Kinetic and Related Models

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GLOBAL WELL-POSEDNESS AND STABILITY OF THE INHOMOGENEOUS KINETIC WAVE EQUATION NEAR VACUUM

Ioakeim Ampatzoglou*

Department of Mathematics, Baruch College, The City University of New York, USA

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Abstract. In this paper, we prove global in time existence, uniqueness and stability of mild solutions near vacuum for the 4-wave inhomogeneous kinetic wave equation, for Laplacian dispersion relation in dimension $d = 2, 3$. We also show that for non-negative initial data, the solution remains non-negative. This is achieved by connecting the inhomogeneous kinetic wave equation, for such dimensions, to the cubic part of the quantum Boltzmann equation for bosons, with Maxwell or hard potential and no collisional averaging.

1. **Introduction.** The problem of understanding the behavior of large systems of nonlinear interacting waves is of fundamental importance in the community of mathematical physics. However, with the size of the system being extremely large, deterministic prediction of its evolution in time is practically impossible and one resorts to a kinetic description. The kinetic theory of waves, referred to as wave turbulence theory, provides a mesoscopic framework for studying averaging quantities of the system e.g. the point energy spectrum, but still obtaining a statistically accurate prediction in time. This is in general achieved through the means of an effective equation, which in the case of wave turbulence is the kinetic wave equation (KWE).

This paper focuses on the global in time well-posedness and stability of the 4-wave space inhomogeneous kinetic wave equation for initial data close to vacuum, Laplacian dispersion relation and physical dimension $d = 2, 3$. The main idea of the paper is that, for such dimensions, it is possible to connect the inhomogeneous kinetic wave equation to the cubic part of the quantum Boltzmann equation for bosons, with Maxwell or hard potential and no collisional averaging, see Lemma 3.1 for more details. Since the well-posedness of Boltzmann-type equations near vacuum has been widely studied [31, 29, 24, 5, 46, 6, 47, 40, 38, 20, 2, 1, 45, 4, 23, 22, 37] in the past, we employ techniques of the classical kinetic theory for Boltzmann-type equations to address existence, uniqueness and stability of global in time mild solutions (see Section 2 for the precise definition of a mild solution)

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*Corresponding author: Ioakeim Ampatzoglou.

to the spatially inhomogeneous (KWE). Up to the author's knowledge, this is the first paper which addresses the global in time well-posedness of the inhomogeneous kinetic wave equation.

1.1. **The inhomogeneous kinetic wave equation and functional spaces.** We study the global in time well-posedness and stability of the space inhomogeneous kinetic wave equation for 4-wave interactions when the initial data are near vacuum in dimension $d=2,3$. The equation for 4-wave interactions is given by

$$\begin{cases}\partial_t f + v\cdot\nabla_x f = \mathcal{C}[f],\\ f(t=0)=f_0,\end{cases} \tag{1}$$

where the collisional kernel is

$$\mathcal{C}[f](t,x,v)=\int_{\mathbb{R}^{3d}}\delta(\Sigma)\delta(\Omega)ff_1f_2f_3\left(\frac{1}{f}+\frac{1}{f_1}-\frac{1}{f_2}-\frac{1}{f_3}\right)dv_1\,dv_2\,dv_3, \tag{2}$$

the resonant manifolds are given by

$$\begin{aligned}\Sigma &= v+v_1-v_2-v_3,\\ \Omega &= \omega(v)+\omega(v_1)-\omega(v_2)-\omega(v_3),\end{aligned}$$

$\omega:\mathbb{R}^d\to\mathbb{R}$ is the dispersion relation, and we denote $f:=f(t,x,v)$, $f_i:=f(t,x,v_i)$ for $i\in\{1,2,3\}$. In this paper we consider the classical Laplacian dispersion relation $\omega(v)=|v|^2$, so

$$\Omega=|v|^2+|v_1|^2-|v_2|^2-|v_3|^2.$$

The initial data f_0 are assumed to be exponential near vacuum. More specifically, given $\alpha,\beta>0$, the initial data will lie in the Banach space of Maxwellian bounded continuous functions:

$$\mathcal{M}_{\alpha,\beta}=\left\{f\in C(\mathbb{R}^d\times\mathbb{R}^d,\mathbb{R}):\sup_{x,v}|f(x,v)|e^{\alpha|x|^2+\beta|v|^2}<\infty\right\},$$

endowed with the norm

$$\|f\|=\sup_{x,v}|f(x,v)|e^{\alpha|x|^2+\beta|v|^2}.$$

Of particular importance will be the set of non-negative initial data:

$$\mathcal{M}^+_{\alpha,\beta}:=\left\{f\in\mathcal{M}_{\alpha,\beta}:\ f(x,v)\geq 0,\quad \forall(x,v)\in\mathbb{R}^d\times\mathbb{R}^d\right\}.$$

The natural space for the mild solutions we will study, is the Banach space

$$\mathcal{S}_{\alpha,\beta}:=\left\{f\in C([0,\infty),\mathcal{M}_{\alpha,\beta}):\ |||f|||<\infty\right\},$$

where the norm is given by

$$|||f|||:=\sup_{t\geq 0}\|f(t)\|.$$

We will also be writing

$$\mathcal{S}^+_{\alpha,\beta}:=\{f\in\mathcal{S}_{\alpha,\beta}\ :\ f(t)\in\mathcal{M}^+_{\alpha,\beta},\ \forall\, t\geq 0\}.$$

We note that the space $\mathcal{S}_{\alpha,\beta}$ continuously embeds in the space $C([0,T],L^1_{x,v})$, since $\mathcal{M}_{\alpha,\beta}$ embeds in $L^1_{x,v}$. Indeed, given $t\geq 0$, we have

$$\|g(t)\|_{L^1_{x,v}}\leq\|g(t)\|\int_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\alpha|x|^2-\beta|v|^2}\,dx\,dv=(\alpha\beta)^{-d/2}\pi^d\|g(t)\|. \tag{3}$$

Hence

$$\|g\|_{C([0,T],L^1_{x,v})}\leq(\alpha\beta)^{-d/2}\pi^d|||g|||. \tag{4}$$

1.2. **Backround.** The kinetic wave equation was first introduced independently by Peierls [39] who worked on solid state physics, and Hasselmann [27, 28] in his work on water waves. Later, the topic was revived by Zakharov and collaborators [48, 49] who provided a broad framework applying to various Hamiltonian systems satisfying weak nonlinearity, high frequency, phase randomness assumptions. Nowadays, the kinetic theory of waves, known as wave turbulence theory, is fundamental to the study of nonlinear waves, having applications e.g. in plasma theory [11], oceanography [30, 21] and crystal thermodynamics [43]. For an introduction to this broad research field, see e.g. Nazarenko [35], Newell-Rumpf [36].

The homogeneous kinetic wave equation. The homogeneous 4-wave (KWE), i.e. equation (1) with no spatial dependence, can be formally derived from the cubic nonlinear Schrödinger equation (NLS) with periodic boundary conditions

$$i\partial_t u + \Delta u = \lambda |u|^2 u, \quad x \in \mathbb{T}_L^d, \tag{5}$$

and asymptotically describes the point-energy distribution of the Fourier modes of the solution to (5) for Gaussian initial data in the weak nonlinearity limit-large box limit $\lambda \to 0$, $L \to \infty$.

