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Research Paper

Geometry of smooth extremal surfaces ☆

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A B S T R A C T

We study the geometry of smooth projective surfaces defined by Frobenius forms, a class of homogenous polynomials in prime characteristic recently shown to have minimal possible F-pure threshold among forms of the same degree. We call these surfaces *extremal surfaces*, and show that their geometry is reminiscent of the geometry of smooth cubic surfaces, especially non-Frobenius split cubic surfaces. For instance, extremal surfaces have many lines but no triangles, hence many "star points" analogous to Eckardt points on a cubic surface. We generalize the classical notion of a double six for cubic surfaces to a double $2d$ on an extremal surface of degree d. We show that, asymptotically in d, smooth extremal surfaces have at least $\frac{1}{16}d^{14}$ double $2d$'s. A key element of the proofs is the large automorphism group of an extremal surface, which we show to act transitively on many associated sets, such as the set of triples of skew lines on the extremal surface.

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1. Introduction

Let k be an algebraically closed field of positive characteristic p. Our goal is to study the geometry of smooth extremal surfaces over k.

An **extremal hypersurface** is a projective hypersurface defined by a reduced degree d polynomial whose *F-pure-threshold* achieves the lower bound $\frac{1}{d-1}$. This lower bound was proved in [29, 1.1], where forms achieving it were classified and dubbed **Frobenius forms**; these exist only when $d-1$ is a power of the characteristic p. The F-pure threshold is a measurement of singularities[1] with smaller thresholds representing "worse singularities," so the affine cone over an extremal hypersurface is "maximally singular" among cones of the same degree. Thus, it is natural to expect the corresponding projective hypersurfaces to exhibit some extremal geometric properties as well.

Unusually beautiful properties of extremal hypersurfaces have been discovered in several different contexts. For example, they are closely related to Hermitian hypersurfaces (§ 2.3) as defined by B. Segre [43] and the finite geometries studied by Hirschfeld [21]. Frobenius forms are given by "p-bilinear forms" analogously to how quadrics correspond to bilinear forms (see § 2); their similarity with quadrics was emphasized in Shimada's and Cheng's study of their geometry, where they are called *p-quadrics* [45], and *q-bics* [9], [10], respectively.

The lowest degree extremal surfaces are the non-Frobenius split cubic surfaces of characteristic two, which were studied in depth in [28]. Geometrically, extremal cubic surfaces can be characterized among all cubic surfaces as those that *admit no triangles*. To understand this statement, recall that each smooth cubic surface admits exactly forty-five plane sections consisting of a union of three lines, typically forming a "triangle". Some special cubic surfaces admit one or more such tri-tangent plane sections in which the three lines are concurrent; in this case, their common intersection point is called an Eckardt point. An extremal cubic surface has the highly unusual property that *each and every one* of the forty-five tri-tangent plane sections consists of three concurrent lines. Such "triangle-free" cubic surfaces do not exist over $\mathbb{C}$ nor indeed over any field of odd characteristic. Extremal cubic surfaces exist only in characteristic two, precisely when the cubic form cutting out the surface is a Frobenius form, or equivalently, when the cubic surface is not Frobenius split. These results are all worked out in [28]; see also [29], [11], [18, 5.5], [24, 1.1] and [21, 20.2] for related work.

The main theme of this paper is that extremal surfaces of any degree exhibit fascinating geometry reminiscent of the geometry of lines on cubic surfaces. Like extremal cubics, extremal surfaces contain a large number of lines but no triangles. In particular, there are a large number of "stars" (collections of concurrent lines) meeting at "star points," analogous to Eckardt points for cubics (Theorem 3.3.1(e)). Extremal surfaces

[1] The F-pure threshold was first defined as a "characteristic p analog" of the log canonical threshold, a well-known invariant of complex singularities, by Takagi and Watanabe [51], who were building on the work of Hara and Yoshida [27]. See also [36], [5] or [4].

also have a large number of lines—exactly $d^2(d^2-3d+3)$ to be precise—which form many interesting configurations beyond stars. For example, extremal surfaces admit quadric configurations—collections of $2d$ lines on the surface all lying on the same quadric. While a generic surface of degree at least four does not contain any line, we show that, like cubic surfaces, a degree d extremal surface contains many quadric configurations—roughly $\frac{d^9}{2}$ for large d (Corollary 4.0.8).

In the final section of the paper, we generalize the classical notion of a "double six" on a cubic surface to any surface, and show that an extremal surface of degree d admits many configurations of "double $2d$"'s—indeed the number of double $2d$'s on an extremal surface of degree d grows at least as fast as d^{14} as d gets large (Theorem 5.0.2, Corollary 5.3.2). In Conjecture 5.0.3, we speculate that, as is classically known for cubic surfaces, every double $2d$ on a smooth surface is a union of two quadric configurations. In Theorem 5.0.4 we prove this conjecture for $d > 10$ and $d < 5$. Interestingly, our proof is purely combinatorial using only the intersection theory of lines on surfaces, so applies to any smooth surface admitting a double $2d$. There are non-extremal surfaces admitting double $2d$'s as we point out in Remark 5.4.3.

The analogy between extremal and cubic hypersurfaces has also been explored recently by Cheng in his PhD thesis, where he chose the eponymous name "q-bic" hypersurface to emphasize this connection [9], [10]. For example, Cheng shows that, like cubic threefolds, smooth extremal threefolds have a smooth Fano surface of lines and a certain intermediate Jacobian closely related to the Albanese variety of its surface of lines [10].

Interestingly, the quartic growth rate for lines on extremal surfaces is a strictly positive characteristic phenomenon: the number of lines on a smooth *complex* surface in $\mathbb{P}^3$ is bounded above by a quadratic function in the degree; see [40,42] or [7]. Bauer and Rams recently showed that a quadratic bound holds even in characteristic p, provided $p > d$ [6]. Their quadratic bound can fail in non-zero characteristic when $p \leq d$ (see *e.g.* [39]). Theorem 3.3.1(e) confirms that it is wildly false in every positive characteristic, even for $d = p + 1$.

Extremal surfaces are highly symmetric. Indeed, we show that the automorphism group of a smooth extremal surface X acts transitively on all of the following sets:

(1) the set of all line-star pairs on X (Theorem 3.2.1(i));
(2) the set of all star chords, that is, lines in $\mathbb{P}^3$ spanned by star points but not on X (Theorem 3.2.1(iii));
(3) the set of all triples of skew lines on X (Theorem 4.0.7(b))[2];
(4) the set of all triples of concurrent lines on X (Theorem 3.2.1(iv));
(5) the set of all quadric configurations on X (Theorem 4.0.7(a));
(6) the set of all pairs of star chords lying in opposite rulings of a quadric configuration for X (Theorem 4.1.4).

[2] This is the optimal result on transitivity of sets of skew lines as the action is not transitive on sets of four skew lines.

These results imply known transitivity results for the set of star points and for the set of lines, but our proofs are independent of existing proofs, which use the machinery of Hermitian or finite geometry. We also count various configurations of geometric objects associated with extremal surfaces; see Theorem 3.3.1 for basic counts of point, line, and star chord configurations, Corollary 4.0.8 for counts of quadric configurations, and Proposition 4.1.2 for counts of star chords associated to quadric configurations. These results are important in our proofs in Section 5 on the existence of double $2d$'s on extremal surfaces.

Extremal varieties are closely connected to finite Hermitian geometry, although our approach is completely independent (see [1,21]). Indeed, a *Hermitian form* is a (very) special type of Frobenius form defined over $\mathbb{F}_{q^2}$, where q is a power of p; see § 2.3. Our work connects extremal varieties to a diverse array of active research groups throughout pure and applied mathematics including in coding and design theory [13,15,50], rational points on curves and varieties [22,23], graph theory [14], cryptology [32], group theory [16,47], and the combinatorics of hyperspace arrangements and generalized quadrangles [38]. Nearly all this research is written from a dramatically different perspective from our paper. We hope to inspire algebraic geometers to investigate some of the many open problems, for example, in [25], and conversely, help researchers in diverse fields gain access to new techniques. A small sample of related literature includes [43], [26], [49], [48], [52], [30, § 35], and the references therein.

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2. Basics of Frobenius forms

This section consolidates facts and terminology about Frobenius forms.

Fix an algebraically closed field k of positive characteristic p, and let q denote p^e for some fixed positive integer e. A Frobenius form (in n variables, say) is a homogeneous polynomial of degree $p^e + 1$ in the "Frobenius power" $\langle x_1^{p^e}, x_2^{p^e}, \dots, x_n^{p^e} \rangle$ of the unique homogenous maximal ideal of the polynomial ring. Put differently, a Frobenius form is a polynomial h that can be written $\sum_{i=1}^n x_i^q L_i$, where L_i are linear forms. In particular, every Frobenius form admits a matrix factorization

$$h = [\, x_1^q \quad x_2^q \quad \dots \quad x_n^q \,]\, A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (\vec{x}^{[q]})^\top A\, \vec{x}, \tag{1}$$

where A is the unique $n \times n$ matrix whose i-th row is made up of the coefficients of the linear form L_i. Here, for a matrix B of any size, the notation $B^{[q]}$ denotes the matrix obtained by raising all entries to the p^e-th power, and $B^\top$ denotes the transpose of B. The notation $\vec{x}$ denotes a column vector with n entries.

2.1. Changes of coordinates

Frobenius forms are taken to Frobenius forms under arbitrary linear changes of coordinates, since both degree and the ideal $\langle x_1^q, x_2^q, \ldots, x_n^q \rangle$ are preserved. Let $g \in GL_n(k)$ be a matrix representing some linear change of coordinates, meaning that

$$g \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} g_1(x_1, \ldots, x_n) \\ g_2(x_1, \ldots, x_n) \\ \vdots \\ g_n(x_1, \ldots, x_n) \end{bmatrix}$$

represents the change of coordinates taking x_i to the linear form g_i. This change of coordinates takes the Frobenius form F represented by the matrix A to the Frobenius form represented by the matrix

$$\left[g^{[p^e]}\right]^\top Ag. \tag{2}$$

See [29, § 5] for details.

A change of coordinates may take a given form to a form in fewer variables; for example, the form $x^q y^q (w + z)$ in four variables is equivalent to $x^q y^q z$ in three variables. Such a form is said to be **degenerate**. Geometrically, a form is degenerate if and only if the projective hypersurface it defines is a cone over some smaller dimensional projective variety. The reader is cautioned that while the matrix of a degenerate Frobenius form is never invertible, the converse is false.

The **rank** of a Frobenius form is the rank of the representing matrix. The rank is the same as the codimension of the singular locus of the corresponding hypersurface [29, 5.3].

There is a unique *smooth* extremal hypersurface of each dimension and allowable degree, a fact that has been discovered in various guises; see [33, Thm 1], [19, Thm 9.10], [3], [45] or [29, 6.1]. More precisely:

Theorem 2.1.1. *Fix an algebraically closed field k, a degree d and number of variables n. All rank n Frobenius forms of degree d are equivalent under linear change of variables.*

More generally, Frobenius forms in n variables (not equivalent to one in fewer variables) are fully classified up to linear changes of coordinates by the partitions of n; see [29, 7.1] for the precise statement.

Example 2.1.2. Considering partitions of 3, there are three equivalence classes of Frobenius forms in three (but no fewer) variables of fixed degree $q+1$. These correspond to the three matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which determine, respectively, the forms $x^{q+1}+y^{q+1}+z^{q+1}$, $x^{q+1}+y^q z$, and $x^q y + y^q z$. See [29, 7.1].

Remark 2.1.3. Example 2.1.2 appears also as Theorem 3 in [20], where the classification of Frobenius forms in n variables of rank $n-1$ is worked out. See also [9] for a different perspective.

2.2. Extremal hypersurfaces

The projective hypersurface defined by a Frobenius form is called an **extremal hypersurface**.[3]

Example 2.2.1. Let X be smooth extremal hypersurface of dimension zero—that is, a reduced extremal configuration of points in $\mathbb{P}^1$. After an appropriate choice of homogeneous coordinates, X is defined by the form $yx^q - xy^q$, so that X is the collection of points $[\mu : 1]$ where $\mu^q = \mu$, together with the "point at infinity" $[1:0]$. That is, the points of X are precisely the $\mathbb{F}_q$-points of $\mathbb{P}^1$.

Smooth extremal surfaces have a large automorphism group:

Proposition 2.2.2. *Let X be a smooth extremal hypersurface defined by a Frobenius form of degree $q+1$ in $n \geq 2$ variables over an algebraically closed field k. The group $\mathrm{Aut}(X)$ of projective linear automorphisms of X is isomorphic to the finite group $\mathrm{PU}_n(\mathbb{F}_{q^2})$, where $\mathrm{PU}_n(\mathbb{F}_{q^2})$ is the quotient of the finite unitary group*

$$\mathrm{U}_n(\mathbb{F}_{q^2}) = \left\{ g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \;\middle|\; (g^{[q]})^\top g = I_n \right\}$$

by its center,

$$\{\lambda I_n \mid \lambda^{q+1} = 1\},$$

the cyclic group of scalar matrices of order $q+1$.

