

Morphisms of Character Varieties

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Let k be a field, let $H \subset G$ be (possibly disconnected) reductive groups over k , and let Γ be a finitely generated group. Vinberg and Martin have shown that the induced morphism $\underline{\mathrm{Hom}}_{k\text{-gp}}(\Gamma, H)/H \rightarrow \underline{\mathrm{Hom}}_{k\text{-gp}}(\Gamma, G)/G$ is finite. In this note, we generalize this result (with a significantly different proof) by replacing k with an arbitrary locally Noetherian scheme, answering a question of Dat. Along the way, we use Bruhat–Tits theory to establish a few apparently new results about integral models of reductive groups over discrete valuation rings.

1 Introduction

Let S be a locally Noetherian scheme, and let H and G be smooth S -affine S -group schemes with reductive fibers and finite étale component groups. Let $f: H \rightarrow G$ be an S -homomorphism and let $h: \Gamma' \rightarrow \Gamma$ be a homomorphism of finitely generated groups. If $\underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma, G)$ denotes the scheme of homomorphisms from Γ to G , then there is a natural S -morphism

$$F: \underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma, H)/H \rightarrow \underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma', G)/G$$

between the GIT quotients. The goal of this note is to prove the following theorem, answering a question of Dat [6, Conjecture 5.16].

Theorem 1.1. If f is a finite morphism and $h(\Gamma')$ is of finite index in Γ , then F is finite.

Theorem 1.1 has been proved in various special cases in the literature.

- 1) If S is the spectrum of a field, $\Gamma = \Gamma'$, and $h = \mathrm{id}_\Gamma$ then Theorem 1.1 is due to Vinberg [17] in characteristic 0 and Martin [13] in positive characteristic.
- 2) If $\Gamma = \Gamma' = \mathbf{Z}$, $h = \mathrm{id}_\mathbf{Z}$, and H and G are reductive, then Theorem 1.1 follows from the Chevalley–Steinberg theorem (for more general H and G , see [9, Lemma 2.2]).
- 3) If $S = \mathrm{Spec} W(k)$ for an algebraically closed field k of characteristic $p > 0$, Γ is a finite group of order prime to p , and H and G are semisimple, then [10, Appendix A] shows that the source and target of F are finite étale, and in particular Theorem 1.1 holds.
- 4) If p is a prime number, $S = \mathrm{Spec} \mathbf{Z}[\frac{1}{p}]$, the function h is injective, and $\Gamma = W_F^0/P$ for a finite extension $F/\mathbf{Q}_p$ (where W_F^0 is the “discretized Weil group” introduced in [8] and P is an open subgroup of wild inertia which is normal in W_F), then Theorem 1.1 is proved with some assumptions on G in [9, Corollary 2.4, Corollary 2.5].

Remark 1.2. The statement of Theorem 1.1 is slightly strange: when $\Gamma = \Gamma'$ is a free group on N letters, we have $\underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma, H) \cong H^N$ and the induced morphism of quotient stacks

$$[H^N/H] \rightarrow [G^N/G]$$

is almost never universally closed. Indeed, if f is monic then pulling back via the map $S \rightarrow [G^N/G]$ corresponding to $(1, \dots, 1)$ gives $G/H \rightarrow S$, which is universally closed if and only if G/H is finite.

We will prove Theorem 1.1 by verifying the valuative criterion of properness. Let us briefly illustrate the strategy, ignoring a few subtleties; we remark that it is quite different from the strategies of [17] and [13] even over a field. Suppose given a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{y} & \underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma, H) // H \\ \downarrow & & \downarrow F \\ \mathrm{Spec}(A) & \xrightarrow{x} & \underline{\mathrm{Hom}}_{S\text{-gp}}(\Gamma', G) // G \end{array}$$

in which A is a DVR with fraction field K . After passing to a local extension of A , which we may assume to be complete, Lemma 2.2 shows that x arises from a homomorphism $\varphi: \Gamma' \rightarrow G(A)$ and y arises from a homomorphism $\psi: \Gamma \rightarrow H(K)$ with closed H_K -orbit such that φ and $f \circ \psi \circ h$ are $G(K)$ -conjugate. We must show that after extending A further, $\psi(\Gamma)$ is $H(K)$ -conjugate to a subgroup of $H(A)$. Using the $G(K)$ -conjugacy of $f \circ \psi \circ h$ and φ , one shows that $\psi(\Gamma)$ is a bounded subgroup of $H(K)$, and using the closedness of the orbit of ψ it follows from [14] that the Zariski closure H_1 of $\psi(\Gamma)$ in H_K is reductive (but possibly disconnected). At this point, the key input is the following theorem (applied to $B = \psi(\Gamma)$), which collects the results of Sections 3 and 4.

