The moduli space of G-algebras

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Let L be a Galois algebra with Galois group G and let x be a normal element of L. The moduli space $\mathcal{X}$ of pairs (L, x) is isomorphic to an open subset of the quotient variety $\mathbb{P}/G$, where $\mathbb{P}$ is the projective space of the regular representation of G. We provide a formula for the height of any pair $(L, x) \in \mathcal{X}(\mathbb{Q})$ in terms of algebraic invariants of L and x with respect to a natural adelic metric on the anticanonical divisor of $\mathbb{P}/G$.

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1. Introduction

Let G be a finite group acting linearly on a finite-dimensional rational vector space V, and let $\mathbb{P} = \mathbb{P}(V)$ denote the projective space of V. In this paper we study heights of rational points on $\mathbb{P}/G$.

Our main result is that for V the regular representation of G and a certain open subset $\mathcal{X}$ of $\mathbb{P}/G$, the height of any rational point P on $\mathcal{X}$ can be expressed in terms of certain objects attached to P. We show that P determines, and is determined by, a Galois $\mathbb{Q}$-algebra L with Galois group G and a trace-one normal element x of L up to Galois conjugacy. The Galois algebra L is isomorphic to $K \times \cdots \times K$ for a number field K. The images of the Galois conjugates of x in K generate a fractional K-ideal J. Let $\|\cdot\|_\infty$ denote the canonical norm on Minkowski space $L_\mathbb{R}$.

Theorem 1.1. *Let $\phi\colon \mathbb{P}/G \to \mathbb{P}^N$ be a non-constant morphism. Let d_ϕ be the degree of the composite map $\mathbb{P} \to \mathbb{P}/G \xrightarrow{\phi} \mathbb{P}^N$. Then for all $P = (L, x) \in \mathcal{X}(\mathbb{Q})$,*

$$h(\phi(P)) = d_\phi \log \|x\|_\infty - \frac{d_\phi}{[K : \mathbb{Q}]} \log N(J) + O(1) \tag{1}$$

for a bounded function $O(1)$.

To prove this theorem we use the method of descent to construct an ample line bundle $\mathscr{L}$ on $\mathbb{P}/G$ which is linearly equivalent to the anticanonical divisor of $\mathbb{P}/G$ up to torsion in the divisor class group. We show that $\mathscr{L}$ is globally generated and its global sections restrict to an immersion on $\mathcal{X}$. These geometric results are used to prove Theorem 1.1 in §4.

In the last section we give a sharper result. We prove that the function

$$h_{AC}(L, x) = |G| \log \|x\|_\infty - \frac{|G|}{[K : \mathbb{Q}]} \log N(J)$$

(with no $O(1)$) is actually the height function h_{AC} associated with a natural adelic metric on $\mathscr{L}$.

1.1. Self-dual elements

In addition to normal elements, we also consider *self-dual* elements. Recall an element $x \in L$ is self-dual if for all $g \in G$ we have

$$\mathrm{tr}^L_\mathbb{Q}(xg(x)) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here we assume that G has odd order since this guarantees the existence of self-dual elements by a theorem of Bayer-Fluckiger–Lenstra [2].

The formula (1) indicates that rational points of $\mathcal{X}$ corresponding to self-dual elements are of particular interest. Suppose $L = K$ for simplicity, and let $x \in K$ be any self-dual element. The first indication is that

$$\|x\|_\infty = \sqrt{\mathrm{tr}_\mathbb{Q}^K(x^2)} = 1.$$

Indeed the left equality holds since K is totally real, and the right since x is self-dual. The second indication involves the norm term $N(J)$. Recall the basic formula relating norms and discriminants:

$$N(J) = [O_K : J] = \sqrt{d_J d_K^{-1}}.$$

In particular, if J is unimodular, then $d_J = 1$ and we have the appealing formula

$$h_{AC}(K, x) = \log \sqrt{d_K}.$$

Naturally it would be of interest to have a direct relationship between heights and discriminants (cf. e.g. [7], [23]) so we should like to understand the extent to which J fails to be unimodular. As x is self-dual the smaller lattice

$$I := \sum_{g \in G} \mathbb{Z}g(x)$$

is unimodular, however in general

$$\sum_{g \in G} \mathbb{Z}g(x) \neq \sum_{g \in G} O_K g(x) = J.$$

Thus we ask whether there is a relationship between $N(I)$ and $N(J)$. In this direction we prove the following result. Let $K/\mathbb{Q}$ be a Galois field extension with Galois group G and let x be a normal element of K, not necessarily self-dual. For each integer $D \geq 1$ consider the following order of K:

$$T_D = \{a \in K : aI^D \subset I^D\}.$$

These orders increase with D and stabilize, and we call

$$T_\infty := \lim_{D \to \infty} T_D$$

the *stable multiplier order* of I. We prove two stability results and also find another interpretation for T_∞.

Theorem 1.2.

(1) $T_\infty = T_D$ if $D \geq |G| - 1$.
(2) $N(I^D) = N(J^D)$ if $D \geq |G| - 1$.
(3) Spec T_∞ is isomorphic to the fiber of $\mathbb{P} \to \mathbb{P}/G$ over the integral point of $\mathbb{P}/G$ determined by (K, x).

2. Construction of orbit parametrizations

In this section we construct orbit parametrizations for G-algebras equipped with normal and self-dual bases. If R is a commutative ring and Spec S is a G-torsor over Spec R in the étale topology, then we call S together with its natural G-action a *G-algebra over R*.

Remark 1. The normal basis theorem guarantees the existence of normal elements if S is a field, however normal elements need not exist for an arbitrary G-algebra (e.g. S might not be free as an R-module or there may be local obstructions due to wild ramification). Self-dual elements exist for Galois field extensions of odd degree in any characteristic [2], [1] but even degree field extensions may not have self-dual elements (e.g. quadratic field extensions with characteristic $\neq 2$).

For fixed R, we consider pairs (S, x), where S/R is a G-algebra and $x \in S$ is a normal element. An isomorphism between pairs (S, x) and (S', x') is a G-equivariant R-algebra isomorphism $\varphi \colon S \xrightarrow{\sim} S'$ satisfying $\varphi(x) = x'$. Given a G-algebra S and a ring homomorphism $f \colon R \to R'$, the base extension $S \otimes R'$ is a G-algebra over R'. If $x \in S$ is normal (resp. self-dual), then $x \otimes 1 \in S \otimes R'$ is normal (resp. self-dual).

Definition 2.1. $\mathfrak{M}$ is the functor taking a commutative ring R to the set of isomorphism classes of pairs (S, x), where S/R is a G-algebra and $x \in S$ is a normal element. We define $\mathfrak{N}$ similarly but with x self-dual.

These functors are representable by affine schemes over $\mathbb{Z}$, which we construct as subquotients of the group of units in the group algebra of G. The functor of commutative rings

$$R \mapsto \left\{ u = \sum_{g \in G} a_g[g] \in R[G]^\times : \sum_{g \in G} a_g = 1 \right\}$$

is representable by an affine group scheme of finite type over $\mathbb{Z}$, which we denote by $\mathcal{G}$. There is an anti-involution $u \mapsto \bar{u}$ of $\mathcal{G}$, given by $\overline{\sum_{g \in G} a_g[g]} = \sum_{g \in G} a_g[g^{-1}]$, and we also consider the subgroup scheme $\mathcal{H} \subset \mathcal{G}$ of norm-one units given by

$$\mathcal{H}(R) = \{ u \in \mathcal{G}(R) : u\bar{u} = 1 \}.$$

The coordinate ring A of $\mathcal{G}$ is the quotient of $\mathbb{Z}[X_g : g \in G][\Delta_G^{-1}]$ by the principal ideal generated by $(\sum_{g \in G} X_g - 1)$ where Δ_G is the determinant of the matrix with rows and columns indexed by G whose g, h component is X_{gh}. The coordinate ring B of $\mathcal{H}$ is the quotient of A by the ideal $(\sum_{h \in G} X_{gh} X_h - \delta_{g,1} : g \in G)$.