The first rigorous result regarding derivation of the homogeneous (KWE) was obtained in the pioneering work of Lukkarinen and Spohn [32], who were able to reach the kinetic timescale, which the time that one expects the kinetic behavior of the system to emerge, for the cubic nonlinear Schrödinger equation (NLS) at statistical equilibrium, leading to a linearized version of the kinetic wave equation (see also [18]). The key idea in [32] is to use Feynmann diagrams in order to control higher order correlations and has inspired most of the subsequent works. The derivation for random data out of statistical equilibrium was first addressed by Buckmaster, Germain, Hani, Shatah [7] using Strichartz estimates to control the error term. However, the derivation was shown to times much smaller than the kinetic timescale. Later, Collot and Germain [9, 10], inspired by the ideas of [32] (construction of an approximate solution, control of the higher order terms via Feynmann diagramms) estimated the error in Bourgain spaces instead of Strichartz spaces and were able reach the kinetic timescale up to arbitrarily small polynomial loss. At the same time, a similar result was obtained independently by Deng and Hani [12]. Later, in a pioneering work, Deng and Hani [13] reached the kinetic timescale for the cubic (NLS), which provides the first full derivation of the homogeneous (KWE) for (NLS). The same authors addressed propagation of chaos and full range of scaling laws in [14, 15]. Recently, they extended the derivation to longer times in [16]. Under the assumption of multiplicative noise, Staffilani and Tran [44] reached the kinetic timescale as well for the Zakharov-Kuznetsov equation.

Regarding the well-posedness of the homogeneous (KWE), the question of local existence and uniqueness for 4-wave interactions was first addressed in [17] for velocity isotropic solutions and Laplacian dispersion relations. It is also proved in [17] that the equation admits global, measure valued, weak solutions, and that condensation can occur. Existence and uniqueness of radial weak solutions to a slightly simplified version of the 4-wave kinetic equation for general power-law dispersion has been proved in [34]. For general solutions, optimal local well-posedness was shown in [19]. The results of [19] hold in L^∞ for more general dispersion relations, and in L^2 for Laplacian dispersion relation. Additionally, stability of solutions in L^2 near equilibrium was recently shown in [33], while stability and cascades of the Kolmogorov-Zakharov spectrum was shown in [8].

The inhomogeneous kinetic wave equation. The inhomogeneous (KWE) appears in the physics literature [48, 49], where existence of a transport term is physically relevant. In particular, this type of equations are widely used in the prediction of wave propagation in the ocean. Moreover, Spohn [43] discusses the emergence of an inhomogeneous kinetic wave equation, which he calls phonon Boltzmann equation, and addresses its connection to nonlinear waves. Therefore, addressing its global well-posedness would be a question of physical interest.

The inhomogeneous 4-wave (KWE) (1) can be formally derived from the cubic nonlinear Schrödinger equation in the whole space

$$i\partial_t u + \Delta u = \lambda |u|^2 u, \quad x \in \mathbb{R}^d, \tag{6}$$

by taking the rescaled Wigner transform

$$W^\epsilon[u](t,x,v) = \frac{1}{(2\pi)^{d/2}} \epsilon^{-d} \mathbb{E} \int \overline{u(t, x + \frac{z}{2})} u(t, x - \frac{z}{2}) e^{i \frac{v}{\epsilon} \cdot z} \, dz,$$

of the solution to (6) for Gaussian initial data exhibiting randomness at a scale $\sim \epsilon$ with an envelope at a scale ~ 1. Upon rescaling to the kinetic time, the solution of (1) is asymptotically described by $W^\epsilon[u](t,x,v)$ in the weak nonlinearity-high frequency limit $\lambda, \epsilon \to 0$. Roughly speaking, the Wigner transform provides a measure of the amount of energy of u (in L^2) localized in phase space at position x and frequency v/ϵ.

Regarding the rigorous derivation of the inhomogeneous (KWE), the first rigorous result justifying a derivation of a 3-wave inhomogeneous kinetic wave equation from dispersive dynamics was recently obtained by the author in collaboration with Collot and Germain [3], who derived the inhomogeneous (KWE) up to an arbitrarily small polynomial loss of the kinetic time scale for dispersion relations close to Laplacian and quadratic nonlinearities. In the stochastic setting, where time-dependent forcing is permitted in the equation, Hannani, Rosenzweig, Staffilani, and Tran [26] have made contributions to the study of a KdV-type equation, reaching the kinetic time, while recently Hani, Shatah and Zhu [25] studied inhomogeneous turbulence for Wick NLS. Up to the author's knowledge, well-posedness of the inhomogeneous (KWE) has not been addressed in the past, and this is the aim of present paper.

1.3. **Connection with the quantum Boltzmann equation.** Although the inhomogeneous (KWE) can be derived from the Schrödinger equation as described above, it can be seen as a simplified model of the quantum Boltzmann equation for bosons

$$\partial_t f + v \cdot \nabla_x f = Q[f], \tag{7}$$

which describes the evolution of the probability density of a gas of quantum particles, where both classical collisional effects as well as the Bose-Einstein condensate for bosons at low temperature are taken into consideration. The collisional operator in (7) is given by

$$Q[f] = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u|^\gamma b(\hat{u} \cdot \omega) \left(f' f_1' (1+f)(1+f_1) - f f_1 (1+f')(1+f_1') \right) d\omega \, dv_1, \tag{8}$$

where $f := f(t,x,v)$, $f_1 := f(t,x,v_1)$, $f' := f(t,x,v')$, $f_1' := f(t,x,v_1')$, $u := v_1 - v$ denotes the relative velocity of the incoming particles with velocities v, v_1, $\omega \in \mathbb{S}^{d-1}$

denotes their unit relative position, v', v_1' are the velocities after the elastic collision given by:

$$\begin{aligned} v' &= v + (\omega \cdot u)\omega, \\ v_1' &= v_1 - (\omega \cdot u)\omega. \end{aligned} \tag{9}$$

In (8), $\gamma \in (1-d, 1]$ represents the type of potential considered. When $\gamma < 0$ the potential is soft, when $\gamma = 0$ we have Maxwell molecules, when $0 < \gamma < 1$ the potential is moderately hard, and when $\gamma = 1$ the potential is hard. Of particular interest to us will be the Maxwell molecules and the hard potentials. The function $b : [-1, 1] \to \mathbb{R}$ is a measurable, non-negative, even function which represents the collisional averaging and is referred as the angular cross-section.

Note that due to cancellations the operator $Q[f]$ is essentially the sum of the classical quadratic Boltzmann operator plus a cubic quantum term, which corresponds to the Bose-Einstein condensate i.e.

$$Q[f] = Q_{cl}(f, f) + Q_{qu}(f, f, f), \tag{10}$$

where

$$Q_{cl}(f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u|^\gamma b(\hat{u} \cdot \omega) \left(f' f_1' - f f_1\right) \, d\omega \, dv_1, \tag{11}$$

$$Q_{qu}(f, f, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |u|^\gamma b(\hat{u} \cdot \omega) \left(f' f_1' \left(f + f_1\right) - f f_1 \left(f' + f_1'\right)\right) \, d\omega \, dv_1. \tag{12}$$

One reason that physicists study equations of the type (1) rather than the full equation (7) is that the distribution function f (i.e. the solution of the quantum Boltzmann equation for bosons) at very low temperature becomes large near the mean velocity: $|v| << 1 \Rightarrow f >> 1$, so the quadratic term $f_2 f_3 - f f_1$ can be omitted in comparison with the cubic term $f_2 f_3 (f + f_1) - f f_1 (f_2 + f_3)$, see e.g. Section C of [41]. For instance, the equilibrium (i.e. the Bose-Einstein distribution) of the original equation at very low temperature is large near $v = 0$:

$$\frac{1}{e^{a + b|v|^2} - 1} \approx \frac{1}{a + b|v|^2}, \quad \text{for } |v| << 1, \quad 0 < a << 1, \quad b > 0,$$

and $\frac{1}{a + b|v|^2}$ is indeed an equilibrium for (1). This type of equilibria for the (KWE) are called Rayleigh-Jeans distributions.

