\ & \searrow \quad \swarrow & \\ & SHI(-M_{T'}, -\gamma_{T'}, (A_2^-, \ldots, A_n^-), (0, \ldots, 0)), & \end{array}$$

which implies that

$$\begin{aligned} &\dim_{\mathbb{C}} SHI(-M_{T'}, -\gamma_{T'}, (A_2^-, \ldots, A_n^-), (0, \ldots, 0)) \\ &\quad \geq \dim_{\mathbb{C}} SHI(-M_{T'}, -\Gamma_{m+1}, (A_2, \ldots, A_n), (0, \ldots, 0)) \\ &\qquad - \dim_{\mathbb{C}} SHI(-M_{T'}, -\Gamma_m, (A_2, \ldots, A_n), (0, \ldots, 0)). \end{aligned} \tag{2.2}$$

From the grading shifting property [Li21, Theorem 1.12] and [Wan20, Proposition 4.1], we have

$$\begin{aligned} &SHI(-M_{T'}, -\gamma_{T'}, (A_2^-, \ldots, A_n^-), (0, \ldots, 0)) \\ &\quad = SHI(-M_{T'}, -\gamma_{T'}, (A_2^+, \ldots, A_n^+), (-1, \ldots, -1)), \end{aligned} \tag{2.3}$$

where A_i^+ is a *positive stabilization* of A_i. Moreover, from the construction of the gradings and stabilizations in [Li21, § 3], we have

$$\begin{aligned} &SHI(-M_{T'}, -\gamma_{T'}, (A_2^+, \ldots, A_n^+), (-1, \ldots, -1)) \\ &\quad = SHI(-M_{T'}, -\gamma_{T'}, (-(A_2^+), \ldots, -(A_n^+)), (1, \ldots, 1)) \\ &\quad = SHI(-M_{T'}, -\gamma_{T'}, ((-A_2)^-, \ldots, (-A_n)^-), (1, \ldots, 1)). \end{aligned} \tag{2.4}$$

By [Li21, Lemma 4.2], this last group is isomorphic to the sutured instanton homology of the manifold obtained from $(-M_{T'}, -\gamma_{T'})$ by decomposing along $(-A_2)^- \cup \cdots \cup (-A_n)^-$. By [Li21, Lemma 3.2], this is the same as the manifold obtained by decomposing along $-A_2 \cup \cdots \cup -A_n$, which is, after reversing orientation, the manifold obtained from $(M_{T'}, \gamma_{T'})$ by decomposing along $A_2 \cup \cdots \cup A_n$. It is straightforward to check that the latter manifold is simply (M_T, γ_T), as indicated in Figure 6 in the case $n = 2$. Thus,

$$SHI(-M_{T'}, -\gamma_{T'}, ((-A_2)^-, \ldots, (-A_n)^-), (1, \ldots, 1)) \cong SHI(-M_T, -\gamma_T). \tag{2.5}$$

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Finally, combining (2.1)–(2.5), we have that

$$\begin{aligned}\dim_{\mathbb{C}} SHI(-M_T,-\gamma_T) &= \dim_{\mathbb{C}} SHI(-M_{T'},-\gamma_{T'},((-A_2)^-,\dots,(-A_n)^-),(1,\dots,1))\\ &= \dim_{\mathbb{C}} SHI(-M_{T'},-\gamma_{T'},(A_2^-,\dots,A_n^-),(0,\dots,0))\\ &\geq \dim_{\mathbb{C}} SHI(-M_{T'},-\Gamma_{m+1},(A_2,\dots,A_n),(0,\dots,0))\\ &\quad - \dim_{\mathbb{C}} SHI(-M_{T'},-\Gamma_m,(A_2,\dots,A_n),(0,\dots,0))\\ &= \dim_{\mathbb{C}} SHI(-M,-\gamma).\end{aligned}$$

Given the symmetry of *SHI* under orientation reversal, this proves Theorem 1.9. □