Lemma 2.2. *Suppose S/R is a G-algebra. Then there is a bijection from the set of G-equivariant ring homomorphisms $\varphi \colon A \to S$ to the set of normal elements of S/R, taking φ to $\varphi(X_1)$. Moreover, φ factors through B if and only if $\varphi(X_1)$ is self-dual.*

Proof. There is a bijection from the set of G-equivariant ring homomorphisms $\varphi \colon \mathbb{Z}[\{X_g\}_{g \in G}] \to S$ to the set of elements of S, given by $\varphi \mapsto \varphi(X_1)$. Fix such a φ, and set $x := \varphi(X_1) \in S$. Then $\varphi(X_g) = g^{-1}(x)$, and $\varphi(\Delta_G)^2$ is the discriminant of the set $\{\varphi(X_g)\}_g = \{g(x)\}_g$. This discriminant is a unit if and only if $\{g(x)\}_g$ is a basis for S/R, and φ kills $\sum_g X_g - 1$ if and only if $\operatorname{tr}_{S/R}(x) = 1$. This proves the first assertion of the lemma.

To prove the second, observe that φ factors through B if and only if

$$\sum_{h \in G} hg(x)h(x) = \begin{cases} 1 & \text{if } g = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $g \in G$, which is precisely the condition that x is self-dual. $\quad\square$

The group G is naturally identified with a constant subgroup scheme of $\mathcal{H}$, and thereby acts freely on $\mathcal{H}$ and $\mathcal{G}$.

Definition 2.3. $\mathcal{X}$ is the quotient scheme $\mathcal{G}/G$, $\mathcal{Y}$ is the quotient scheme $\mathcal{H}/G$.

The following theorem says that the scheme $\mathcal{X}$ is a fine moduli space for G-algebras equipped with a normal element, and $\mathcal{Y}$ is a fine moduli space for G-algebras equipped with a self-dual element.

Proposition 2.4. *For any commutative ring R there are bijections $\mathcal{X}(R) \xrightarrow{\sim} \mathfrak{M}(R)$ and $\mathcal{Y}(R) \xrightarrow{\sim} \mathfrak{N}(R)$ which are functorial in R.*

First proof of Proposition 2.4. The scheme $\mathcal{X} = \mathcal{G}/G$ represents the stack quotient $[\mathcal{G}/G]$ as the G-action is free. An R-point of $[\mathcal{G}/G]$ is, by definition, a G-torsor $\operatorname{Spec} S \to \operatorname{Spec} R$ together with a G-equivariant morphism $\operatorname{Spec} S \to \mathcal{G}$. The G-equivariant morphisms $\operatorname{Spec} S \to \mathcal{G}$ are in bijection with G-equivariant ring homomorphisms $\mathcal{O}(\mathcal{G}) = A \to S$, which are in bijection with normal elements of S/R by Lemma 2.2. $\quad\square$

Here is a more elementary proof.

Second proof of Proposition 2.4. Since $\mathcal{G} \to \mathcal{X}$ is a G-torsor, A/A^G is a G-algebra. Then Lemma 2.2 applied to the identity map $A \to A$ implies $X_1 \in A$ is a normal element.

Now, fix a ring R. There is a function

$$\gamma_1 \colon \operatorname{Hom}(A^G, R) \to \mathfrak{M}(R)$$

$$f \mapsto \left(A \otimes_{A^G, f} R, X_1 \otimes 1 \right).$$

We also define a function $\gamma_2 \colon \mathfrak{M}(R) \to \operatorname{Hom}(A^G, R)$ as follows. Given $(S, x) \in \mathfrak{M}(R)$, Lemma 2.2 implies there is a unique G-equivariant homomorphism $f \colon A \to S$ such that $f(X_1) = x$. Then f takes A^G into $S^G = R$, and we define $\gamma_2(S, x) = f\big|_{A^G}$. We claim that γ_1 and γ_2 are inverses.

In one direction, given $f \colon A^G \to R$, we see directly that the natural map $A \to A \otimes_{A^G, f} R$ is G-equivariant and takes X_1 to $X_1 \otimes 1$. It follows that $\gamma_2(\gamma_1(f))$ is the natural map $A^G \to A^G \otimes_{A^G, f} R = R$, so that $\gamma_2(\gamma_1(f)) = f$. In the other direction, given $(S, x) \in \mathfrak{M}(R)$, let $f \colon A \to S$ be the G-equivariant homomorphism satisfying $f(X_1) = S$. Then

$$\gamma_1(\gamma_2(S, x)) = \left(A \otimes_{A^G, f|_{A^G}} R, X_1 \otimes 1 \right).$$

Now, S is an R-algebra, so f extends uniquely to an R-algebra homomorphism

$$\tilde{f} \colon A \otimes_{A^G, f|_{A^G}} R \to S.$$

One sees directly that $\tilde{f}$ is G-equivariant, and that $\tilde{f}(X_1 \otimes 1) = f(X_1) = x$. Finally, every G-equivariant morphism of torsors is an isomorphism, so we conclude $\gamma_1(\gamma_2(S, x)) \simeq (S, x)$. This proves that γ_1 and γ_2 are inverse bijections. Moreover, it is clear that γ_1 and γ_2 are natural in R.

To prove the statement about $\mathcal{Y}(R)$, we observe that γ_1 takes $\operatorname{Hom}(B^G, R)$ into $\mathfrak{N}(R)$ because $X_1 \in B$ is self-dual, and γ_2 takes $\mathfrak{N}(R)$ into $\operatorname{Hom}(B^G, R)$ by Lemma 2.2. $\quad \square$

Remark 2. Gundlach [9] independently constructed $\mathcal{X}$ and its orbit parametrization for G-algebras with a normal element. Gundlach's construction uses the constraints on the structure constants of a G-algebra to cut out $\mathcal{X}$ inside $\mathbb{A}^{|G|^2}$. In special cases the varieties $\mathcal{X}$ have appeared before in the literature [20, §VI.2], [21], [15], [6], [22], [3], [4] and [5]. See also [19] for a related construction.

It is well-known that a G-torsor is trivial if and only if it admits a section. For G-torsors S/R obtained by pulling back $\mathcal{G} \to \mathcal{X}$ along an R-point (S, x) of $\mathcal{X}$, the next proposition gives a formula for such a section after a suitable base change.

Proposition 2.5. *Let $(S, x) \in \mathcal{X}(R)$. Let R' be an R-algebra and suppose there is an R-algebra homomorphism $\varphi \colon S \to R'$. We have the following diagram:*

$$\begin{array}{ccc} \text{Spec } S & \longrightarrow & \mathcal{G} \\ & \downarrow & \downarrow \\ \text{Spec } R' \longrightarrow \text{Spec } R & \xrightarrow{(S,x)} & \mathcal{X}. \end{array}$$

Then $(S \otimes_R R', x \otimes 1) \in \mathcal{X}(R')$ *is the image of the* R'*-valued unit*

$$u = \sum_{g \in G} \varphi(g(x))[g^{-1}] \in \mathcal{G}(R')$$

under the natural morphism $\mathcal{G} \to \mathcal{X}$.