Theorem 1.3. Let A be an excellent Henselian DVR with fraction field K .

- 1) (Lemmas 3.4, 4.3) Let H_1 be a reductive K -group, and let $B \subset H_1(K)$ be a bounded subgroup. There exists a finite local extension of DVRs $A \subset A'$ with fraction field K' and a smooth affine model $\mathcal{H}_1$ of $(H_1)_K$ over A' such that $\mathcal{H}_1^0$ is a reductive group scheme, $\mathcal{H}_1/\mathcal{H}_1^0$ is finite étale, and $B \subset \mathcal{H}_1(A')$.
- 2) (Lemma 4.4) Let $\mathcal{H}_1$ and H be smooth affine A -group schemes such that H^0 is reductive and H/H^0 is finite étale, and let $f_1: (\mathcal{H}_1)_K \rightarrow H_K$ be a K -homomorphism. There exists a finite local extension of DVRs $A \subset A'$ with fraction field K' and $h \in H^0(K')$ such that $\mathrm{Ad}(h) \circ (f_1)_K$ extends to an A' -homomorphism $f: (\mathcal{H}_1)_{A'} \rightarrow H_{A'}$.

In the case that $S = \mathrm{Spec} W(k)$ for an algebraically closed field k of characteristic $p > 0$, $\mathcal{H}_1$ is constant of order prime to p , and H is semisimple (with connected fibers), Theorem 1.3(2) is proved in [10, Lemma A.8].

The proof of Theorem 1.3(1) proceeds by first constructing (after a finite local extension of A) a single integral model of H_1 with good properties. The group of A -points of any such model can be realized as the stabilizer of a hyperspecial point in the Bruhat-Tits building $\mathcal{B}(H_1^0)$. Next, we show that for any bounded subgroup B of $H_1(K)$, there is a finite extension K'/K such that B stabilizes a hyperspecial point of $\mathcal{B}((H_1^0)_K)$, so we obtain (1) from the conjugacy of hyperspecial points. Item (2) follows from (1) applied to $B = f_1(H(A))$, the conjugacy of hyperspecial points, and a simple lemma (Lemma 4.1).

2 Preliminaries

We recall first the existence part of the (Noetherian) valuative criterion of properness for morphisms of stacks. For simplicity, let $f: \mathcal{X} \rightarrow Y$ be a finite type morphism of algebraic stacks such that Y is a locally Noetherian scheme. We say that f satisfies the existence part of the valuative criterion of properness if, for every DVR A with fraction field K and every solid commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec} K' & \cdots \rightarrow & \mathrm{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \mathrm{Spec} A' & \cdots \rightarrow & \mathrm{Spec} A & \longrightarrow & Y \end{array}$$

there exists a local extension $A \subset A'$ of DVRs with fraction field K' such that the dotted diagram can be filled in to become commutative. (Note that this differs slightly from the definition in [16, Tag0CLA], in which A and A' are only required to be valuation rings; we have also simplified the definition by assuming that Y has no stabilizers.)

The relevance of this definition for our purposes is that if f is finite type and quasi-separated, then [16, Tag0H2C, Tag0CLX] shows that f satisfies the existence part of the valuative criterion of properness if and only if f is universally closed. (Strictly speaking, [16, Tag0CLX] is only a statement about the non-Noetherian valuative criterion, but its proof works mutatis mutandis for the Noetherian valuative criterion by using [16, Tag0H2B] in place of [16, Tag0CL2].)

We recall also the notion of geometric reductivity from [1, Definition 9.1.1]. The precise definition is of no relevance to us, but we recall the following two facts which we will use without comment in what follows.

- 1) [1, Theorem 9.7.6] If S is a scheme, then a smooth affine S -group scheme G is geometrically reductive if and only if the relative identity component G^0 is a reductive group scheme and the quotient G/G^0 is finite étale over S . (The published version of [1, Theorem 9.7.6] omits the word “affine” in this statement. Without this assumption, the statement is false: indeed, abelian schemes are geometrically reductive. In the 2018 arXiv version of [1, Theorem 9.7.6], this is corrected, but there is an additional assumption that G/G^0 is separated. This latter assumption is superfluous: all group schemes over a field are separated, so if G is geometrically reductive smooth affine then the theorem shows G has reductive fibers, and thus [4, Proposition 3.1.3] shows that G/G^0 is a separated étale S -group of finite presentation. I thank Vytautas Paškūnas and Julian Quast for inquiring about this discrepancy.)
- 2) [1, Theorems 5.3.1, 6.3.3, 9.1.4] If S is a locally Noetherian scheme, G is a geometrically reductive S -group scheme, and $p: X \rightarrow S$ is a finite type S -affine S -scheme equipped with a G -action, then the GIT quotient $X//G := \text{Spec}_S(p_*\mathcal{O}_X)^G$ is of finite type over S and the natural map $[X/G] \rightarrow X//G$ from the quotient stack is surjective and universally closed.

