

Analysis of an embedded-hybridizable discontinuous Galerkin method for Biot's consolidation model

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Abstract We present an embedded-hybridizable discontinuous Galerkin finite element method for the total pressure formulation of the quasi-static poroelasticity model. Although the displacement and the Darcy velocity are approximated by discontinuous piece-wise polynomials, $H(\text{div})$ -conformity of these unknowns is enforced by Lagrange multipliers. The semi-discrete problem is shown to be stable and the fully discrete problem is shown to be well-posed. Additionally, space-time a priori error estimates are derived, and confirmed by numerical examples, that show that the proposed discretization is free of volumetric locking.

Keywords Biot's Consolidation Model · Poroelasticity · Discontinuous Galerkin · Finite Element Methods · Hybridization

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1 Introduction

Poroelasticity models are systems of partial differential equation that describe the physics of deformable porous media saturated by fluids. They were originally developed for geophysics applications in petroleum engineering but nowadays they

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are also widely used for biomechanical modeling. The first poroelasticity models were derived by Biot [4, 5]. Since then, mathematical properties and numerical methods for these models have been widely studied. Here we give a brief literature review.

Early studies on linear poroelasticity models include well-posedness analysis and finite element discretizations for quasi-static [44, 43] and dynamic [39, 40, 49] models. For quasi-static models with incompressible elastic grains, Murad et al. [30] observed spurious pressure oscillations of certain finite element discretizations for small time and studied their asymptotic behavior. Phillips and Wheeler [33] connected these pressure oscillations to volumetric locking due to incompressibility of the displacement. They further developed numerical methods in [32, 33] coupling mixed methods and discontinuous Galerkin methods that do not show pressure oscillations. Yi [46, 47, 48] proposed numerical methods coupling mixed and non-conforming finite elements that are also free of pressure oscillations. An analysis to address the volumetric locking problem for poroelasticity was first presented in [26] adopting mixed methods for linear elasticity. Various numerical methods avoiding this locking problem have since been studied using nonconforming or stabilized finite elements [27, 38, 21, 6], the total pressure formulation [28, 31, 14], and exactly divergence-free finite element spaces [22, 19]. A non-symmetric interior penalty discontinuous Galerkin method was numerically shown to be locking free for small enough penalty parameter in [37].

Discontinuous Galerkin methods are known to be computationally expensive. A remedy for this was provided by Cockburn et al. [10] by introducing the hybridizable discontinuous Galerkin (HDG) framework for elliptic problems. Indeed, element unknowns can be eliminated from the problem resulting in a global problem for facet unknowns only. The number of globally coupled degrees-of-freedom can be reduced even further using the embedded discontinuous Galerkin (EDG) framework [11, 17]; where the HDG method uses a discontinuous trace approximation, the EDG method uses a continuous trace approximation. HDG, and related hybrid high-order (HHO), methods have recently been introduced for the poroelasticity problem [15, 25, 7]. These discretizations consider the primal bilinear form for linear elasticity. In contrast, in this paper we adopt the total pressure formulation [28, 31] and present novel HDG and EDG-HDG methods for the quasi-static poroelasticity models. (It is possible to also consider an EDG method for the poroelasticity model, however, such a discretization is sub-optimal.) The total pressure formulation provides a natural decoupling of the linear elasticity and Darcy equations in the incompressible limit. Indeed, in this limit our discretizations reduce to the exactly divergence-free HDG and EDG-HDG discretizations of [34, 36] for the Stokes problem and the hybridized formulation of [3] for the Darcy problem. We further remark that the total pressure formulation has been applied also in the context of magma/mantle dynamics problems [23, 24] where it was shown to be advantageous in the context of coupled physics problems beyond quasi-static poroelasticity problems.

We present an analysis of the proposed HDG and EDG-HDG methods in which we show that the space-time discretizations are well-posed. We further determine an a priori error estimate for all unknowns that is robust in the incompressible limit and for arbitrarily small specific storage coefficient. We remark that the standard approach of analyzing time-dependent problems is to use discrete Grönwall inequalities. However, this results in error bounds with a coefficient that grows

exponentially in time. We present an alternative approach that avoids this exponential term.

The remainder of this paper is organized as follows. We present Biot's consolidation model in Section 2. The HDG and EDG-HDG methods for Biot's model is presented in Section 3 together with a stability proof for the semi-discrete problem. Well-posedness and a priori error estimates for the fully discrete problem are shown in Section 4. The analysis is verified by numerical examples in Section 5 and conclusions are drawn in Section 6.

2 Biot's consolidation model

To introduce Biot's consolidation model, let us introduce the following notation. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polygonal domain with a boundary partitioned as $\partial\Omega = \bar{\Gamma}_P \cup \bar{\Gamma}_F$ and $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_T$, where $\Gamma_P \cap \Gamma_F = \emptyset$, $|\Gamma_P| > 0$, $\Gamma_D \cap \Gamma_T = \emptyset$, and $|\Gamma_D| > 0$. We denote the unit outward normal to $\partial\Omega$ by n and we denote by $I = (0, T]$ the time interval of interest.

Let $f : \Omega \times I \rightarrow \mathbb{R}^d$ be a given body force and let $g : \Omega \times I \rightarrow \mathbb{R}$ be a given source/sink term. Furthermore, let $\kappa > 0$ be a scalar constant that represents the permeability of the porous media, $c_0 \geq 0$ the specific storage coefficient, and $0 < \alpha < 1$ the Biot-Willis constant. Denoting Young's modulus of elasticity by E and Poisson's ratio by ν , in the case of plane strain, the Lamé constants are given by $\lambda = E\nu/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$.

Biot's consolidation model describes a system of equations for the displacement of the porous media, $u : \Omega \times I \rightarrow \mathbb{R}^d$, and the pore pressure of the fluid $p : \Omega \times I \rightarrow \mathbb{R}$. Denoting by $\sigma = 2\mu\varepsilon(u) + \lambda\nabla \cdot u\mathbb{I} - \alpha p\mathbb{I}$ the total Cauchy stress, where $\mathbb{I}$ is the $d \times d$ -dimensional identity matrix, this model is given by

$$-\nabla \cdot \sigma = f, \quad \partial_t(c_0 p + \alpha \nabla \cdot u) - \nabla \cdot (\kappa \nabla p) = g, \quad \text{in } \Omega \times I. \quad (2.1)$$

Following [28], by introducing the total pressure $p_T := -\lambda \nabla \cdot u + \alpha p$ and the Darcy velocity $z := -\kappa \nabla p$, we may write Biot's consolidation model also as:

$$-\nabla \cdot 2\mu\varepsilon(u) + \nabla p_T = f \quad \text{in } \Omega \times I, \quad (2.2a)$$

$$-\nabla \cdot u - \lambda^{-1}(p_T - \alpha p) = 0 \quad \text{in } \Omega \times I, \quad (2.2b)$$

$$\partial_t(c_0 p + \lambda^{-1}\alpha(\alpha p - p_T)) + \nabla \cdot z = g \quad \text{in } \Omega \times I, \quad (2.2c)$$

$$\kappa^{-1}z + \nabla p = 0 \quad \text{in } \Omega \times I, \quad (2.2d)$$

which will be the formulation studied in this article. Noting that $\sigma = 2\mu\varepsilon(u) - p_T\mathbb{I}$, we close the model by imposing the following boundary and initial conditions:

$$u = 0 \quad \text{on } \Gamma_D \times I, \quad (2.3a)$$

$$p = 0 \quad \text{on } \Gamma_P \times I, \quad (2.3b)$$

$$z \cdot n = 0 \quad \text{on } \Gamma_F \times I, \quad (2.3c)$$

$$\sigma n = 0 \quad \text{on } \Gamma_T \times I, \quad (2.3d)$$

$$p(x, 0) = p_0(x) \quad \text{in } \Omega, \quad (2.3e)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega. \quad (2.3f)$$

In the remainder of this article we assume that c_0 , μ^{-1} , κ , and μ are bounded above by a constant C . We furthermore assume that there exists a ν_* such that $0 < \nu_* \leq \nu < 0.5$ on Ω . As a consequence, $C_*\mu \leq \lambda$ with $C_* = 2\nu^*/(1 - 2\nu^*)$.