For the proof, we will use the natural action of $\mathcal{G}$ on the homogeneous space $\mathcal{G}/G = \mathcal{X}$. For $u = \sum_g a_g[g] \in \mathcal{G}(R)$ and $(S, x) \in \mathcal{X}(R)$ this action is given by

$$u(S, x) = \left(S, \sum_{g \in G} a_g g(x) \right).$$

Proof. The set of morphisms Spec $R' \to$ Spec S over Spec R is in bijection with the set of sections of Spec $S \times_R$ Spec $R' \to$ Spec R' by the universal property of the fiber product. Thus the existence of φ implies that the pullback of the G-torsor S/R to R' is isomorphic to the trivial G-torsor over R'. Recall the coordinate ring of the trivial G-torsor over R' is the R'-algebra $\mathcal{A}^{\mathrm{spl}}_{R'}$ of set-theoretic functions $f \colon G \to R'$ under pointwise operations with G-action given by $g(f)(h) = f(hg)$. We have the isomorphism of G-algebras over R' given by

$$S \otimes_R R' \xrightarrow{\sim} \mathcal{A}^{\mathrm{spl}}_{R'}$$
$$x \otimes r \mapsto \left[g \mapsto r\varphi(g(x)) \right].$$

This isomorphism maps $x \otimes 1$ to the function $g \mapsto \varphi(g(x))$, which is equal to $u\chi_{\{1\}}$ where u is the R'-valued unit given by $u = \sum_{g \in G} \varphi(g(x))[g^{-1}] \in \mathcal{G}(R')$ and $\chi_{\{1\}}$ is the characteristic function of the singleton $\{1\}$ containing the identity element $1 \in G$. In terms of points of $\mathcal{X}$, this means that

$$(S \otimes_R R', x \otimes 1) = \left(\mathcal{A}^{\mathrm{spl}}_{R'}, g \mapsto \varphi(g(x)) \right) = u \left(\mathcal{A}^{\mathrm{spl}}_{R'}, \chi_{\{1\}} \right).$$

The result now follows from the fact that $(\mathcal{A}^{\mathrm{spl}}_{R'}, \chi_{\{1\}}) \in \mathcal{X}(R')$ is the image of $1 \in \mathcal{G}(R')$ under the $\mathcal{G}$-equivariant quotient morphism $\mathcal{G} \to \mathcal{X}$. $\square$

3. Descent for G-line bundles

In this section we prove there is a unique line bundle $\mathscr{L}$ over $\mathbb{P}/G$ whose pullback to $\mathbb{P}$ is equal to $-K_{\mathbb{P}}$ (Theorem 3.5). We show that $\mathrm{Pic}(\mathbb{P}/G) \cong \mathbb{Z}$ and $\mathscr{L}$ generates

the subgroup of index n/e where n (resp. e) denotes the order (resp. exponent) of G (Theorem 3.7). We also prove that $\mathscr{L}$ is globally generated and its global sections define an immersion of $\mathcal{X}$ into projective space (Theorem 3.8).

3.1. Descent for G-line bundles

Let Y denote a projective variety equipped with an action by a finite group G, all defined over a characteristic zero field K. We assume the G-action on the structure sheaf of Y is $\mathcal{O}(Y)$-linear. Suppose Y admits the action of another finite group Γ commuting with the action of G. Consider a G-line bundle $\mathscr{L}$ over Y/Γ. When is $\mathscr{L}$ isomorphic to the pullback of a line bundle from $Y/(G \times \Gamma)$? Equivalently, when does $\mathscr{L}$ vanish under the map

$$\mathrm{Pic}_G(Y/\Gamma) \to \frac{\mathrm{Pic}_G(Y/\Gamma)}{\mathrm{im}(\mathrm{Pic}(Y/(G \times \Gamma)) \to \mathrm{Pic}_G(Y/\Gamma))}?$$

If Γ acts freely on Y/G, the next lemma shows this can be determined by pulling back to $Y \to Y/G$ and resolving the question there.

Lemma 3.1. *Suppose $Y/G \to Y/(G \times \Gamma)$ is a Galois covering with Galois group Γ. Then a G-line bundle on Y/Γ descends to $Y/(G \times \Gamma)$ if and only if its pullback to Y descends to Y/G.*

Equivalently, the following natural map is injective:

$$\frac{\mathrm{Pic}_G(Y/\Gamma)}{\mathrm{im}(\mathrm{Pic}(Y/(G \times \Gamma)) \to \mathrm{Pic}_G(Y/\Gamma))} \to \frac{\mathrm{Pic}_G(Y)}{\mathrm{im}(\mathrm{Pic}(Y/G) \to \mathrm{Pic}_G(Y))} \qquad (2)$$

Proof. Let $\mathscr{L}$ be a G-line bundle over Y/Γ which represents some class in the kernel of (2). We have diagrams:

where $\mathscr{L}_1$ is the pullback of $\mathscr{L}$, $\mathscr{L}_2$ is any line bundle which pulls back to $\mathscr{L}_1$, $\mathscr{L}_3$ is the quotient of $\mathscr{L}_2$ by Γ, and $\mathscr{L}_4$ is the pullback of $\mathscr{L}_3$ (the quotient of $\mathscr{L}_2$ is for its Γ-linearization coming from $\mathscr{L}_1$, and the quotient exists by descent along torsors since Γ acts freely). As the left diagram commutes, $\mathscr{L}_4$ pulls back to $\mathscr{L}_1$ in $\mathrm{Pic}_G(Y)$; however $\mathscr{L}$ also pulls back to $\mathscr{L}_1$, so to prove (2) is injective it suffices to show that $\mathrm{Pic}_G(Y/\Gamma) \to \mathrm{Pic}_G(Y)$ is injective.

There is a commutative diagram with exact rows [13, 2.2]:

$$
\begin{array}{ccccc}
H^1(G,\mathcal{O}(Y/\Gamma)^\times) & \longrightarrow & \mathrm{Pic}_G(Y/\Gamma) & \longrightarrow & \mathrm{Pic}(Y/\Gamma) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(G,\mathcal{O}(Y)^\times) & \longrightarrow & \mathrm{Pic}_G(Y) & \longrightarrow & \mathrm{Pic}(Y).
\end{array}
$$

The action of G on $\mathcal{O}(Y)^\times$ and $\mathcal{O}(Y/\Gamma)^\times$ is trivial by assumption. Thus the left column is injective, so it now suffices to show that $\mathrm{Pic}(Y/\Gamma) \to \mathrm{Pic}(Y)$ is injective. Now the Hochschild–Serre spectral sequence

$$
H^p(\Gamma, H^q_{\mathrm{et}}(Y,\mathbb{G}_m)) \Rightarrow H^{p+q}_{\mathrm{et}}(Y/\Gamma,\mathbb{G}_m)
$$

yields the exact sequence

$$
1 \longrightarrow H^1(\Gamma,\mathcal{O}(Y)^\times) \longrightarrow \mathrm{Pic}(Y/\Gamma) \longrightarrow \mathrm{Pic}(Y)^\Gamma \longrightarrow H^2(\Gamma,\mathcal{O}(Y)^\times) \longrightarrow \cdots .
$$

From this it suffices to show that $H^1(\Gamma,\mathcal{O}(Y)^\times) = 1$. As Y is projective, $\mathcal{O}(Y)/\mathcal{O}(Y/\Gamma)$ is a field extension with Galois group Γ, so $H^1(\Gamma,\mathcal{O}(Y)^\times) = 1$ by Hilbert's theorem 90. $\square$

Using the lemma we can give a simple criterion for descent for line bundles even when the group action is not free.

Proposition 3.2. *The image of $\mathrm{Pic}(Y/G) \to \mathrm{Pic}_G(Y)$ is the subset of isomorphism classes of G-line bundles $\mathscr{L}$ over Y satisfying the following condition:*

$(*)$ *the stabilizer subgroup G_P acts trivially on $\mathscr{L}_P$ for every $P \in Y(\overline{K})$.*

Remark 3. A result of Mumford [16, Cor. 1.6] says that if Y is normal and proper, with an action of a connected linear group G, and $\mathscr{L}$ is a G-line bundle on Y, then some positive power $\mathscr{L}^{\otimes e}$ is G-linearizable. Proposition 3.2 is the analogous result for finite G, with e the exponent of G.

Proof. In the algebraically closed setting this is [13, Prop. 4.2]. We will reduce to this case using Lemma 3.1. Pulling back $\mathscr{L}$ to $Y_{\overline{K}}$ shows the condition $(*)$ is clearly satisfied if $\mathscr{L}$ descends to Y/G.