3. Proof of Theorem 1.1

3.1 Full tangles

Let (M,γ) be a balanced sutured manifold. Let

$$\mathcal{H} = \big(\Sigma, \alpha = \{\alpha_1,\dots,\alpha_k\}, \beta = \{\beta_1,\dots,\beta_k\}\big)$$

be any (not necessarily admissible) sutured Heegaard diagram for (M,γ). This means that M is obtained from $\Sigma\times[-1,1]$ by attaching 3-dimensional 2-handles

$$\mathbb{D}_{\alpha_i} = D^2_{\alpha_i}\times I,$$
$$\mathbb{D}_{\beta_i} = D^2_{\beta_i}\times I,$$

along $A_i\times\{-1\}$ and $B_i\times\{+1\}$, where A_i and B_i are annular neighborhoods of α_i and β_i, respectively, for $i=1,\dots,k$. The suture γ is given by

$$\gamma = \partial\Sigma\times\{0\}.$$

We next define a special class of vertical tangles in (M,γ) associated to $\mathcal{H}$.

Let $R_1,\dots,R_n$ be the regions of Σ–α–β disjoint from $\partial\Sigma$. For each $i=1,\dots,n$, let $p_{i1},\dots,p_{ia_i}$ be a_i distinct points in R_i, for some integer $a_i\geq 1$. Let

$$T_{ij} = p_{ij}\times[-1,1]\subset\Sigma\times[-1,1],$$

and let

$$T = \bigcup_{i=1}^{n}\bigcup_{j=1}^{a_i} T_{ij}.$$

Then T is a vertical tangle in (M,γ), oriented from $R_+(\gamma)$ to $R_-(\gamma)$. Let D^2_{ij} be a tubular neighborhood of the point $p_{ij}\in R_i$, let

$$N_{ij} = D^2_{ij}\times[-1,1]$$

be a tubular neighborhood of the component T_{ij} in M, and let

$$\gamma_{ij} = \partial D^2_{ij}\times\{0\}\subset\partial N_{ij}$$

be a positively oriented meridian of this component; see Figure 7. Let (M_T,γ_T) be the balanced sutured manifold obtained from M by removing these tubular neighborhoods,

$$M_T = M - \bigcup_{i=1}^{n}\bigcup_{j=1}^{a_i} N_{ij},$$

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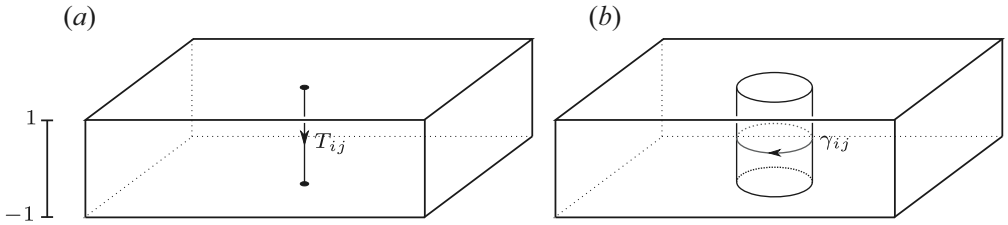

Figure 7. (*a*) The component $T_{ij} = p_{ij} \times [-1, 1] \subset M$. (*b*) The complement of N_{ij} with the meridian γ_{ij}.

where γ_T is the union of γ with meridians of the T_{ij},

$$\gamma_T = \gamma \cup \bigcup_{i=1}^{n} \bigcup_{j=1}^{a_i} \gamma_{ij},$$

as in § 2. We refer to any tangle obtained in this way as a *full tangle for* $\mathcal{H}$.

The main result of this section is the following.

Proposition 3.1. *If T is a full tangle for $\mathcal{H}$, then* $\dim_{\mathbb{C}} SHI(M_T, \gamma_T) = |\mathfrak{S}(\mathcal{H})|$.

Remark 3.2. The analogue of this proposition for *SFH* is immediate. One forms a sutured Heegaard diagram $\mathcal{H}_T$ for (M_T, γ_T) from $\mathcal{H}$ by removing neighborhoods of the $p_{ij} \in \Sigma$. Since there is at least one such point in every region of Σ–α–β not intersecting $\partial\Sigma$, this ensures that $\mathcal{H}_T$ is admissible and that the differential on $SFC(\mathcal{H}_T)$ is zero, so that

$$\mathrm{rk}_{\mathbb{Z}} SFH(M_T, \gamma_T) = \mathrm{rk}_{\mathbb{Z}} SFC(\mathcal{H}_T) = |\mathfrak{S}(\mathcal{H}_T)| = |\mathfrak{S}(\mathcal{H})|.$$

This was the inspiration for our result above.