3 In other contexts, these are called p-quadric hypersurfaces [45] or q-bic hypersurfaces [9].

Remark 2.2.3. An arbitrary automorphism of a projective hypersurface of degree d in $\mathbb{P}^{n-1}$ is given by a projective linear change of coordinates, provided that $d \neq n$; see *e.g.* [35, Thm 2]. Thus in most cases, Proposition 2.2.2 describes the full group of *all* automorphisms of an extremal hypersurface. For surfaces, this is so except when $d = 4$ and $p = 3$.

Proof of Proposition 2.2.2. The proof follows straightforwardly from considering the action of $\mathrm{PGL}_n(k)$ on the identity matrix (representing the Frobenius form $\sum_{i=1}^{n} x_i^{q+1}$) as described in Section 2.1. Alternatively, the reader may consult [44, p. 97], [11, § 5.1], and [46, p. 102] to see proof of this statement in various generalities. □

In light of Example 2.2.1, the automorphism group of a reduced extremal collection of points in $\mathbb{P}^1$ is isomorphic to $\mathrm{PGL}_2(\mathbb{F}_q)$. Thus Proposition 2.2.2 confirms the well-known fact that $\mathrm{PGL}_2(\mathbb{F}_q) \cong \mathrm{PU}_2(\mathbb{F}_{q^2})$.

Corollary 2.2.4. *Let $X \subset \mathbb{P}^1$ be a reduced extremal configuration of points. Then the automorphism group of X acts three-transitively on the points of X.*

Proof. This is immediate from Example 2.2.1, after identifying X with the set of all $\mathbb{F}_q$-points of $\mathbb{P}^1$ and $\mathrm{Aut}(X)$ with $\mathrm{PGL}_2(\mathbb{F}_q)$. □

2.3. Hermitian forms over finite fields

A *Hermitian form* is a special kind of Frobenius form in which the representing matrix A satisfies $(A^{[q]})^\top = A$. In this case, all entries of A satisfy $a_{ij}^{q^2} = a_{ij}$, which means they are in the finite field $\mathbb{F}_{q^2}$. Thus, a Hermitian form is defined over the finite field $\mathbb{F}_{q^2}$. In this case, the Frobenius map ($x \mapsto x^q$) is an involution on the set of $\mathbb{F}_{q^2}$-points, so can play a role analogous to complex conjugation. See [21, § 19.1] and [45].

Hermitian forms are taken to Hermitian forms under any change of coordinates defined over $\mathbb{F}_{q^2}$, and conversely, any change of coordinates taking one Hermitian form to another is defined over $\mathbb{F}_{q^2}$; This is readily checked using Formula (2). In particular, the group of projective linear transformations of a Hermitian hypersurface is contained in $\mathrm{PGL}_n(\mathbb{F}_{q^2})$. The proof of Proposition 2.2.2 shows that the automorphism group of the smooth extremal hypersurface of degree $q+1$ in $\mathbb{P}_k^{n-1}$ defined by the *Hermitian* form $\sum_{i=1}^{n} x_i^{q+1}$ is literally the subgroup of projective unitary matrices in $\mathrm{PGL}_n(\mathbb{F}_{q^2})$ as defined above in Proposition 2.2.2. In general, the automorphism group of an arbitrary smooth extremal hypersurface is conjugate to $\mathrm{PGL}_n(\mathbb{F}_{q^2})$ by the appropriate coordinate change.

The classification of Hermitian forms is well-known and simple: there is only one invariant, rank [1, 4.1]. The classification of Frobenius forms is more subtle [29, 7.1]. On the other hand, every *smooth* projective hypersurface defined by a Frobenius form is projectively equivalent (over the algebraically closed field k) to one defined by the

Hermitian form $\sum_{i=1}^{n} x_i^{q+1}$ (Theorem 2.1.1); of course, the needed change of coordinates is not usually defined over $\mathbb{F}_{q^2}$.

2.4. Stars and other plane sections of extremal surfaces

It is easy to see that hyperplane sections of extremal hypersurfaces are extremal [29, 8.1]. Throughout this paper, we will make frequent use of the following classification of plane sections of smooth extremal surfaces:

Proposition 2.4.1. *[28] A plane section of a smooth extremal surface is one of the following types of divisors, all defined by Frobenius forms:*

(1) A smooth extremal curve;
(2) A singular extremal curve with an isolated cuspidal singularity;
(3) The reduced sum of a line and an irreducible curve tangent at one point; or
*(4) A **star** of lines on the surface, meaning a reduced configuration of lines meeting at one point.*

Proof of Proposition 2.4.1. Let X be a smooth extremal surface in $\mathbb{P}^3$ and let H be an arbitrary plane in $\mathbb{P}^3$. The plane section $X \cap H$ is given by a Frobenius form $\overline{F}$ in three variables [29, 8.1]. If $X \cap H$ is not given by a degenerate form,[4] then it must be one of those described in Example 2.1.2; these three cases produce the first three types of divisors listed above. Otherwise, the Frobenius form $\overline{F}$ defining $X \cap H$ can be written in two (or fewer) variables after changing coordinates. Now, invoking the classification of Frobenius forms in two variables, we see $\overline{F}$ can be assumed to be $x^q y + y^q x$, $x^q y$, or x^{q+1}. But since plane sections of smooth surfaces are reduced (by *e.g.* [53, 1.15]), $\overline{F}$ can be assumed given by $x^q y + y^q x$, which means that $X \cap H$ is a union of $q+1$ coplanar lines meeting at one point—a star. □

Example 2.4.2. Consider the smooth extremal surface X defined by $x^q w + w^q x + y^{q+1} + z^{q+1}$. Intersecting with the plane H defined by w, we see a star $X \cap H$ consisting of $q+1$ distinct lines

$$\{\mathbb{V}(w, y - \nu z) \mid \nu^{q+1} = -1\},$$

all intersecting in the point $p = [1 : 0 : 0 : 0]$. These lines are indistinguishable up to projective transformation since, as μ ranges through the $q+1$-roots of unity in k, the projective transformations $[x : y : z : w] \mapsto [x : y : \mu z : w]$ stabilize the surface X and its star plane H while transitively permuting around the lines in the star $H \cap X$.

[4] meaning that we can not write $\overline{F}$ as a form in fewer than three variables.

Terminology 2.4.3. Even outside the context of extremal surfaces, one can define a **star** on a degree d surface X to be any configuration of d lines on X all meeting at one point p called the **center** of the star, or a **star point.** If X is smooth, any set of d lines forming a star on a degree d surface X are *coplanar*, because all lie in the tangent plane T_pX to the center of the star. In this case, the plane section $T_pX \cap X$ is the reduced union of the d lines of the star. A plane containing a star of X is called a **star plane.** Star planes are uniquely determined by their centers and vice versa, since each star plane is the tangent plane to X at the center of its star. Stars are defined and studied for higher dimensional hypersurfaces in [8].

Remark 2.4.4. Any important point gleaned from the proof of Proposition 2.4.1 is that a plane section $H \cap X$ of a smooth extremal surface X is a star if and only if the Frobenius form $\overline{F}$ defining $X \cap H$ in H is degenerate—that is, if and only if $\overline{F}$ can be written as a Frobenius form in two (of three) homogeneous coordinates for the projective plane H.

Proposition 2.4.1 has the following useful consequences.

Corollary 2.4.5. *[29, 8.11] Let X be a smooth extremal surface.*

(i) Any collection of coplanar lines on X is concurrent. In particular, X contains no triangles;
(ii) Every line on X is in some star.

Proof. Statement (i) follows immediately from Proposition 2.4.1, by considering the plane section spanned by the lines. For (ii), fix a line L on X. Without loss of generality X is as in Example 2.4.2 (Theorem 2.1.1); in particular, X admits a plane section $H \cap X$ that is a star. If L lies in H, then (ii) is proved. If L does not lie in H, then L meets H at some point p'. The point p' is in the star $X \cap H$, so p' lies on some line $L' \subset X \cap H$. The two lines L and L' on X intersect at p', so the plane H' they span is a star plane centered at p' (Proposition 2.4.1). Clearly L is in the star $X \cap H'$, establishing (ii). □

2.5. Star points on Hermitian surfaces

A Hermitian surface is a smooth extremal surface defined by a Hermitian form (*cf.* § 2.3). These were studied by Segre [43], Hirschfeld [21], and many others, notably Shimada [45]. We recall and give a brief proof of the following result.

Proposition 2.5.1. *The star points on a Hermitian surface of degree $q+1$ are precisely its $\mathbb{F}_{q^2}$-points.*

Proof of Proposition 2.5.1. Since the change of coordinates taking an arbitrary Hermitian form to another is defined over $\mathbb{F}_{q^2}$ (see § 2.3), it suffices to consider the Hermitian surface X defined by $x^qw + w^qx + y^qz + z^qy$.

First, we show that the $\mathbb{F}_{q^2}$-points of X are star points. By symmetry, it suffices show that any point $p = [1 : a : b : c] \in X$ where $a, b, c \in \mathbb{F}_{q^2}$ is a star point. The tangent plane T_pX at p is $c^q x + b^q y + a^q z + w$, so the plane section $T_pX \cap X$ is defined by the Frobenius form in x, y, z

$$-x^q(c^q x + b^q y + a^q z) - x(c^q x + b^q y + a^q z)^q + y^q z + z^q y. \tag{3}$$

Thus, p is a star point if and only if (3) is a *degenerate* Frobenius form (Remark 2.4.4). Because the change of coordinates

$$x \mapsto x, \qquad y \mapsto y + ax, \qquad z \mapsto z + bx$$

transforms the form (3) into the degenerate form $y^q z + z^q y$ (remember that $c + c^q + a^q b + b^q a = 0$), we conclude that all $\mathbb{F}_{q^2}$-points of X are star points.

For the converse, we use the fact that the automorphism group of any extremal surface acts transitively on stars, to be proved in the next section (Theorem 3.2.1 (ii)), and that all projective linear automorphisms of a Hermitian surface are defined over $\mathbb{F}_{q^2}$ (§ 2.3). Since $[0 : 0 : 0 : 1]$ is a star, we see that all other stars are the image of $[0 : 0 : 0 : 1]$ under some projective linear change of coordinates over $\mathbb{F}_{q^2}$. In particular, all stars are defined over $\mathbb{F}_{q^2}$. □

Remark 2.5.2. Shimada defines *special points* on Hermitian surfaces and proves every special point is a star point [45, Prop 2.20(1)]. He also proves that special points are precisely the $\mathbb{F}_{q^2}$-rational points [45, Prop 2.12]. Thus by Proposition 2.5.1, the converse is true: star points are the same as special points in the sense of Shimada.

3. Geometry of extremal surfaces

In this section, we study the basic projective geometry of extremal surfaces.

3.1. Star chords

Star chords are important auxiliary lines *not* on the extremal surface:

Definition 3.1.1. A **star chord** for a smooth extremal surface X is a line in $\mathbb{P}^3$ *not on* X which passes through (at least) two star points of X.

Remark 3.1.2. In the special case where the extremal surface is defined by a Hermitian form over $\mathbb{F}_{q^2}$, star chords are *Baer sublines* or *hyperbolic lines* in the terminology of finite geometry (see *e.g.* [2, p. 4] or [34, p. 102]).

Remark 3.1.3. A star chord ℓ through star point p is *never* in the star plane T_pX centered at p. For if $\ell \subset T_pX$, then there is another star point $p' \in \ell$ necessarily on some line L

in the star $T_pX \cap X$. But then both lines L and ℓ contain the points p and p', so $\ell = L$, contrary to the fact that $\ell \not\subset X$.

The next result tells us star chords come in pairs:

Theorem 3.1.4. *Let ℓ be an arbitrary star chord for a smooth extremal surface X. Then*

- *(i) The stars centered at points on ℓ share no lines.*
- *(ii) The intersection of all star planes centered along ℓ is a star chord ℓ' for X skew to ℓ.*
- *(iii) The intersection of all star planes centered along ℓ' is the star chord ℓ.*
- *(iv) The star chords ℓ and ℓ' each intersect X in $q+1$ distinct star points.*

Before proving Theorem 3.1.4, we observe that it ensures that the next definition makes sense.

Definition 3.1.5. The **dual chord of a star chord** ℓ for an extremal surface is the unique star chord ℓ' contained in all star planes centered along ℓ, or equivalently, the intersection of all star planes centered along ℓ.

Duality between star chords is a symmetric relationship by Theorem 3.1.4 (iii).