We first observe that $Y_L/G = (Y/G)_L$ for any K-algebra L. Indeed $Y_L \to (Y/G)_L$ is G-invariant so we get a map $Y_L/G \to (Y/G)_L$. To show this affine map is an isomorphism we must show that $\mathcal{O}_{(Y/G)_L} \to \mathcal{O}_{Y_L/G}$ is an isomorphism of $\mathcal{O}_{(Y/G)_L}$-algebras. In fact these sheaves are already equal on the level of presheaves, namely the presheaves $U \mapsto H^0(G,\mathcal{O}_Y(\pi^{-1}(U))) \otimes L$ and $U \mapsto H^0(G,\mathcal{O}_Y(\pi^{-1}(U)) \otimes L)$ sheafify to $\mathcal{O}_{(Y/G)_L}$ and

$\mathcal{O}_{Y_L/G}$, respectively, and these are isomorphic by the universal coefficient theorem: for any G-module N and $n \geq 0$ we have the exact sequence [11, Prop. 4.18]:

$$0 \longrightarrow H^n(G, N) \otimes L \longrightarrow H^n(G, N \otimes L) \longrightarrow \mathrm{Tor}_1^K(H^{n+1}(G, N), L) \overset{=}{\longrightarrow} 0.$$

Now if $(*)$ holds then $\mathscr{L} \otimes \overline{K}$ is the pullback of a line bundle on $Y_{\overline{K}}/G$. This bundle on $Y_{\overline{K}}/G = (Y/G)_{\overline{K}}$ descends to $(Y/G)_L = Y_L/G$ for some finite Galois extension L/K. Applying Lemma 3.1 to Y_L (regarded over K via the structure map $Y_L \to \mathrm{Spec}\, K \to \mathrm{Spec}\, K$) and $\Gamma = \mathrm{Gal}(L/K)$ shows $\mathscr{L}$ is in the image of $\mathrm{Pic}(Y/G) \to \mathrm{Pic}_G(Y)$. $\square$

When the action of G on Y is free, the proof of Lemma 3.1 shows that $\mathrm{Pic}(Y/G) \to \mathrm{Pic}(Y)$ is injective. Unfortunately the spectral sequence there is unavailable for non-free actions, but there is nonetheless an easily checked criterion for injectivity.

Lemma 3.3. *Assume that Y is geometrically integral. Suppose that for each nontrivial character $c \colon G \to K^\times$ there is a point $P \in Y(\overline{K})$ with $c(G_P) \neq 1$. Then the homomorphism*

$$\mathrm{Pic}(Y/G) \to \mathrm{Pic}(Y)$$

is injective.

The hypothesis holds for instance if $Y \to Y/G$ has a totally ramified $\overline{K}$-point.

Proof. The pullback homomorphism $\mathrm{Pic}(Y/G) \to \mathrm{Pic}_G(Y)$ is injective (since the G-linearization suffices to undo the pullback), while the kernel of the homomorphism $\mathrm{Pic}_G(Y) \to \mathrm{Pic}(Y)$ which forgets the G-linearization can be identified with the group of G-linearizations of the trivial bundle, namely $H^1(G, \mathcal{O}(Y)^\times)$ [13, (2.2)]. As Y is proper and geometrically integral, this group can be identified with $\mathrm{Hom}(G, K^\times)$. However the pullback of a line bundle on Y/G to Y carries a unique G-linearization for which it descends. Indeed, if $\mathscr{L}$ is a G-line bundle on Y which descends to Y/G, and c is a nontrivial K-valued character, then the G_P-action on $(\mathscr{L} \otimes c)_P$ is nontrivial, so $\mathscr{L} \otimes c$ does not descend to Y/G (Proposition 3.2). $\square$

3.2. Quotient of the regular representation

Let V denote the regular representation of G, and write $\mathbb{P}$ for the projective space $\mathbb{P}(V) = \mathrm{Proj}\, \mathrm{Sym}\, V^\vee$ of V regarded as a variety over $\mathbb{Q}$. Let π denote the quotient map $\mathbb{P} \to \mathbb{P}/G$.

Proposition 3.4. $\pi^* K_{\mathbb{P}/G} = K_\mathbb{P}$ *if* $|G| \neq 2$ *and* $\pi^* K_{\mathbb{P}/G} = 2K_\mathbb{P}$ *if* $|G| = 2$.

Proof. First observe $\mathbb{P}/G$ is normal since it is the quotient of a normal variety by a finite group. Now suppose $\mathbb{P} \to \mathbb{P}/G$ is not étale in codimension one, i.e. there is some irreducible codimension one subvariety D in the support of $\Omega^1_{\mathbb{P} \to \mathbb{P}/G}$. Then some $g \neq 1$ fixes the generic point η of D. In fact D is a hyperplane since otherwise g would fix $|G| - 1$ linearly independent vectors in the affine cone of D, any line passing through $0 \in V$ would intersect the affine cone of D at a nonzero point, and g would be 1. We see that g fixes a codimension one hyperplane (the affine cone of D), so G contains a pseudoreflection.

Suppose $|G| \neq 2$. Then the regular representation contains no pseudoreflections and so $\mathbb{P} \to \mathbb{P}/G$ is étale in codimension one. We may take an open subset $U \subset \mathbb{P}$ whose complement has codimension at least two such that $\pi|_U$ is étale and $\pi(U)$ is contained in the smooth locus V of $\mathbb{P}/G$. Then $\pi^*\Omega^1_{V/\mathbb{Q}}$ and $\Omega^1_{\mathbb{P}/\mathbb{Q}}$ are isomorphic over U which means $\pi^*\Omega^{|G|-1}_{(\mathbb{P}/G)/\mathbb{Q}}$ and $\Omega^{|G|-1}_{\mathbb{P}/\mathbb{Q}}$ are isomorphic over U. This shows their supports $\pi^* K_{\mathbb{P}/G}$ and $K_{\mathbb{P}}$ are equal away from a codimension two closed subset and are therefore equal everywhere.

For $|G| = 2$ the map $\mathbb{P} \to \operatorname{Proj} \mathbb{Q}[u, v]$ given by $(u, v) = ((x + y)^2, (x - y)^2)$ induces an isomorphism $\mathbb{P}/G \cong \operatorname{Proj} \mathbb{Q}[u, v]$, and the anticanonical bundle on $\operatorname{Proj} \mathbb{Q}[u, v]$ pulls back to $-2K_{\mathbb{P}}$. $\square$

Theorem 3.5. *There is a line bundle $\mathscr{L}$ on $\mathbb{P}/G$, unique up to isomorphism, satisfying $\pi^*\mathscr{L} = -K_{\mathbb{P}}$. If $|G| \neq 2$ then $\mathscr{L} + K_{\mathbb{P}/G}$ is torsion in the divisor class group of $\mathbb{P}/G$ and if $|G| = 2$ then $2\mathscr{L} = -K_{\mathbb{P}/G}$.*

Proof. Let e be the exponent of G. Then $\operatorname{Pic}(\mathbb{P})^{\times e}$ is contained in the image of $\operatorname{Pic}(\mathbb{P}/G) \to \operatorname{Pic}(\mathbb{P})$ by Proposition 3.2, and we conclude the existence of $\mathscr{L}$. Uniqueness follows from the existence of the totally ramified point $[1 : \cdots : 1]$ for π and Lemma 3.3. For the second assertion note that $K_{\mathbb{P}/G}$ is $\mathbb{Q}$-Cartier [14, Prop. 5.20] so if $nK_{\mathbb{P}/G}$ is Cartier and $|G| \neq 2$ then

$$\pi^*(n(\mathscr{L} + K_{\mathbb{P}/G})) = 0.$$

However π is totally ramified at $[1 : \cdots : 1]$ so π^* is injective (Lemma 3.3). $\square$

Remark 4. With a bit more work one can show that $\mathscr{L} + K_{\mathbb{P}/G}$ is 2-torsion and that $\mathscr{L} + K_{\mathbb{P}/G}$ is trivial if the Sylow 2-subgroup of G is trivial or non-cyclic. This shows that $\mathbb{P}/G$ is $\mathbb{Q}$-Gorenstein of index ≤ 2.

Corollary 3.6. *The Picard group of $\mathbb{P}/G$ is $\mathbb{Z}$.*

Proof. π has a totally ramified point so $\operatorname{Pic}(\mathbb{P}/G) \to \operatorname{Pic}(\mathbb{P})$ is injective by Lemma 3.3, however the pullback of $\mathscr{L}$ to $\mathbb{P}$ is nontrivial. $\square$

Let n (resp. e) denote the order (resp. exponent) of G.