1.4. **Statement of the main results.** We now state the main results of this paper. We first prove global in time existence, uniqueness and stability of mild solutions, when the initial data are near vacuum:

Theorem 1.1. *Let $\alpha, \beta > 0$ and $0 < R \leq \frac{\alpha^{1/4}}{4\sqrt{6} K_{d,\beta}^{1/2}}$, where $K_{d,\beta} > 0$ is the constant given in (28). Let $f_0 \in \mathcal{M}_{\alpha,\beta}$ with $\|f_0\| \leq R$. Then equation (1) has a unique mild solution f satisfying the bound*

$$|f(t, x, v)| \leq 2R e^{-\alpha|x - tv|^2 - \beta|v|^2}, \quad \forall (t, x, v) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{13}$$

Additionally, if $f_0, g_0 \in \mathcal{M}_{\alpha,\beta}$ with $\|f_0\|, \|g_0\| \leq R$, and f, g are the corresponding mild solutions to (1), the following stability estimate holds:

$$|f(t, x, v) - g(t, x, v)| \leq 2\|f_0 - g_0\| e^{-\alpha|x - tv|^2 - \beta|v|^2} \quad \forall (t, x, v) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{14}$$

Remark 1.2. The uniqueness claimed holds in the class of solutions of (1) satisfying the bound (13).

When the initial data are non-negative, we show that the corresponding solution of (1) remains non-negative in time:

Theorem 1.3. *Let $\alpha, \beta > 0$ and $0 < R \leq \frac{\alpha^{1/4}}{4\sqrt{6}K_{d,\beta}^{1/2}}$, where $K_{d,\beta} > 0$ is the constant given in (28). Let $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ with $\|f_0\| \leq R$. Then, there exists a unique non-negative mild solution of (1) with*

$$0 \leq f(t,x,v) \leq 2Re^{-\alpha|x-tv|^2-\beta|v|^2}, \quad \forall (t,x,v) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Theorem 1.1 is proved in Section 4, while Theorem 1.3 is proved in Section 5.

Remark 1.4. As described in Subsection 1.3 a solution f of the inhomogeneous (KWE) (1) approximates a solution of the quantum Boltzmann equation (7) when $f >> 1$. However, in this paper the solutions we obtain are small. In the future, we plan addressing the well posedness of (1) for initial data which become large near $v = 0$, in order for these solutions to be relevant for (7) as well.

1.5. **Strategy of the proofs.** The main idea of the present paper is to connect equation (1) to the cubic part of the quantum Boltzmann equation for bosons with hard potential and no collisional averaging, and then employ techniques used in the context of kinetic theory of particles.

In particular, we show that (1) is connected to (8) follows: the collisional operator (2) is equivalent to the cubic part of (8) for $\gamma = d - 2$ and constant angular cross-section b. Thus for $d = 2$, (1) corresponds to Maxwell molecules, while for $d = 3$ to hard potentials.

After establishing the above connection in Lemma 3.1, and the appropriate a-priori bounds, we prove global well-posedness and stability using the contraction mapping principle. However, to prove existence of a non-negative solution for non-negative initial data, we use a more delicate argument which takes advantage of the monotonicity properties of the equation. We achieve that by employing a strong tool from the kinetic theory of particles, namely the Kaniel-Shinbrot iteration, which is an iterative scheme constructing monotone sequences of subsolutions and supersolutions which in turn converge to the solution of the nonlinear equation, as long as an appropriate beginning condition is satisfied.

The Kaniel-Shinbrot iteration was introduced for the first time by Kaniel and Shinbrot in [31] for local in time mild solutions to the Boltzmann equation and used by Illner and Shinbrot [29] to provide global in time mild solutions to the Boltzmann equation for small initial data. Later, it has been further used in the context of the Boltzmann equation as well as for Boltzmann-type equations such as inelastic Boltzmann equation, Boltzmann-Enskog equation, relativistic Boltzmann equation, binary-ternary Boltzmann equation, gas mixtures, see e.g. [24, 5, 46, 6, 47, 40, 38, 20, 2, 1, 45, 4, 23, 22]. Recently, the Kaniel-Shinbrot iteration has been used for the quantum Boltzmann equation for hard spheres in [37].

This is the first paper employing the Kaniel-Shinbrot iteration in the context of wave turbulence and in particular the inhomogeneous kinetic wave equation (1). We should mention that this technique relies on the the inhomogeneous nature of the problem, i.e. the fact that, for small enough initial data, the transport dominates the collisions under time evolution. Therefore, we would not expect such techniques to apply in the space homogeneous problem.

2. **Gain and loss operators and the notion of a mild solution.** In this section, we first write the kinetic wave equation in gain and loss form and introduce the notion of a mild solution to (1).

2.1. **Gain and loss operators.** Namely, notice that (1) can be equivalently written as

$$\partial_t f + v \cdot \nabla_x f = G(f,f,f,f) - L(f,f,f,f), \tag{15}$$

where the generalized gain operator G and loss operator L are given by

$$G(f,g,h,k)(t,x,v) = \int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) h_2 k_3 (f + g_1)\, dv_1\, dv_2\, dv_3,$$
$$L(f,g,h,k)(t,x,v) = \int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) f g_1 (h_2 + k_3)\, dv_1\, dv_2\, dv_3.$$

We note that the gain and loss operators are increasing with respect to non-negative inputs. Moreover, the gain operator is linear with respect to h, k and linear with respect to the vector (f, g). Similarly, the loss operator is linear with respect to f, g and linear with respect to the vector (h, k). In particular, we have the linearity decompositions

$$\begin{aligned} &G(f,g,h,k) - G(\tilde{f},\tilde{g},\tilde{h},\tilde{k}) \\ &= G(f,g,h-\tilde{h},k) + G(f,g,\tilde{h},k-\tilde{k}) + G(f-\tilde{f},g-\tilde{g},\tilde{h},\tilde{k}), \end{aligned} \tag{16}$$

and

$$\begin{aligned} &L(f,g,h,k) - L(\tilde{f},\tilde{g},\tilde{h},\tilde{k}) \\ &= L(f-\tilde{f},g,h,k) + L(\tilde{f},g-\tilde{g},h,k) + L(\tilde{f},\tilde{g},h-\tilde{h},k-\tilde{k}). \end{aligned} \tag{17}$$

Finally, we note that the loss term is local with respect to the first input i.e. we can write

$$L(f,g,h,k) = f R(g,h,k),$$

where

$$R(g,h,k) = \int_{\mathbb{R}^{3d}} \delta(\Sigma)\delta(\Omega) g_1 (h_2 + k_3)\, dv_1\, dv_2\, dv_3.$$

The operator R is also clearly increasing for non-negative inputs.