Example 3.1.6. The lines $\ell = \mathbb{V}(x, y)$ and $\ell' = \mathbb{V}(z, w)$ are a pair of dual star chords on the Fermat extremal surface $X = \mathbb{V}(x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1})$. Indeed, ℓ is not on X but contains the $q+1$ star points $p_a = [0 : 0 : a : 1]$, where $a^{q+1} = -1$. To check that p_a is a star point, observe that the tangent plane to p_a is $T_{p_a}X = \mathbb{V}(a^q z + w) = \mathbb{V}(z - aw)$, which intersects X in a star. These star planes $\mathbb{V}(z - aw)$ all obviously contain ℓ', so ℓ' is their common intersection. Dually, the star points on ℓ' are the points $p'_b = [b : 1 : 0 : 0]$ where $b^{q+1} = -1$, and the corresponding star planes $\mathbb{V}(x - by)$ intersect in ℓ.

Proof of Theorem 3.1.4. Since ℓ is a star chord, we can fix two star points p_1 and p_2 on ℓ. Since p_1 is on *every* line in the star centered at p_1, and likewise for p_2, any shared line between these stars would contain both p_1 and p_2 and hence be ℓ itself. But by definition, the star chord ℓ is not on X. So stars centered on ℓ can not share any lines, proving (i).

Now, let $\ell' = T_{p_1}X \cap T_{p_2}X$. Note that $\ell' \not\subset X$: otherwise, $\ell' \subset T_{p_1}X \cap X$ and $\ell' \subset T_{p_2}X \cap X$, making ℓ' a shared line between these stars, which would contradict (i).

We claim that ℓ' is skew to ℓ. First note that $\ell \neq \ell'$, for otherwise the star chord ℓ lies in the star plane $T_{p_1}X$, contradicting Remark 3.1.3. So at least one of p_1 or p_2—say p_1—is not on ℓ'. Now, if ℓ and ℓ' are not skew, the unique plane they span is necessarily the plane $T_{p_1}X$, since both planes contain ℓ' and $p_1 \notin \ell'$. But now the star chord ℓ is in the star plane $T_{p_1}X$, again contradicting Remark 3.1.3.

We now claim ℓ' is a star chord intersecting X in $q+1$ distinct star points. Observe that because $\ell' \subset T_{p_1}X$, it meets each line in the star $T_{p_1}X \cap X$. But since the center p_1 is not on ℓ', we know ℓ' must meet each of the $q+1$ lines in the star $T_{p_1}X \cap X$ in a *distinct* point. These $q+1$ points make up the full intersection $\ell' \cap X$, since X has degree $q+1$. Similarly, since also $p_2 \notin \ell'$, the points of $\ell' \cap X$ are the $q+1$ distinct intersection points of ℓ' with the lines in the star $T_{p_2}X \cap X$. Thus, each p' in $\ell' \cap X$ lies on at least two lines of X so it is a star point. So ℓ' is a star chord and meets X in $q+1$ distinct star points.

Next, we show that $\ell \subset T_{p'}X$ for all $p' \in \ell' \cap X$, which will establish (iii). As we saw in the preceding paragraph, the star $X \cap T_{p'}X$ contains a line in each of the two stars $T_{p_1}X \cap X$ and $T_{p_2}X \cap X$. In particular, both p_1 and p_2 are in $T_{p'}X$, so also $\ell = \overline{p_1p_2} \subset T_{p'}X$.

We now claim ℓ meets X in $q+1$ distinct star points, which will prove (iv). To see this, take an arbitrary $p' \in \ell' \cap X$. Since $\ell \subset T_{p'}X$ (using (iii)) but $p' \notin \ell$ (by skewness of ℓ and ℓ'), each line in the star $T_{p'}X \cap X$ meets ℓ in a distinct point. These are the $q+1$ points of $X \cap \ell$. They are star points because each lies on a line in every other star $T_{p''}X \cap X$ with $p'' \in \ell'$.

To conclude (ii), it suffices to show that $\ell' \subset T_pX$ for each star point p on ℓ since $\ell' = T_{p_1}X \cap T_{p_2}X$. By (iv), we can fix two star points, q_1 and q_2, on ℓ'. By (iii), $\ell = T_{q_1}X \cap T_{q_2}X$, so p is contained in $X \cap T_{q_1}$ and $X \cap T_{q_2}$. Thus, p lies in one of the lines in each of the stars centered at q_1 and q_2. Thus, the lines $\overline{pq_1}$ and $\overline{pq_2}$ are lines in the star at p so $q_1, q_2 \in X \cap T_pX$ and $\ell' \subset T_pX$ as desired. □

3.2. Symmetry of extremal surfaces

Extremal surfaces are highly symmetric, as evidenced by the following result:

Theorem 3.2.1. *The automorphism group of a smooth extremal surface X acts transitively on each of the following sets:*

- *(i) the set of all pairs (H, L), where L is any line on X and H is any star plane containing L;*
- *(ii) the set of all pairs (p, L), where L is any line on X and p is any star point on L;*
- *(iii) the set of all ordered pairs (p_1, p_2) of (distinct) star points spanning a star chord;*
- *(iv) the set of ordered triples of concurrent lines on X.*

In particular, the automorphism group acts transitively on the set of star points, on the set of lines, and on the set of star chords of any smooth extremal surface. Transitivity on star points and on lines can also be deduced from the existing literature in light of Proposition 2.5.1; see, in particular, [45, Thm 2.19].

The proof uses the following lemma.

Lemma 3.2.2. *Given an arbitrary star* $X \cap H$ *with center* p *on a smooth extremal surface* X*, we may choose coordinates for* $\mathbb{P}^3$ *so that*

$$p = [0:0:0:1], \quad H = \mathbb{V}(x), \quad \textit{and} \quad X = \mathbb{V}(x^q \ell + xw^q + y^q z + z^q y), \tag{4}$$

for some linear form $\ell = ax + by + cz + w$*.*

Proof. Choose coordinates so that the star plane H is defined by $x = 0$. In this case, the form F defining X is

$$F = xG + G'(y, z, w)$$

where G is some form of degree q and G' is a Frobenius form in the variables y, z, w. The form G' defines the star $X \cap H$ in the plane H, and hence G' is *degenerate* (Remark 2.4.4). So by a change of coordinates involving only y, z, w, without loss of generality

$$F = xG + yz^q + zy^q,$$

in which case the star point p has coordinates $[0 : 0 : 0 : 1]$.

Observe that $xG \in \langle x^q, y^q, z^q, w^q \rangle$, which implies that $G \in \langle x^{q-1}, y^q, z^q, w^q \rangle$. Because $\deg G = q$, we can write

$$G = x^{q-1}\ell + (\alpha_1 y + \alpha_2 z + \alpha_3 w)^q$$

for some scalars α_i and linear form ℓ. That is,

$$F = x^q \ell + x(\alpha_1 y + \alpha_2 z + \alpha_3 w)^q + zy^q + yz^q,$$

where the star $H \cap X$ is given by $x = 0$ and the star point is $p = [0 : 0 : 0 : 1]$ in these coordinates. The scalar α_3 cannot be zero, for in that case F would be rank 3, so would not define a smooth surface [29, 5.3]. Therefore, we may replace the form $\alpha_1 y + \alpha_2 z + \alpha_3 w$ by w (which changes ℓ but nothing else) to assume without loss of generality that

$$F = x^q \ell + xw^q + zy^q + yz^q. \tag{5}$$

The linear form $\ell = ax + by + cz + dw$ must satisfy $d \neq 0$, for otherwise F would have rank at most 3. Finally, the change of coordinates

$$x \mapsto \lambda x, \quad y \mapsto y, \quad z \mapsto z, \quad w \mapsto \lambda^{-1/q} w$$

where $\lambda^{q^2-1} = \frac{1}{d^q}$ transforms F (formula (5)) into

$$(\lambda x)^q (a\lambda x + by + cz + d\lambda^{-1/q} w) + xw^q + zy^q + yz^q$$

without changing p or H. This has the desired form since the coefficient of $x^q w$ is $d\lambda^{q-\frac{1}{q}} = 1$. □

Proof of Theorem 3.2.1. The first two statements are equivalent via the bijection between star points and star planes given by the correspondence $p \leftrightarrow T_pX$ (see § 2.4.3).

Fix an arbitrary pair (H, L). Let p be the center of the star $H \cap X$, so that $H = T_pX$. Use Lemma 3.2.2 to choose coordinates so that $p = [0 : 0 : 0 : 1]$, H is defined by $x = 0$ and the Frobenius form defining X looks like

$$F = x^q(ax + by + cz + w) + w^q x + y^q z + z^q y. \tag{6}$$

Apply the change of coordinates

$$x \mapsto x, \quad y \mapsto y, \quad z \mapsto z + \lambda^q x, \quad w \mapsto w - \lambda y$$

where $\lambda \in k$ satisfies $\lambda^{q^2} - \lambda + b = 0$. This transformation fixes p and H but replaces form (6) by one in which $b = 0$ and c is unchanged. Interchanging the roles of y and z, without loss of generality $b = c = 0$. Finally, if $a \neq 0$, fix $\gamma \in k$ such that $\gamma^q + \gamma + a = 0$, and apply the transformation

$$x \mapsto x, \quad y \mapsto y, \quad z \mapsto z, \quad w \mapsto w + \gamma x$$

to transform (6) to $x^q w + w^q x + y^q z + z^q y$ without changing the star point $p = [0 : 0 : 0 : 1]$ or the star plane H. Likewise, we can change coordinates, fixing x and w but taking the rank 2 Frobenius form $y^q z + z^q y$ to $y^{q+1} + z^{q+1}$, without changing the star point $p = [0 : 0 : 0 : 1]$ or the star plane H.

The line L is taken to some other line L' in the same star $X \cap H$. But we can then compose with an automorphism of X preserving H while taking the image of L to *any* line in the star $H' \cap X$ (Example 2.4.2). This proves (i), and equivalently (ii).

For (iii), fix an arbitrary ordered pair (p_1, p_2) of star points spanning a star chord of X. Since $\mathrm{Aut}(X)$ acts transitively on star points (by (a)), there is no loss of generality in assuming

$$X = \mathbb{V}(x^q w + w^q x + y^q z + z^q y) \qquad \text{and} \qquad p_1 = [0 : 0 : 0 : 1].$$

The theorem will be proved if we show that, in addition, we can choose coordinates so that p_2 is the star point $[1 : 0 : 0 : 0]$.

First note that we can assume that $p_2 = [1 : a : b : c]$. Indeed, otherwise $p_2 \in \mathbb{V}(x) = T_{p_1}X$, so that p_2 would be in the star centered at p_1. In this case, the line $\overline{p_1 p_2}$ is in that star and hence on X, contrary to the assumption that p_1 and p_2 span a star chord. Note also that $a, b, c \in \mathbb{F}_{q^2}$ (Proposition 2.5.1).

Consider the change of coordinates

$$x \mapsto x, \quad y \mapsto y - ax, \quad z \mapsto z - bx, \quad w \mapsto w + c^q x + b^q y + a^q z.$$

Since $a^{q^2} = a$, $b^{q^2} = b$, $c^{q^2} = c$, and $c^q + c = -(a^q b + b^q a)$, we easily verify that this change of coordinates fixes $x^q w + w^q x + y^q z + z^q y$. It also fixes the ideal (x, y, z) of $[0 : 0 : 0 : 1]$ and sends the ideal (y, z, w) to $(y - ax, z - bx, w - cx)$, so that the corresponding projective transformation of $\mathbb{P}^3$ induces an automorphism of X that fixes the point $p_1 = [0 : 0 : 0 : 1]$ and sends the point $p_2 = [1 : a : b : c]$ to $[1 : 0 : 0 : 0]$.

For (iv): any three concurrent lines intersect in a star point, so by (i), we can assume that $X = \mathbb{V}(x^q w + w^q x + y^{q+1} + z^{q+1})$ and move the three lines by an automorphism to three lines in the star $X \cap H$ where $H = \mathbb{V}(x)$ is the star plane centered at $[0 : 0 : 0 : 1]$. These three lines can be moved to any other three lines in $X \cap H$ by a change of coordinates fixing x, w and $y^{q+1} + z^{q+1}$ since this is equivalent to the three-transitivity of the automorphism group of the points defined by $y^{q+1} + z^{q+1}$ in $\mathbb{P}^1$ (Corollary 2.2.4). This completes the proof of Theorem 3.2.1. □

3.3. Counting stars and lines on an extremal surface

We gather together various counts of configurations on extremal surfaces for future reference, some of which can be found or deduced from results scattered throughout the existing literature (making use of Proposition 2.5.1); see especially [43], [21], and [45]. To keep the paper self-contained, we provide straightforward projective-geometric arguments proofs independent of the theory of q^2-rational points on Hermitian surfaces.