Theorem 3.7. *Let $\mathscr{L}_0$ be the ample generator of $\mathrm{Pic}(\mathbb{P}/G)$. Then $\mathscr{L}_0^{\frac{n}{e}} = \mathscr{L}$.*

Proof. The pullback of $\mathscr{L}_0$ to $\mathbb{P}$ will be $\mathcal{O}(t)$ for the smallest positive integer t such that $\mathcal{O}(t)$ descends to $\mathrm{Pic}(\mathbb{P}/G)$ for some G-linearization. First equip $\mathcal{O}(t)$ with the G-linearization coming from the natural G-action on $\mathcal{O}(1)$. If c is any nontrivial character of G, then G will act on the fiber of $\mathcal{O}(t) \otimes c$ over $[1 : \cdots : 1]$ by c so $\mathcal{O}(t) \otimes c$ cannot descend (Proposition 3.2). This means that if $\mathcal{O}(t)$ descends for some G-linearization it must be its natural G-linearization. Now if g is an element of G with order d then $\langle g \rangle$ acts as ζ^{-1} on the line spanned by $\sum_{j=0}^{d-1} \zeta^j [g^j]$ (ζ a primitive dth root of unity), so g fixes the corresponding point of $\mathbb{P}(\overline{K})$ and acts by ζ^t on the fiber of $\mathcal{O}(t)$ over this point. The only way that $\zeta^t = 1$ for all $g \in G$ is if e divides t, and this condition is also sufficient for descent. We conclude that $\mathcal{O}(e)$ descends to a line bundle $\mathscr{L}_0$ on $\mathbb{P}/G$ and $\mathscr{L}_0^{\frac{n}{e}} = \mathscr{L}$. $\square$

3.3. $\mathscr{L}$ determines an immersion of $\mathcal{X}$

Theorem 3.8. *$\mathscr{L}$ is globally generated and its global sections restrict to an immersion of $\mathcal{X}$ into $\mathbb{P}^N$, where $N + 1$ is the dimension of the linear subspace of homogeneous degree $|G|$ G-invariants in $\mathrm{Sym}\, V^\vee$.*

Remark 5. Using Molien's theorem one can show that $N + 1$ is equal to the sum over divisors d of $|G|$ of $|G|^{-1} \binom{2d-1}{d} \#\{g \in G : g \text{ has order } |G|/d\}$.

Proof. To show that the rational map $\varphi \colon \mathbb{P}/G \dashrightarrow \mathbb{P}^N$ determined by $\mathscr{L}$ is defined on all of $\mathbb{P}/G$, we may assume without loss of generality that K is algebraically closed. Furthermore, as the property of being a quasi-compact immersion is stable under faithfully flat descent we may also assume that K is algebraically closed for the claim that φ restricts to an immersion on $\mathcal{X}$.

Let n denote the order of G. We will show that for any two distinct closed points of $\mathbb{P}/G$ there exists a global section of $\mathscr{L}$ vanishing at one point but not the other. Since pullback along π induces an isomorphism $H^0(\mathbb{P}/G, \mathscr{L}) \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}(n))^G$, it is the same to find a G-invariant global section of $\mathcal{O}(n)$ separating two arbitrary closed G-orbits in $\mathbb{P}$. Suppose $P, Q \in \mathbb{P}(\overline{K})$ are not in the same G-orbit. Choose a hyperplane $H \subset \mathbb{P}$ passing through P but disjoint from the orbit of Q, and let t be a global section of $\mathcal{O}(1)$ whose zero locus is H. Then

$$\prod_{g \in G} {}^g t$$

is a G-invariant global section of $\mathcal{O}(n)$ vanishing at P but not at Q. The existence of such sections also shows that $\mathscr{L}$ is globally generated so φ is defined on all of $\mathbb{P}/G$. The same fact shows that φ is injective on closed points of $\mathbb{P}/G$.

We next show that φ separates tangent vectors away from isotropy. It is equivalent to show $\tilde{\varphi}$ separates tangent vectors away from isotropy since the quotient map $\mathbb{P} \to \mathbb{P}/G$ is an isomorphism on tangent spaces away from isotropy. Suppose $P \in \mathbb{P}(\overline{K})$ has trivial isotropy group and $0 \neq v \in T_P(\mathbb{P})$. Choose a hyperplane $H \subset \mathbb{P}$ passing through P, disjoint from $\{g(P) : g \in G - \{1\}\}$, and not tangent to v. Let t be a global section of $\mathcal{O}(1)$ whose zero locus is H. Then

$$\prod_{g \in G} {}^g t$$

is a G-invariant global section of $\mathcal{O}(n)$ vanishing at P such that v is not tangent to its zero locus. This implies $\tilde{\varphi}_* v$ is not zero (cf. Remark II.7.8.2 of [10]). We conclude $\tilde{\varphi}_*$ is injective on $T_P(\mathbb{P})$. Since G acts freely on $\mathcal{G}$ we conclude that φ separates tangent vectors on $\mathcal{X}$.

To conclude the proof it suffices to show that the restriction $\varphi|_{\mathcal{X}} \colon \mathcal{X} \to \varphi(\mathcal{X})$ of φ to $\mathcal{X}$ is proper and that $\varphi(\mathcal{X})$ is a locally closed subset of $\mathbb{P}^N$. This will imply that $\varphi|_{\mathcal{X}}$ is a closed immersion of $\mathcal{X}$ into an open subset of $\mathbb{P}^N$ by [8, Prop. 12.94], obtaining the desired conclusion that the composition $\mathcal{X} \to \varphi(\mathcal{X}) \to \mathbb{P}^N$ is an immersion. Observe that

$$\begin{array}{ccc} \mathcal{X} & \overset{\subset}{\longrightarrow} & \mathbb{P}/G \\ {\scriptstyle \varphi|_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \varphi(\mathcal{X}) & \overset{\subset}{\longrightarrow} & \mathbb{P}^N \end{array}$$

is a pullback diagram because φ is injective. As the property of being proper is preserved under basechange this shows that $\varphi|_{\mathcal{X}}$ is proper. To see that $\varphi(\mathcal{X})$ is locally closed we first observe that the subset of $\mathbb{P}$ where the group determinant Δ_G is invertible is equal to $\mathcal{G}$. As $\mathcal{G}$ is a G-stable affine open subset, its quotient $\mathcal{X}$ is an open subset of $\mathbb{P}/G$. Then we have that $\varphi(\mathcal{X} \sqcup \mathcal{X}^c) = \varphi(\mathcal{X}) \sqcup \varphi(\mathcal{X}^c)$ which shows that

$$\varphi(\mathcal{X}) = \varphi(\mathbb{P}/G) \cap (\mathbb{P}^N - \varphi(\mathcal{X}^c)). \tag{3}$$

As φ is a proper morphism it is closed so (3) shows $\varphi(\mathcal{X})$ is locally closed. $\quad\square$

4. Height formulas

In this section we prove Theorem 1.1 and Theorem 1.2. For each rational prime p let $|\cdot|_p$ denote the norm on $\overline{\mathbb{Q}_p}$ satisfying $|p|_p = 1/p$ and let $|\cdot|_\infty$ denote the usual complex absolute value. If $K/\mathbb{Q}$ is any finite extension, then the set of places w of K over a fixed place v of $\mathbb{Q}$ is naturally in bijection with field homomorphisms $j \colon K \to \overline{\mathbb{Q}_v}$ up to isometry, and we write $|\cdot|_w$ for $|j(\cdot)|_v$. We write $\|\cdot\|_\infty$ for the canonical norm on Minkowski space $K_\mathbb{R}$. For any $y \in K \subset K_\mathbb{R}$ it is given by

$$\|y\|_\infty = \left(\sum_{j:K\to\mathbb{C}} d_j |j(y)|_\infty^2 \right)^{1/2}$$

where j runs over the complex embeddings of K up to isometry (one j for each pair of complex embeddings) and d_j is 1 if j is real and 2 if j is complex (cf. [17]). For a product $L = K \times \cdots \times K$ we extend the norm to $L_\mathbb{R}$ by setting

$$\|y\|_\infty = \left(\sum_{i=1}^{\dim_K L} \|y_i\|_\infty^2 \right)^{1/2}.$$

Theorem *(Theorem 1.1). Let $\phi\colon \mathbb{P}/G \to \mathbb{P}^N$ be a non-constant morphism. Let d_ϕ be the degree of the composite map $\mathbb{P} \to \mathbb{P}/G \xrightarrow{\phi} \mathbb{P}^N$. Then for all $P = (L, x) \in \mathcal{X}(\mathbb{Q})$,*

$$h(\phi(P)) = d_\phi \log \|x\|_\infty - \frac{d_\phi}{[K:\mathbb{Q}]} \log N(J) + O(1)$$

for a bounded function $O(1)$.