2.2. **The transport operator.** We now define the transport operator, which is a composition of function $g(t,x,v)$ with the Hamiltonian flow, and will be the fundamental operation for constructing mild solutions. Namely, we define $\# : C^0([0,\infty), L^1_{x,v}) \to C^0([0,\infty), L^1_{x,v})$ as:

$$g^{\#}(t,x,v) := g(t, x+tv, v). \tag{18}$$

This operator is an invertible isometry, since the free flow is measure preserving. Indeed, fixing arbitrary $t \geq 0$, we have

$$\|g^{\#}(t)\|_{L^1_{x,v}} = \int_{\mathbb{R}^d \times \mathbb{R}^d} |g(t,x+tv,v)|\, dx\, dv = \int_{\mathbb{R}^d \times \mathbb{R}^d} |g(t,x,v)|\, dx\, dv = \|g(t)\|_{L^1_{x,v}},$$

which after taking supremum in time implies that $\#$ is an isometry on $C^0([0,\infty), L^1_{x,v})$ i.e.

$$\|g^{\#}\|_{C^0([0,\infty),L^1_{x,v})} = \|g\|_{C^0([0,\infty),L^1_{x,v})}. \tag{19}$$

The inverse operator $-\# : C^0([0,\infty), L^1_{x,v}) \to C^0([0,\infty), L^1_{x,v})$ is clearly given by

$$g^{-\#}(t,x,v) := g(t, x-tv, v), \tag{20}$$

and is an isometry on $C^0([0,\infty),L^1_{x,v})$ as well.

2.3. **Notion of a mild solution.** Consider a formal solution f of (1). Using (15), the chain rule and integrating in time, we obtain

$$f^\#(t)=f_0+\int_0^t G^\#(f,f,f,f)(\tau)\,d\tau-\int_0^t L^\#(f,f,f,f)(\tau)\,d\tau,\quad t\geq 0, \tag{21}$$

where we denote $L^\#(f,g,h,k):=(L[f,g,h,k])^\#$, $G^\#(f,g,h,k):=(G[f,g,h,k])^\#$. One can easily verify that $L^\#[f,g,h,k]=f^\# R^\#[g,h,k]$, where $R^\#[g,h,k]:=(R[g,h,k])^\#$. Clearly, the operators $G^\#,L^\#,R^\#$ share the same monotonicity and linearity properties as G,L,R respectively. In particular, there hold the linearity decompositions:

$$\begin{aligned}&G^\#(f_1,g_1,h_1,k_1)-G^\#(f_2,g_2,h_2,k_2)\\&=G^\#(f_1,g_1,h_1-h_2,k_1)+G^\#(f_1,g_1,h_2,k_1-k_2)\\&\quad+G^\#(f_1-f_2,g_1-g_2,h_2,k_2),\end{aligned}\tag{22}$$

and

$$\begin{aligned}&L^\#(f_1,g_1,h_1,k_1)-L^\#(f_2,g_2,h_2,k_2)\\&=L^\#(f_1-f_2,g_1,h_1,k_1)+L^\#(f_2,g_1-g_2,h_1,k_1)\\&\quad+L^\#(f_2,g_2,h_1-h_2,k_1-k_2).\end{aligned}\tag{23}$$

Motivated by (21), we give the definition of a mild solution to (1) with as follows:

Definition 2.1. Let $\alpha,\beta>0$ and $f_0\in\mathcal{M}_{\alpha,\beta}$. We say that a function $f\in C^0([0,\infty),L^1_{x,v})$ is a mild solution of (1) with initial data f_0 if $f^\#\in\mathcal{S}_{\alpha,\beta}$, and the following integral equation holds

$$f^\#(t)=f_0+\int_0^t G^\#(f,f,f,f)(\tau)\,d\tau-\int_0^t L^\#(f,f,f,f)(\tau)\,d\tau,\quad t\geq 0. \tag{24}$$

3. **A-priori estimates.**

The goal of this section is to establish the basic global in time a-priori estimates, namely Proposition 3.4, which will be of fundamental importance for proving well-posedness and stability for equation (1).

We first provide a key computation which connects the wave kinetic kernel (2) with the cubic part of the quantum Boltzmann kernel (8) for $d=2,3$, and will allow us to use estimates used in the context of the Boltzmann equation, namely Lemma 3.2 and Lemma 3.3 in order to prove Proposition 3.4.

Lemma 3.1. *Let $v,v_1\in\mathbb{R}^d$, and denote*

$$I(v,v_1)=\int_{\mathbb{R}^{2d}}\delta(v+v_1-v_2-v_3)\delta(|v|^2+|v_1|^2-|v_2|^2-|v_3|^2)\,dv_2\,dv_3.$$

Then

$$I(v,v_1)=\frac{\omega_{d-1}}{2^d}|v-v_1|^{d-2},$$

where ω_{d-1} denotes the area of the $(d-1)$-dimensional unit sphere.

Proof. Substituting $v_3=v+v_1-v_2$, we get

$$I(v,v_1)=\int_{\mathbb{R}^d}\delta(|v_2|^2+|v+v_1-v_2|^2-|v|^2-|v_1|^2)\,dv_2.$$

But notice that

$$|v_2|^2+|v+v_1-v_2|^2-|v|^2-|v_1|^2 = 2\left(|v_2-\alpha|^2 - R^2\right), \quad \alpha = \frac{v+v_1}{2}, \quad R = \frac{|v-v_1|}{2}.$$

So

$$\begin{aligned} I(v,v_1) &= \int_{\mathbb{R}^d} \delta(2(|v_2-\alpha|^2 - R^2))\, dv_2 = \frac{\omega_{d-1}}{2}\int_0^\infty r^{d-1}\delta(r^2-R^2)\, dr \\ &= \frac{\omega_{d-1}}{2}\int_0^\infty \frac{\delta(r-R)+\delta(r+R)}{2R} r^{d-1}\, dr \\ &= \frac{\omega_{d-1}R^{d-2}}{4} = \frac{\omega_{d-1}}{2^d}|v-v_1|^{d-2}. \end{aligned}$$

□

Now, we present the two estimates which will be useful to us in the proof of Proposition 3.4 and have been used in the context of Boltzmann-type equations, see e.g. [29, 2, 4, 37]. For convenience of the reader, we provide the proofs below. The first estimate is on the time integral of a traveling Maxwellian:

Lemma 3.2. *Let $x_0, u_0 \in \mathbb{R}^d$, with $u_0 \neq 0$ and $\alpha > 0$. Then, the following estimate holds*

$$\int_0^\infty e^{-\alpha|x_0+\tau u_0|^2}\, d\tau \le \sqrt{\pi}\alpha^{-1/2}|u_0|^{-1}.$$

Proof. By triangle inequality, we have

$$\tau|u_0| - |x_0| \le |x_0+\tau u_0| \Rightarrow e^{-\alpha|x_0+\tau u_0|^2} \le e^{-\alpha(\tau|u_0|-|x_0|)^2}, \quad \forall \tau \ge 0.$$

Therefore integrating in τ, we obtain

$$\begin{aligned} \int_0^\infty e^{-\alpha|x_0-\tau u_0|^2}\, d\tau \le \int_{-\infty}^\infty e^{-\alpha(\tau|u_0|-|x_0|)^2}\, d\tau &\le \alpha^{-1/2}|u_0|^{-1}\int_{-\infty}^\infty e^{-y^2}\, dy \\ &\le \sqrt{\pi}\alpha^{-1/2}|u_0|^{-1}, \end{aligned}$$

and the estimate is proved. □

The second estimate is a convolution-type bound:

Lemma 3.3. *Let $q \in (-d, 0]$. Then for any $v \in \mathbb{R}^d$ there holds the uniform convolution estimate*

$$\int_{\mathbb{R}^d} |v-v_1|^q e^{-\beta|v_1|^2}\, dv_1 \le \beta^{-d/2}\pi^{d/2} + \frac{\omega_{d-1}}{d+q},$$

where ω_{d-1} denotes the area of the $(d-1)$-dimensional unit sphere.