3 The embedded-hybridizable discontinuous Galerkin method

3.1 Notation

On a Lipschitz domain D in $\mathbb{R}^d$, we denote by $W^{l,p}(D)$ the usual Sobolev spaces for $l \geq 0$ and $1 \leq p \leq \infty$ (see, for example, [1]). When $p = 2$, we define on $H^l(D) = W^{l,2}(D)$ the norm $\|\cdot\|_{l,D}$ and semi-norm $|\cdot|_{l,D}$. We note that $L^2(D) = H^0(D)$ is the Lebesgue space of square integrable functions with norm $\|\cdot\|_D = \|\cdot\|_{0,D}$ and inner product $(\cdot, \cdot)_D$. Vector-valued function spaces will be denoted by $[L^2(D)]^d$ and $[H^l(D)]^d$. The L^2 -inner product over a surface $S \subset \mathbb{R}^{d-1}$ will be denoted by $\langle \cdot, \cdot \rangle_S$.

Let X be a Banach space and $J = (0, T]$, $T > 0$ a time interval. We denote by $C^0(J; X)$ the space of continuous functions $f : J \rightarrow X$, which is equipped with the norm $\|f\|_{C^0(J; X)} := \sup_{t \in \bar{J}} \|f(t)\|_X$. By $C^k(J; X)$, $k \geq 0$, we denote the space of continuous functions $f : J \rightarrow X$ such that $\partial_t^i f \in C^0(J, X)$ for $1 \leq i \leq k$. For $1 \leq p < \infty$, $W^{k,p}(J; X)$ is defined to be the closure of $C^k(J; X)$ with respect to the norm

$$\|f\|_{W^{k,p}(J; X)}^p := \int_0^T \sum_{i=0}^k \|\partial_t^i f(t)\|_X^p dt.$$

We note that for $k = 0$, $W^{k,p}(J; X) = L^p(J; X)$.

Let $\mathcal{T}_h$ be a family of shape-regular simplicial triangulations of the domain Ω . We will denote the diameter of an element $K \in \mathcal{T}_h$ by h_K , the meshsize by $h := \max_{K \in \mathcal{T}_h} h_K$, and the sets of interior facets and facets that lie on Γ_D , Γ_P , Γ_F , and Γ_T by, respectively, $\mathcal{F}_h$, $\mathcal{F}_h^D$, $\mathcal{F}_h^P$, $\mathcal{F}_h^F$, and $\mathcal{F}_h^T$. The set of all facets is denoted by $\mathcal{F}_h$ and their union is denoted by Γ_0 . On the boundary of an element K , we denote by n_K the outward unit normal vector, although, where no confusion will occur we drop the subscript K . On the mesh and skeleton we define the inner products

$$\begin{aligned} (\phi, \psi)_\Omega &:= \sum_{K \in \mathcal{T}_h} (\phi, \psi)_K, & \langle \phi, \psi \rangle_{\partial \mathcal{T}_h} &:= \sum_{K \in \mathcal{T}_h} \langle \phi, \psi \rangle_{\partial K}, & \text{if } \phi, \psi \text{ are scalar,} \\ (\phi, \psi)_\Omega &:= \sum_{i=1}^d (\phi_i, \psi_i)_\Omega, & \langle \phi, \psi \rangle_{\partial \mathcal{T}_h} &:= \sum_{i=1}^d \langle \phi_i, \psi_i \rangle_{\partial \mathcal{T}_h}, & \text{if } \phi, \psi \text{ are vector-valued.} \end{aligned}$$

The norms induced by these inner products are denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_{\partial \mathcal{T}_h}$, respectively.

Sets of polynomials of degree not larger than $l \geq 0$ defined on, respectively, an element $K \in \mathcal{T}_h$ and a facet $F \in \mathcal{F}_h$ will be denoted by $\mathbb{P}_l(K)$ and $\mathbb{P}_l(F)$. As

approximation spaces we then use:

$$\begin{aligned}
V_h &:= \{v_h \in [L^2(\Omega)]^d : v_h \in [\mathbb{P}_k(K)]^d, \forall K \in \mathcal{T}_h\}, \\
\bar{V}_h &:= \{\bar{v}_h \in [L^2(\Gamma_0)]^d : \bar{v}_h \in [\mathbb{P}_k(F)]^d \forall F \in \mathcal{F}_h, \bar{v}_h = 0 \text{ on } \Gamma_D\}, \\
Q_h &:= \{q_h \in L^2(\Omega) : q_h \in \mathbb{P}_{k-1}(K), \forall K \in \mathcal{T}_h\}, \\
\bar{Q}_h &:= \{\bar{q}_h \in L^2(\Gamma_0) : \bar{q}_h \in \mathbb{P}_k(F) \forall F \in \mathcal{F}_h\}, \\
\bar{Q}_h^0 &:= \{\bar{\psi}_h \in \bar{Q}_h : \bar{\psi}_h = 0 \text{ on } \Gamma_P\}.
\end{aligned} \tag{3.1}$$

Element and facet function pairs will be denoted by boldface, for example,

$$\begin{aligned}
\mathbf{v}_h &= (v_h, \bar{v}_h) \in \mathbf{V}_h := V_h \times \bar{V}_h, & \mathbf{q}_h &= (q_h, \bar{q}_h) \in \mathbf{Q}_h := Q_h \times \bar{Q}_h, \\
\boldsymbol{\psi}_h &= (\psi_h, \bar{\psi}_h) \in \mathbf{Q}_h^0 := Q_h \times \bar{Q}_h^0,
\end{aligned}$$

and it will also be useful to define $\mathbf{X}_h := \mathbf{V}_h \times \mathbf{Q}_h \times V_h \times \mathbf{Q}_h^0$.

Remark 3.1 The HDG method seeks an approximation in $\mathbf{X}_h$ with $V_h, \bar{V}_h, Q_h, \bar{Q}_h$, and $\bar{Q}_h^0$ defined in eq. (3.1). If $\bar{V}_h$ is replaced by $\bar{V}_h \cap C^0(\Gamma_0)$ then we obtain the EDG-HDG method. The analysis in this paper holds for both the HDG and EDG-HDG methods. For notational purposes, in the analysis, $\mathbf{X}_h$ and $\mathbf{V}_h$ will refer both to the HDG and EDG-HDG spaces.