We need the following for the proof of Proposition 3.1; see [GL19, Corollary 4.3].

Proposition 3.3. *Suppose (M, γ) is an irreducible balanced sutured manifold and $D \subset M$ is a properly embedded disk which intersects γ in four points. Then*

$$SHI(M, \gamma) \cong SHI(M', \gamma') \oplus SHI(M'', \gamma''),$$

where (M', γ') and (M'', γ'') are the decompositions of (M, γ) along D and $-D$, respectively.

Before proving this proposition, we record the following lemma. This lemma is well-known (Kronheimer and Mrowka prove the much harder converse in [KM10, Theorem 7.12]), but since we could not find a concrete reference in the literature we provide a short proof here.

Lemma 3.4. *Suppose (M, γ) is an irreducible balanced sutured manifold. If (M, γ) is not taut, then $SHI(M, \gamma) = 0$.*

Proof. As mentioned in the proof of Lemma 2.2, $SHI(M, \gamma)$ is defined in [KM10, § 7] in terms of a *closure* (Y, R, ω) of (M, γ), where Y is a closed oriented 3-manifold, $R \subset Y$ is a connected closed oriented surface which we can take to have genus $g(R) > 1$, and $\omega \subset Y$ is a simple closed curve with $|\omega \cap R| = 1$. Specifically, $SHI(M, \gamma)$ is the generalized $(2g(R)-2)$-eigenspace of the operator $\mu(R)$ on $I_*(Y)_\omega$.

Now suppose that (M, γ) is irreducible but not taut. Then, by the definition of tautness (e.g. [Juh06, Definition 9.18]), either one of $R_\pm(\gamma)$ is compressible or $R_\pm(\gamma)$ do not minimize the Thurston-norm in their homology classes. In either case, there exists a surface $R' \subset Y$ such

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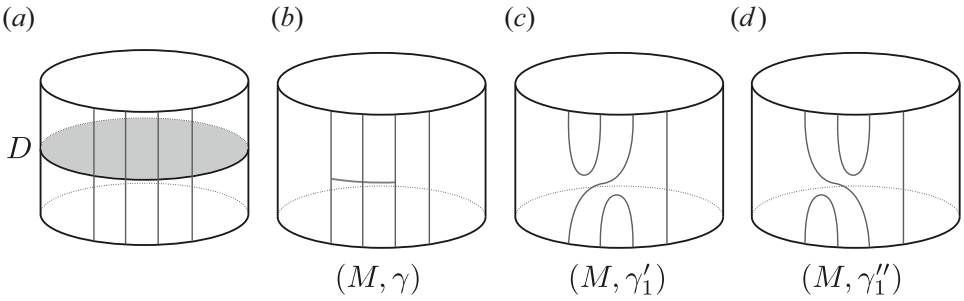

FIGURE 8. (*a*) A neighborhood of the disk $D \subset M$ whose boundary intersects the suture γ in four points. (*b–d*) The arc of attachment for the initial bypass in the triangle.

that $[R'] = [R] \in H_2(Y)$ but $1 \leq g(R') < g(R)$. This implies that $SHI(M, \gamma) = 0$ by [KM10, Proposition 7.5]. □

Proof of Proposition 3.3. Ghosh and Li prove this in [GL19, Corollary 4.3] under the additional assumption that (M, γ) is taut and at least one of (M', γ') and (M'', γ'') is taut. We show that this additional assumption is unnecessary, by showing that the proposition still holds when the assumption is not true.

First, suppose (M, γ) is not taut. Then neither (M', γ') nor (M'', γ'') is taut, by [Gab87, Lemma 0.4]. Note that the irreducibility of M implies the irreducibility of any manifold obtained by cutting M open along a properly embedded disk; in particular, M' and M'' are irreducible. Therefore,

$$SHI(M, \gamma) \cong SHI(M', \gamma') \cong SHI(M'', \gamma'') = 0,$$

by Lemma 3.4, and the proposition holds.