Lemma 2.1. Let S be a locally Noetherian scheme, let X be a finite type S -affine S -scheme, and let G be a geometrically reductive S -group scheme acting on X . The natural map $\pi: [X/G] \rightarrow X//G$ satisfies the existence part of the valuative criterion of properness.

Proof. As noted above, π is universally closed. Since $G \rightarrow S$ is quasi-compact, the quotient stack $[X/G]$ is evidently quasi-separated and of finite type over S . By cancellation, it follows that π is quasi-separated and of finite type, and thus the result follows from [16, Tag0CLX] (suitably adapted to the Noetherian setting, as indicated above). ■

Lemma 2.2. Let A be a DVR, let G be a geometrically reductive smooth affine A -group scheme, and let X be an affine A -scheme on which G acts. If $x \in (X//G)(A)$, then there is a local extension of DVRs $A \subset A'$ such that $x_{A'}$ lies in the image of $X(A') \rightarrow (X//G)(A')$.

Proof. By Lemma 2.1 and surjectivity, we may pass to a local extension of A to assume that there exists $x_0 \in [X/G](A)$ mapping to x . By definition, $X \rightarrow [X/G]$ is an étale G -torsor, so we may pass to a further local extension of A to assume that x_0 lifts to $x_1 \in X(A)$. ■

3 Extending Reductive Groups Over DVRs

The goal of this section is to show that, after passing to a quasi-finite local extension of A , any (possibly disconnected) reductive K -group G admits a geometrically reductive smooth affine model over A . This result (Lemma 3.4) will be complemented by Lemma 4.3, which will show that, *provided a single such model exists*, one can choose a geometrically reductive smooth affine model whose set of A -points contains any given bounded subgroup of $G(K)$.

To deal with issues of disconnectedness, we will perform a pushout construction with certain finite flat (not necessarily étale) A -group schemes. To this end, we first develop a small amount of theory for

extensions of such group schemes. In fact, we will develop a bit more than is necessary for the proof of Theorem 1.1.

Lemma 3.1. Let A be a DVR with fraction field K , and let

$$1 \rightarrow M \rightarrow E_i \rightarrow \Gamma \rightarrow 1$$

($i = 1, 2$) be short exact sequences of finite flat A -group schemes such that M is of multiplicative type and Γ is étale. If $f_1: (E_1)_K \rightarrow (E_2)_K$ is a K -homomorphism preserving M_K , then f_1 extends uniquely to an A -homomorphism $f: E_1 \rightarrow E_2$. If f_1 is an isomorphism, then f is also an isomorphism.

Proof. If f_1 extends to an A -morphism $f: E_1 \rightarrow E_2$, then by flatness and schematic density of $\text{Spec } K$ in $\text{Spec } A$, this extension is unique and it is a homomorphism. Thus by descent, we may and do extend A to assume that there are scheme-theoretic sections $s_i: \Gamma \rightarrow E_i$. From this we obtain isomorphisms of A -schemes $\varphi_i: M \times_A \Gamma \rightarrow H$ given functorially by $\varphi(m, \gamma) = ms_i(\gamma)$. Using the φ_i , we see that f_1 induces a K -morphism $g_1: M_K \times_K \Gamma_K \rightarrow M_K \times_K \Gamma_K$ given functorially by $g_1(m, \gamma) = (f_1(m)d(\gamma), \tilde{f}_1(\gamma))$, where $d: \Gamma_K \rightarrow M_K$ is a K -morphism. Thus it suffices to extend $f_1|_{M_K}$, d , and $\tilde{f}_1$ over A . Since M is finite and Γ is étale, the fact that d and $\tilde{f}_1$ extend comes from the valuative criterion of properness. The fact that $f_1|_{M_K}$ extends comes from Cartier duality, noting that the Cartier dual of M is finite étale.

Now suppose that f_1 is an isomorphism with inverse g_1 , and let f and g be the unique extensions. By uniqueness of extensions, we see that $f \circ g$ and $g \circ f$ are the respective identities. ■

Lemma 3.2. Let A be a DVR with fraction field K , and let H be a finite K -group scheme. Suppose that we are given finite flat A -group schemes H_0 and H_1 and a short exact sequence

$$1 \rightarrow (H_0)_K \rightarrow H \rightarrow (H_1)_K \rightarrow 1. \quad (1)$$

Suppose moreover that H_0 is of multiplicative type and H_1 is étale.