For the analysis of the HDG and EDG-HDG methods we assume that the exact solution is such that:

$$\begin{aligned}
u(t) &\in V := \{v \in [H^1(\Omega)]^d : v|_{\Gamma_D} = 0\} \cap [H^2(\Omega)]^d, \\
p_T(t) &\in Q := H^1(\Omega), \\
z(t) &\in Z := \{v \in [H^1(\Omega)]^d : v \cdot n = 0 \text{ on } \Gamma_F\}, \\
p(t) &\in Q^0 := \{q \in H^1(\Omega) : q|_{\Gamma_P} = 0 \text{ on } \Gamma_P\} \cap H^2(\Omega).
\end{aligned}$$

Denoting by $\bar{V}, \bar{Q}, \bar{Z}$, and $\bar{Q}^0$ the trace spaces of, respectively, V, Q, Z , and Q^0 to the mesh skeleton, we introduce the extended spaces

$$\begin{aligned}
\mathbf{V}(h) &:= \mathbf{V}_h + V \times \bar{V}, & Z(h) &:= V_h + Z, \\
\mathbf{Q}(h) &:= \mathbf{Q}_h + Q \times \bar{Q}, & \mathbf{Q}^0(h) &:= \mathbf{Q}_h^0 + Q^0 \times \bar{Q}^0.
\end{aligned}$$

Norms on the extended spaces $\mathbf{V}(h), \mathbf{Q}(h)$, and $\mathbf{Q}^0(h)$ are defined as:

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{V}}^2 &:= \|\varepsilon(v)\|_{\Omega}^2 + \sum_{K \in \mathcal{T}_h} h_K^{-1} \|v - \bar{v}\|_{\partial K}^2 & \forall \mathbf{v} \in \mathbf{V}(h), \\
\|\mathbf{v}\|_{\mathbf{V}'}^2 &:= \|\mathbf{v}\|_{\mathbf{V}}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{2,K}^2, & \forall \mathbf{v} \in \mathbf{V}(h), \\
\|\mathbf{q}_h\|_{\mathbf{Q}}^2 &:= \|q_h\|_{\Omega}^2 + \sum_{K \in \mathcal{T}_h} h_K \|\bar{q}_h\|_{\partial K}^2 & \forall \mathbf{q}_h \in \mathbf{Q}(h).
\end{aligned}$$

To conclude this section, we remark that $C > 0$ will denote a constant independent of h and the model parameters.

3.2 The semi-discrete problem

In this section, we present the semi-discrete problem and provide an energy estimate for this discretization. The fully-discrete problem is presented in Section 3.3 which is analysed in Section 4.

The semi-discrete HDG method for Biot's consolidation model eqs. (2.2) and (2.3) is given by: Find $(\mathbf{u}_h, z_h) \in C^0(I; \mathbf{V}_h \times V_h)$ and $(\mathbf{p}_{T,h}, \mathbf{p}_h) \in C^1(I; \mathbf{Q}_h \times \mathbf{Q}_h^0)$ such that for all $(\mathbf{v}_h, \mathbf{q}_{T,h}, w_h, \mathbf{q}_h) \in \mathbf{V}_h \times \mathbf{Q}_h \times V_h \times \mathbf{Q}_h^0$:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_{T,h}) = (f, v_h)_\Omega, \quad (3.2a)$$

$$b_h(\mathbf{u}_h, \mathbf{q}_{T,h}) - (\lambda^{-1}(p_{T,h} - \alpha p_h), q_{T,h})_\Omega = 0, \quad (3.2b)$$

$$(\kappa^{-1} z_h, w_h)_\Omega + b_h((w_h, 0), \mathbf{p}_h) = 0, \quad (3.2c)$$

$$(\partial_t(c_0 p_h + \lambda^{-1} \alpha (\alpha p_h - p_{T,h})), q_h)_\Omega - b_h((z_h, 0), \mathbf{q}_h) = (g, q_h)_\Omega, \quad (3.2d)$$

where

$$a_h(\mathbf{u}, \mathbf{v}) := (2\mu \varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_\Omega + \sum_{K \in \mathcal{T}} \langle \frac{2\beta\mu}{h_K} (u - \bar{u}), v - \bar{v} \rangle_{\partial K} \quad (3.3a)$$

$$- \langle 2\mu \varepsilon(\mathbf{u}) \mathbf{n}, v - \bar{v} \rangle_{\partial \mathcal{T}} - \langle 2\mu \varepsilon(\mathbf{v}) \mathbf{n}, u - \bar{u} \rangle_{\partial \mathcal{T}},$$

$$b_h(\mathbf{v}, \mathbf{q}) := - (q, \nabla \cdot \mathbf{v})_\Omega + \langle \bar{q}, (v - \bar{v}) \cdot \mathbf{n} \rangle_{\partial \mathcal{T}}. \quad (3.3b)$$

To analyze the HDG and EDG-HDG methods, let us recall some properties of the bilinear forms a_h and b_h . It was shown in [34, Lemma 4.2] and [9, Lemma 2] that there exist constants C and $\beta_0 > 0$ such that for $\beta > \beta_0$,

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \geq C\mu \|\mathbf{v}_h\|_v^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (3.4)$$

Additionally, a_h satisfies the following continuity result [9, Lemma 3]:

$$a_h(\mathbf{u}, \mathbf{v}) \leq C\mu \|\mathbf{u}\|_{v'} \|\mathbf{v}\|_{v'} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}(h). \quad (3.5)$$

The bilinear form b_h satisfies the following stability results:

$$\inf_{\mathbf{q}_h \in \mathbf{Q}_h} \sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, \mathbf{q}_h)}{\|\mathbf{v}_h\|_v \|\mathbf{q}_h\|_q} \geq C, \quad (3.6a)$$

$$\inf_{\mathbf{q}_h \in \mathbf{Q}_h^0} \sup_{0 \neq w_h \in V_h} \frac{b_h((w_h, 0), \mathbf{q}_h)}{\|w_h\|_\Omega \|\mathbf{q}_h\|_q} \geq C, \quad (3.6b)$$

where the first inequality was shown in [35, Lemma 1] and [36, Lemma 8] and the second is proven in Appendix A. Continuity of the bilinear form b_h was established in [34]:

$$|b_h(\mathbf{v}, \mathbf{q})| \leq C \|\mathbf{v}\|_v \|\mathbf{q}\|_q \quad \forall \mathbf{v} \in \mathbf{V}(h), \mathbf{q} \in \mathbf{Q}(h). \quad (3.7)$$

Lemma 3.1 (Consistency) *Let (u, p_T, z, p) be a solution to eqs. (2.2) and (2.3) and let $\bar{u}$, $\bar{p}_T$, and $\bar{p}$ be the traces of, respectively, u , p_T , and p on the mesh skeleton. Then $(\mathbf{u}, \mathbf{p}_T, z, \mathbf{p})$ satisfies eq. (3.2).*

Proof The proof is standard and follows by integration by parts, smoothness of the solution to eqs. (2.2) and (2.3), single-valuedness of $\bar{v}_h$ and $\bar{q}_h$ on element boundaries, and using that $\bar{v}_h = 0$ on Γ_D and $\bar{q}_h = 0$ on Γ_P . $\square$

The following theorem now shows energy stability of the semi-discrete problem.