Next, suppose neither (M', γ') nor (M'', γ'') is taut. Consider the bypass exact triangle of [BS22]:

$$\begin{array}{ccc} SHI(-M, -\gamma) & \longrightarrow & SHI(-M, -\gamma'_1) \\ & \nwarrow \quad \swarrow & \\ & SHI(-M, -\gamma''_1), & \end{array}$$

determined by an initial bypass attachment to (M, γ) along an arc in ∂D as shown in Figure 8. The other manifolds (M, γ'_1) and (M, γ''_1) in the triangle product disk decompose (along a copy of D which intersects the new sutures in two points) to (M', γ') and (M'', γ''), respectively. Since SHI is invariant under product disk decomposition, we therefore have

$$SHI(M, \gamma'_1) \cong SHI(M', \gamma') \cong 0,$$
$$SHI(M, \gamma''_1) \cong SHI(M'', \gamma'') \cong 0,$$

by Lemma 3.4. It then follows from the bypass triangle, and the symmetry of SHI under orientation reversal, that $SHI(M, \gamma) = 0$ as well, so the proposition holds. □

Proof of Proposition 3.1. Recall that D^2_{ij} denotes a tubular neighborhood of $p_{ij} \in R_i$. There exists a (possibly empty) set of disjoint, properly embedded arcs

$$d_1, \ldots, d_m \subset \Sigma - \bigcup_{i=1}^{n} \bigcup_{j=1}^{a_i} D_{ij}$$

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which satisfy the following three conditions:

(1) for every $e = 1, \dots, m$, the arc d_e is contained in some region of Σ–α–β;
(2) for every e, either both endpoints of d_e are on $\partial\Sigma$, or each is on some ∂D^2_{ij}; and
(3) $\Sigma - \bigcup_{i=1}^{n}\bigcup_{j=1}^{a_i} D_{ij} - d_1 - \ldots - d_m$ deformation retracts onto $\alpha \cup \beta$.

Now consider the disk

$$\delta_e = d_e \times [-1,1] \subset \left(\Sigma - \bigcup_{i=1}^{n}\bigcup_{j=1}^{a_i} D_{ij}\right) \times [-1,1] \subset \partial M_T,$$

for $e = 1, \dots, m$. The boundary of each δ_e intersects γ_T in two points; hence, δ_e is a product disk. Since *SHI* is invariant under product disk decomposition, let (M_T, γ_T) henceforth refer to the balanced sutured manifold obtained after decomposing along $\delta_1, \dots, \delta_m$.

Then M_T admits the following description. Let $q_1, \dots, q_t$ denote the intersection points between the α and β curves. If q_ℓ is an intersection point between α_i and β_j, let $r_\ell \subset \Sigma$ denote the rectangular component of $A_i \cap B_j$ which contains q_ℓ. Then M_T is the given by the union

$$M_T = \mathbb{D}_{\alpha_1} \cup \cdots \cup \mathbb{D}_{\alpha_k} \cup \mathbb{D}_{\beta_1} \cup \cdots \cup \mathbb{D}_{\beta_k} \cup \tau_1 \cup \cdots \cup \tau_t$$

of the usual 2-handles with the *tubes*

$$\tau_\ell = r_\ell \times [-1,1],$$

as shown in Figure 9. Let c_ℓ denote the union of the four corners of the rectangle r_ℓ. Then the suture γ_T is given by

$$\gamma_T = \bigcup_{i=1}^{k}(\partial D^2_{\alpha_i} \times \partial I) \cup \bigcup_{i=1}^{k}(\partial D^2_{\beta_i} \times \partial I) \cup \bigcup_{\ell=1}^{t}(c_\ell \times [-1,1]) - \bigcup_{\ell=1}^{t}(\partial r_\ell \times \{-1,1\})$$

as shown and oriented in the figure near a tube τ_ℓ. Let

$$m_\ell = r_\ell \times \{0\} \subset \tau_\ell$$

denote the meridional disk of τ_ℓ, oriented as in Figure 9, for $\ell = 1, \dots, t$. Note that the boundary of each m_ℓ intersects the suture γ_T in four points.