Proof. Since $q \in (-d, 0]$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |v-v_1|^q e^{-\beta|v_1|^2}\, dv_1 &\le \int_{|v-v_1|>1} e^{-\beta|v_1|^2}\, dv_1 + \int_{|v-v_1|<1} |v-v_1|^q\, dv_1 \\ &\le \beta^{-d/2}\int_{\mathbb{R}^d} e^{-|x|^2}\, dx + \int_{|y|<1} |y|^q\, dy \\ &= \beta^{-d/2}\left(\int_{-\infty}^{+\infty} e^{-r^2}\, dr\right)^d + \omega_{d-1}\int_0^1 r^{d-1+q}\, dr \\ &= \beta^{-d/2}\pi^{d/2} + \frac{\omega_{d-1}}{d+q}. \end{aligned} \tag{25}$$

□

We are now ready to prove the necessary a-priori estimates on the gain and the loss operators:

Proposition 3.4. *Let $f,g,h,k\in C([0,\infty),L^1_{x,v})$. Then, there hold the estimates:*

$$\left|\left|\left|\int_0^t L^\#(f,g,h,k)(\tau)\,d\tau\right|\right|\right| \leq K_{d,\beta}\alpha^{-1/2}|||f^\#|||\cdot|||g^\#|||(|||h^\#|||+|||k^\#|||) \tag{26}$$

$$\left|\left|\left|\int_0^t G^\#(f,g,h,k)(\tau)\,d\tau\right|\right|\right| \leq K_{d,\beta}\alpha^{-1/2}|||h^\#|||\cdot|||k^\#|||(|||f^\#|||+|||g^\#|||), \tag{27}$$

where

$$K_{d,\beta}=\frac{\omega_{d-1}\sqrt{\pi}}{2^{d-1}}\left(\beta^{-d/2}\pi^{d/2}+\frac{\omega_{d-1}}{2d-3}\right), \tag{28}$$

and ω_{d-1} denotes the area of the $(d-1)$-unit sphere.

Proof. We estimate the gain first. Notice that by the definition of the norm of $\mathcal{M}_{\alpha,\beta}$ we have

$$|f^\#(t,x,v)|\leq e^{-\alpha|x|^2-\beta|v|^2}|||f^\#|||,$$

and the same is true for $g^\#,h^\#,k^\#$.

Now on the resonant manifold, there holds $v+v_1=v_2+v_3$ and $|v|^2+|v_1|^2=|v_2|^2+|v_3|^2$ which readily implies

$$\begin{aligned}|v-v_2|^2+|v-v_3|^2&=2|v|^2-2\langle v,v_2+v_3\rangle+|v_2|^2+|v_3|^2\\&=3|v|^2+|v_1|^2-2\langle v,v+v_1\rangle\\&=|v-v_1|^2.\end{aligned}$$

Hence

$$\begin{aligned}&|x+\tau(v-v_2)|^2+|x+\tau(v-v_3)|^2\\&=|x|^2+|x|^2+2\tau\langle x,2v-v_2-v_3\rangle+\tau^2(|v-v_2|^2+|v-v_3|^2)\\&=|x|^2+|x|^2+2\tau\langle x,v-v_1\rangle+\tau^2|v-v_1|^2\\&=|x|^2+|x+\tau(v-v_1)|^2.\end{aligned}$$

Then, using Fubini's theorem, we take

$$\begin{aligned}&\left|\int_0^t G^\#(f,g,h,k)(\tau,x,v)\,d\tau\right|\\&\leq|||h^\#|||\cdot|||k^\#|||(|||f^\#|||+|||g^\#|||)\int_0^t\int_{\mathbb{R}^{3d}}\delta(\Sigma)\delta(\Omega)\\&\quad\times e^{-\alpha(|x+\tau(v-v_2)|^2+|x+\tau(v-v_3)|^2}e^{-\beta(|v_2|^2+|v_3|^2)}\\&\quad\times\left(e^{-\alpha|x|^2-\beta|v|^2}+e^{-\alpha|x+t(v-v_1)|^2-\beta|v_1|^2}\right)dv_{1,2,3}\,d\tau\\&\leq 2|||h^\#|||\cdot|||k^\#|||(|||f^\#|||+|||g^\#|||)e^{-\alpha|x|^2-\beta|v|^2}\\&\quad\times\int_{\mathbb{R}^{3d}}\delta(\Sigma)\delta(\Omega)e^{-\beta|v_1|^2}\int_0^\infty e^{-\alpha|x+\tau(v-v_1)|^2}\,d\tau\,dv_{1,2,3}\\&\leq 2\sqrt{\pi}\alpha^{-1/2}|||h^\#|||\cdot|||k^\#|||(|||f^\#|||+|||g^\#|||)e^{-\alpha|x|^2-\beta|v|^2}\\&\quad\times\int_{\mathbb{R}^d}|v-v_1|^{-1}e^{-\beta|v_1|^2}I(v,v_1)\,dv_1 \qquad (29)\\&\leq\frac{\omega_{d-1}\sqrt{\pi}}{2^{d-1}}\alpha^{-1/2}|||h^\#|||\cdot|||k^\#|||(|||f^\#|||+|||g^\#|||)e^{-\alpha|x|^2-\beta|v|^2}\end{aligned}$$

$$\times \int_{\mathbb{R}^d} |v-v_1|^{d-3} e^{-\beta|v_1|^2}\, dv_1 \tag{30}$$
$$\leq \frac{\omega_{d-1}\sqrt{\pi}}{2^{d-1}} \left(\beta^{-d/2}\pi^{d/2} + \frac{\omega_{d-1}}{2d-3}\right) \alpha^{-1/2}$$
$$\times |||h^{\#}||| \cdot |||k^{\#}|||(|||f^{\#}||| + |||g^{\#}|||) e^{-\alpha|x|^2-\beta|v|^2} \tag{31}$$
$$:= K_{d,\beta}\alpha^{-1/2} |||h^{\#}||| \cdot |||k^{\#}|||(|||f^{\#}||| + |||g^{\#}|||) e^{-\alpha|x|^2-\beta|v|^2},$$

where to obtain (29) we use Lemma 3.2, to obtain (30) we use Lemma 3.1, and to obtain (31) we use Lemma 3.3 for $q = d-3 \in (-d, 0]$. Bringing the exponential to the other side and taking supremum over x, v and $t \geq 0$, we obtain estimate (27). The argument for the loss is similar but simpler since it does not require use of the conservation laws, so we omit the proof. □

4. **Well-posedness and stability for arbitrary data near vacuum.** In this section, we will use the a-priori estimates stated in Proposition 3.4 to prove Theorem 1.1 through a fixed point argument.

Proof of Theorem 1.1. We will use the contraction mapping principle. Define the closed set

$$E = \{g \in \mathcal{S}_{\alpha,\beta} \,:\, |||g||| \leq 2R\} \subset \mathcal{S}_{\alpha,\beta}.$$

Recalling the inverse transport operator $-\#$ given in (20), we define the operator $\mathcal{T} : E \to E$ by

$$\mathcal{T}g(t) = f_0 + \int_0^t G^{\#}(g^{-\#}, g^{-\#}, g^{-\#}, g^{-\#})(\tau)\, d\tau - \int_0^t L^{\#}(g^{-\#}, g^{-\#}, g^{-\#}, g^{-\#})(\tau)\, d\tau.$$