Then there is a finite flat A -group scheme $\mathcal{H}$ and a short exact sequence

$$1 \rightarrow H_0 \rightarrow \mathcal{H} \rightarrow H_1 \rightarrow 1 \quad (2)$$

whose base change to K is isomorphic to (1).

Proof. Note that by Lemma 3.1, the pair of the extension (2) and its isomorphism with (1) is unique up to unique isomorphism if it exists. Thus, by descent, we may pass to a quasi-finite extension of A to assume that H_1 is constant there is a scheme-theoretic section $s: (H_1)_K \rightarrow H$. Since H_1 is constant, Lemma 3.1 also shows that the induced conjugation map $(H_1)_K \times_K (H_0)_K \rightarrow (H_0)_K$, $(h_1, h_0) \mapsto s(h_1)h_0s(h_1)^{-1}$ extends uniquely to an action map $\alpha: H_1 \times_A H_0 \rightarrow H_0$.

Define $\mathcal{H}$ scheme-theoretically as the product $H_0 \times H_1$. Note that $\mathcal{H}_K \cong H$ via the map $f(h_0, h_1) = h_0s(h_1)$. Under this isomorphism, the multiplication map on H corresponds to the map $\mu_K: \mathcal{H}_K \times_K \mathcal{H}_K \rightarrow \mathcal{H}_K$ given by

$$\mu_K((h_0, h_1), (h'_0, h'_1)) = (h_0 \cdot s(h_1)h'_0s(h_1)^{-1} \cdot c(h_1, h'_1), h_1h'_1),$$

where $c(h_1, h'_1) = s(h_1)s(h'_1)s(h_1h'_1)^{-1}$. Since H_1 is étale and H_0 is finite, c extends to an A -morphism $H_1 \times_A H_1 \rightarrow H_0$ which we will also denote by c . Similarly, $s(1) \in H_0(A)$.

Define an A -morphism $\mu: \mathcal{H} \times_A \mathcal{H} \rightarrow \mathcal{H}$ extending μ_K by

$$\mu((h_0, h_1), (h'_0, h'_1)) = (h_0 \cdot \alpha(h_1, h'_0) \cdot c(h_1, h'_1), h_1h'_1).$$

Diagrams of flat A -schemes may be checked to be commutative after passage to K , so it follows that μ is a monoid law (with identity $(s(1)^{-1}, 1)$). Moreover, μ is actually a group law: to see this functorially, let B be an A -algebra and let $(h_0, h_1) \in \mathcal{H}(B)$. By assumption on α , there is a unique $h'_0 \in H_0(B)$ such that $\alpha(h_1, h'_0) = h_0^{-1}s(1)^{-1}c(h_1, h_1^{-1})^{-1}$. The right inverse of (h_0, h_1) under μ is clearly (h'_0, h_1^{-1}) , and by the Yoneda lemma this gives a right inverse morphism $r: \mathcal{H} \rightarrow \mathcal{H}$. Note that r is also a left inverse map (as one can check over K), so indeed $\mathcal{H}$ is an A -group scheme.

Finally, there are A -homomorphisms $i: H_0 \rightarrow \mathcal{H}$ and $\pi: \mathcal{H} \rightarrow H_1$ given by $i(h_0) = (h_0 s(1)^{-1}, 1)$ and $\pi(h_0, h_1) = h_1$, and it is straightforward to check that these form a short exact sequence whose base change to K is isomorphic to (1). ■

The following lemma is a mild variant of [2, Lemma 2.23].

Lemma 3.3. Let S be a connected scheme, and let

$$1 \rightarrow M \rightarrow E \rightarrow \Gamma \rightarrow 1 \quad (3)$$

be a short exact sequence of finitely presented S -group schemes such that M is of multiplicative type, Γ is constant of order n , and $E \rightarrow \Gamma$ admits a scheme-theoretic section. There exists a short exact sequence

$$1 \rightarrow M[n] \rightarrow H \rightarrow \Gamma \rightarrow 1 \quad (4)$$

such that (3) is obtained from (4) by pushing forward along the inclusion $M[n] \rightarrow M$. Moreover, the pushout of (4) by the inclusion $M[n] \rightarrow M[n^2]$ is unique up to isomorphism.

Proof. Since $E \rightarrow \Gamma$ admits a section, (3) corresponds to an element α of the Hochschild cohomology group $H^2(\Gamma, M)$ (see [7, Proposition 2.3.6]). Since Γ is constant and S is connected, $H^i(\Gamma, M)$ agrees with the ordinary group cohomology $H^i(\Gamma(S), M(S))$ for all $i \geq 0$. This latter group is killed by n by classical theory, so α is the image of a class in $H^2(\Gamma(S), M[n](S)) = H^2(\Gamma, M[n])$; this corresponds to the desired extension (4). To see the final claim, consider the commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(\Gamma, M) & \longrightarrow & H^2(\Gamma, M[n]) & \longrightarrow & H^2(\Gamma, M) \\ \downarrow n & & \downarrow & & \downarrow \\ H^1(\Gamma, M) & \longrightarrow & H^2(\Gamma, M[n^2]) & \longrightarrow & H^2(\Gamma, M) \end{array}$$

Since $H^1(\Gamma, M)$ is killed by n (as above), a diagram chase concludes the proof. ■

The following lemma is analogous to the fact that a (real) Lie group G with finitely many connected components admits a maximal compact subgroup which meets every component of G .