Theorem 3.1 (Stability) *Suppose that $(\mathbf{u}_h, \mathbf{p}_{Th}, z_h, \mathbf{p}_h) \in C^1(I; \mathbf{X}_h)$ is a solution to eq. (3.2) with $f \in W^{1,1}(I; L^2(\Omega))$ and $g \in L^2(I; L^2(\Omega))$. Let $X(t) \geq 0$ and $Y(t) \geq 0$ be defined by:*

$$\begin{aligned} X(t)^2 &= a_h(\mathbf{u}_h(t), \mathbf{u}_h(t)) \\ &\quad + (\lambda^{-1} (p_{Th}(t) - \alpha p_h(t)), p_{Th}(t) - \alpha p_h(t))_{\Omega} + (c_0 p_h(t), p_h(t))_{\Omega}, \\ Y(t)^2 &= (\kappa^{-1} z_h(t), z_h(t))_{\Omega}. \end{aligned}$$

Then, there exists $C > 0$, independent of $t > 0$, such that

$$\begin{aligned} X(t) &\leq X(0) + C \left[\mu^{-1/2} \max_{0 \leq s \leq t} \|f(s)\|_{\Omega} \right. \\ &\quad \left. + \mu^{-1/2} \int_0^t \|\partial_t f(s)\|_{\Omega} ds + \kappa^{-1/2} \left(\int_0^t \|g(s)\|_{\Omega}^2 ds \right)^{\frac{1}{2}} \right], \end{aligned} \quad (3.8a)$$

and

$$\begin{aligned} \left(\int_0^t Y(s)^2 ds \right)^{\frac{1}{2}} &\leq C \left[X(0) + \mu^{-1/2} \max_{0 \leq s \leq t} \|f(s)\|_{\Omega} \right. \\ &\quad \left. + \mu^{-1/2} \int_0^t \|\partial_t f(s)\|_{\Omega} ds + \kappa^{-1/2} \left(\int_0^t \|g(s)\|_{\Omega}^2 ds \right)^{\frac{1}{2}} \right]. \end{aligned} \quad (3.8b)$$

Proof We first note that by the inf-sup condition eq. (3.6b), eq. (3.2c), and the Cauchy–Schwarz inequality,

$$\begin{aligned} \|p_h(t)\|_{\Omega} &\leq C \sup_{0 \neq w_h \in V_h} \frac{|b_h((w_h, 0), \mathbf{p}_h)|}{\|w_h\|_{\Omega}} \\ &\leq C \sup_{0 \neq w_h \in V_h} \frac{|(\kappa^{-1} z_h, w_h)|}{\|w_h\|_{\Omega}} \leq C \kappa^{-1} \|z_h(t)\|_{\Omega}. \end{aligned} \quad (3.9)$$

Now, in eqs. (3.2a), (3.2c) and (3.2d) set $(\mathbf{v}_h, w_h, \mathbf{q}_h) = (\partial_t \mathbf{u}_h, z_h, \mathbf{p}_h)$. Take the time derivative of eq. (3.2b) and set $\mathbf{q}_{Th} = -\mathbf{p}_{Th}$. Adding the resulting equations we find:

$$\frac{1}{2} \frac{d}{dt} X(t)^2 + Y(t)^2 = (f(t), \partial_t u_h(t))_{\Omega} + (g(t), p_h(t))_{\Omega}. \quad (3.10)$$

Integrating eq. (3.10) in time from 0 to t results in

$$\frac{1}{2} (X(t)^2 - X(0)^2) + \int_0^t Y(s)^2 ds = \int_0^t [(f(s), \partial_t u_h(s))_{\Omega} + (g(s), p_h(s))_{\Omega}] ds.$$

Integration by parts, Young's inequality and eq. (3.9) imply

$$\begin{aligned} \frac{1}{2} (X(t)^2 - X(0)^2) + \int_0^t Y(s)^2 ds &\leq (f(t), u_h(t))_{\Omega} - (f(0), u_h(0))_{\Omega} \\ &\quad - \int_0^t (\partial_t f(s), u_h(s))_{\Omega} ds \\ &\quad + C \int_0^t \|g(s)\|_{\Omega}^2 ds + \frac{1}{2} \int_0^t Y(s)^2 ds. \end{aligned}$$

Coercivity of a_h eq. (3.4) and a discrete Korn's inequality imply that

$$\|u_h(t)\|_{\Omega} \leq C\mu^{-1/2} a_h(\mathbf{u}_h, \mathbf{u}_h)^{1/2} \leq C\mu^{-1/2} X(t).$$

Therefore, by the Cauchy-Schwarz inequality, for any $t \geq 0$,

$$\begin{aligned} X(t)^2 + \int_0^t Y(s)^2 \, ds &\leq X(0)^2 + C\mu^{-1/2} \left(\|f(t)\|_{\Omega} X(t) + \|f(0)\|_{\Omega} X(0) \right) \\ &\quad + C\mu^{-1/2} \int_0^t \|\partial_t f(s)\|_{\Omega} X(s) \, ds + C \int_0^t \|g(s)\|_{\Omega}^2 \, ds. \end{aligned} \quad (3.11)$$

To obtain eq. (3.8a), we may assume, without loss of generality, that

$$\max_{0 \leq s \leq t} X(s) = X(t) > 0. \quad (3.12)$$

Note that if eq. (3.12) does not hold, then there exists a t_M such that $\max_{0 \leq s \leq t} X(s) = X(t_M)$ for $0 \leq t_M < t$. The estimate eq. (3.8a) for $X(t_M)$ then implies eq. (3.8a) for $X(t)$. From eqs. (3.11) and (3.12), we then find

$$\begin{aligned} &X(t)^2 + \int_0^t Y(s)^2 \, ds \\ &\leq \left(X(0) + C\mu^{-1/2} (\|f(t)\|_{\Omega} + \|f(0)\|_{\Omega} + \int_0^t \|\partial_t f(s)\|_{\Omega} \, ds) \right) X(t) \\ &\quad + C \int_0^t \|g(s)\|_{\Omega}^2 \, ds. \end{aligned} \quad (3.13)$$

Define $\alpha(t) := (C \int_0^t \|g(s)\|_{\Omega}^2 \, ds)^{1/2} > 0$. If $\alpha(t) \leq X(t)$, dividing eq. (3.13) by $X(t)$ implies

$$X(t) \leq X(0) + C\mu^{-1/2} (\|f(t)\|_{\Omega} + \|f(0)\|_{\Omega} + \int_0^t \|\partial_t f(s)\|_{\Omega} \, ds) + \alpha(t).$$

Note that this inequality holds trivially if $X(t) < \alpha(t)$. Proceeding, we find

$$\begin{aligned} X(t) &\leq X(0) + C\mu^{-1/2} \left(2 \max_{0 \leq s \leq t} \|f(s)\|_{\Omega} + \int_0^t \|\partial_t f(s)\|_{\Omega} \, ds \right) + \alpha(t) \\ &= X(0) + C\mu^{-1/2} \left(2 \max_{0 \leq s \leq t} \|f(s)\|_{\Omega} + \int_0^t \|\partial_t f(s)\|_{\Omega} \, ds \right) + C \left(\int_0^t \|g(s)\|_{\Omega}^2 \, ds \right)^{1/2}, \end{aligned}$$

so that eq. (3.8a) follows. Equation (3.8b) follows by combining eq. (3.8a) and eq. (3.13). $\square$