Proof. Recall that $\mathbb{P}/G$ has Picard rank one (Corollary 3.6). As we are only proving (1) up to $O(1)$, the claimed formula follows for any morphism ϕ once we have proven it for a single morphism ϕ. We use $\mathscr{L}$ to determine such a ϕ as $\mathscr{L}$ is globally generated (Theorem 3.8). Then $d_\phi = |G|$ and the formula to be shown is that

$$h(\phi(P)) = |G| \log \|x\|_\infty - \frac{|G|}{[K:\mathbb{Q}]} \log N(J) + O(1).$$

Let $\varphi\colon L \to \mathbb{C}$ be any $\mathbb{Q}$-algebra homomorphism. By Proposition 2.5, the fiber of π over (L, x) consists of the G-translates of the unit given by

$$u = \sum_{g\in G} \varphi(g(x))[g^{-1}] \in \mathcal{G}(K).$$

We will use the Weil height H on $\mathbb{P}(\overline{\mathbb{Q}})$ associated to $\mathcal{O}(1)$ for which $H(u)$ is

$$\prod_{w\in M_K^\infty} \left(\sum_{g\in G} |\varphi(g(x))|_w^2 \right)^{d_w/(2[K:\mathbb{Q}])} \prod_{w\in M_K^f} \max_{g\in G} |\varphi(g(x))|_w^{d_w/[K:\mathbb{Q}]} = H_\infty(u) H_f(u)$$

where d_w is the local degree of K at w. (We call this the *standard metric on $\mathcal{O}(1)$*.) By the theory of heights,

$$h(\phi(L, x)) = |G| \log H(u) + O(1)$$

for a bounded function $O(1)$.

Let d_∞ be the local degree of K at any of its infinite places. The infinite component H_∞ of the height is none other than the canonical norm on Minkowski space $K_\mathbb{R}$ (when evaluated on a unit of the form u). Indeed, $H_\infty(u)$ is equal to

$$\prod_{w \in M_K^\infty} \left(\sum_{g \in G} |\varphi(g(x))|_w^2 \right)^{\frac{d_\infty}{2[K:\mathbb{Q}]}} = \prod_{j:K \to \mathbb{C}} \left(\sum_{g \in G} |j(\varphi(g(x)))|_\infty^2 \right)^{\frac{d_\infty}{2[K:\mathbb{Q}]}}$$

$$= \prod_{j:K \to \mathbb{C}} \left(\sum_{i=1}^{\dim_K L} \sum_{g \in \mathrm{Gal}(K/\mathbb{Q})} |j(g(x_i))|_\infty^2 \right)^{\frac{d_\infty}{2[K:\mathbb{Q}]}}$$

$$= \prod_{j:K \to \mathbb{C}} \left(\sum_{i=1}^{\dim_K L} \sum_{j':K \to \mathbb{C}} d_\infty |j'(x_i)|_\infty^2 \right)^{\frac{d_\infty}{2[K:\mathbb{Q}]}}$$

$$= \left(\sum_{i=1}^{\dim_K L} \sum_{j:K \to \mathbb{C}} d_\infty |j(x_i)|_\infty^2 \right)^{\frac{1}{2}}$$

$$= \|x\|_\infty.$$

For the finite components observe that $H_f(u)$ is equal to

$$\prod_{w \in M_K^f} \max_{g \in G} |\varphi(g(x))|_w^{d_w/[K:\mathbb{Q}]} = \prod_{w \in M_K^f} \max_{y \in J} |y|_w^{d_w/[K:\mathbb{Q}]} = N(J)^{-1/[K:\mathbb{Q}]}. \quad \square$$

Let $T_\infty = \lim\limits_{D \to \infty} T_D$ denote the stable multiplier order of K attached to a rational point (L, x) of $\mathcal{X}$.

Theorem *(Theorem 1.2).*

(1) $\mathrm{Spec}\, T_\infty \sqcup \cdots \sqcup \mathrm{Spec}\, T_\infty$ *for* $\dim_K L$ *many copies are isomorphic to the fiber of* $\mathbb{P} \to \mathbb{P}/G$ *over the* $\mathbb{Z}$*-point* $(L, x) \to \mathbb{P}/G$.
(2) $T_\infty = T_D$ *if* $D \geq |G| - 1$.

Proof. Let $\mathrm{Spec}\, A$ denote the fiber over $\mathbb{P} \to \mathbb{P}/G$ over $(L, x) \in \mathcal{X}(\mathbb{Q})$, where we regard (L, x) as a $\mathbb{Z}$-point of $\mathbb{P}/G$ by the valuative criterion of properness. Let $q: \mathrm{Spec}\, A \to \mathbb{P}$ denote the natural projection map. We have identifications

$$H^0(\mathbb{P}, \mathcal{O}(D)) = \bigoplus_{m \in Mon_D} \mathbb{Z}m$$

and

$$H^0(\text{Spec } A, q^*\mathcal{O}(D)) = \sum_{m \in Mon_D} Am(\{g(x)\}_g).$$

We claim the natural map

$$\phi \colon H^0(\mathbb{P}, \mathcal{O}(D)) \to H^0(\text{Spec } A, q^*\mathcal{O}(D))$$

is surjective if $D \geq |G| - 1$. Indeed its cokernel C is finite, since its image and codomain are both lattices (full rank abelian groups) of L. However C also has trivial p-torsion for each prime p by [18, Lemma 2.1]. Thus $C = 0$. We see that

$$\text{im } \phi = \sum_{m \in Mon_D} \mathbb{Z}m(\{g(x)\}_g) = \sum_{m \in Mon_D} Am(\{g(x)\}_g). \tag{4}$$

This equality shows that $\text{im } \phi$ is closed under multiplication by A. Since $q^*\mathcal{O}(D)$ is a line bundle over Spec A, the A-module $\text{im } \phi$ is projective.

Now apply φ to both sides of (4) where $\varphi \colon L \to K$ is any $\mathbb{Q}$-algebra homomorphism. The homomorphism $\text{im } \phi \to \varphi(\text{im } \phi) = I^D$ splits, which implies that I^D is a direct summand of $\text{im } \phi$ and is therefore projective as an A-module. The action of A on I^D factors through $\varphi(A)$, which shows I^D is a projective $\varphi(A)$-module. This implies that $\varphi(A) = (I^D : I^D)$ by localization. As $T_\infty = \lim_{D \to \infty}(I^D : I^D)$ we have $\varphi(A) = T_\infty$. Since $A \cong \varphi(A)^{\dim_K L}$ the result follows. □

Corollary 4.1. *For any integer $D \geq |G| - 1$, $N(I^D) = N(J^D)$.*

Proof. I^D is the module of global sections of the line bundle on Spec T_∞ obtained by pullback of $\mathcal{O}(D)$ along Spec $T_\infty \to \mathbb{P}$ and is therefore an invertible T_∞-module. By localization we see that

$$N(I^D) = [T_\infty : I^D] = [O_K : O_K I^D] = [O_K : J^D] = N(J^D). \quad \square$$

Remark 6. An elementary argument shows that $N(J)N(I)^{-1}$ is bounded from above by the index of T_1 in O_K and from below by 1.