Theorem 3.3.1. *Let X be a smooth extremal surface X of degree $q + 1$. Then*

- *(a) There are exactly q^2+1 star points on each line on X. Equivalently, there are exactly $q^2 + 1$ stars on the surface X containing any given line. [21, Table 19.2], [45, Cor 2.14]*
- *(b) There are exactly $q(q^2 + 1)$ lines on X that intersect any fixed line on X.*
- *(c) There are exactly q^4 lines on X skew to any given line on X.*
- *(d) Each star plane of X contains exactly $q^3 + q^2 + 1$ star points—that is, each star contains $q^3 + q^2$ star points other than its center. [21, 19.1.5]*
- *(e) There are a total of $q^4 + q^3 + q + 1 = (q^3 + 1)(q + 1)$ distinct lines on X, each containing exactly $q^2 + 1$ star points. [21, 19.1.5]*
- *(f) There are a total of $q^5 + q^3 + q^2 + 1 = (q^3 + 1)(q^2 + 1)$ distinct stars on X, each containing exactly $q + 1$ lines. [43], [21, 19.1.5]*
- *(g) There are $q^4(q^2 - q + 1)(q^2 + 1)$ star chords of X. [21, Table 19.2]*
- *(h) For each pair of skew lines on X, there are exactly $q^2 + 1$ lines on X that meet both. [21, 19.3.4]*

Remark 3.3.2. The reader can compute that the order of $\mathrm{PU}_n(\mathbb{F}_{q^2})$ is $q^6(q^4 - 1)(q^3 + 1)(q^2 - 1)$ [12, pp. 131–144] using the orbit-stabilizer theorem applied to the actions in Theorem 3.2.1 with the counts in Theorem 3.3.1.

Proof. (a). Fix a line L on X. We know L belongs to some star $H \cap X$ (Corollary 2.4.5(ii)). Choose coordinates so that X is defined by $F = x^q w + xw^q + y^q z + z^q y$, the plane H is cut out by x, and the line L is cut out by x and y (Theorem 3.2.1(i)).

Consider the pencil of planes containing the line L. Each plane H_λ in the pencil is defined by the vanishing of some linear form $\lambda x - y$, with the star plane H itself the case where $\lambda = \infty$. Restricting F to the plane H_λ, we can set $y = \lambda x$, so that the plane section $X \cap H_\lambda$ is defined by

$$\overline{F} = x^q w + xw^q + \lambda^q x^q z + \lambda x z^q.$$

The plane section $X \cap H_\lambda$ is a star if and only if the form $\overline{F}$ is degenerate (cf. Remark 2.4.4). The change of coordinates

$$x \mapsto x, \quad z \mapsto z, \quad w \mapsto w - \lambda^q z$$

transforms $\overline{F}$ to

$$\overline{F_1} = x^q w + xw^q + (\lambda - \lambda^{q^2}) x z^q = x(w^q + x^{q-1} w + (\lambda - \lambda^{q^2}) z^q),$$

which is clearly degenerate if $\lambda^{q^2} - \lambda = 0$. Conversely, if $\lambda^{q^2} - \lambda \neq 0$, then $\overline{F}$ is *not degenerate* because it defines the union of the line L and an irreducible curve of degree q rather than a star. Thus, there are precisely q^2 planes H_λ (besides H) whose intersection with X is a star containing L.

(b). Fix a line L on X. A line M on X intersects L if and only if L and M appear together in a star. There are q^2+1 stars containing L and each of them contains q distinct lines (other than L). Of course, a pair L and M can not appear together in more than one star, since the plane producing a star is uniquely determined by any two lines in it. So there must be $q(q^2+1)$ distinct lines M which intersect L on our extremal surface.

(c). Fix a line L on X. Fix a star $H \cap X$ containing L (Corollary 2.4.5 (ii)). Every line on X intersects H, and hence some line in the star $H \cap X$. Thus it suffices to count the lines skew to L that intersect lines in $H \cap X$. There are q other lines in this star. Pick one, M. Now M appears in exactly q^2 other stars besides $X \cap H$ by (a). For each of these stars, each of the other q lines in the star is a line L' which does not meet L. Indeed, if L' meets L, then the lines L, L', M form a triangle, contradicting Corollary 2.4.5 (i). In this way, we produce q^3 distinct lines L' on X which meet M but not L. Now, varying over each of the q lines M in the star $H \cap X$ (other than L), we produce q^3 new lines for each of the q choices of line M. In total, we found q^4 lines skew to L.

(d). Let p be the center of the star $H \cap X$. Each of the $q+1$ lines in this contains exactly q^2 star points other than p by (a). So H contains exactly $q^2(q+1)+1$ star points.

(e). Fix one line L on X. There are exactly $q(q^2+1)$ lines on X which intersect L by (b). On the other hand, there are q^4 lines on X disjoint from L by (c). So the total number of lines, counting L, is $q^4 + q^3 + q + 1$.

(f). By (e), there are a total of q^4+q^3+q+1 lines on X, and each line is contained in exactly q^2+1 stars by (a). So there are $(q^4+q^3+q+1)(q^2+1)$ pairs (L, H) consisting of a line L in a star $H \cap X$. On the other hand, each star contains exactly $q+1$ lines, so the total number of stars is

$$\frac{(q^4+q^3+q+1)(q^2+1)}{q+1} = \frac{(q^3+1)(q+1)(q^2+1)}{q+1} = (q^3+1)(q^2+1) = q^5+q^3+q^2+1.$$

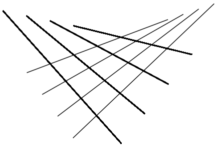
(g). A star chord is determined by any two star points on it, so we first count the number of ordered pairs of star points spanning a star chord. There are $q^5+q^3+q^2+1$ star points by (f). Fix a star point p. Any other star point spans a star chord with p unless it lies on one of the $q+1$ lines in the star centered at p; in particular, there are q^3+q^2+1 star points that do not span a star chord with p by (d). In other words, there are q^5 choices of star points p' such that $\overline{pp'}$ is a star chord, so $q^5(q^3+1)(q^2+1)$ ordered pairs of stars spanning star chords. Finally, since each star chord contains $q+1$ star points (Theorem 3.1.4 (iv)), there are $(q+1)q$ ordered pairs of star points determining each star chord. Thus, there are $\frac{q^5(q^3+1)(q^2+1)}{q(q+1)} = q^4(q^2-q+1)(q^2+1)$ star chords.

(h). Fix arbitrary skew lines L and L' on X. Because there are exactly q^2+1 star points on L and any intersection point of lines on X is a star point, it suffices to prove that for each star point p on L, there is exactly one line through p meeting L'. To this end, observe that L' is not in the star plane H centered at p, since that would imply L' meets L. Thus, L' meets H at a unique point p'; the point p' is in the star $X \cap H$ and hence in (exactly) one of its lines, M. The line M meets both L and L'. There is no other line through p meeting both L and L', for if M' is another, then M, M', L' form a triangle, contrary to Corollary 2.4.5 (i). □

4. Quadric configurations

Extremal surfaces contain interesting line configurations we call **quadric configurations:**

Definition 4.0.1. A **quadric configuration** on a surface of degree $d \geq 3$ in projective three space is a collection of $2d$ lines on the surface consisting of two sets of d skew lines with the property that each line in either set meets every line of the other set.



$d = 4$

The next proposition justifies the name:

Proposition 4.0.2. *A quadric configuration on an irreducible surface X is equal to $X \cap Q$ for some unique smooth quadric surface Q.*

Proof. Let $\mathcal{L} \cup \mathcal{M}$ be a configuration of lines, where $\mathcal{L}$ (respectively $\mathcal{M}$) consists of d skew lines intersecting every line in $\mathcal{M}$ (respectively $\mathcal{L}$). Choose any three skew lines $L_1, L_2, L_3 \in \mathcal{L}$, and let Q be the unique smooth quadric they determine [17, 2.12]. The lines of $\mathcal{M}$ intersect *all* lines in $\mathcal{L}$, including L_1, L_2, and L_3, which lie on Q. So each line $M \in \mathcal{M}$ intersects the quadric Q in at least three points, which means $M \subset Q$. But now each line $L \in \mathcal{L}$ intersects all lines in $\mathcal{M}$, so L intersects Q in at least three points. Again, we conclude $L \subset Q$. So $\mathcal{L} \cup \mathcal{M} \subset Q$.

Now if $\mathcal{L} \cup \mathcal{M} \subset X$, then $\mathcal{L} \cup \mathcal{M} \subset X \cap Q$. So since $X \cap Q$ and $\mathcal{L} \cup \mathcal{M}$ both have degree $2d$ and $X \cap Q$ is a complete intersection, we conclude that $X \cap Q$ is precisely the reduced union of the $2d$ lines in $\mathcal{L} \cup \mathcal{M}$. □

Example 4.0.3. Let Q_μ be the quadric surface $Q_\mu = \mathbb{V}(\mu xw - yz)$, where $\mu \in k$ is a fixed $(q+1)$-st root of unity. The quadric Q_μ defines a quadric configuration on the Fermat extremal surface. Indeed, the lines in the sets

$$\mathcal{L}_\mu = \{\mathbb{V}(x - \alpha y, z - \mu\alpha w) \mid \alpha^{q+1} = -1\}$$
$$\mathcal{M}_\mu = \{\mathbb{V}(x - \beta z, y - \mu\beta w) \mid \beta^{q+1} = -1\}$$

all lie on the quadric Q_μ (with the lines in $\mathcal{L}_\mu$ and $\mathcal{M}_\mu$ in opposite rulings), as well as on the extremal surface $X = \mathbb{V}(x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1})$. Thus, $X \cap Q_\mu$ is the quadric configuration $\mathcal{L}_\mu \cup \mathcal{M}_\mu$.

Quadric configurations are rare on an arbitrary surface—for example, a generic surface of degree greater than three admits *no lines at all* [17, 12.8]. Extremal surfaces, however, contain many quadric configurations.

Theorem 4.0.4. *Any triple of skew lines on a smooth extremal surface determines a unique quadric configuration.*

For the proof, we need the following lemma, which will be generalized to triples of skew lines in the next section.

Lemma 4.0.5. *The automorphism group of a smooth extremal surface X acts transitively on the set of pairs of skew lines on X.*

Proof of Lemma 4.0.5. Fix a pair of skew lines L and L' on X. There are $(q^2+1)^2$ lines in $\mathbb{P}^3$ connecting star points on L to star points on L' (Theorem 3.3.1(a)) but only q^2+1 of them lie on X (Theorem 3.3.1(h)). Thus we can pick star points $p \in L$ and $p' \in L'$ that span a star chord ℓ.

Choose coordinates so that $X = \mathbb{V}(x^q w + w^q x + y^{q+1} + z^{q+1})$, $p = [0 : 0 : 0 : 1]$, and $p' = [1 : 0 : 0 : 0]$ (Theorem 3.2.1(iii)). In this case, the star chord $\ell = \overline{pp'}$ is $\mathbb{V}(y, z)$ and the star planes at p and p', respectively, are defined by x and w. The line L is therefore in the star $T_pX \cap X = \mathbb{V}(x, y^{q+1} + z^{q+1})$ and the line L' is in the star $T_{p'}X \cap X = \mathbb{V}(w, y^{q+1} + z^{q+1})$. In particular, $L = \mathbb{V}(x, y - \nu_1 z)$ and $L' = \mathbb{V}(w, y - \nu_2 z)$ where $\nu_1^{q+1} = \nu_2^{q+1} = -1$. The assumption that L and L' are skew means that $\nu_1 \neq \nu_2$.