Note that M_T is a handlebody and therefore irreducible. We may thus apply Proposition 3.3 to it, as well as to the manifolds obtained by decomposing M_T along any collection of the meridional disks m_ℓ, as these manifolds are simply disjoint unions of handlebodies and, hence, also irreducible. For each t-tuple of signs

$$I = (\epsilon_1, \dots, \epsilon_t) \in \{+,-\}^t,$$

let (M_T^I, γ_T^I) be the sutured manifold obtained by decomposing (M_T, γ_T) along the disks

$$\epsilon_1 m_1 \cup \cdots \cup \epsilon_t m_t.$$

Then

$$SHI(M_T, \gamma_T) \cong \bigoplus_{I \in \{+,-\}^t} SHI(M_T^I, \gamma_T^I),$$

by Proposition 3.3. Each (M_T^I, γ_T^I) is simply a union of 3-balls

$$M_T^I \cong \mathbb{D}_{\alpha_1} \cup \cdots \cup \mathbb{D}_{\alpha_k} \cup \mathbb{D}_{\beta_1} \cup \cdots \cup \mathbb{D}_{\beta_k},$$

which means that $SHI(M_T^I, \gamma_T^I)$ is either $\mathbb{C}$ or trivial, according as whether γ_T^I has exactly one component on the boundary of each of these 3-balls or not. We claim that the nonzero

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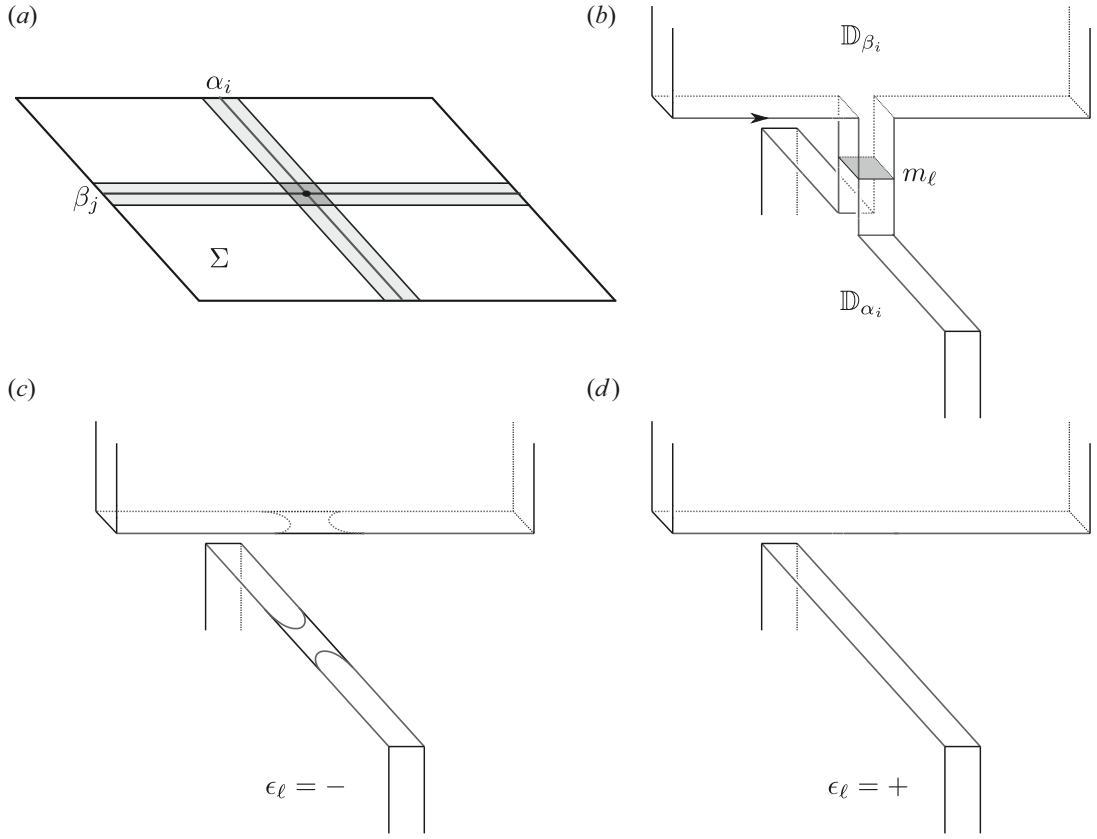