5. Descent for metrics

In this final section we take up the problem of descending metrized line bundles through finite quotient maps. As is well-known, the theory of metrized line bundles refines the theory of heights and is important for comparing height functions not just up to an $O(1)$. With an eye towards future applications, we take a step in this direction. The main result in this section is Theorem 5.4, which implies a refinement of Theorem 1.1. Let Y be a projective G-variety over $\mathbb{Q}$. Our result says that if $\mathscr{E}$ is any metrized G-line bundle over Y for which G acts by isometries, then $\mathscr{E}^{\otimes |G|}$ descends to a metrized line bundle $\mathscr{L}$ on Y/G such that the associated height functions satisfy

$$h_{\mathscr{L}}(x) = |G|h_{\mathscr{E}}(y)$$

where y is any preimage of $x \in (Y/G)(\overline{\mathbb{Q}})$.

Defining this metric on $\mathscr{L}$ as a set-theoretic function is easy: the fiber $x^*\mathscr{L}$ can be identified with the fiber of $\mathscr{E}^{\otimes|G|}$ over any preimage of x and inherits the metric we already have there. This will be independent of the choice of preimage if G acts by isometries on the metric on $\mathscr{E}$. The difficulty is showing that this function satisfies standard technical requirements on metrics, specifically the so-called adelic compatibility condition (see (3) in the definition below).

Definition 5.1. A *v-adic metric* on $\mathscr{E}$ is a collection $\|\cdot\|_v = (\|\cdot\|_{v,y})_{y \in Y(\overline{\mathbb{Q}_v})}$ where $\|\cdot\|_{v,y}$ is a norm on $y^*\mathscr{E}$ satisfying $\|cu\|_{v,y} = |c|_v\|u\|_{v,y}$ for all $c \in \overline{\mathbb{Q}_v}$ and $u \in y^*\mathscr{E}$. A collection $(\|\cdot\|_v)_{v \in M_{\mathbb{Q}}}$ of v-adic metrics is called an *adelic metric* on $\mathscr{E}$ if the following conditions are satisfied for all $v \in M_{\mathbb{Q}}$ and $y \in Y(\overline{\mathbb{Q}_v})$:

(1) $(y^*\mathscr{E}, \|\cdot\|_{v,y}) \xrightarrow{\sigma} ((\sigma y)^*\mathscr{E}, \|\cdot\|_{v,\sigma y})$ is an isometry for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$,

(2) $U(\overline{\mathbb{Q}_v}) \to \mathbb{R}: y \mapsto \|s(y)\|_{v,y}$ is continuous for any open subset U of Y and $s \in \mathscr{E}(U)$,

(3) $\mathscr{E}$ admits a generating set of global sections $\{s_1, \ldots, s_m\}$, independent of y and v, with the property that for all but finitely many $v \in M_{\mathbb{Q}}$,

$$\|s(y)\|_{v,y} = \left(\max_{1 \leq j \leq m} \left| \frac{s_j(y)}{s(y)} \right|_v \right)^{-1} \tag{5}$$

for any local section s that is nonzero at y.

When there is no need for disambiguation we write $\|\cdot\|_v$ for $\|\cdot\|_{v,y}$.

Now let $K/\mathbb{Q}$ be a number field. The set of places w of K over a fixed place v of $\mathbb{Q}$ is naturally in bijection with the set of field homomorphisms $j\colon K \to \overline{\mathbb{Q}_v}$ up to isometry. Given a point $y \in Y(K)$ let $j(y) \in Y(\overline{\mathbb{Q}_v})$ denote the point $\mathrm{Spec}\,\overline{\mathbb{Q}_v} \to \mathrm{Spec}\,K \xrightarrow{y} Y$. Let us write $\|\cdot\|_{w,y}$ for the norm obtained on the fiber $y^*\mathscr{L}$ (non-canonically isomorphic with K) by pulling back $\|\cdot\|_{v,j(y)}$ along the natural map

$$y^*\mathscr{L} \to y^*\mathscr{L} \otimes_j \overline{\mathbb{Q}_v} = j(y)^*\mathscr{L}.$$

The height of $y \in Y(\overline{\mathbb{Q}})$ is defined by

$$H_{\mathscr{L}}(y) = \prod_{w \in M_K} \|s(y)\|_{w,y}^{-d_w/[K:\mathbb{Q}]}$$

where $s \in \mathscr{L}$ is any local section that is nonzero at y and $K/\mathbb{Q}$ is any finite extension which s and y are defined over. (The d_w in the exponent ensures this is independent of s by the product formula and the $[K:\mathbb{Q}]$ ensures this is independent of K.)

The following lemma helps with finding global sections which satisfy (5).

Lemma 5.2. *Let* $\|\cdot\|_v$ *be a v-adic metric on* $\mathscr{L}$. *If* $\{s_1, \ldots, s_m\}$ *is a set of global sections of* $\mathscr{L}$ *satisfying*

$$\max_{1 \le i \le m} \|s_i(x)\|_v = 1 \tag{6}$$

for all $x \in Y(\overline{\mathbb{Q}_v})$, *then* $\{s_1, \ldots, s_m\}$ *globally generates* $\mathscr{L}$. *Furthermore, for any local section* s *that is nonzero at* x,

$$\|s(x)\|_v = \left(\max_{1 \le i \le m} \left| \frac{s_i(x)}{s(x)} \right|_v \right)^{-1}.$$

Proof. It is clear that the $\{s_1, \ldots, s_m\}$ globally generate $\mathscr{L}$ from (6). For any $i \in \{1, \ldots, m\}$ with $s_i(x) \ne 0$ we have

$$\|s(x)\|_v = \left| \frac{s(x)}{s_i(x)} \right|_v \|s_i(x)\|_v \le \left| \frac{s(x)}{s_i(x)} \right|_v.$$

By (6), equality is obtained for some i and therefore

$$\|s(x)\|_v = \min_{i:s_i(x) \ne 0} \left| \frac{s(x)}{s_i(x)} \right|_v = \left(\max_{1 \le i \le m} \left| \frac{s_i(x)}{s(x)} \right|_v \right)^{-1}. \quad \square$$

Let $\mathscr{E}$ be a metrized G-line bundle over Y.

Definition 5.3. We say that G *acts by isometries* if for all $v \in M_{\mathbb{Q}}$, $g \in G$, and $x \in Y(\overline{\mathbb{Q}_v})$, the linear map on fibers

$$(x^*\mathscr{E}, \|\cdot\|_{w,x}) \xrightarrow{g} ((xg)^*\mathscr{E}, \|\cdot\|_{w,xg})$$

is an isometry.

Let $\pi \colon Y \to Y/G$ denote the quotient map. By Proposition 3.2, there is a unique line bundle $\mathscr{L}$ on Y/G such that

$$\pi^*\mathscr{L} \cong \mathscr{E}^{\otimes |G|}$$

as G-line bundles. Since $\mathscr{E}^{\otimes |G|}$ carries a natural metric induced from $\mathscr{E}$, we can ask whether this metric descends to $\mathscr{L}$.

Theorem 5.4. *Suppose G acts by isometries on $\mathscr{E}$. Then there is a unique adelic metric on $\mathscr{L}$ which pulls back to the induced metric on $\mathscr{E}^{\otimes |G|}$.*

Thus if $x \in (Y/G)(\overline{\mathbb{Q}})$ and $y \in Y(\overline{\mathbb{Q}})$ is any preimage, then $H_{\mathscr{L}}(x) = H_{\mathscr{E}}(y)^{|G|}$.

Remark 7. The reason for descending the metric on the $|G|$-fold tensor product rather than the original metric is that the global sections in (5) have no reason to be G-invariant. Our remedy is to symmetrize the global sections by taking the product (11) over their G-orbits which necessitates replacing $\mathscr{E}$ with $\mathscr{E}^{\otimes|G|}$. This suffices for our application to $\mathscr{E} = \mathcal{O}(1)$ and the line bundle $\mathscr{L}$ constructed in §3.