Lemma 3.4. Let A be a DVR with fraction field K , and let G be a (possibly disconnected) reductive K -group. There exists a quasi-finite local extension of DVRs $A \subset A'$ inducing the extension of fraction fields $K \subset K'$ and a geometrically reductive smooth affine A' -integral model $\mathcal{G}'$ of $G_{K'}$.

Proof. By extending A , we may assume G^0 is split and G/G^0 is constant. First let $\mathcal{G}_0$ be a split reductive A -group scheme with generic fiber G^0 , which exists by the Existence and Isomorphism Theorems [15, Exp. XXV, Théorème 1.1]. Let $(\mathcal{B}_0, \mathcal{T}_0)$ be a Borel-torus pair of $\mathcal{G}_0$ over A , and let $N = N_G((\mathcal{B}_0)_K, (\mathcal{T}_0)_K)$, the normalizer of the pair $((\mathcal{B}_0)_K, (\mathcal{T}_0)_K)$ in G . By Lemma 3.3, if the constant group scheme $N/(\mathcal{T}_0)_K \cong G/G^0$ is of order n , then after extending K there is an extension

$$1 \rightarrow (\mathcal{T}_0)_K[n] \rightarrow H \rightarrow N/(\mathcal{T}_0)_K \rightarrow 1 \quad (5)$$

whose pushout along $(\mathcal{T}_0)_K[n] \subset (\mathcal{T}_0)_K$ is the tautological extension for $(\mathcal{T}_0)_K \subset N$. If Γ is the constant A -group scheme corresponding to $N/(\mathcal{T}_0)_K \cong G/G^0$, then using Lemma 3.2 we find that there exists a finite flat A -group scheme $\mathcal{H}$ and a short exact sequence

$$1 \rightarrow \mathcal{T}_0[n] \rightarrow \mathcal{H} \rightarrow \Gamma \rightarrow 1$$

whose generic fiber is (5).

Now note that G is isomorphic to the pushout $G^0 \times^{(\mathcal{T}_0)_K[n]} H := (G^0 \times_K H)/(\mathcal{T}_0)_K[n]$. Thus if we define $\mathcal{G} = \mathcal{G}_0 \times^{\mathcal{T}_0[n]} \mathcal{H}$, we find that $\mathcal{G}_K \cong G$. Moreover, $\mathcal{G}^0 = \mathcal{G}_0$ is reductive and $\mathcal{G}/\mathcal{G}^0 \cong \Gamma$ is finite étale, whence $\mathcal{G}$ is a geometrically reductive smooth affine model of G . ■

4 Extending Homomorphisms

In this section we aim to establish, essentially, that the quotient stack $[\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)/G$ satisfies the existence part of the valuative criterion of properness when S is a locally Noetherian scheme, H and G are smooth S -affine S -group schemes, and G is geometrically reductive. (This stack is often algebraic but it is usually not proper; it is locally of finite type and quasi-separated, but rarely quasi-compact or separated. See [2, Section 2.1] or [5] for a discussion of $\underline{\mathrm{Hom}}_{S\text{-gp}}(H, G)$.) We begin with a simple lemma which mildly generalizes [11, Corollary 2.10.10].

Lemma 4.1. Let $A \subset A'$ be a finite local extension of strictly Henselian DVRs inducing the extension $K \subset K'$ of fraction fields. Let X be a smooth A -scheme, and let X' be an affine A' -scheme. A K' -morphism $f: X_{K'} \rightarrow X'_{K'}$ extends (uniquely) to an A' -morphism $X_{A'} \rightarrow X'$ if and only if $f(X(A)) \subset X'(A')$. In particular, if $f(X(A)) \subset X'(A')$ then $f(X(A')) \subset X'(A')$.