3.3 The fully discrete problem

To define the fully discrete scheme, let $\{t^n\}_{0 \leq n \leq N}$ be a uniform partition of I and let $\Delta t > 0$ be the corresponding time step. We will denote the value of a function $f(t)$ at $t = t^n$ by $f^n := f(t^n)$. For a sequence $\{f^n\}_{n \geq 1}$, $d_t f^n := \frac{f^n - f^{n-1}}{\Delta t}$ defines a first order difference operator. Note that we use the superscript n to denote the time level. This is not to be confused with the normal vector n . Using Backward Euler time stepping, the fully discrete problem reads: Find $(\mathbf{u}_h^{n+1}, \mathbf{p}_{Th}^{n+1}, z_h^{n+1}, \mathbf{p}_h^{n+1}) \in \mathbf{X}_h$, with $n \geq 0$, such that

$$a_h(\mathbf{u}_h^{n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_{Th}^{n+1}) = (f^{n+1}, \mathbf{v}_h)_\Omega, \quad (3.14a)$$

$$b_h(\mathbf{u}_h^{n+1}, \mathbf{q}_{Th}) + \lambda^{-1} (\alpha p_h^{n+1} - p_{Th}^{n+1}, \mathbf{q}_{Th})_\Omega = 0, \quad (3.14b)$$

$$(\kappa^{-1} z_h^{n+1}, w_h)_\Omega + b_h((w_h, 0), \mathbf{p}_h^{n+1}) = 0, \quad (3.14c)$$

$$\begin{aligned} & \frac{1}{\Delta t} ((c_0 p_h^{n+1}, q_h)_\Omega + \lambda^{-1} (\alpha p_h^{n+1} - p_{Th}^{n+1}, \alpha q_h)_\Omega) - b_h((z_h^{n+1}, 0), \mathbf{q}_h), \\ & = \frac{1}{\Delta t} ((c_0 p_h^n, q_h)_\Omega + \lambda^{-1} (\alpha p_h^n - p_{Th}^n, \alpha q_h)_\Omega) + (g^{n+1}, q_h)_\Omega, \end{aligned} \quad (3.14d)$$

for all $(\mathbf{v}_h, \mathbf{q}_{Th}, w_h, \mathbf{q}_h) \in \mathbf{X}_h$. We first show that eq. (3.14) is well-posed.

Theorem 3.2 *There exists a unique solution to eq. (3.14).*

Proof It is sufficient to show that if the data is equal to zero then the solution is zero. As such, suppose that $f^{n+1} = 0$, $g^{n+1} = 0$, $\mathbf{p}_{Th}^n = \mathbf{0}$, and $\mathbf{p}_h^n = \mathbf{0}$. Then, setting $\mathbf{v}_h = \mathbf{u}_h^{n+1}$, $\mathbf{q}_{Th} = -\mathbf{p}_{Th}^{n+1}$, $w_h = \Delta t z_h^{n+1}$, and $\mathbf{q}_h = \Delta t \mathbf{p}_h^{n+1}$ in eq. (3.14) and adding the equations, we obtain:

$$a_h(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + c_0 \|p_h^{n+1}\|_\Omega^2 + \lambda^{-1} \|\alpha p_h^{n+1} - p_{Th}^{n+1}\|_\Omega^2 + \kappa^{-1} \Delta t \|z_h^{n+1}\|_\Omega^2 = 0.$$

Coercivity of a_h eq. (3.4), positivity of κ and λ , and nonnegativity of c_0 directly imply that $\mathbf{u}_h^{n+1} = \mathbf{0}$ and $z_h^{n+1} = 0$. Substituting $\mathbf{u}_h^{n+1} = \mathbf{0}$ in eq. (3.14a), $\mathbf{p}_{Th}^{n+1} = 0$ follows from the inf-sup condition eq. (3.6a). This then implies $p_h^{n+1} = 0$ since $\alpha, \lambda > 0$.

Using a BDM local lifting of the normal trace [12, Proposition 2.10], there exists $\tilde{w}_h \in V_h$ such that $\langle \tilde{w}_h \cdot n, \bar{p}_h^{n+1} \rangle_{\partial \mathcal{T}_h} = \|\bar{p}_h^{n+1}\|_{\partial \mathcal{T}_h}^2$. Setting $z_h^{n+1} = 0$, $p_h^{n+1} = 0$ and choosing $w_h = \tilde{w}_h$ in eq. (3.14c), we obtain $\bar{p}_h^{n+1} = 0$. This completes the proof. $\square$

Let us also note that the fully-discrete scheme eq. (3.14) results in divergence-conforming solutions for the displacement $\mathbf{u}_h^n$ and velocity z_h^n . To see this, set $\mathbf{v}_h = \mathbf{0}$, $\mathbf{q}_h = \mathbf{0}$, $w_h = 0$, and $\mathbf{q}_{Th} = 0$ in eq. (3.2) and note that since $u_h^n \cdot n \in P_k(F)$ and $\bar{u}_h^n = 0$ on Γ_D ,

$$[[u_h^n \cdot n]] = 0, \quad \forall x \in F, \quad \forall F \in \mathcal{F} \setminus \mathcal{F}_T, \quad (3.15a)$$

$$u_h^n \cdot n = \bar{u}_h^n \cdot n, \quad \forall x \in F, \quad \forall F \in \mathcal{F}_T. \quad (3.15b)$$

Similarly, by setting $\mathbf{v}_h = \mathbf{0}$, $\mathbf{q}_{Th} = \mathbf{0}$, $w_h = 0$, and $q_h = 0$ in eq. (3.2), and noting that $z_h^n \cdot n \in P_k(F)$, we find that

$$[[z_h^n \cdot n]] = 0, \quad \forall x \in F, \quad \forall F \in \mathcal{F} \setminus \mathcal{F}_P. \quad (3.16)$$

4 A priori error estimates

To facilitate the a priori error analysis, we introduce various interpolation operators. First, let $\Pi_V : [H^1(\Omega)]^d \rightarrow V_h$ be the BDM interpolation operator [8, Section III.3], [18, Lemma 7] with the following interpolation estimate:

$$\|z - \Pi_V z\|_K \leq Ch_K^\ell \|z\|_{\ell,K}, \quad 1 \leq \ell \leq k+1. \quad (4.1)$$

The elliptic interpolation operator, $\boldsymbol{\Pi}_V^{\text{ell}} := (\Pi_V^{\text{ell}}, \bar{\Pi}_V^{\text{ell}}) : [H^1(\Omega)]^d \rightarrow \mathbf{V}_h$ is defined by:

$$a_h(\boldsymbol{\Pi}_V^{\text{ell}} u, \mathbf{v}_h) = a_h((u, u), \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

Standard a priori error estimate theory for second order elliptic equations imply

$$a_h(\mathbf{u} - \boldsymbol{\Pi}_V^{\text{ell}} u, \mathbf{u} - \boldsymbol{\Pi}_V^{\text{ell}} u)^{\frac{1}{2}} \leq C\mu^{1/2} h_K^{\ell-1} \|u\|_{\ell,\Omega} \quad 1 \leq \ell \leq k+1. \quad (4.2)$$

By Π_Q , $\bar{\Pi}_Q$, and $\bar{\Pi}_{Q^0}$ we denote the L^2 -projections onto, respectively, Q_h and the trace spaces $\bar{Q}_h$ and $\bar{Q}_h^0$. Given the interpolation/projection operators, the numerical initial data is set by imposing the interpolation/projection of continuous initial data as follows:

$$\begin{aligned} & ((u_h^0, \bar{u}_h^0), (p_{Th}^0, \bar{p}_{Th}^0), z_h^0, (p_h^0, \bar{p}_h^0)) \\ &= ((\Pi_V^{\text{ell}} u(0), \bar{\Pi}_V^{\text{ell}} u(0)), (\Pi_Q p_{Th}(0), \bar{\Pi}_Q p_{Th}(0)), \Pi_V z(0), (\Pi_{Q^0} p(0), \bar{\Pi}_{Q^0} p(0))). \end{aligned} \quad (4.3)$$