Finally, observe that the lines L and L' can be taken to any other two lines in their respective stars by an automorphism of X that fixes p and p'. Indeed, there is a linear change of coordinates that fixes x and w but sends the factors $\{y - \nu_1 z, y - \nu_2 z\}$ of $y^{q+1} + z^{q+1}$ to any other two distinct factors $\{y - \mu_1 z, y - \mu_2 z\}$ of $y^{q+1} + z^{q+1}$ (Corollary 2.2.4). □

Proof of Theorem 4.0.4. Fix three skew lines, L, L', and L'' on the extremal surface X of degree $d = q + 1$. Without loss of generality, assume X is defined by the form $x^q w + w^q x + y^q z + z^q y$, L by $x = y = 0$, and L' by $z = w = 0$ (Lemma 4.0.5). In this case, L'' can be defined by linear equations of the form

$$x = az + bw \ \text{ and } \ y = cz + dw,$$

where the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is full rank, and is parametrized as $\{[as + bt : cs + dt : s : t] \mid [s : t] \in \mathbb{P}^1\}$. Furthermore, the condition that L'' lies on X means that

$$(as + bt)^q t + t^q (as + bt) + (cs + dt)^q s + s^q (cs + dt) = 0$$

for all s, t. This imposes the constraints

$$c^q + c = b^q + b = a^q + d = a + d^q = 0. \tag{7}$$

The quadric Q defined by

$$cxz + dxw - ayz - byw \tag{8}$$

contains L, L', and L''. Note that Q is the image of the Segre map

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma} \mathbb{P}^3 \qquad ([s_1 : s_2], [t_1, t_2]) \mapsto [(as_1 + bs_2)t_1 : (cs_1 + ds_2)t_1 : s_1 t_2 : s_2 t_2].$$

Now, consider an arbitrary line in one of the rulings on Q, say

$$\ell = \{[(a\lambda_1 + b\lambda_2)t_1 : (c\lambda_1 + d\lambda_2)t_1 : \lambda_1 t_2 : \lambda_2 t_2] \quad | \quad [t_1 : t_2] \in \mathbb{P}^1\}.$$

The line ℓ is on X if and only if, plugging into the Frobenius form defining X, the form

$$\lambda_2(a\lambda_1 + b\lambda_2)^q t_1^q t_2 + \lambda_2^q (a\lambda_1 + b\lambda_2) t_1 t_2^q + \lambda_1 (c\lambda_1 + d\lambda_2)^q t_1^q t_2 + \lambda_1^q (c\lambda_1 + d\lambda_2) t_1 t_2^q, \tag{9}$$

is uniformly zero for all values of t_1, t_2. Equivalently, ℓ is on X precisely when the coefficients of $t_1^q t_2$ and of $t_1 t_2^q$ in expression (9) satisfy

$$\lambda_2(a\lambda_1 + b\lambda_2)^q + \lambda_1(c\lambda_1 + d\lambda_2)^q = 0$$
$$\lambda_2^q(a\lambda_1 + b\lambda_2) + \lambda_1^q(c\lambda_1 + d\lambda_2) = 0.$$

In light of the constraints (7), these equations simplify to

$$c\lambda_1^{q+1} + d\lambda_1^q\lambda_2 + a\lambda_1\lambda_2^q + b\lambda_2^{q+1} = 0 \tag{10}$$

Because the form in (10) is a Frobenius form in λ_1, λ_2 with the full rank matrix $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$, there are precisely $q+1$ distinct solutions to (10) in $\mathbb{P}^1$. We conclude that there are precisely $q+1$ lines ℓ of the form $\sigma([\lambda_1 : \lambda_2] \times \mathbb{P}^1)$ lying on both X and Q. These are $q+1$ different skew lines on the extremal surface.

A similar argument shows that a line

$$m = \{[(as_1 + bs_2)\lambda_1 : (cs_1 + ds_2)\lambda_1 : s_1\lambda_2 : s_2\lambda_2] \mid [s_1 : s_2] \in \mathbb{P}^1\}$$

in the other ruling of the quadric lies on X if and only if $\lambda_1^q\lambda_2 - \lambda_1\lambda_2^q$. These form a set of $q+1$ skew lines, each of which meets every line in the other set of $q+1$ skew lines on X. □

Remark 4.0.6. In the finite geometry setting, Hirschfeld proves an analog of Theorem 4.0.4 for Hermitian geometries using different techniques and language [21, 19.3.1].

Theorem 4.0.7. *The automorphism group of a smooth extremal surface X acts transitively*

(a) on the set of all quadric configurations on X; and
(b) on the set of triples of skew lines on X.

Proof of Theorem 4.0.7. In light of Theorem 4.0.4, it is enough to prove (b). For this, it suffices to show that $\mathrm{Aut}(X)$ acts transitively on the set $\mathcal{S}$ of all ordered sextuples $(L_1, L_2, L_3, M_1, M_2, M_3)$ of lines on X, consisting of two triples of skew lines $\{L_1, L_2, L_3\}$ and $\{M_1, M_2, M_3\}$ with $L_i \cap M_j \neq \emptyset$ for all i, j.

Fix an ordered sextuple $(L_1, L_2, L_3, M_1, M_2, M_3) \in \mathcal{S}$. First note that its stabilizer, even in $\mathrm{PGL}_4(k)$, is trivial. Indeed, the intersection points $p_{ij} = L_i \cap M_j$ must be fixed by any element in the stabilizer of $(L_1, L_2, L_3, M_1, M_2, M_3)$. These nine points contain five points in general linear position (no three on a line, no four on a plane). But an automorphism of $\mathbb{P}^3$ fixing five points in general linear position is trivial.

Next, we compute the cardinality of $\mathcal{S}$. There are $(q^3+1)(q+1)$ choices for L_1 by Theorem 3.3.1(e), and fixing L_1, there are q^4 choices for a skew line L_2 on X by Theorem 3.3.1(c). The number of choices for L_3 is the total number of lines on X

minus the number of lines meeting L_1 *or* L_2. Accounting for the double-counting of lines meeting *both* L_1 and L_2, the number of choices for L_3 is

$$\begin{aligned}&\big[(q^3+1)(q+1)\big]-2\big[q^3+q+1\big]+\big[q^2+1\big]\\&\quad=q(q^2+1)(q-1),\end{aligned}$$

using Theorem 3.3.1 (b), (e), and (h). The choice of the triple L_1, L_2, L_3 determines the quadric, and hence $q+1$ lines in $Q \cap X$ that all intersect L_1, L_2, L_3 by Theorem 4.0.4. There are $(q+1)q(q-1)$ ways to choose the triple M_1, M_2, M_3. In total, the number of ordered sextuples is thus

$$[(q^3+1)(q+1)]\cdot[q^4]\cdot[q(q^2+1)(q-1)]\cdot[(q+1)q(q-1)]=q^6(q^4-1)(q^3+1)(q^2-1).$$

This is precisely the order of $\mathrm{Aut}(X)$ (Remark 3.3.2). So $\mathrm{Aut}(X)$ acts transitively on the set $\mathcal{S}$, and hence on the set of all triples of skew lines on X. □

For future reference, we record the following corollary of the proof of Theorem 4.0.7. This is also in [21], using more complicated techniques:

Corollary 4.0.8. *[21, 19.3.1(ii)] A smooth extremal surface X of degree $q+1$ contains exactly $\frac{1}{2}(q^3+1)(q^2+1)q^4$ quadric configurations.*

Proof of Corollary. By Theorem 4.0.4, each quadric configuration on a smooth extremal surface X is uniquely determined by an ordered triple of skew lines (L_1, L_2, L_3) on X. The number of such ordered triples is

$$(q^3+1)(q+1)\cdot q^4\cdot q(q^2+1)(q-1),$$

as we computed in the proof of Theorem 4.0.7. To determine the number of quadric configurations, then, we must determine the number of ordered triples determining the *same* quadric. To this end, first note that there are $2(q+1)$ choices of a line L_1 in $\mathcal{Q}$. Once L_1 is fixed, the lines L_2 and L_3 are among the q lines in same ruling of $\mathcal{Q}$, so there are $q(q-1)$ choices for (L_2, L_3). We conclude that there are

$$\frac{(q^3+1)(q+1)q^5(q-1)(q^2+1)}{2(q+1)q(q-1)}=\frac{1}{2}(q^3+1)(q^2+1)q^4$$

quadric configurations on a smooth extremal surface. □

4.1. Star chords in quadric configurations

We record some observations about star chords and quadric configurations that will be useful in Section 5.

Lemma 4.1.1. *Let Q be a quadric defining a quadric configuration on a smooth extremal surface X. Let ℓ be a line on Q but not on X. Then ℓ intersects X in $q+1$ distinct points, and if any one of these intersection points is a star point of X, then they all are.*

Proof. Because the automorphism group of X acts transitively on quadric configurations (Theorem 4.0.7), we may assume that X is given by the Fermat Frobenius form and Q by $xw = yz$. The lines on Q have the following parametrizations

$$\{[\lambda s : s : \lambda t : t] \mid [s:t] \in \mathbb{P}^1\} \quad \text{and} \quad \{[\lambda s : \lambda t : s : t] \mid [s:t] \in \mathbb{P}^1\}.$$

Without loss of generality, let $\ell = \{[\lambda s : s : \lambda t : t] \mid [s:t] \in \mathbb{P}^1\}$ for some fixed λ. The condition that a point $[\lambda s_0 : s_0 : \lambda t_0 : t_0]$ of ℓ lies on X is that

$$(\lambda s_0)^{q+1} + s_0^{q+1} + (\lambda t_0)^{q+1} + t_0^{q+1} = (\lambda^{q+1}+1)(s_0^{q+1} + t_0^{q+1}) = 0. \tag{11}$$

There are two ways this can happen. Either $\lambda^{q+1} = -1$, which means (11) holds for all values of $[s_0 : t_0]$, so the line ℓ lies on X. Or $\lambda^{q+1} \neq -1$, and there are exactly $q+1$ points $[s_0 : t_0]$ satisfying $s_0^{q+1} + t_0^{q+1} = 0$. In this case, there are exactly $q+1$ distinct points of $\ell \cap X$, all of the form $[\lambda\mu : \mu : \lambda : 1]$ where μ ranges through the $q+1$ distinct roots of -1. In particular, $\mu \in \mathbb{F}_{q^2}$. Now if one of these points $[\lambda\mu : \mu : \lambda : 1]$ is a star point, then it is defined over $\mathbb{F}_{q^2}$ (Proposition 2.5.1), so $\lambda \in \mathbb{F}_{q^2}$ as well. Thus, all $q+1$ points of $X \cap \ell$ are defined over $\mathbb{F}_{q^2}$, and hence all are star points. □

Proposition 4.1.2. *Let Q be a smooth quadric defining a quadric configuration on a smooth extremal surface X. Then there are exactly $q^2 - q$ star chords in each ruling of Q, and those in opposite rulings meet off X.*

Proof. Consider a star chord ℓ on Q. Write $Q \cap X = \mathcal{L} \cup \mathcal{M}$ where $\mathcal{L}$ and $\mathcal{M}$ are the two skew sets of lines on X in opposite rulings of Q.

Because ℓ must lie in one of the rulings of Q, it intersects each of the $q+1$ lines in, say, $\mathcal{M}$. For each $M \in \mathcal{M}$, the intersection point $\ell \cap M$ is a star point (Theorem 3.1.4(iv)). Conversely, through each star point on M, the unique line in the opposite ruling of Q is either a line in $\mathcal{L}$, or a star chord, depending on whether or not it is on X (Lemma 4.1.1). Since there are q^2+1 total star points on M (Theorem 3.3.1(a)), this leaves $q^2 - q$ possible points of intersection of the star chord ℓ with M. Thus, there are exactly $q^2 - q$ possibilities for the star chord ℓ in this ruling of Q. By symmetry, the same holds in the other ruling.

Now suppose ℓ and m are star chords in opposite rulings on Q. If $p = \ell \cap m$ lies on X, then it must be one of the $q+1$ points on $\ell \cap X$, and hence p is some star point on some line $M \subset Q \cap X$ in the ruling opposite ℓ. In this case, M is the unique line through p on Q in the ruling opposite ℓ, forcing $m = M$. This contradicts our assumption that m is not on X. □

Remark 4.1.3. Proposition 4.1.2 and Lemma 4.1.1 together say the complete set of lines on Q passing through star points of X consists of two sets of q^2+1 skew lines (one on each ruling); in each of these skew sets, there are $q+1$ lines on X and q^2-q star chords.

Theorem 4.1.4. *The automorphism group of a smooth extremal surface acts transitively on the set of triples (Q,ℓ,m) consisting of a quadric Q defining a quadric configuration, together with a choice star chords ℓ and m, one in each ruling of Q.*

Proof. We may assume that the extremal surface X is defined by $x^{q+1}+y^{q+1}+z^{q+1}+w^{q+1}$ and Q by $xw-yz$ (Theorem 4.0.7 (a)). Let ℓ and m be an arbitrary pair of star chords on Q, lying in opposite rulings. It suffices to show that there is an automorphism of X which stabilizes Q and sends ℓ and m to the star chords $\mathbb{V}(x,z)$ and $\mathbb{V}(z,w)$, respectively.

The lines in the two rulings of Q have the form

$$\mathbb{V}(\lambda x-\mu y,\lambda z-\mu w) \qquad \text{and} \qquad \mathbb{V}(\alpha x-\beta z,\alpha y-\beta w);$$

the star chords among them are precisely those where $[\lambda:\mu]$ (respectively $[\alpha:\beta]$) is an $\mathbb{F}_{q^2}$-point of $\mathbb{P}^1$ not on $\mathbb{V}(s^{q+1}+t^{q+1})$ (see Example 4.0.3 and the proof of Lemma 4.1.1). Indeed, all such lines are on Q, but not on X, and since there are q^2-q in each ruling, we have found the complete list of star chords on Q (Proposition 4.1.2).