Proof. Let $x \in (Y/G)(\overline{\mathbb{Q}_v})$. Let s be any local nonvanishing section at x and suppose $y \in Y(\overline{\mathbb{Q}_v})$ maps to x. In order for the metric on $\mathscr{L}$ to pull back to the metric on $\mathscr{E}^{\otimes|G|}$ it is clear that we must define the norm on $x^*\mathscr{L}$ to be

$$\|s(x)\|_{v,x} := \|(\pi^*s)(y)\|_{v,y},$$

and so uniqueness is clear. This is independent of the choice of y since G acts by isometries. It is straightforward to show that the collection of v-adic metrics this defines satisfies the first two conditions (Galois invariance and continuity) of an adelic metric.

For the third condition we will construct a generating set of global sections $\{s_1, \ldots, s_m\}$ of $\mathscr{L}$ such that

$$\|(\pi^*s)(y)\|_{v,y} = \left(\max_{1 \leq i \leq m} \left| \frac{s_i(x)}{s(x)} \right|_v \right)^{-1} \tag{7}$$

Let $\{t_1, \ldots, t_n\} \subset H^0(Y, \mathscr{E})$ be a generating set of global sections that defines by (5) the v-adic components of the adelic metric on $\mathscr{E}$ for all places $v \notin S$, where $S \subset M_{\mathbb{Q}}$ is any finite set containing the archimedean place. Fix a place $v \notin S$ and a point $y \in Y(\overline{\mathbb{Q}_v})$. Then

$$\max_{1 \leq i \leq n} \|t_i(y)\|_v = \max_{i: t_i(y) \neq 0} \min_{j: t_j(y) \neq 0} \left| \frac{t_j(y)}{t_i(y)} \right|_v = 1. \tag{8}$$

Consider the set

$$A_y := \{(i, g) \in [n] \times G : \|(^g t_i)(y)\|_v = 1\}. \tag{9}$$

Since G acts by isometries,

$$\|(^g t_i)(y)\|_v = \|g(t_i(g^{-1}y))\|_{v,y} = \|t_i(g^{-1}y)\|_{v,g^{-1}y}$$

so from (8),

$$\max_{i,g} \|(^g t_i)(y)\|_{v,y} = \max_{i,g} \|t_i(g^{-1}y)\|_{v,g^{-1}y} = 1. \tag{10}$$

This shows A_y is not empty so let i be any fixed element of $[n]$ with $(i, g) \in A_y$ for some $g \in G$. Now define the sets

$$B_y := \{g : (i, g) \in A_y\}, \quad H_y := \{h : B_y h = B_y\}.$$

For each orbit $gH_y \in B_y/H_y$ choose a positive integer a_{gH_y} subject to the condition that

$$\sum_{gH_y \in B_y/H_y} a_{gH_y} = [G : H_y].$$

This is always possible since H_y acts freely on B_y. It is easily verified that

$$\prod_{h \in B_y} ({}^h t_i)^{a_{h^{-1}H_y}}$$

is an H_y-invariant global section of $\mathscr{E}^{\otimes |G|}$. We define the global section

$$s'_y := \sum_{gH_y \in G/H_y} \prod_{h \in B_y} ({}^{gh} t_i)^{a_{h^{-1}H_y}} \tag{11}$$

which is manifestly G-invariant. Since $H^0(Y, \mathscr{E}^{\otimes |G|})^G \cong H^0(Y/G, \mathscr{L})$ (by pulling back along π) the section s'_y determines a unique global section s''_y of $\mathscr{L}$ such that $\pi^* s''_y = s'_y$. *A priori* the section s''_y depends on y, i, v, and the integers $(a_{gH_y})_{gH_y \in B_y/H_y}$, but as y varies over $Y(\overline{\mathbb{Q}_v})$ for all $v \notin S$ there are only finitely many possibilities for A_y, i, and the $(a_{gH_y})_{gH_y \in B_y/H_y}$, and these suffice to determine s''_y. Let $\{s_1, \ldots, s_m\}$ denote the set of global sections of $\mathscr{L}$ arising in this way.

Let $s'_i = \pi^* s_i$ for $1 \le i \le m$. First note that $\frac{s_i(x)}{s(x)} = \frac{s'_i(y)}{(\pi^* s)(y)}$ and thus

$$\max_{1 \le i \le m} \left| \frac{s_i(x)}{s(x)} \right|_v = \max_{1 \le i \le m} \left| \frac{s'_i(y)}{(\pi^* s)(y)} \right|_v.$$

Thus by Lemma 5.2, if

$$\max_{1 \le i \le m} \| s'_i(y) \|_v = 1$$

for all $y \in Y(\overline{\mathbb{Q}_v})$ then (7) holds. By construction, for each $s' \in \{s'_1, \ldots, s'_m\}$ there is a global section $t \in \{t_1, \ldots, t_n\}$, a subset $B \subset G$, and a subgroup $H \le G$ acting freely on B, and integers a_{gH} for each coset $gH \in B/H$ such that

$$s' = \sum_{gH \in G/H} \prod_{h \in B} ({}^{gh} t)^{a_{h^{-1}H}}. \tag{12}$$

Since the set A_y defined by (9) is nonempty, there are $g \in B_y$ and $1 \le i \le n$ such that $\|({}^g t_i)(y)\|_v = 1$. By (10), $\|({}^g t_i)(y)\|_v < 1$ for any $g \notin B_y$, and therefore a single monomial in the sum (12) realizes the maximum. By the ultrametric inequality we have

$$\max_{1 \le i \le m} \| s'_i(y) \|_v = \prod_{h \in B_y} \left\| ({}^h t_i)(y) \right\|_v^{a_{h^{-1}H_y}} = 1. \quad \square$$

Let $\mathscr{L}$ denote the line bundle on $\mathbb{P}/G$ constructed in §3.

Corollary 5.5. *There is a unique adelic metric on $\mathscr{L}$ which pulls back to the standard metric on $\mathcal{O}(1)^{\otimes|G|}$. The height h_{AC} induced by this metric satisfies*

$$h_{AC}(L, x) = |G| \log \|x\|_\infty - \frac{|G|}{[K:\mathbb{Q}]} \log N(J)$$

on any rational point $(L, x) \in \mathcal{X}(\mathbb{Q})$.

Proof. The proof of Theorem 1.1 shows that if $u \in \mathcal{G}$ is any preimage of (L, x), then

$$h(u) = \log \|x\|_\infty - \frac{1}{[K:\mathbb{Q}]} \log N(J)$$

where h is the height on $\mathbb{P}$ associated to the standard metric on $\mathcal{O}(1)$. By the last theorem there is a unique adelic metric on $\mathscr{L}$ which pulls back to the metric on $\mathcal{O}(1)^{\otimes|G|}$ and the associated height h_{AC} will satisfy $h_{AC}(L, x) = |G|h(u)$. $\quad\square$

In the next two examples we take $H_{AC} = e^{h_{AC}}$.

Example 1 *(T Gorenstein, $L = K$)*. If $T = (I : I)$ is Gorenstein, then I is T-projective (cf. e.g. [12, 4.2]). (For instance, if T is monogenic then it is Gorenstein. More generally, T is Gorenstein if and only if its different is invertible.) Then $N(I) = [T : I] = \sqrt{d_I d_T^{-1}}$ and

$$H_{AC}(L, x) = \|x\|_\infty^{|G|} \sqrt{d_T d_I^{-1}}.$$

If additionally L is totally real and x is self-dual, then

$$H_{AC}(L, x) = \sqrt{d_T}.$$

Example 2 *($L = \mathbb{Q} \times \cdots \times \mathbb{Q}$)*. $T = \mathbb{Z}$ and $I = \ell\mathbb{Z}$ for some $\ell \in \mathbb{Q}^{>0}$. Then

$$H_{AC}(\mathbb{Q} \times \cdots \times \mathbb{Q}, x) = \left(\frac{\sqrt{x_1^2 + \cdots + x_{|G|}^2}}{\ell}\right)^{|G|}.$$

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