In the error analysis it will be convenient to split the error into approximation and interpolation errors:

$$\omega - \omega_h = e_\omega^I - e_\omega^h, \quad \omega = u, p_T, z, p, \quad (4.4a)$$

$$\xi|_{\Gamma^0} - \bar{\xi}_h = \bar{e}_\xi^I - \bar{e}_\xi^h, \quad \xi = u, p_T, p, \quad (4.4b)$$

where

$$\begin{aligned} e_u^I &= u - \Pi_V^{\text{ell}} u, & e_u^h &= u_h - \Pi_V^{\text{ell}} u, & e_z^I &= z - \Pi_V z, & e_z^h &= z_h - \Pi_V z, \\ e_p^I &= p - \Pi_Q p, & e_p^h &= p_h - \Pi_Q p, & e_{p_T}^I &= p_T - \Pi_Q p_T, & e_{p_T}^h &= p_{Th} - \Pi_Q p_T, \end{aligned}$$

and where

$$\begin{aligned} \bar{e}_u^I &= u|_{\Gamma^0} - \bar{\Pi}_V^{\text{ell}} u, & \bar{e}_u^h &= \bar{u}_h - \bar{\Pi}_V^{\text{ell}} u, & \bar{e}_p^I &= p|_{\Gamma^0} - \bar{\Pi}_{Q^0} p, & \bar{e}_p^h &= \bar{p}_h - \bar{\Pi}_{Q^0} p, \\ \bar{e}_{p_T}^I &= p_T|_{\Gamma^0} - \bar{\Pi}_Q p_T, & \bar{e}_{p_T}^h &= \bar{p}_{Th} - \bar{\Pi}_Q p_T. \end{aligned}$$

Following the convention introduced earlier in this paper, we use boldface notation for element/facet error pairs, i.e., $\mathbf{e}_\xi^I = (e_\xi^I, \bar{e}_\xi^I)$ and $\mathbf{e}_\xi^h = (e_\xi^h, \bar{e}_\xi^h)$ for $\xi = u, p_T, p$.

It will also be useful to introduce the following error estimates: let ψ be a regular enough function defined on $[0, T] \times D$, for some domain $D \subset \mathbb{R}^d$, then, as a consequence of Taylor's theorem (see Appendix B),

$$\sum_{m=1}^n \Delta t \|\partial_t \psi^m - d_t \psi^m\|_{0,D} \leq \Delta t \|\partial_{tt} \psi\|_{L^1(I; L^2(D))}, \quad (4.5a)$$

$$\Delta t \sum_{m=1}^n \|d_t e_\psi^{I,m}\|_{0,D} \leq Ch^l \|\partial_t \psi\|_{L^1(I; H^l(D))}, \quad \psi = p, p_T. \quad (4.5b)$$

Lemma 4.1 *Let $\{A_i\}_i$, $\{B_i\}_i$, $\{E_i\}_i$, and $\{D_i\}_i$ be nonnegative sequences. Suppose these sequences satisfy*

$$A_n^2 + \sum_{i=0}^n B_i^2 \leq A_0^2 + \sum_{i=1}^n E_i A_i + \sum_{i=0}^n D_i, \quad (4.6)$$

for all $n \geq 0$. Then for any $n \geq 0$,

$$A_n \leq A_0 + \sum_{i=1}^n E_i + \left(\sum_{i=0}^n D_i \right)^{1/2}, \quad (4.7a)$$

$$\left(\sum_{i=0}^n B_i^2 \right)^{1/2} \leq C \left(A_0 + \sum_{i=1}^n E_i + \left(\sum_{i=0}^n D_i \right)^{1/2} \right), \quad (4.7b)$$

with $C > 0$ independent of n .

Proof First, note that eq. (4.6) and eq. (4.7a) directly imply eq. (4.7b). It is therefore sufficient to prove eq. (4.7a). Similar to the assumption made in the proof of Theorem 3.1, we assume without loss of generality that $0 < A_n = \max_{0 \leq i \leq n} A_i$. If $A_n \leq \left(\sum_{i=0}^n D_i \right)^{1/2}$, then eq. (4.7a) is satisfied trivially. On the other hand, if $A_n > \left(\sum_{i=0}^n D_i \right)^{1/2}$, then eq. (4.6) implies

$$A_n^2 + \sum_{i=0}^n B_i^2 \leq A_0 A_n + A_n \sum_{i=0}^n E_i + A_n \left(\sum_{i=0}^n D_i \right)^{1/2}.$$

The result now follows by dividing by A_n . $\square$

Before addressing the main result (Theorem 4.1) of this section, we first determine the error equation.

Lemma 4.2 (Error equation) *Suppose that $\{(\mathbf{u}_h^n, \mathbf{p}_{Th}^n, z_h^n, \mathbf{p}_h^n)\}_{n \geq 1}$ is the solution of (3.14) with the numerical initial data (4.3). The approximation and interpolation errors satisfy*

$$\begin{aligned} & a_h(\mathbf{e}_u^{h,n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{e}_{p_T}^{h,n+1}) \\ & + b_h(d_t \mathbf{e}_u^{h,n+1}, \mathbf{q}_{Th}) + \lambda^{-1}(d_t(\alpha \mathbf{e}_p^{h,n+1} - \mathbf{e}_{p_T}^{h,n+1}), \alpha \mathbf{q}_h + \mathbf{q}_{Th})_\Omega \\ & + \kappa^{-1}(\mathbf{e}_z^{h,n+1}, w_h)_\Omega - b_h((\mathbf{e}_z^{h,n+1}, 0), \mathbf{q}_h) \\ & + b_h((w_h, 0), \mathbf{e}_p^{h,n+1})_\Omega + (c_0 d_t \mathbf{e}_p^{h,n+1}, \mathbf{q}_h)_\Omega \\ & = b_h(d_t \mathbf{e}_u^{I,n+1}, \mathbf{q}_{Th}) + \kappa^{-1}(\mathbf{e}_z^{I,n+1}, w_h)_\Omega + (c_0(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}), \mathbf{q}_h)_\Omega \\ & + \lambda^{-1}(\alpha(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}) - (\partial_t p_T^{n+1} - d_t \Pi_Q p_T^{n+1}), \alpha \mathbf{q}_h)_\Omega, \end{aligned}$$

for any $(\mathbf{v}_h, \mathbf{q}_{Th}, w_h, \mathbf{q}_h) \in \mathbf{X}_h$.

Proof By Lemma 3.1, we can substitute $(u^{n+1}, p_T^{n+1}, z^{n+1}, p^{n+1})$, the solution of eqs. (2.2) and (2.3) evaluated at $t = t^{n+1}$, into eq. (3.2). Then, subtracting

eq. (3.14), applying d_t to the second equation of the resulting set of equations, and adding all the equations, we obtain using eq. (4.4):

$$\begin{aligned}
& a_h(e_u^{h,n+1}, v_h) + b_h(v_h, e_{p_T}^{h,n+1}) \\
& + b_h(d_t e_u^{h,n+1}, q_{Th}) + \lambda^{-1}(d_t(\alpha e_p^{h,n+1} - e_{p_T}^{h,n+1}), \alpha q_h + q_{Th})_\Omega \\
& + \kappa^{-1}(e_z^{h,n+1}, w_h)_\Omega - b_h((e_z^{h,n+1}, 0), q_h) \\
& + b_h((w_h, 0), e_p^{h,n+1})_\Omega + (c_0 d_t e_p^{h,n+1}, q_h)_\Omega \\
& = a_h(e_u^{I,n+1}, v_h) + b_h(v_h, e_{p_T}^{I,n+1}) \\
& + b_h(d_t e_u^{I,n+1}, q_{Th}) + \lambda^{-1}(d_t(\alpha e_p^{I,n+1} - e_{p_T}^{I,n+1}), \alpha q_h + q_{Th})_\Omega \\
& + \kappa^{-1}(e_z^{I,n+1}, w_h)_\Omega - b_h((e_z^{I,n+1}, 0), q_h) \\
& + b_h((w_h, 0), e_p^{I,n+1}) + (c_0(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}), q_h)_\Omega \\
& + \lambda^{-1}(\alpha(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}) - (\partial_t p_T^{n+1} - d_t \Pi_Q p_T^{n+1}), \alpha q_h)_\Omega.
\end{aligned}$$