Proof. By passing to an open cover, we may and do assume that X is affine. If f extends to an A' -morphism $X_{A'} \rightarrow X'$, then clearly $f(X(A)) \subset X'(A')$, so we need only show the converse. Note that $A \subset A'$ is flat, so the Weil restriction $\mathbf{R}_{A'/A}(X')$ exists as an affine A -scheme. The K' -morphism $f: X_{K'} \rightarrow X'_{K'}$ corresponds to a K -morphism $f': X \rightarrow \mathbf{R}_{K'/K}(X'_{K'})$ satisfying $f'(X(A)) \subset \mathbf{R}_{A'/A}(X')(A)$. By [11, Corollary 2.10.10], the K -morphism f' extends to an A -morphism $X \rightarrow \mathbf{R}_{A'/A}(X')$, which in turn corresponds to an A' -morphism $X_{A'} \rightarrow X'$ extending f . ■

Recall that, if K is a discretely valued field with valuation v and X is an affine K -scheme, then a subset $B \subset X(K)$ is *bounded* if, for every $f \in \Gamma(X, \mathcal{O}_X)$, the function $b \mapsto v(f(b))$ is bounded below on B . For a detailed discussion of boundedness, see [11, Section 2.2]. We need also standard properties of the (reduced) Bruhat–Tits building, which are summarized in [11, Chapter 4].

Lemma 4.2. Let A be a Henselian DVR with fraction field K , and let G be a (possibly disconnected) reductive K -group. If $B \subset G(K)$ is a bounded subgroup, then there is a facet F of the Bruhat–Tits building $\mathcal{B}(G^0)$ such that B stabilizes the barycenter of F .

Proof. Note first that since $G(K)$ acts on $G^0(K)$ by conjugation, it also acts on $\mathcal{B}(G^0)$, and the restriction of this action to $G^0(K)$ is the usual action. By [11, Corollary 4.2.14], there is a point x_0 of the (restricted) Bruhat–Tits building $\mathcal{B}(G^0)$ such that $B \cap G^0(K)$ stabilizes x_0 . Since $B \cap G^0(K)$ is of finite index in B , it follows that the set Bx_0 is finite. If X is the convex hull of Bx_0 in $\mathcal{B}(G^0)$, then X is closed and bounded by [11, Lemma 1.1.13], and [11, Theorem 4.2.12] shows that X has a unique barycenter x_1 , invariant under all isometries of $\mathcal{B}(G^0)$ which preserve X . In particular, the action of B on $\mathcal{B}(G^0)$ preserves x_1 . If F is the (open) facet of $\mathcal{B}(G^0)$ containing x_1 , then the action of B preserves the barycenter of F , as desired. ■

If G is a geometrically reductive smooth affine group scheme over a Henselian DVR A with fraction field K and valuation v , then we let $G(K)^1$ be the open subgroup $G^0(K)^1 \cdot G(A)$ of $G(K)$, where $G^0(K)^1$ is defined as in [11, Section 2.6(d)]:

$$G^0(K)^1 := \{g \in G^0(K) : v(\chi(g)) = 0 \text{ for all } \chi \in X^*(G^0)\},$$

where $X^*(G^0)$ is the group of characters $\chi: G^0 \rightarrow \mathbf{G}_m$. As in the connected case, every bounded subgroup of $G(K)$ is contained in $G(K)^1$ because $G(K)/G(K)^1$ is topologically isomorphic to a subgroup of the torsion-free discrete group $X^*(T)$, where T is the maximal central K -torus of G^0 . Notice that $G^0(K)^1 = G(K)^1 \cap G^0(K)$.

Lemma 4.3. Let A be an excellent Henselian DVR with fraction field K , and let G be a geometrically reductive smooth affine A -group scheme. If $B \subset G(K)$ is a bounded subgroup, then there exists a finite local extension of DVRs $A \subset A'$ with fraction field K' and a geometrically reductive smooth affine A' -integral model G' of $G_{K'}$ such that $B \subset G'(A')$. Moreover, for any such G' , there is a further extension of A' such that the subgroup $G'(A')$ is $G^0(K')$ -conjugate to $G(A')$.

Proof. By spreading out, we may and do assume that A is strictly henselian. By Lemma 4.2, there is a facet F of the Bruhat–Tits building $\mathcal{B}(G^0)$ such that B stabilizes the barycenter x of F . Since x is the

barycenter of a facet, the argument of [12, Lemma 2.4] shows that there exists a finite extension K'/K with valuation ring A' such that the image x' of x in the building $\mathcal{B}(G_{K'}^0)$ is hyperspecial. (The proof of [12, Lemma 2.4] appears to have a small gap, which can be fixed by noting that one may first pass to a finite extension K'/K such that x' is a vertex. Indeed, if $\mathcal{A}$ is an apartment containing x and corresponding to a maximal split K -torus T , then $\mathcal{A}$ is an affine space under $V(T) := \mathbf{R} \otimes_{\mathbf{Z}} X_*(T)$, and the metric on $\mathcal{B}(G^0)$ restricts to the Euclidean metric on $\mathcal{A}$ (see [11, Section 4.2]). After choosing an identification of $\mathcal{A}$ and $V(T)$, one sees that x is a $\mathbf{Q}$ -linear combination of vertices, and the claim follows from the discussion in [11, Section 6.5]. The phrase “As x_1 is a facet” from [12] should then be replaced with “As x_1 is a vertex”.) Let U' be the stabilizer of x' in $G(K')^1$, so $B \subset U'$. By [11, Propositions 8.3.1, A.7.1], there is a smooth affine model G' of $G_{K'}$ over A' such that $G'(A') = U'$.