Suppose that $\ell=\mathbb{V}(\lambda x-\mu y,\lambda z-\mu w)$. The change of coordinates given by the matrix $\begin{bmatrix}\lambda^q & \mu & 0 & 0\\ -\mu^q & \lambda & 0 & 0\\ 0 & 0 & \lambda^q & \mu\\ 0 & 0 & -\mu^q & \lambda\end{bmatrix}$ scales the defining equation of both X and Q by a non-zero scalar (remember $\lambda^{q+1}+\mu^{q+1}\neq 0$), so the corresponding projective transformation g is in $\mathrm{Aut}(X)\cap\mathrm{Aut}(Q)$. In addition, g sends ℓ to $\mathbb{V}(x,z)$, as

$$\begin{aligned} g(\ell) &= \mathbb{V}\left(\lambda(\lambda^q x+\mu y)-\mu(-\mu^q x+\lambda y),\ \lambda(\lambda^q z+\mu w)-\mu(-\mu^q z+\lambda w)\right)\\ &= \mathbb{V}\left((\lambda^{q+1}+\mu^{q+1})x,\ (\lambda^{q+1}+\mu^{q+1})z\right) \ =\ \mathbb{V}(x,z). \end{aligned}$$

Of course, g sends m to some star chord on Q in the opposite ruling from $g(\ell)$. So $g(m)=\mathbb{V}(\alpha x-\beta z,\alpha y-\beta w)$ for some $\mathbb{F}_{q^2}$ point $[\alpha:\beta]\in\mathbb{P}^1$ not on $\mathbb{V}(s^{q+1}+t^{q+1})$. Now observe that the change of coordinates given by the matrix $\begin{bmatrix}\beta & 0 & \alpha^q & 0\\ 0 & \beta & 0 & \alpha^q\\ \alpha & 0 & -\beta^q & 0\\ 0 & \alpha & 0 & -\beta^q\end{bmatrix}$ preserves the Fermat Frobenius form and the quadric polynomial $xw-yz$ defining Q, so that the corresponding projective transformation h is an automorphism of both X and Q. In addition, h preserves the line $\mathbb{V}(x,z)$, since the matrix sends both x and z to forms in only x and z. Finally, the line $g(m)=\mathbb{V}(\alpha x-\beta z,\alpha y-\beta w)$ is sent to

$$\begin{aligned} h(g(m)) &= \mathbb{V}(\alpha(\beta x+\alpha^q z)-\beta(\alpha x-\beta^q z),\ \alpha(\beta y+\alpha^q w)-\beta(\alpha y-\beta^q w))\\ &= \mathbb{V}((\alpha^{q+1}+\beta^{q+1})z,\ (\alpha^{q+1}+\beta^{q+1})w) \ =\ \mathbb{V}(z,w). \end{aligned}$$

We conclude that the composition $h \circ g$ is an automorphism of X which preserves Q, and takes ℓ and m to $\mathbb{V}(x, z)$ and $\mathbb{V}(z, w)$, respectively. This completes the proof. □

Corollary 4.1.5. *Let X be a smooth extremal surface and suppose Q is a quadric defining a quadric configuration on X.*

(i) If ℓ is star chord of X lying on Q, then its dual chord ℓ' also lies on Q, in the same ruling.
(ii) If ℓ and m are star chords in opposite rulings of Q, then there are exactly $q + 1$ quadrics (including Q) containing ℓ and m that define quadric configurations on X.

Proof. We can assume that the extremal surface is the Fermat surface, $Q = \mathbb{V}(xw - yz)$, and ℓ and m are the star chords $\mathbb{V}(x, y)$ and $m = \mathbb{V}(x, z)$, respectively (Theorem 4.1.4). For (i), recall that the dual chord of ℓ is $\ell' = \mathbb{V}(z, w)$ (Example 3.1.6), which clearly lies on Q as well. The lines ℓ and ℓ' lie in the same ruling of Q because they are skew (Theorem 3.1.4(ii)).

For (ii), note that the quadrics defining quadric configurations that contain ℓ and m must also contain their dual star chords, $\ell' = \mathbb{V}(z, w)$ and $m' = \mathbb{V}(y, w)$, respectively, by (i). The quadrics containing $\{\ell, \ell', m, m'\}$ are defined by degree two polynomials in the ideal

$$\langle x, z\rangle \cap \langle y, w\rangle \cap \langle z, w\rangle \cap \langle x, y\rangle = \langle xw, yz\rangle.$$

But a quadratic form $\mu\, xw - yz$ (where μ is a non-zero scalar) defines a quadric containing lines of X if and only if $\mu^{q+1} = 1$. Indeed, the lines in one of the rulings are parametrized by $[a : b] \in \mathbb{P}^1$:

$$L_{ab} = \{[as : \mu bs : at : bt] \mid [s : t] \in \mathbb{P}^1\},$$

which lies on the Fermat surface only if $\mu^{q+1} = 1$ and $a^{q+1} + b^{q+1} = 0$. Thus, there are $q + 1$ quadrics that contain the four star chords $\{\ell, \ell', m, m'\}$. □

5. Double $2d$ configurations

One fascinating classical feature of the geometry of a cubic surface is the existence of thirty six "double sixes" [41]. A double six consists of two collections of six skew lines on the cubic, with the property that each line in one collection intersects exactly five lines in the other. A choice of double six is equivalent to a labeling of the twenty-seven lines on the cubic so that one of the collections of six skew lines is the set of six exceptional divisors, thinking of the cubic surface as the blow up of the plane at six points, and the other collection is the set of strict transforms of the six conics through five of the points.

In this section, we present a generalization of a double six which exists on all extremal surfaces.

Definition 5.0.1. For any $d \geq 2$, a **double 2d** is a collection of two sets, $\mathcal{A}$ and $\mathcal{B}$, each consisting of $2d$ lines in projective three space, such that

(1) Each line in $\mathcal{A}$ (resp. $\mathcal{B}$) is skew to every other line in $\mathcal{A}$ (resp. $\mathcal{B}$); and
(2) Each line in $\mathcal{A}$ (resp. $\mathcal{B}$) intersects exactly $d+2$ lines in $\mathcal{B}$ (resp. $\mathcal{A}$).

Typically, we do not an expect a surface of degree d to contain *any* double $2d$—for example, a general surface in $\mathbb{P}^3$ of degree greater than three contains no line [17, 12.8]. The next result guarantees, however, that like cubic surfaces, extremal surfaces always contain double $2d$'s.

Theorem 5.0.2. *Every smooth extremal surface of degree d contains double $2d$ configurations of lines.*

In fact, there are a great many double $2d$'s on an extremal surface of degree d: we show in Corollary 5.3.2 that their number grows asymptotically like $\frac{1}{16}d^{14}$ as d grows large.

We will prove Theorem 5.0.2 by constructing explicit pairs of quadric configurations whose union is a double $2d$. First, we speculate that every double $2d$ arises from pairs of quadrics:

Conjecture 5.0.3. *Every double $2d$ on a smooth surface X of degree d consists of $4d$ lines that are the union of two quadric configurations on X.*

Towards Conjecture 5.0.3, we have proven the following:

Theorem 5.0.4. *Every double $2d$ on a degree d smooth surface is a union of two quadric configurations when $d > 10$ or $d < 5$. Moreover, for $d \geq 5$, if two quadrics determine some double $2d$, then no other pair of quadrics determines the same double $2d$.*

Remark 5.0.5. Our proof of Theorem 5.0.4 is almost entirely combinatorial: given a double $2d$ of lines $(\mathcal{A}, \mathcal{B})$ in $\mathbb{P}^3$, the corresponding $(2d) \times (2d)$ incidence matrix has the property that every row and column contains exactly $d+2$ ones and $d-2$ zeros (see Definition 5.0.1); we give a combinatorial argument that when $d > 10$, this forces the matrix to contain a 5×3 block of ones, which in turn, forces the lines to come from a pair of quadric configurations when they lie on a smooth surface (Lemma 5.4.1). Interestingly, one can write down a 10×10 matrix with seven ones (and three zeros) in each row and column, which does not contain a 5×3 block of ones; however, we have verified this is not the incidence matrix of lines lying on any smooth extremal surface. Indeed, we show by computer that Conjecture 5.0.3 is true when $d = 5$ for smooth extremal surfaces.

Before proving Theorems 5.0.2 and 5.0.4, we review the motivating example of cubic surfaces.

5.1. Double sixes on cubics

Every double six on a cubic surface—whether extremal or not—is a union of two quadric configurations. For an arbitrary double six $\mathcal{A} \cup \mathcal{B}$ on a cubic surface X, there is a choice of coordinates making X the blowup of six points on $\mathbb{P}^2$ (no three on a line, not all on a conic), and so that $\mathcal{A}$ consists of the six lines of exceptional divisors $\{E_1, \ldots, E_6\}$ and $\mathcal{B}$ consists of the proper transforms $\{\tilde{C}_1, \ldots, \tilde{C}_6\}$ of the six conics in $\mathbb{P}^2$ through five of the six points [37, Thm 8, p. 366]. Here, $\tilde{C}_i$ denotes the proper transform of the conic that misses the point blown up to E_i.

Now, given any three lines in $\mathcal{A}$, say $\{E_1, E_2, E_3\}$, there are three lines, $\{\tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}$, in $\mathcal{B}$ that meet all of them. This says that the unique quadric surface Q containing $\{E_1, E_2, E_3\}$ must also contain $\{\tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}$. Likewise, the unique quadric Q' containing $\{E_4, E_5, E_6\}$ must contain $\{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3\}$. So the quadrics Q and Q' both produce quadric configurations on X:

$$Q \cap X = \{E_1, E_2, E_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6\} \qquad \text{and} \qquad Q' \cap X = \{E_4, E_5, E_6, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3\},$$

which together produce the double six

$$\mathcal{A} = \{E_1, E_2, E_3, E_4, E_5, E_6\} \qquad \text{and} \qquad \mathcal{B} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4, \tilde{C}_5, \tilde{C}_6\}.$$

So every double six on a cubic surface is the union of two quadric configurations.

Remark 5.1.1. The quadric configurations Q and Q' determining the double six $\mathcal{A} \cup \mathcal{B}$ on a cubic surface are not unique: there is a quadric containing any three of the six skew lines in $\mathcal{A}$ and another containing the remaining three, and the lines of $\mathcal{B}$ lie three in each of these two quadrics. Thus, there are $\frac{1}{2}\binom{6}{3} = 10$ different pairs of quadrics determining the double six $\mathcal{A} \cup \mathcal{B}$. This confirms that *some* restriction on d is necessary in the uniqueness statement in Theorem 5.0.4 above.

Remark 5.1.2. The previous discussion applies to an arbitrary smooth cubic surface: each of its thirty-six double sixes is a union of two quadric configurations. However, for an *extremal* cubic surface, the double sixes come from two quadrics of a particular form. Specifically, if Q and Q' are quadrics on an extremal cubic surface which together give a double six, then $Q \cap Q'$ is the union of four lines.

To see this, observe that $Q \cap Q' \cap X$ consists of twelve *distinct* points—otherwise, one line of $Q \cap X$ would intersect a line from both rulings of $Q' \cap X$ (or vice versa), violating the skewness condition for a double six. These twelve points are star (Eckardt) points as they lie at the intersection of a line in $Q \cap X$ with a line in $Q' \cap X$. Now, each of these twelve star points lies on only one line in $X \cap Q$, again by skewness, so these twelve star points lie on star chords of $Q \cap X$. Since there are only two star chords in each ruling (Remark 4.1.3), each containing exactly $q + 1 = 3$ star points, these twelve points lie three each on the four star chords on Q. Likewise, the same argument replacing Q by Q' shows that the twelve star points lie three each on the four star chords on Q'. We conclude that $Q \cap Q'$ consists of the four shared star chords for X.

5.2. The existence of double 2d's on extremal surfaces

Proof of Theorem 5.0.2. Choose coordinates so that the extremal surface X is defined by $x^{q+1} + y^{q+1} + z^{q+1} + w^{q+1}$.

Fix μ, a $(q + 1)$-st root of unity. As we saw in Example 4.0.3, the lines

$$\mathcal{L}_\mu := \{\mathbb{V}(x - \alpha y, z - \mu\alpha w) \mid \alpha^{q+1} = -1\}$$

and

$$\mathcal{M}_\mu = \{\mathbb{V}(x - \beta z, y - \mu\beta w) \mid \beta^{q+1} = -1\}$$

form a quadric configuration cut out by the quadric $Q_\mu = \mathbb{V}(\mu xw - yz)$.

We claim that if μ_1 and μ_2 are *distinct* $(q + 1)$ roots of unity, then the sets

$$\mathcal{A} := \mathcal{L}_{\mu_1} \cup \mathcal{M}_{\mu_2} \quad \text{and} \quad \mathcal{B} := \mathcal{L}_{\mu_2} \cup \mathcal{M}_{\mu_1}$$

together form a double $2(q + 1)$*.*

To see that $\mathcal{A}$ consists of skew lines, first observe that the lines of $\mathcal{L}_{\mu_1}$ are mutually skew, as they lie in the same ruling of a quadric. To see that each $L \in \mathcal{L}_{\mu_1}$ is skew to every $M \in \mathcal{M}_{\mu_2}$, we check that the ideal of their intersection, $\langle x - \alpha y,\ z - \mu_1\alpha w,\ x - \beta z,\ y - \mu_2\beta w\rangle$, is generated by four linearly independent linear forms. For this, it suffices to show that the matrix

$$\begin{bmatrix} 1 & -\alpha & 0 & 0 \\ 0 & 0 & 1 & -\mu_1\alpha \\ 1 & 0 & -\beta & 0 \\ 0 & 1 & 0 & -\mu_2\beta \end{bmatrix},$$

whose rows are the coefficients of the linear forms, has full rank. But this is clear, since its determinant is $\alpha\beta(\mu_2 - \mu_1)$. A symmetric argument shows that also $\mathcal{B}$ consists of skew lines.