The result follows by noting that: $a_h(e_u^{I,n+1}, v_h) = 0$ by definition of Π_V^{ell} ; $b_h(v_h, e_{p_T}^{I,n+1}) = b_h((w_h, 0), e_p^{I,n+1}) = 0$ because Π_Q and $\bar{\Pi}_Q$ are L^2 projections into Q_h and $\bar{Q}_h$, respectively, and $\nabla \cdot V_h = Q_h$; $b_h((e_z^{I,n+1}, 0), q_h) = 0$ by the commuting property of the BDM interpolation operator $\Pi_Q \nabla \cdot v = \nabla \cdot \Pi_V v$ for $v \in [H^1(\Omega)]^d$, the $H(\text{div})$ -conformity of $e_z^{I,n+1}$, and the boundary conditions on Γ_F ; and $\lambda^{-1}(d_t(\alpha e_p^{I,n+1} - e_{p_T}^{I,n+1}), \alpha q_h + q_{Th})_\Omega = 0$ because Π_Q is the L^2 -projection into Q_h . $\square$

We are now ready to prove an a priori error estimate for the HDG and EDG-HDG methods eq. (3.14).

Theorem 4.1 *Let (u, p_T, z, p) be a solution to eqs. (2.2) and (2.3) on the time interval $I = (0, T]$ and let $\bar{u}$, $\bar{p}_T$, and $\bar{p}$ be the traces of, respectively, u , p_T , and p on the mesh skeleton. Let $(\mathbf{u}_h^n, \mathbf{p}_{Th}^n, z_h^n, \mathbf{p}_h^n) \in \mathbf{X}_h$ be the solution to eq. (3.14). Suppose the numerical initial data is imposed according to eq. (4.3). The following error estimates hold:*

$$\begin{aligned}
& c_0^{1/2} \|p^n - p_h^n\|_\Omega + \lambda^{-1/2} \|\alpha(p^n - p_h^n) - (p_T^n - p_{Th}^n)\|_\Omega \\
& + \mu^{1/2} \|\mathbf{u}^n - \mathbf{u}_h^n\|_v + \kappa^{-1/2} \left(\sum_{i=1}^n \Delta t \|z^i - z_h^i\|_\Omega^2 \right)^{1/2} \leq C_1 \Delta t + C_2 h^l, \quad (4.8a)
\end{aligned}$$

$$\mu^{-1/2} \|\mathbf{p}_T^n - \mathbf{p}_{Th}^n\|_q \leq C_1 \Delta t + C_3 h^l, \quad (4.8b)$$

where

$$\begin{aligned}
C_1 &= C \max \{c_0^{1/2}, \lambda^{-1/2}\} \|p, p_T\|_{W^{2,1}(I; L^2(\Omega))}, \\
C_2 &= C \left(\mu^{1/2} \|u\|_{W^{1,1}(I; H^{l+1}(\Omega))} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right. \\
& \quad \left. + \max \{c_0^{1/2}, \lambda^{-1/2}\} \|p, p_T\|_{W^{1,1}(I; H^l(\Omega))} \right), \\
C_3 &= C \left(\mu^{1/2} \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \max \{c_0^{1/2}, \mu^{-1/2}\} \|p, p_T\|_{W^{1,1}(I; H^l(\Omega))} \right. \\
& \quad \left. + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right).
\end{aligned}$$

Proof Choose $\mathbf{v}_h = d_t \mathbf{e}_u^{h,n+1}$, $\mathbf{q}_{Th} = -\mathbf{e}_{p_T}^{h,n+1}$, $w_h = e_z^{h,n+1}$, and $\mathbf{q}_h = \mathbf{e}_p^{h,n+1}$ in the error equation in Lemma 4.2. Then,

$$\begin{aligned} & a_h(\mathbf{e}_u^{h,n+1}, d_t \mathbf{e}_u^{h,n+1}) + \kappa^{-1}(e_z^{h,n+1}, e_z^{h,n+1})_\Omega \\ & + (c_0 d_t \mathbf{e}_p^{h,n+1}, \mathbf{e}_p^{h,n+1})_\Omega + \lambda^{-1}(d_t(\alpha \mathbf{e}_p^{h,n+1} - \mathbf{e}_{p_T}^{h,n+1}), \alpha \mathbf{e}_p^{h,n+1} - \mathbf{e}_{p_T}^{h,n+1})_\Omega \\ = & -b_h(d_t \mathbf{e}_u^{I,n+1}, \mathbf{e}_{p_T}^{h,n+1}) + \kappa^{-1}(e_z^{I,n+1}, e_z^{h,n+1})_\Omega \\ & + (c_0(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}), \mathbf{e}_p^{h,n+1})_\Omega \\ & + \lambda^{-1}(\alpha(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}) - (\partial_t p_T^{n+1} - d_t \Pi_Q p_T^{n+1}), \alpha \mathbf{e}_p^{h,n+1})_\Omega. \end{aligned}$$

Using $a(a-b) = \frac{1}{2}(a^2 + (b-a)^2 - b^2) \geq \frac{1}{2}(a^2 - b^2)$ and multiplying both sides of the resulting inequality by Δt , we arrive at

$$\begin{aligned} & \frac{c_0}{2} \left(\|\mathbf{e}_p^{h,n+1}\|_\Omega^2 - \|\mathbf{e}_p^{h,n}\|_\Omega^2 \right) + \frac{\lambda^{-1}}{2} \left(\|\alpha \mathbf{e}_p^{h,n+1} - \mathbf{e}_{p_T}^{h,n+1}\|_\Omega^2 - \|\alpha \mathbf{e}_p^{h,n} - \mathbf{e}_{p_T}^{h,n}\|_\Omega^2 \right) \\ & + \frac{1}{2} (a_h(\mathbf{e}_u^{h,n+1}, \mathbf{e}_u^{h,n+1}) - a_h(\mathbf{e}_u^{h,n}, \mathbf{e}_u^{h,n})) + \kappa^{-1} \Delta t \|e_z^{h,n+1}\|_\Omega^2 \\ \leq & -\Delta t b_h(d_t \mathbf{e}_u^{I,n+1}, \mathbf{e}_{p_T}^{h,n+1}) + \Delta t \kappa^{-1} (e_z^{I,n+1}, e_z^{h,n+1})_\Omega \\ & + \Delta t (c_0(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}), \mathbf{e}_p^{h,n+1})_\Omega \\ & + \lambda^{-1} \Delta t (\alpha(\partial_t p^{n+1} - d_t \Pi_Q p^{n+1}) - (\partial_t p_T^{n+1} - d_t \Pi_Q p_T^{n+1}), \alpha \mathbf{e}_p^{h,n+1})_\Omega \\ =: & I_1^{n+1} + I_2^{n+1} + I_3^{n+1} + I_4^{n+1}. \end{aligned} \tag{4.9}$$

We define

$$\begin{aligned} A_i^2 &:= \frac{c_0}{2} \|\mathbf{e}_p^{h,i}\|_\Omega^2 + \frac{\lambda^{-1}}{2} \|\alpha \mathbf{e}_p^{h,i} - \mathbf{e}_{p_T}^{h,i}\|_\Omega^2 + \frac{1}{2} a_h(\mathbf{e}_u^{h,i}, \mathbf{e}_u^{h,i}), \\ B_i^2 &:= \frac{\kappa^{-1}}{2} \Delta t \|e_z^{h,i}\|_\Omega^2, \end{aligned}$$

so that eq. (4.9) can be written as

$$A_{n+1}^2 + 2B_{n+1}^2 \leq A_n^2 + I_1^{n+1} + I_2^{n+1} + I_3^{n+1} + I_4^{n+1}. \tag{4.10}$$

We proceed by bounding I_1^i , I_2^i , I_3^i , and I_4^i , starting with I_1^i .