FIGURE 9. (*a*) An intersection point $q_\ell \in \alpha_i \cap \beta_j$. The rectangular component r_ℓ of $A_i \cap B_j$ containing q_ℓ is shown in darker gray. (*b*) The 2-handles $\mathbb{D}_{\alpha_i}$ and $\mathbb{D}_{\beta_j}$ glued together by the tube τ_ℓ. The meridional disk m_ℓ is shown in gray; it intersects the suture γ_T in four points, and its oriented normal points upwards. (*c*) The result of decomposing along $-m_\ell$. (*d*) The result of decomposing along m_ℓ.

summands $SHI(M_T^I, \gamma_T^I)$ are in one-to-one correspondence with the elements of $\mathfrak{S}(\mathcal{H})$, which will then complete the proof.

For this claim, we consider the restriction of γ_T^I to the ball $\mathbb{D}_{\alpha_i}$. Let

$$q_{\ell_1}, \ldots, q_{\ell_p} \in \{q_1, \ldots, q_t\}$$

denote the intersection points between α_i and β. Then γ_T^I restricts to exactly one component on the boundary of $\mathbb{D}_{\alpha_i}$ if and only if exactly one of $\epsilon_{\ell_1}, \ldots \epsilon_{\ell_p}$ is $-$ and the rest are $+$. The analogous statement holds for the restriction of γ_T^I to $\mathbb{D}_{\beta_j}$. Thus, if we let

$$q(I) = \{q_\ell \mid \epsilon_\ell = -\},$$

then γ_T^I restricts to exactly one component on each 3-ball in M_T^I if and only if $q(I) \in \mathfrak{S}(\mathcal{H})$. □

Remark 3.5. Proposition 3.1 also follows from the fact that *SHI* and *SFH* obey the same decomposition laws (Proposition 3.3, and the invariance under product disk decomposition), agree in rank for sutured 3-balls, and

$$\mathrm{rk}_{\mathbb{Z}} SFH(M_T, \gamma_T) = |\mathfrak{S}(\mathcal{H})|,$$

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per Remark 3.2. Our original proof has the advantages that it does not rely on the definition of the differential in SFC, and it establishes a very concrete bijection between the nonzero summands $SHI(M_T^I, \gamma_T^I)$ and elements of $\mathfrak{S}(\mathcal{H})$.

3.2 The proof

Recall that a sutured Heegaard diagram $\mathcal{H}$ for a balanced sutured manifold (M, γ) is *admissible* if and only if every nontrivial periodic domain has both positive and negative multiplicities [Juh06]. This is automatically true of any $\mathcal{H}$ when $H_1(M, \partial M; \mathbb{Q}) = 0$ (there are no nontrivial periodic domains in this case), though every balanced sutured manifold admits an admissible diagram.

Proof of Theorem 1.1. Let $\mathcal{H} = (\Sigma, \alpha, \beta)$ be an admissible sutured Heegaard diagram for (M, γ). Then we can assign a positive integer area a_i to each region R_i of Σ–α–β disjoint from $\partial\Sigma$, so that the signed area of every periodic domain is zero; see [OS04b, Lemma 4.12].[4] Fix a_i distinct points $p_{i1}, \dots, p_{ia_i} \in R_i$ for each i, and let

$$T = \bigcup_{i=1}^{n} \bigcup_{j=1}^{a_i} T_{ij} \subset M$$

be the corresponding full tangle for $\mathcal{H}$, as in § 3.1.