Now that we know $\mathcal{A}$ and $\mathcal{B}$ are skew sets, the proof of Theorem 5.0.2 will be complete once we have proved the following general lemma.

Lemma 5.2.1. *Let X be a smooth extremal surface of degree d. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadric configurations on X that do not share a line (on X). Write $\mathcal{Q}_1 = \mathcal{L}_1 \cup \mathcal{M}_1$ and $\mathcal{Q}_2 = \mathcal{L}_2 \cup \mathcal{M}_2$ for the decomposition of each quadric configuration into the lines of the* *two rulings. Then $\mathcal{A} = \mathcal{L}_1 \cup \mathcal{M}_2$ and $\mathcal{B} = \mathcal{L}_2 \cup \mathcal{M}_1$ form a double $2d$ on X if (and only if) both $\mathcal{A}$ and $\mathcal{B}$ are skew sets.*

Proof of Lemma 5.2.1. Since $\mathcal{Q}_1$ and $\mathcal{Q}_2$ have no common line, there are $4d$ lines in $\mathcal{Q}_1 \cup \mathcal{Q}_2$, and $2d$ lines in each of $\mathcal{A}$ and $\mathcal{B}$. Because we are given that $\mathcal{A}$ and $\mathcal{B}$ are each skew sets, we need only check condition (2) of Definition 5.0.1 to verify that $\mathcal{A} \cup \mathcal{B}$ is a double $2d$.

To this end, take any $N \in \mathcal{A}$. Without loss of generality, assume $N \in \mathcal{L}_1$. We need to show that N intersects exactly $d+2$ lines in $\mathcal{B}$. Since N lies in one ruling of the quadric Q_1 determining $\mathcal{Q}_1$, the line N intersects the d lines of the opposite ruling $\mathcal{M}_1 \subset \mathcal{B}$. Thus, we need to show that N intersects exactly two lines of $\mathcal{L}_2$.

Since N does not lie on the quadric Q_2 determining $\mathcal{Q}_2$ (remember $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$), its intersection multiplicity with Q_2 is two. If N meets Q_2 in two distinct points, we are done: N must meet exactly two of the lines in the ruling $\mathcal{L}_2$ since it does not meet any line of the ruling $\mathcal{M}_2$ by our assumption that $\mathcal{A}$ is a skew set.

It remains to show that N can not be tangent to Q_2. If, on the contrary, N is tangent to Q_2 at some point p, then $N \subset T_pQ_2$. Because $p \in Q_2 \cap X$, and $Q_2 \cap X$ is a union of lines, the point p lies on some line M in $Q_2 \cap X$. In particular, p is a star point since it is the intersection of the two lines M and N on X. Furthermore, since both N and M are in the tangent plane T_pQ_2, as well as in the star plane T_pX, we have $T_pX = T_pQ$. But now consider the unique line ℓ through the star point p on Q_2 in the opposite ruling from M. We know ℓ is not on X, for otherwise, $p \in \ell \subset Q_2 \cap X$, which means p lies on lines in *both* rulings of $\mathcal{Q}_2$, violating skewness. By Lemma 4.1.1, we conclude that ℓ is a star chord through p, and being on Q_2, also $\ell \subset T_pQ_2 = T_pX$. But no star chord through a star point p can lie in the star plane T_pX (Remark 3.1.3). This contradiction ensures that N is not tangent to Q_2, and the proof is complete. □ □

5.3. Pairs of quadrics containing a common line

The double $2d$ constructed in the proof of Theorem 5.0.2 is obtained from two quadric configurations whose quadric surfaces intersect in four lines. These are an abundant type of double $2d$'s—encompassing all the double sixes in the case of extremal cubics.

Theorem 5.3.1. *Let X be a smooth extremal surface of degree d, and let Q and Q' be distinct quadrics defining quadric configurations on X.*

Assume that Q and Q' share a common line, but share no line on X. Then

- *(i) The $4d$ lines of $(Q \cap X) \cup (Q' \cap X)$ can be split into two sets of $2d$ lines forming a double $2d$;*
- *(ii) The intersection $Q \cap Q'$ consists of four star chords $\{\ell, m, \ell', m'\}$, where $\{\ell, \ell'\}$ and $\{m, m'\}$ are dual chord pairs in opposite rulings.*

Importantly, not all double $2d$'s are of the type guaranteed by Theorem 5.3.1; see Example 5.3.4.

Before proving Theorem 5.3.1, we deduce the following corollary bounding below the total number of double $2d$s on an extremal surface.

Corollary 5.3.2. *An extremal surface of degree $d = q + 1 \geq 5$ contains at least*

$$\frac{1}{16}(q^3 + 1)(q^2 + 1)(q - 1)^2 q^7$$

collections of double $2d$'s.

Proof of Corollary. By Theorem 5.0.4, if two quadrics determine a double $2d$ on an extremal surface of degree $d \geq 5$, then they are unique. So we can prove Corollary 5.3.2 by counting the pairs of quadrics $\{Q, Q'\}$ determining quadric configurations whose intersection consists of four star chords (Theorem 5.3.1).

Fix one quadric Q giving a quadric configuration on X. There are $(q^2 - q)^2$ choices of pairs of star chords $\{\ell, m\}$ on Q, one in each ruling, by Proposition 4.1.2. Since the dual of each star chord on Q is also on Q, there are $\frac{(q^2-q)^2}{4}$ choices for sets of star chords $\{\ell, \ell', m, m'\}$ on Q, where ℓ and m are in opposite rulings and ℓ', m' are their duals.

There are exactly q additional quadrics, besides Q, that contain $\{\ell, \ell', m, m'\}$ and define a quadric configuration (Corollary 4.1.5(ii)). So there are exactly $\frac{q^3(q-1)^2}{4}$ quadrics Q' defining quadric configurations such that $Q \cap Q'$ is the union of two star chords and their duals.

Finally, multiplying by the total number of choices for Q (provided by Corollary 4.0.8), we get

$$\frac{1}{2}(q^3 + 1)(q^2 + 1)q^4 \cdot \frac{1}{4}q^3(q - 1)^2 = \frac{1}{8}(q^3 + 1)(q^2 + 1)(q - 1)^2 q^7$$

ordered pairs of quadric configurations whose intersection is four star chords. This counts each pair twice so the result follows. □

Proof of Theorem 5.3.1. Suppose $\ell \subset Q \cap Q'$ but $\ell \not\subset X$. Because ℓ is in some ruling on each of Q and Q', ℓ must intersect d lines on $X \cap Q$ and d lines on $X \cap Q'$. By hypothesis, these lines are distinct, so ℓ intersects $2d$ lines on X. Now because $\ell \cap X$ can be at most d points, ℓ simultaneously intersects X at a line on $X \cap Q$ and a line on $X \cap Q'$, so ℓ intersects X at a star point. So ℓ passes through d star points and is a star chord.

Let ℓ' be the dual star chord to ℓ. We know $\ell' \subset Q \cap Q'$, by Corollary 4.1.5(i). Since ℓ and ℓ' are skew, they are in the same ruling on Q and also in the same ruling on Q', which means that $\ell \cup \ell'$ is a curve of bi-degree $(2, 0)$ on each quadric. Since $Q \cap Q'$ is a curve of bidegree $(2, 2)$ on each quadric, the residual intersection curve has bidegree $(0, 2)$ in each quadric. Since homogeneous polynomials in two variables over an algebraically closed field factor into linear terms, this residual curve is either two distinct lines, or a double line. In particular, it contains some line m, which, by the argument above, must be a star chord. Now again by Corollary 4.1.5(i), the residual intersection must be two dual star chords m and m'. This proves (ii).

To prove (i), we use Theorem 4.1.4 to chose coordinates so that X is the Fermat extremal surface, Q is the quadric defined by $xw = yz$, and ℓ and m are the lines $\mathbb{V}(x, z)$ and $\mathbb{V}(w, z)$, respectively. In this case, we have already computed (in the proof of Corollary 4.1.5(ii)) that the quadrics containing $\{\ell, \ell', m, m'\}$ and defining quadric configurations are all of the form $\mathbb{V}(\mu xw - yz)$ where $\mu^d = 1$, and that any two such quadrics define a double $2d$ (in the proof of Theorem 5.0.2). □

Remark 5.3.3. The bound in Corollary 5.3.2 is not valid when d is less than 5 because in this case, there can be multiple pairs of quadric configurations that determine the same double $2d$. For example, every double six on a cubic surface can be split into the union of two quadric configurations in ten different ways (Remark 5.1.1). Note that dividing the bound provided by Corollary 5.3.2 by ten, we get a lower bound of 36 double sixes on a cubic surface, recovering the fact that *all* double sixes on an extremal cubic come from quadrics sharing star chords (Remark 5.1.2).

Similarly, when $d = 4$, there are double eights that split into the union of two quadric configurations in multiple ways. For example, the double eight on the Fermat quartic defined by the two quadrics $Q_1 = \mathbb{V}(xw - yz)$ and $Q_2 = \mathbb{V}(xw + yz)$ can also be given by two different quadrics Q_3 and Q_4, as one can check by examining the intersection matrix for the sixteen lines of $(Q_1 \cap X) \cup (Q_2 \cap X)$ to find a different grouping into lines in two quadrics.

Example 5.3.4. We now construct an example of a double eight on a quartic extremal surface that can not be given by two quadrics sharing a line. This shows that not every double $2d$ on an extremal surface is of the special type in Theorem 5.3.1.

We work on the Fermat quartic, $X = \mathbb{V}(x^4 + y^4 + z^4 + w^4)$ in characteristic three. The quadrics $Q_1 = \mathbb{V}(xw - yz)$ and $Q_2 = \mathbb{V}(x^2 + xy + xz - xw - y^2 + yz + yw + z^2 - zw - w^2)$ both give quadric configurations on X. The quadric configuration $X \cap Q_1$ is the union $\mathcal{L} \cup \mathcal{M}$ where

$$\mathcal{L} = \{\mathbb{V}(x - \alpha y, z - \alpha w) \mid \alpha^4 = -1\} \quad \text{and} \quad \mathcal{M} = \{\mathbb{V}(x - \alpha z, y - \alpha w) \mid \alpha^4 = -1\},$$

as we computed in Example 4.0.3. The quadric configuration $X \cap Q_2$ is the union $\mathcal{N} \cup \mathcal{P}$ where

$$\mathcal{N} = \{\mathbb{V}(x - aw, y - az), \mathbb{V}(x - \overline{a}w, y - \overline{a}z), \mathbb{V}(-x - y + w, x - y - z), \mathbb{V}(-x - y - w, x - y + z)\}$$

and

$$\mathcal{P} = \{\mathbb{V}(x + ay, z - \overline{a}w), \mathbb{V}(x + \overline{a}y, z - aw), \mathbb{V}(-x + y + w, -x - y + z), \mathbb{V}(-x - y + w, x - y + z)\},$$

where a and $\overline{a}$ are the roots in k of the polynomial $T^2 - T - 1$ over $\mathbb{F}_3$. We leave it to the reader to directly verify these eight lines all lie on both Q_2 and X.

The set $\mathcal{A} \cup \mathcal{B}$ is a double eight, where $\mathcal{A} = \mathcal{L} \cup \mathcal{N}$ and $\mathcal{B} = \mathcal{M} \cup \mathcal{P}$. To check this, it suffices to check that the lines in $\mathcal{L}$ and skew to those in $\mathcal{N}$, and similarly that the lines in $\mathcal{M}$ are skew to those in $\mathcal{P}$ (Lemma 5.2.1), which can be directly verified.

It remains to check that Q_1 and Q_2 do not share any line. If they did, then there are two shared lines in each ruling (Theorem 5.3.1). So it suffices to show an arbitrary line $\ell = \{\lambda s : s : \lambda t : t] \ [s : t] \in \mathbb{P}^1\}$ in one of the rulings of Q_1 can not lie on Q_2. If $\ell \subset Q_2$, then the points $[0 : 0 : \lambda : 1]$, $[\lambda : 1 : 0 : 0]$ and $[\lambda : 1 : -\lambda : -1]$ in ℓ must all lie on Q_2. Plugging into the equation for Q_2 produces the constraints

$$\lambda^2 - \lambda - 1 = 0, \quad \lambda^2 + \lambda - 1 = 0, \quad \text{and} \quad \lambda^2 = 0.$$

Because these three equations are inconsistent, we conclude that ℓ does not lie on Q_2.