We first prove that $\mathcal{T}$ maps into E. Indeed, for $g \in E$, using triangle inequality and Proposition 3.4, we have

$$\begin{aligned} |||\mathcal{T}g||| &\leq \|f_0\| + \left|\left|\left| \int_0^t G^{\#}(g^{-\#}, g^{-\#}, g^{-\#}, g^{-\#})(\tau)\, d\tau \right|\right|\right| \\ &\quad + \left|\left|\left| \int_0^t L^{\#}(g^{-\#}, g^{-\#}, g^{-\#}, g^{-\#})(\tau)\, d\tau \right|\right|\right| \\ &\leq R + 4K_{d,\beta}\alpha^{-1/2}|||g|||^3 \\ &\leq R + 32K_{d,\beta}\alpha^{-1/2}R^3 \\ &= (1 + 32K_{d,\beta}\alpha^{-1/2}R^2)R \\ &< 2R, \end{aligned}$$

so $\mathcal{T} : E \to E$. Now, by the triangle inequality, the trilinearity decompositions (16)-(17), and Proposition 3.4, for $h, g \in E$, we obtain

$$\begin{aligned} |||\mathcal{T}h - \mathcal{T}g||| &\leq 4K_{d,\beta}\alpha^{-1/2}(|||h|||^2 + |||h||| \cdot |||g||| + |||g|||^2)|||h-g||| \\ &\leq 48K_{d,\beta}\alpha^{-1/2}R^2|||h-g||| \\ &\leq \frac{1}{2}|||h-g|||, \end{aligned} \tag{32}$$

thus $\mathcal{T} : E \to E$ is a contraction. By the contraction mapping principle, $\mathcal{T}$ has a unique fixed point $g \in E$. Then $f := g^{-\#}$ is clearly the unique mild solution to (1), corresponding to the initial data f_0.

To prove (33), let f, g be the solutions corresponding to f_0, g_0 respectively. Then, we have

$$f^{\#}(t)-g^{\#}(t)=f_0-g_0+\int_0^t \left(G^{\#}\left(f,f,f,f\right)(\tau)-G^{\#}\left(g,g,g,g\right)(\tau)\right)\,d\tau$$
$$-\int_0^t \left(L^{\#}\left(f,f,f,f\right)(\tau)-L^{\#}\left(g,g,g,g\right)(\tau)\right)\,d\tau \quad . \tag{33}$$

Now, by the triangle inequality, and an estimate similar to (32), we obtain

$$|||f^{\#}-g^{\#}||| \leq \|f_0-g_0\|+\frac{1}{2}|||f^{\#}-g^{\#}|||,$$

thus $|||f^{\#}-g^{\#}||| \leq 2\|f_0-g_0\|$, and (33) follows. □

5. **Existence of a non-negative solution.** In this section, we prove existence of a non-negative solution, when the initial data are non-negative. In order to achieve that, instead of using a fixed point argument, we will take advantage of the gain and loss form of the equation. We will rely on a strong tool from classical kinetic theory, which preserves the monotonicity properties of the equation, namely the Kaniel-Shinbrot iteration. More specifically, let $f_0 \in \mathcal{M}^+_{\alpha,\beta}$ and $(u_0^{\#}, l_0^{\#}) \in \mathcal{M}^+_{\alpha,\beta}\times\mathcal{M}^+_{\alpha,\beta}$. For $n\in\mathbb{N}$, consider the initial value problems:

$$\begin{cases} \dfrac{dl_n^{\#}}{dt}+l_n^{\#}R^{\#}(u_{n-1},u_{n-1},u_{n-1}) &= G^{\#}(l_{n-1},l_{n-1},l_{n-1},l_{n-1}),\\ l_n^{\#}(0)=f_0, \end{cases} \tag{34}$$

$$\begin{cases} \dfrac{du_n^{\#}}{dt}+u_n^{\#}R^{\#}(l_{n-1},l_{n-1},l_{n-1}) &= G^{\#}(u_{n-1},u_{n-1},u_{n-1},u_{n-1}),\\ u_n^{\#}(0)=f_0. \end{cases} \tag{35}$$

By basic ODE theory, solutions to (34)-(35) are given inductively by

$$\begin{aligned} l_n^{\#}(t)=f_0\exp\left(-\int_0^t R^{\#}(u_{n-1},u_{n-1},u_{n-1})(\tau)\,d\tau\right)&\\ +\int_0^t G^{\#}(l_{n-1},l_{n-1},l_{n-1},l_{n-1})(\tau)&\\ \times\exp\left(-\int_\tau^t R^{\#}(u_{n-1},u_{n-1},u_{n-1})(s)\,ds\right)\,d\tau,& \end{aligned} \tag{36}$$

and

$$\begin{aligned} u_n^{\#}(t)=f_0\exp\left(-\int_0^t R^{\#}(l_{n-1},l_{n-1},l_{n-1})(\tau)\,d\tau\right)&\\ +\int_0^t G^{\#}(u_{n-1},u_{n-1},u_{n-1},u_{n-1})(\tau)&\\ \times\exp\left(-\int_\tau^t R^{\#}(l_{n-1},l_{n-1},l_{n-1})(s)\,ds\right)\,d\tau.& \end{aligned} \tag{37}$$

Proposition 5.1. *Let $\alpha,\beta>0$, $f_0\in\mathcal{M}^+_{\alpha,\beta}$ and $(l_0^{\#},u_0^{\#})\in\mathcal{M}^+_{\alpha,\beta}\times\mathcal{M}^+_{\alpha,\beta}$. Let $l_1^{\#}, u_1^{\#}$ be the corresponding solutions to (36)-(37) for $n=1$, and assume that the following beginning condition holds for any $t\geq 0$:*

$$0\leq l_0^{\#}\leq l_1^{\#}(t)\leq u_1^{\#}(t)\leq u_0^{\#}\leq \frac{\alpha^{1/4}}{4K_{d,\beta}^{1/2}}e^{-\alpha|x|^2-\beta|v|^2}. \tag{38}$$

Then for all $n \in \mathbb{N}$ and $t \geq 0$, we have

$$\begin{aligned} 0 &\leq l_0^\# \leq l_1^\#(t) \leq \cdots \leq l_{n-1}^\#(t) \leq l_n^\#(t) \\ &\leq u_n^\#(t) \leq u_{n-1}^\#(t) \leq \cdots \leq u_1^\#(t) \\ &\leq u_0^\# \leq \frac{\alpha^{1/4}}{4K_{d,\beta}^{1/2}} e^{-\alpha|x|^2-\beta|v|^2}. \end{aligned} \tag{39}$$

Additionally, the sequences $l_n^\#$, $u_n^\#$ pointwise converge to a common limit $f^\#$ such that f is a non-negative mild solution of the (1) with initial data f_0.

Proof. We first will prove (39) inductively. For $n = 1$, (39) holds due to the beginning condition (38). Assume (39) holds for n, we will show it also holds for $n + 1$. It suffices to show

$$l_n^\#(t) \leq l_{n+1}^\#(t) \leq u_{n+1}^\#(t) \leq u_n^\#(t).$$

By the induction's assumption, and the monotonicity properties of $R^\#, G^\#$, we have

$$\begin{aligned} R^\#(u_n, u_n, u_n) &\leq R^\#(u_{n-1}, u_{n-1}, u_{n-1}), \\ G^\#(l_{n-1}, l_{n-1}, l_{n-1}, l_{n-1}) &\leq G^\#(l_n, l_n, l_n, l_n), \end{aligned}$$

so the solution's formula (36) implies $l_n^\# \leq l_{n+1}^\#$. Similarly,

$$\begin{aligned} R^\#(l_{n-1}, l_{n-1}, l_{n-1}) &\leq R^\#(l_n, l_n, l_n), \\ G^\#(u_n, u_n, u_n, u_n) &\leq G^\#(u_{n-1}, u_{n-1}, u_{n-1}, u_{n-1}), \end{aligned}$$

so (37) implies $u_{n+1}^\# \leq u_n^\#$, while

$$R^\#(l_n, l_n, l_n) \leq R^\#(u_n, u_n, u_n), \quad G^\#(l_n, l_n, l_n, l_n) \leq G^\#(u_n, u_n, u_n, u_n),$$

thus (36)-(37) imply $l_{n+1}^\# \leq u_{n+1}^\#$. Hence, (39) follows by induction.