Since G^0 is a reductive group scheme, [11, Theorem 9.9.3(2)] shows that $G^0(A')$ stabilizes a hyperspecial point y of $\mathcal{B}(G_{K'}^0)$. By [11, Proposition 1.3.43(3), Corollary 7.4.8], the point y is the *unique* one stabilized by $G^0(A')$. Moreover, Lemma 4.2 shows that $G(A')$ stabilizes some point of $\mathcal{B}(G^0)$, so in fact $G(A')$ stabilizes y . By [11, Proposition 10.2.2], we may pass to a finite extension of K' to find some $g \in G^0(K')$ such that $g \cdot x = y$, and thus $gG'(A')g^{-1} = G(A')$. By Lemma 4.1, the K' -isomorphism $G_{K'} \rightarrow G_{K'}$ given by g -conjugation induces an A' -isomorphism $G' \rightarrow G$. Because A is excellent Henselian, the extension $A \subset A'$ is finite. ■

Lemma 4.4. Let A be an excellent Henselian DVR with fraction field K , and let G and H be smooth affine A -group schemes such that G is geometrically reductive. If $f_1: H_K \rightarrow G_K$ is a K -homomorphism, then there exists a finite local extension of DVRs $A \subset A'$ with fraction field K' and $g \in G(K')$ such that $\text{Ad}(g) \circ f_1$ extends to an A' -homomorphism $f': H_{A'} \rightarrow G_{A'}$.

Proof. By spreading out, we may and do assume that A is strictly Henselian with algebraically closed residue field. Note that $f_1(H(A)) \subset G(K)$ is a bounded subgroup, so Lemma 4.3 shows that there is a finite local extension $A \subset A'$, a geometrically reductive smooth affine A' -group scheme G' such that $f_1(H(A)) \subset G'(A')$, and $g \in G^0(K')$ such that $gG'(A')g^{-1} = G(A')$. By Lemma 4.1, it follows that $\text{Ad}(g) \circ (f_1)_{K'}: H_{K'} \rightarrow G'_{K'}$ extends to an A' -homomorphism $f': H_{A'} \rightarrow G'$, as desired. ■

We need one more technical lemma before proving Theorem 1.1.

Lemma 4.5. Let K be a discretely valued field, let G be a reductive K -group, let Γ be a finitely generated group, and let $\varphi, \psi: \Gamma \rightarrow G(K)$ be two homomorphisms whose G -orbits have intersecting closures in $\underline{\text{Hom}}_{K\text{-gp}}(\Gamma, G)$. Suppose moreover that $\varphi(\Gamma)$ is bounded in $G(K)$ and that the Zariski closure $G_1 = \overline{\psi(\Gamma)}$ is reductive. Then $\psi(\Gamma)$ is bounded in $G(K)$.

Proof. Note first that if $\Gamma' \subset \Gamma$ is a finite index subgroup, then $\psi(\Gamma')$ is bounded if and only if $\psi(\Gamma)$ is bounded; thus we may shrink Γ to assume that G_1 is connected. Let $\rho: G \rightarrow \text{GL}(V)$ be a faithful K -representation, and note that boundedness of $\varphi(\Gamma)$ implies that for each $\gamma \in \Gamma$, all of the eigenvalues of $\rho(\varphi(\gamma))$ on $V_{\overline{K}}$ are integral. Because the G -orbits of φ and ψ have intersecting closures, the eigenvalues of $\rho(\psi(\gamma))$ on $V_{\overline{K}}$ are the same as those of $\rho(\varphi(\gamma))$, and in particular they are all integral. Thus, by [11, Lemma 2.2.11] (applied to the connected reductive group G_1), we find that $\psi(\Gamma)$ is bounded, as desired. ■

5 Proof of Theorem 1.1

As in the introduction, let S be a locally Noetherian scheme, and let G and H be geometrically reductive smooth affine S -group schemes. Fix a finite S -homomorphism $f: H \rightarrow G$ and a homomorphism $h: \Gamma' \rightarrow \Gamma$ of finitely generated groups such that $h(\Gamma')$ is of finite index in Γ . For simplicity, write $\mathbf{H}_H = \underline{\text{Hom}}_{S\text{-gp}}(\Gamma, H)$ and $\mathbf{H}_G = \underline{\text{Hom}}_{S\text{-gp}}(\Gamma', G)$. We will show in this section that the S -morphism

$$F: \mathbf{H}_H // H \rightarrow \mathbf{H}_G // G$$

is finite.