We claim that $[T] = 0$ in $H_1(M, \partial M; \mathbb{Q})$. To see this, note that for every periodic domain P of $\mathcal{H}$, the intersection number of T with the 2-cycle in M represented by P is negative the signed area of P, which is zero. Since the homology classes represented by periodic domains span $H_2(M)$, the claim follows. Theorem 1.9 therefore implies that

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leq \dim_{\mathbb{C}} SHI(M_T, \gamma_T).$$

Theorem 1.1 then follows from the fact that

$$\dim_{\mathbb{C}} SHI(M_T, \gamma_T) = |\mathfrak{S}(\mathcal{H})|,$$

by Proposition 3.1. □

Remark 3.6. It is not true that the inequality

$$\dim_{\mathbb{C}} SHI(M, \gamma) \leq |\mathfrak{S}(\mathcal{H})|$$

holds for *any* sutured Heegaard diagram $\mathcal{H}$ for an arbitrary balanced sutured (M, γ). For example, consider the diagram $\mathcal{H} = (T^2 - D^2, \alpha_1, \beta_1)$ for

$$(M, \gamma) = ((S^1 \times S^2)(1), \delta)$$

in which α_1 and β_1 are disjoint curves on the punctured torus. In this case, we know that

$$\dim_{\mathbb{C}} SHI(M, \gamma) = 2,$$

while $|\mathfrak{S}(\mathcal{H})| = 0$. The issue here is that $\mathcal{H}$ is not admissible.

4. Further directions

Let $\mathcal{H}$ be an admissible sutured Heegaard diagram for a balanced sutured manifold (M, γ). Let T be a full tangle for $\mathcal{H}$, as defined in § 3. In ongoing work, we prove that

$$SHI(-M_T, -\gamma_T) \cong \mathbb{C}^{|\mathfrak{S}(\mathcal{H})|}$$

[4] Ozsváth and Szabó state this for real-valued areas, but proof shows the same is true for integer areas.

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has a basis given by the contact invariants of the tight contact structures on (M_T, γ_T). In particular, this sutured instanton homology group is naturally graded by homotopy classes of 2-plane fields. We discuss potential applications of this fact in the following.

Let T' be a mixed tangle for T, as defined in § 2. Let V_T and V_m be the groups

$$V_T = SHI(-M_{T'}, -\gamma_{T'}, (A_2^-, \dots, A_n^-), (0, \dots, 0)),$$
$$V_m = SHI(-M_{T'}, -\Gamma_m, (A_2, \dots, A_n), (0, \dots, 0))$$

from § 2, for $m \in \mathbb{N}$. To prove Theorem 1.1, we proved that $V_T \cong SHI(-M_T, -\gamma_T)$ and

$$\dim_{\mathbb{C}} SHI(-M, -\gamma) = \dim_{\mathbb{C}} V_{m+1} - \dim_{\mathbb{C}} V_m \leq \dim_{\mathbb{C}} V_T$$

for sufficiently large m. However, in fact, this inequality can be viewed as coming from a spectral sequence similar to that in [LY22, § 4]. This spectral sequence can also be described as follows. From the bypass exact triangle used in § 2, we have that

$$V_T \cong H_*(\mathrm{Cone}(\psi_- : V_m \to V_{m+1})),$$

where ψ_- is the map associated to the bypass attachment along the arc η_-. One can prove, on the other hand, that

$$SHI(-M, -\gamma) \cong H_*(\mathrm{Cone}(\psi_- - \psi_+ : V_m \to V_{m+1}))$$

for m large, where ψ_+ is a related bypass attachment map. The groups V_m and V_{m+1} can be graded using the rational Seifert surface for T', as in [Li21]. After adjusting this grading by an overall shift, the map ψ_- is grading-preserving while ψ_+ decreases the grading by 1, for m large. The complex $\mathrm{Cone}(\psi_- - \psi_+)$ is then filtered, and the E_1 page of the associated spectral sequence is $H_*(\mathrm{Cone}(\psi_-))$. In sum, we have a spectral sequence

$$\mathbb{C}^{|\mathfrak{S}(\mathcal{H})|} \cong V_T \cong H_*(\mathrm{Cone}(\psi_-)) \implies H_*(\mathrm{Cone}(\psi_- - \psi_+)) \cong SHI(-M, -\gamma). \tag{4.1}$$

The first potential application of these ideas involves defining a grading on $SHI(-M, -\gamma)$ by homotopy classes of plane fields. Indeed, $SHI(-M_T, -\gamma_T)$ has such a grading, as mentioned previously, as it is generated by contact invariants of contact structures. The manifold $(M_{T'}, \gamma_{T'})$ is obtained by gluing (M_T, γ_T) along annuli, as in § 2, and we believe that the tight contact structures on the latter glue to give tight contact structures on the former whose invariants form a basis for V_T. Thus, there should be a natural grading by homotopy classes of 2-plane fields on V_T as well. The bypass maps ψ_-, ψ_+ are natural from a contact-geometric standpoint, and should therefore shift plane field gradings in a sensible way. We expect that one can then use the relation between V_T and $\mathrm{Cone}(\psi_-)$ and the structure of the latter to define a plane field grading on $\mathrm{Cone}(\psi_-)$, and then on $\mathrm{Cone}(\psi_- - \psi_+)$.