Now, fixing $t \geq 0$. By (39), the sequence $(l_n^\#(t))_n$ is increasing and upper bounded so $l_n^\#(t) \nearrow l^\#(t)$, while the sequence $(u_n^\#(t))_n$ is decreasing and lower bounded so $u_n^\#(t) \searrow u^\#(t)$. Moreover, by (39) we have

$$0 \leq l^\#(t) \leq u^\#(t) \leq u_0^\#.$$

Integrating (36)-(37) in time and using the dominated convergence theorem to let $n \to \infty$, we obtain

$$l^\#(t) + \int_0^t l^\#(\tau) R^\#(u, u, u)(\tau)\, \tau = f_0 + \int_0^t G^\#(l, l, l, l)(\tau)\, d\tau, \tag{40}$$

$$u^\#(t) + \int_0^t u^\#(\tau) R^\#(l, l, l)(\tau)\, \tau = f_0 + \int_0^t G^\#(u, u, u, u)(\tau)\, d\tau. \tag{41}$$

Subtracting (40)-(41), using the facts that

$$l^\# R^\#(u, u, u) = L^\#(l, u, u, u), \quad u^\# R^\#(l, l, l) = L^\#(u, l, l, l),$$

and the triangle inequality, we obtain

$$\begin{aligned} |u^\#(t) - l^\#(t)| \leq& \int_0^t \left| G^\#(u, u, u, u)(\tau) - G^\#(l, l, l, l)(\tau) \right| d\tau \\ &+ \int_0^t \left| L^\#(l, u, u, u)(\tau) - L^\#(u, l, l, l)(\tau) \right| d\tau. \end{aligned} \tag{42}$$

By the trilinearity decomposition (16)-(17) of $L^\#$ and $G^\#$, we have the expansions

$$\begin{aligned}&G^\#(u,u,u,u)-G^\#(l,l,l,l)\\&=G^\#(u,u,u,u-l)+G^\#(u,u,u-l,l)+G^\#(u-l,u-l,l,l),\end{aligned}$$

and

$$\begin{aligned}&L^\#(l,u,u,u)-L^\#(u,l,l,l)\\&=L^\#(l-u,u,u,u)+L^\#(u,u-l,u,u)+L^\#(u,l,u-l,u-l).\end{aligned}$$

Then, (42), triangle inequality, (3.4), and inequality of (38) imply

$$\begin{aligned}|||u^\#-l^\#|||&\leq K_{d,\beta}\alpha^{-1/2}\left(6|||u^\#|||^2+4|||u^\#|||\cdot|||l^\#|||+2|||l^\#|||^2\right)|||u^\#-l^\#|||\\&\leq K_{d,\beta}\alpha^{-1/2}\|u_0^\#\|^2|||u^\#-l^\#|||\\&\leq\frac{3}{4}|||u^\#-l^\#|||,\end{aligned}$$

thus $u=l$. Clearly $f:=u=l$, is a mild solution of (1). □

Now, with the aid of Proposition 5.1, we will prove Theorem 1.3:

Proof of Theorem 1.3. To prove existence, we aim to use Proposition 5.1 for an appropriate choice of $(l_0^\#,u_0^\#)\in\mathcal{M}^+_{\alpha,\beta}\times\mathcal{M}^+_{\alpha,\beta}$. Namely, we define $l_0^\#=0$ and $u_0^\#=Ce^{-\alpha|x|^2-\beta|v|^2}$, where

$$C=2R(1-\sqrt{1-R^{-1}\|f_0\|})\geq 0, \tag{43}$$

which is well-defined since $\|f_0\|\leq R$. Moreover, notice that C satisfies the equation

$$\|f_0\|+\frac{1}{2}K_{d,\beta}^{1/2}\alpha^{-1/4}C^2=C, \tag{44}$$

and is estimated as follows:

$$C=2R(1-\sqrt{1-R^{-1}\|f_0\|})=\frac{2\|f_0\|}{1+\sqrt{1-R^{-1}\|f_0\|}}\leq 2\|f_0\|\leq 2R. \tag{45}$$

Solving (36)-(37) for $l_1^\#,u_1^\#$, we obtain

$$\begin{aligned}l_1^\#(t)&=f_0\exp\left(-\int_0^t R^\#(u_0,u_0,u_0)(\tau)\,d\tau\right),\\u_1^\#(t)&=f_0+\int_0^t G^\#(u_0,u_0,u_0,u_0)(\tau)\,d\tau.\end{aligned}$$

We clearly have $0=l_0^\#\leq l_1^\#(t)\leq u_1^\#(t)$. Moreover, by (45) we have

$$\|u_0^\#\|=C\leq 2R<\frac{\alpha^{1/4}}{4K_{d,\beta}^{1/2}}.$$

Therefore, in order to apply Proposition 5.1, it suffices to show that $u_1^\#(t)\leq u_0^\#$ for (38) to hold. Indeed, by Lemma 3.4, bound (27), and equation (44), we have

$$\begin{aligned}u_1^\#(t)&\leq e^{-\alpha|x|^2-\beta|v|^2}\left(\|f_0\|+2K_{d,\beta}\alpha^{-1/2}\|u_0^\#\|^3\right)\\&\leq e^{-\alpha|x|^2-\beta|v|^2}\left(\|f_0\|+\frac{1}{2}K_{d,\beta}^{1/2}\alpha^{-1/4}\|u_0^\#\|^2\right)\end{aligned}$$

$$
\begin{aligned}
&= e^{-\alpha|x|^2-\beta|v|^2}\left(\|f_0\| + \frac{1}{2}K_{d,\beta}^{1/2}\alpha^{-1/4}C^2\right) \\
&= Ce^{-\alpha|x|^2-\beta|v|^2} \\
&= u_0^{\#},
\end{aligned}
$$

so (38) is satisfied for this choice $(l_0^{\#}, u_0^{\#})$. Thus, existence of a non-negative mild solution to (1) is guaranteed by Proposition 5.1. Moreover, by (45), there holds the bound

$$f^{\#}(t) \leq u_0^{\#} = Ce^{-\alpha|x|^2-\beta|v|^2} \leq 2Re^{-\alpha|x|^2-\beta|v|^2}.$$

Thus, existence of such a solution is proved. Uniqueness follows immediately from Theorem 1.1. □

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REFERENCES

[1] R. Alonso, Existence of global solutions to the Cauchy problem for the inelastic Boltzmann equation with near-vacuum data, *Indiana University Mathematics Journal*, **58** (2009), 999-1022.

[2] R. Alonso and I. M. Gamba, Distributional and classical solutions to the Cauchy Boltzmann problem for soft potentials with integrable angular cross section, *J. Stat. Phys.*, **137** (2009), 1147-1165.

[3] I. Ampatzoglou, C. Collot and P. Germain, Derivation of the kinetic wave equation for quadratic dispersive problems in the inhomogeneous setting, to appear, *Amer. J. Math.*

[4] I. Ampatzoglou, I. M. Gamba, N. Pavlović and M. Tascović, Global well-posedness for a binary-ternary Boltzmann equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **39** (2022), 327-369.

[5] N. Bellomo and G. Toscani, On the Cauchy problem for the nonlinear Boltzmann equation: Global existence, uniqueness and asymptotic stability, *J. Math. Phys.*, **26** (1985), 334-338.