The map F is evidently affine, and it is of finite type by [1, Theorem 6.3.3], so to show that F is finite it suffices (by [16, Tag01WM]) to show that it is universally closed, or equivalently to verify the existence part of the valuative criterion of properness for F . In other words, we must show that if A is a DVR with fraction field K and we have a solid diagram

$$\begin{array}{ccccc} \mathrm{Spec} K' & \cdots \rightarrow & \mathrm{Spec} K & \xrightarrow{y} & \mathbf{H}_H // H \\ \downarrow & \nearrow & \downarrow & & \downarrow F \\ \mathrm{Spec} A' & \cdots \rightarrow & \mathrm{Spec} A & \xrightarrow{x} & \mathbf{H}'_G // G \end{array}$$

then there is a local extension $A \subset A'$ of DVRs inducing the extension $K \subset K'$ of fraction fields such that the above dashed diagram can be filled in to become commutative. We will extend A in stages; note that at every stage we are free to replace A by its completion to assume that A is excellent and Henselian.

From now on, fix a solid diagram as above. By [1, Theorem 5.3.1(5)], every element of $(\mathbf{H}_H // H)(\bar{K})$ lifts to a homomorphism $\psi: \Gamma \rightarrow H(\bar{K})$ with closed $H_{\bar{K}}$ -orbit in $\mathbf{H}_{H_{\bar{K}}}$. Since Γ is finitely generated, we may pass to a local extension of A to assume that y lifts to such a homomorphism ψ valued in $H(K)$ with closed H -orbit in $\mathbf{H}_H$. Concretely, we must show that there is a local extension $A \subset A'$ of DVRs with fraction field K' such that ψ is $H(K')$ -conjugate to a homomorphism $\Gamma \rightarrow H(A')$, or equivalently that $\psi(\Gamma)$ is $H(K')$ -conjugate to a subgroup of $H(A')$. Let H_1 be the closed K -subgroup of H_K which is topologically generated by $\psi(\Gamma)$. A surjection $F_N \rightarrow \Gamma$ from a free group F_N induces an H -equivariant closed embedding $\mathbf{H}_H \rightarrow H^N$. In particular, the image of ψ under such an embedding still has closed H -orbit, so [14, Lemma 16.3, Theorem 16.4] shows that H_1 is reductive. (Strictly speaking, this reference requires H to be connected, but its proof works in the non-connected case because of the robustness of the results of [3] on which it relies.)

By Lemma 2.2, we may extend A to assume that x is the image of a homomorphism $\varphi: \Gamma' \rightarrow G(A)$ under the natural map $\mathbf{H}'_G(A) \rightarrow (\mathbf{H}'_G // G)(A)$. Because $f \circ \psi \circ h$ and φ have the same image in $(\mathbf{H}'_G // G)(K)$, [1, Theorem 5.3.1(5)] shows that the closures of their orbits intersect. Moreover, if G_1 is the Zariski closure of $f(\psi(h(\Gamma')))$, then there is a finite surjective map $H_1^0 \rightarrow G_1^0$, so G_1 is reductive because H_1 is reductive. By Lemma 4.5, it follows that $f(\psi(h(\Gamma')))$ is bounded. Since $f(\psi(\Gamma))$ contains $f(\psi(h(\Gamma')))$ as a finite index subgroup, it follows that $f(\psi(\Gamma))$ is bounded in $G(K)$. By [11, Lemma 2.2.10], the fact that f is finite implies that $\psi(\Gamma)$ is bounded in $H(K)$.

By the previous paragraph, Lemma 3.4, and Lemma 4.3, after extending A further we may and do assume that there is a geometrically reductive smooth affine A -group scheme $\mathcal{H}_1$ with generic fiber H_1 such that $\psi(\Gamma) \subset \mathcal{H}_1(A)$. By construction, there is a K -morphism $i_1: H_1 \rightarrow H_K$, and by Lemma 4.4 we may pass to a local extension of A and conjugate by an element of $H(K)$ to assume that i_1 extends to an A -homomorphism $i: \mathcal{H}_1 \rightarrow H$. Thus $\psi(\Gamma)$ is $H(K)$ -conjugate to a subgroup of $H(A)$, verifying the existence part of the valuative criterion of properness and proving that F is finite, as desired.

Remark 5.1. If S is not required to be locally Noetherian, then it follows from Theorem 1.1, [1, Proposition 5.2.9], and spreading out that F is integral. However, I do not know whether the schemes involved are finitely presented (or even of finite type) in this case, so *finiteness* of the morphism is not clear.

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