A grading by homotopy classes of 2-plane fields on *SHI* would enable one to define Spin^c decompositions of these groups, as well as an analogue of the Maslov grading in Heegaard Floer homology (see [LY22, § 4] for another approach to such a decomposition). The current lack of such structure makes it difficult to translate arguments from the Heegaard Floer setting to the instanton Floer setting.

A related second application is towards proving the isomorphism (1.1). Indeed, there is some hope that one could understand the spectral sequence (4.1) purely in terms of contact geometry, and thereby obtain a more axiomatic proof that

$$SHI(M, \gamma) \cong SFH(M, \gamma) \otimes \mathbb{C} \cong SHM(M, \gamma) \otimes \mathbb{C}, \tag{4.2}$$

since the analogous spectral sequences can be defined in the Heegaard Floer and monopole Floer settings by the same contact-geometric means.

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A more pedestrian approach to (4.2) is the following: first, prove that one can understand the spectral sequence

$$\mathbb{C}^{|\mathfrak{S}(\mathcal{H})|} \implies SHI(-M,-\gamma)$$

as coming from a differential

$$\partial = \partial_1 + \partial_2 + \ldots : \mathbb{C}^{|\mathfrak{S}(\mathcal{H})|} \to \mathbb{C}^{|\mathfrak{S}(\mathcal{H})|},$$

where ∂_k shifts a grading (coming from homotopy classes of 2-plane fields) on $\mathbb{C}^{|\mathfrak{S}(\mathcal{H})|} \cong V_T$ by k, such that

$$SHI(-M,-\gamma) \cong H_*(\mathbb{C}^{|\mathfrak{S}(\mathcal{H})|}, \partial).$$

Then, for generators $x, y \in \mathfrak{S}(\mathcal{H})$ and the corresponding basis elements $e_x, e_y \in \mathbb{C}^{|\mathfrak{S}(\mathcal{H})|}$, perhaps one could use the 2-plane field gradings to show that the coefficient

$$\langle \partial e_x, e_y \rangle \tag{4.3}$$

is nonzero only if there is a homotopy class of Whitney disks

$$\varphi \in \pi_2(x,y) \tag{4.4}$$

with positive domain in $\mathcal{H}$ and Maslov index one. If even this were true, then one could prove, for example, that the inequality in Corollary 1.7 is an equality,

$$\dim_{\mathbb{C}} KHI(L(p,q),K) \leq \mathrm{rk}_{\mathbb{Z}} \widehat{HFK}(L(p,q),K)$$

for $(1,1)$-knots $K \subset L(p,q)$.

More generally, the hope would be that for a *nice* diagram $\mathcal{H}$ (one in which the regions of Σ–α–β disjoint from $\partial\Sigma$ are bigons or rectangles), one could show that the coefficient (4.3) is nonzero if and only if there is a class as in (4.4) with positive domain and Maslov index one (the domain of such a class is necessarily an empty embedded bigon or rectangle in this case). These are precisely the domains counted in the differential on $SFC(\mathcal{H})$ in this case, by [SW10]. If one could further show that these domains are counted with the *same* nonzero complex coefficients in both the sutured instanton and Heegaard Floer settings, then this would prove (1.1) and then (4.2) by the same methods. If one could work with coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, then the last step would be unnecessary, as all nonzero elements of this field are equal. Unfortunately, *SHI* is not defined over $\mathbb{F}$. On the other hand, *SHM* is, and therefore the strategy outlined above minus the last step would be sufficient to give an alternative, more axiomatic proof that

$$SFH(M,\gamma;\mathbb{F}) \cong SHM(M,\gamma;\mathbb{F}).$$

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Conflicts of Interest

None.

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