[6] N. Bellomo and G. Toscani, The Enskog-Boltzmann equation in the whole space $\mathbb{R}^3$: some global existence, uniqueness and stability results, *Comput. Math. Appl.*, **13** (1987), 851-859.

[7] T. Buckmaster, P. Germain, Z. Hani and J. Shatah, Onset of the kinetic wave turbulence description of the long-time behavior of the nonlinear Schrödinger equation, *Invent. Math.*, **225** (2021), 787-855.

[8] C. Collot, H. Dietert and P. Germain, Stability and cascades for the Kolmogorov-Zakharov spectrum of wave turbulence, *Arch. Rational Mech. Anal.*, **248** (2024).

[9] C. Collot and P. Germain, On the derivation of the homogeneous kinetic wave equation, preprint, 2019, arXiv:1912.10368.

[10] C. Collot and P. Germain, Derivation of the homogeneous kinetic wave equation: Longer time scales, preprint, 2020, arXiv:2007.03508.

[11] R. Davidson, *Methods in Nonlinear Plasma Theory*, First Edition, Academic Press, 1972.

[12] Y. Deng and Z. Hani, On the derivation of the wave kinetic equation for NLS, *Forum of Mathematics, Pi*, **9** (2021), 37 pp.

[13] Y. Deng and Z. Hani, Full derivation of the wave kinetic equation, *Invent. Math.*, **233** (2023), 543-724.

[14] Y. Deng and Z. Hani, Propagation of chaos and the higher order statistics in the wave kinetic theory, preprint, 2021, arXiv:2110.04565.

[15] Y. Deng and Z. Hani, Derivation of the wave kinetic equation: full range of scaling laws, preprint, 2021, arXiv:2301.07063.

[16] Y. Deng and Z. Hani, Long time justification of wave turbulence theory, preprint, 2021, arXiv:2311.10082.

[17] M. Escobedo and J. J. L. Velázquez, On the theory of weak turbulence for the nonlinear Schrödinger equation, *Mem. Amer. Math. Soc.*, **238** (2015).

[18] E. Faou, Linearized wave turbulence convergence results for three-wave systems, *Comm. Math. Phys.*, **378** (2020), 807-849.

[19] P. Germain, A. Ionescu and M.-B. Tran, Optimal local well-posedness theory for the kinetic wave equation, *J. Funct. Anal.*, **279** (2020), 108570, 28 pp.

[20] T. Goudon, Generalized invariant sets for the Boltzmann equation, *Math. Models Methods Appl. Sci.*, **7** (1997), 457-476.

[21] *Guide to Wave Analysis and Forecasting*, Secretariat of the World Meteorological Organization, Geneva, Switzerland, 1998.

[22] S.-Y. Ha and S.-E. Noh, Global existence and stability of mild solutions to the inelastic Boltzmann system for gas mixtures, *Quart. Appl. Math.*, **68** (2010), 671-699.

[23] S.-Y. Ha, S.-E. Noh and S.-B. Yun, Global existence and stability of mild solutions to the Boltzmann system for gas mixtures, *Quart. Appl. Math.*, **65** (2007), 757-779.

[24] K. Hamdache, Existence in the large and asymptotic behavior for the Boltzmann equation, *Japan J. Appl. Math.*, **2** (1985), 1-15.

[25] Z. Hani, J. Shatah and H. Zhu, Inhomogeneous turbulence for Wick NLS, preprint, 2023, arXiv:2309.12037.

[26] A. Hannani, M. Rosenzweig, G. Staffilani and M.-B. Tran, Inhomogeneous turbulence for Wick NLS, preprint, 2022, arXiv:2210.17445.

[27] K. Hasselmann, On the non-linear energy transfer in a gravity-wave spectrum. I. General theory, *J. Fluid Mech.*, **12** (1962), 481-500.

[28] K. Hasselmann, On the non-linear energy transfer in a gravity-wave spectrum. II. Conservation theorems; wave particle analogy; irreversibility, *J. Fluid Mech.*, **15** (1963), 273-281.

[29] R. Illner and M. Shinbrot, The Boltzmann Equation: Global Existence for a Rare Gas in an Infinite Vacuum, *Comm. Math. Phys.*, **95** (1984), 217-226.

[30] P. A. Janssen, Progress in ocean wave forecasting, *Journal of Computational Physics*, **227** (2008), 3572-3594.

[31] S. Kaniel and M. Shinbrot, The Boltzmann Equation: Uniqueness and local existence, *Comm. Math. Phys.*, **58** (1978), 65-84.

[32] J. Lukkarinen and H. Spohn, Weakly nonlinear Schrödinger equation with random initial data, *Invent. Math.*, **183** (2011), 79-188.

[33] A. Menegaki, L^2-stability near equilibrium for the 4 waves kinetic equation, *Kinet. Relat. Models*, (2023).

[34] S. Merino-Aceituno, Isotropic wave turbulence with simplified kernels: Existence, uniqueness, and mean-field limit for a class of instantaneous coagulation-fragmentation processes, *J. Math. Phys.*, **57** (2016), 121501, 26 pp.

[35] S. Nazarenko, *Wave Turbulence*, Lecture Notes in Physics, 825. Springer, Heidelberg, 2011.

[36] A. C. Newell and B. Rumpf, *Advances in Wave Turbulence*, Volume 83 of World Sci. Ser. Nonlinear Sci. Ser. A Monogr. Treatises, 1-51. World Sci. Publ., Hackensack, NJ, 2013

[37] Z. Ouyang and L. Wu, On the quantum Boltzmann equation near Maxwellian and vacuum, *J. Differential Equations*, **316** (2022), 471-551.

[38] A. Palczewski and G. Toscani, Global solution of the Boltzmann equation for rigid spheres and initial data close to a local Maxwellian, *J. Math. Phys.*, **30** (1989), 2445-2450.

[39] R. Peierls, Zur kinetischen theorie der Wärmeleitung in kristallen, *Annalen der Physik*, **395** (1929), 1055-1101.

[40] J. Polewczak, Classical solution of the nonlinear Boltzmann equation in all $\mathbb{R}^3$: asymptotic behavior of solutions, *J. Stat. Phys.*, **50** (1988), 611-632.

[41] D.V.Semikov and I.I.Tkachev Condensation of Bose in the kinetic regime, *Phys. Rev. D*, **55** (1997), 489–502.

[42] H. Spohn, Derivation of the transport equation for electrons moving through random impurities, *J. Stat. Phys.*, **17** (1977), 385-412.

[43] H. Spohn, The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics, *J. Stat. Phys.*, **124** (2006), 1041-1104.

[44] G. Staffilani and M.-B. Tran, On the wave turbulence theory for stochastic and random multidimensional KdV type equations, preprint, 2021, arXiv:2106.09819.

[45] R. M. Strain, Global Newtonian limit for the relativistic Boltzmann equation near vacuum, *SIAM J. Math. Anal.*, **42** (2010), 1568-1601.

[46] G. Toscani, On the nonlinear Boltzmann equation in unbounded domains, *Arch. Rational Mech. Anal.*, **95** (1986), 37-49.

[47] G. Toscani, Global solution of the initial value problem for the Boltzmann equation near a local Maxwellian, *Arch. Rational Mech. Anal.*, **102** (1988), 231-241.

[48] V. Zakharov and V. L'vov, The statistical description of nonlinear wave fields, *RaF*, **18** (1975), 1470-1487.

[49] V. E. Zakharov, V. S. L'vov and G. Falkovich *Kolmogorov Spectra of Turbulence I: Wave Turbulence*, Springer Science and Business Media, 2012.

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