Finally, we must show that the double eight $\mathcal{A} \cup \mathcal{B}$ can not be given by *any other* pair of quadrics that *do* share a line. To this end, assume on the contrary that $\mathcal{A} \cup \mathcal{B}$ is given by quadrics Q_3 and Q_4, and that $\ell \subset Q_3 \cap Q_4$ for some line ℓ. Furthermore, since Q_1 and Q_2 share no line, we may assume that $\ell \not\subset Q_1$; in particular, ℓ intersects two lines in each ruling of Q_1. Because ℓ lies in one ruling of each of Q_3 and of Q_4, ℓ meets each in a set of *eight* skew lines in the double eight $(Q_3 \cap X) \cup (Q_4 \cap X) = \mathcal{A} \cup \mathcal{B}$. Since at most two of these eight intersection points are on Q_1, we know ℓ intersects at least six of the lines in Q_2, and so $\ell \subset Q_2$. But this is impossible: ℓ lies in one of the rulings of Q_2 (and is not on X), so it intersects exactly four of the lines on $Q_2 \cap X$.

5.4. Progress towards Conjecture 5.0.3

We now prove Theorem 5.0.4. The proof uses the combinatorics of the intersection matrix between the two skew sets of size $2d$ and the properties of quadrics.

Lemma 5.4.1. *Let $\mathcal{A} \cup \mathcal{B}$ be a double $2d$ on a smooth surface X of degree $d \geq 5$. If $\mathcal{A}$ contains three lines A_1, A_2, A_3 and $\mathcal{B}$ contains five lines that all meet each A_i for $i = 1, 2$ and 3, then the double $2d$ is the union of two unique quadric configurations.*

Proof. Let Q be the unique smooth quadric containing the three skew lines A_1, A_2, A_3. Let B_1, B_2, B_3, B_4, and $B_5 \in \mathcal{B}$ be the five lines meeting each of A_1, A_2, A_3. Since each B_i meets Q in three points—namely $B_i \cap A_1, B_i \cap A_2$, and $B_i \cap A_3$—B_i lies on Q for $i = 1, 2, \ldots, 5$.

Label the lines in $\mathcal{A}$ so that A_i lies on Q if and only if $i \leq t$. We first show that $t \geq 5$. For any $A \in \mathcal{A}$, note that A meets all five $\{B_1, \ldots, B_5\}$ if A lies on Q and at most two of $\{B_1, \ldots, B_5\}$ if A is not on Q. So

$$\sum_{i=1}^{2d} A_i \cdot \left(\sum_{j=1}^{5} B_j\right) \leq 5t + 2(2d-t) = 3t + 4d.$$

On the other hand, each B_j must intersect exactly $d+2$ lines in $\mathcal{A}$, so

$$\sum_{i=1}^{2d} A_i \cdot \left(\sum_{j=1}^{5} B_j\right) = 5(d+2) = 5d + 10.$$

Thus, $3t + 4d \geq 5d + 10$ so $t \geq 3 + \frac{d+1}{3}$. Since $d \geq 5$, we get $t \geq 5$. Thus, the double $2d$ must contain at least five lines of each ruling of Q.

Next we show that in fact each set $\mathcal{A}$ and $\mathcal{B}$ contains d lines on Q. If not, let k be maximal such that $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ lie on Q, and assume without loss of generality that $A_{k+1} \not\subset Q$. Since each B_j intersects exactly $d+2$ lines in $\mathcal{A}$, k of which are $A_1, \ldots, A_k$, we have that

$$\sum_{j=1}^{k} B_j \cdot \left(\sum_{i=k+1}^{2d} A_i\right) = k(d+2) - k^2 = k(d+2-k). \tag{12}$$

Since A_i lies on Q if and only if $i \leq k$, each line A_j for $j > k$ can meet at most two of $\{B_1, \ldots, B_k\}$. So

$$\sum_{j=1}^{k} B_j \cdot \left(\sum_{i=k+1}^{2d} A_i\right) \leq 2(2d-k) = 4d - 2k. \tag{13}$$

Comparing (12) and (13), we have $(4d-2k) - k(d+2-k) = (k-d)(k-4) \geq 0$, which (since we've shown $k \geq 5$) is a contradiction unless $k \geq d$. We conclude that $\mathcal{L} = \{A_1, \ldots, A_d\}$ and $\mathcal{M} = \{B_1, \ldots, B_d\}$ lie on the same quadric Q. That is, half the lines of $\mathcal{A}$, together with half the lines of $\mathcal{B}$, form a quadric configuration $\mathcal{L} \cup \mathcal{M}$ on X.

It suffices to show that the remaining halves of $\mathcal{A}$ and $\mathcal{B}$ also form a quadric configuration. Now fix $B \in \mathcal{B} \setminus \mathcal{M}$. Because B intersects at most two of the lines of $\mathcal{L}$, we know B intersects every line in $\mathcal{A} \setminus \mathcal{L}$. So each line of the skew set $\mathcal{B} \setminus \mathcal{M}$ meets every line of the skew set $\mathcal{A} \setminus \mathcal{L}$—that is $\mathcal{A} \setminus \mathcal{L}$ and $\mathcal{B} \setminus \mathcal{M}$ form a quadric configuration, as desired. □

Remark 5.4.2. Even if d is three or four, the final paragraph of the proof of Lemma 5.4.1 shows that if half the lines of a double $2d$ lie on some quadric Q, then the complementary half lies on some other quadric Q'. The quadrics $\{Q, Q'\}$ may not be the only quadrics determining the double $2d$ in this case, however. See Remark 5.1.1 and Remark 5.3.4.

Proof of Theorem 5.0.4. Let X be a smooth surface of degree d. The case where $d = 3$ is dealt with in § 5.1. We next handle the case $d \geq 11$.

Let $\mathcal{A} := \{A_1, \ldots, A_{2d}\}$ and $\mathcal{B} := \{B_1, \ldots, B_{2d}\}$ denote the two skew sets of the double $2d$. By Lemma 5.4.1, it suffices to show that there are three skew lines in $\mathcal{A}$ all intersecting each of five skew lines in $\mathcal{B}$. Let M denote the intersection matrix $M_{ij} = A_i \cdot B_j$. By definition of a double $2d$, M has exactly $d+2$ ones and exactly $d-2$ zeros in every row and column.

For any subset $S \subset \mathcal{A}$, let

$$\text{IntersectionSet}(S, \mathcal{B}) := \{B_i \in \mathcal{B} \mid B_i \cdot A_j = 1 \text{ for all } A_j \in S\}.$$

We want to show that there exists some A_i, A_j, A_k such that $|\text{IntersectionSet}(\{A_i, A_j, A_k\}, \mathcal{B})| \geq 5$. After a possible relabeling of the B_i, we may assume

$$\text{IntersectionSet}(A_1, \mathcal{B}) = \{B_1, \ldots B_{d+2}\}.$$

Let

$$k := \max_{2 \leq i \leq 2d} \{|\text{IntersectionSet}(\{A_1, A_i\}, \mathcal{B})|\}.$$

Then by assumption, the number of ones in rows $2, \ldots, 2d$ and columns $1, \ldots, d+2$ of M is at most $k(2d-1)$ since there are at most k ones in each of these rows. However, by looking at columns, we see that there are exactly $(d+1)(d+2)$ ones in this submatrix of M. Thus, we see

$$\frac{(d+1)(d+2)}{(2d-1)} \leq k$$

and since we may assume k is an integer, we have

$$\left\lceil \frac{(d+1)(d+2)}{(2d-1)} \right\rceil \leq k.$$

By relabeling $A_2, \ldots, A_{2d}$, we may assume $|\text{IntersectionSet}(\{A_1, A_2\}, \mathcal{B})| = k$. By relabeling $B_1, \ldots, B_{d+2}$, we may assume $\text{IntersectionSet}(\{A_1, A_2\}, \mathcal{B}) = \{B_1, \ldots, B_k\}$. Now note that M has exactly kd ones in columns $1, \ldots, k$ and rows $3, \ldots, 2d$.

Let

$$\ell := \max_{3 \leq i \leq 2d} \{|\text{IntersectionSet}(\{A_1, A_2, A_i\}, \mathcal{B})|\}.$$

Then the number of ones in columns $1, \ldots, k$ and rows $3, \ldots, 2d$ is at most $\ell(2d-2)$, so we have

$$\ell \geq \frac{kd}{2d-2} \geq \left\lceil \frac{(d+1)(d+2)}{(2d-1)} \right\rceil \frac{d}{2d-2}$$

and since ℓ is an integer, we have

$$\ell \geq \left\lceil \left\lceil \frac{(d+1)(d+2)}{(2d-1)} \right\rceil \frac{d}{2d-2} \right\rceil \tag{14}$$

From (14), it follows that when $d \geq 11$, we have $\ell \geq 5$, as desired.

Finally, when $d = 4$, formula (14) implies that $\ell \geq 4$, so that there exists a set of three skew lines A_1, A_2, and $A_3 \in \mathcal{A}$ that all intersect four skew lines B_1, B_2, B_3, and $B_4 \in \mathcal{B}$. In particular, each of the four B_i must lie on the unique quadric Q determined by A_1, A_2, and A_3, since they each intersect this quadric in 3 points. We claim one more line in $\mathcal{A}$ lies on Q, in which case it follows that every double eight is the union of two quadric configurations (Remark 5.4.2). To verify the claim, observe that if no A_i lies on Q for $i > 3$, then these A_i intersect at most two of B_1, B_2, B_3, and $B_4 \in \mathcal{B}$. Thus,

$$\sum_{i=1}^{4} B_i \cdot \sum_{A_i \in \mathcal{A}} A_i \;=\; \sum_{i=1}^{4} B_i \cdot \sum_{i=1}^{3} A_i \;+\; \sum_{i=1}^{4} B_i \cdot \sum_{i=4}^{8} A_i \;\leq\; 12 + 10 \;=\; 22,$$

contrary to the fact that $\sum_{i=1}^{4} B_i \cdot \sum_{A_i \in \mathcal{A}} A_i = 24$, since each line in $\mathcal{B}$ intersects exactly six lines in $\mathcal{A}$. $\square$

Remark 5.4.3. It is worth emphasizing that Theorem 5.0.4 is valid for *any* smooth surface of degree containing a double 2d. While a generic surface certainly contains none, there always exist smooth surfaces, including *non-extremal* ones, that contain double $2d$'s over an algebraically closed field of characteristic p and of every degree $d = p^e + 1 > 6$.

To see the existence of such non-extremal surfaces, we do a simple dimension count. Fix a double $2d$ $\{\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2\}$ on an extremal surface arising as the union of two quadric configurations, $\mathcal{Q}_1 = \{\mathcal{A}_1, \mathcal{B}_1\}$ and $\mathcal{Q}_2 = \{\mathcal{A}_2, \mathcal{B}_2\}$.

We claim that the set $\mathcal{X}$ of all degree d surfaces in $\mathbb{P}^3$ containing the lines in $\{\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{B}_1 \cup \mathcal{B}_2\}$ forms a subvariety of $\mathbb{P}(\mathrm{Sym}^d(k^4)^*)$ of dimension exceeding the dimension of the subvariety of extremal surfaces. That latter dimension is 15, since it is equal to the dimension of $\mathrm{PGL}_4(k)$. Since there is a smooth degree surface in $\mathcal{X}$ (our original extremal surface), then provided $\dim \mathcal{X} > 15$, we can conclude that an open subset of $\mathcal{X}$ consists of smooth non-extremal surfaces containing the given double $2d$.

To see that the dimension of $\mathcal{X}$ exceeds 15, label the lines in $\mathcal{A}_i$ (respectively $\mathcal{B}_i$) by A_{ij} (respectively B_{ij}) for $1 \leq j \leq d$. One can show that if a surface S contains every intersection point $A_{1i} \cap B_{1j}$ for $1 \leq i, j \leq d$, one additional point on each line of $\mathcal{Q}_1$, and all points of the form $A_{2i} \cap B_{2j}$ with $1 \leq i, j \leq d-1$, then S will contain the entire double $2d$. Let $\mathcal{D}$ be the linear space in $\mathbb{P}(\mathrm{Sym}^d(k^4)^*)$ parametrizing surfaces of degree d containing these $d^2 + 2d + (d-1)^2 = 2d^2 + 1$ points, and note that $\mathcal{X} \supset \mathcal{D}$. Since each point imposes a linear condition, the dimension of $\mathcal{D}$, and hence the dimension of $\mathcal{X}$, is at least $\binom{d+3}{d} - 1 - (2d^2 + 1) = \frac{1}{6}(d-1)(d-2)(d-3)$. This exceeds 15 whenever $d > 6$.

Data availability

No data was used for the research described in the article.

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