Restricting the error equation in Lemma 4.2 for $\mathbf{v}_h$ with general index i , we find the error equation

$$a_h(\mathbf{e}_u^{h,i}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{e}_{p_T}^{h,i}) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

By eq. (3.6a), the above equality, eq. (3.5), the equivalence between $\|\cdot\|_v$ and $\|\cdot\|_{v'}$ [45, eq. (5.5)], and eq. (3.4),

$$\begin{aligned} C \|\mathbf{e}_{p_T}^{h,i}\|_q &\leq \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, \mathbf{e}_{p_T}^{h,i})}{\|\mathbf{v}_h\|_v} = \sup_{\mathbf{0} \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{-a_h(\mathbf{e}_u^{h,i}, \mathbf{v}_h)}{\|\mathbf{v}_h\|_v} \\ &\leq C \mu \|\mathbf{e}_u^{h,i}\|_v \leq C \mu^{1/2} (a_h(\mathbf{e}_u^{h,i}, \mathbf{e}_u^{h,i}))^{1/2}, \end{aligned} \tag{4.11}$$

implying that

$$\mu^{-1/2} \|\mathbf{e}_{p_T}^{h,i}\|_q \leq C A_i. \tag{4.12}$$

We may now bound I_1^i using eq. (3.7) and eq. (4.12):

$$I_1^i \leq \Delta t |b_h(d_t e_u^{I,i}, e_{p_T}^{h,i})| \leq C \Delta t \|d_t e_u^{I,i}\|_v \|e_{p_T}^{h,i}\|_q \leq C \Delta t \mu^{1/2} \|d_t e_u^{I,i}\|_v A_i. \quad (4.13)$$

A bound for I_2^i follows from the Cauchy–Schwarz and Young’s inequalities:

$$I_2^i \leq \kappa^{-1} \Delta t \|e_z^{I,i}\|_\Omega \|e_z^{h,i}\|_\Omega \leq \frac{\kappa^{-1}}{2} \Delta t \|e_z^{I,i}\|_\Omega^2 + B_i^2.$$

Using the Cauchy–Schwarz and triangle inequalities, we bound I_3^i as follows:

$$\begin{aligned} I_3^i &\leq c_0 \Delta t \|\partial_t p^i - d_t \Pi_Q p^i\|_\Omega \|e_p^{h,i}\|_\Omega \\ &\leq c_0 \Delta t (\|\partial_t p^i - d_t p^i\|_\Omega + \|d_t e_p^{I,i}\|_\Omega) \|e_p^{h,i}\|_\Omega \\ &\leq (2c_0)^{1/2} \Delta t (\|\partial_t p^i - d_t p^i\|_\Omega + \|d_t e_p^{I,i}\|_\Omega) A_i. \end{aligned}$$

To estimate I_4^i , we first derive an auxiliary result. By the assumption that $C_* \mu \leq \lambda$ (see Section 2),

$$\begin{aligned} \lambda^{-1} \alpha^2 \|e_p^{h,i}\|_\Omega^2 &\leq 2\lambda^{-1} \left(\|\alpha e_p^{h,i} - e_{p_T}^{h,i}\|_\Omega^2 \right) + 2\lambda^{-1} \|e_{p_T}^{h,i}\|_\Omega^2 \\ &\leq 2\lambda^{-1} \left(\|\alpha e_p^{h,i} - e_{p_T}^{h,i}\|_\Omega^2 \right) + 2(C_* \mu)^{-1} \|e_{p_T}^{h,i}\|_\Omega^2. \end{aligned}$$

Combining this estimate with eq. (4.12) we obtain:

$$\lambda^{-1/2} \alpha \|e_p^{h,i}\|_\Omega \leq C A_i. \quad (4.14)$$

The Cauchy–Schwarz and triangle inequalities, together with eq. (4.14) now imply

$$\begin{aligned} I_4^i &\leq \lambda^{-1} \alpha \Delta t \|\alpha(\partial_t p^i - d_t \Pi_Q p^i) - (\partial_t p_T^i - d_t \Pi_Q p_T^i)\|_\Omega \|e_p^{h,i}\|_\Omega \\ &\leq C \lambda^{-1/2} \Delta t \left(\alpha \|\partial_t p^i - d_t p^i\|_\Omega + \alpha \|d_t e_p^{I,i}\|_\Omega + \|\partial_t p_T^i - d_t p_T^i\|_\Omega + \|d_t e_{p_T}^{I,i}\|_\Omega \right) A_i. \end{aligned}$$

If we define

$$\begin{aligned} E_i &= C \mu^{1/2} \Delta t \|d_t e_u^{I,i}\|_v + (2c_0)^{1/2} \Delta t (\|\partial_t p^i - d_t p^i\|_\Omega + \|d_t e_p^{I,i}\|_\Omega) \\ &\quad + C \lambda^{-1/2} \Delta t \left(\alpha \|\partial_t p^i - d_t p^i\|_\Omega + \alpha \|d_t e_p^{I,i}\|_\Omega + \|\partial_t p_T^i - d_t p_T^i\|_\Omega + \|d_t e_{p_T}^{I,i}\|_\Omega \right), \\ D_i &= \frac{\kappa^{-1}}{2} \Delta t \|e_z^{I,i}\|_\Omega^2, \end{aligned}$$

we find, using eq. (4.10), that

$$(A_{i+1}^2 - A_i^2) + B_{i+1}^2 \leq E_{i+1} A_{i+1} + D_{i+1}. \quad (4.15)$$

Summing now for $i = 0$ to $i = n - 1$, and using $A_0 = 0$, we obtain, after shifting indices and using $D_0 \geq 0$ and $B_0 = 0$,

$$A_n^2 + \sum_{j=1}^n B_j^2 \leq \sum_{j=1}^n E_j A_j + \sum_{j=1}^n D_j. \quad (4.16)$$

Then, by Lemma 4.1 we obtain

$$A_n + \left(\sum_{i=1}^n B_i^2 \right)^{1/2} \leq C \left(\sum_{i=1}^n E_i + \left(\sum_{i=1}^n D_i \right)^{1/2} \right). \quad (4.17)$$

To prove eq. (4.8a), we therefore need to estimate $\sum_{i=1}^n E_i$ and $(\sum_{i=1}^n D_i)^{1/2}$.

By eqs. (3.4) and (4.2), we note that

$$\begin{aligned} \mu^{1/2} \Delta t \sum_{i=1}^n \|d_t e_u^{I,i}\|_v &\leq C \Delta t \sum_{i=1}^n (a_h(d_t e_u^{I,i}, d_t e_u^{I,i}))^{1/2} \\ &\leq C \Delta t \mu^{1/2} \sum_{i=1}^n h^l \|d_t u^i\|_{l+1, \Omega} \leq C \mu^{1/2} h^l \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))}, \end{aligned}$$

where we used $\Delta t d_t u^i = \int_{t^{i-1}}^{t^i} \partial_t u(s) ds$ for the last inequality. Using this estimate, together with eqs. (4.5a) and (4.5b), we find:

$$\begin{aligned} \sum_{i=1}^n E_i &= C \sum_{i=1}^n \left[\mu^{1/2} \Delta t \|d_t e_u^{I,i}\|_v + (2c_0)^{1/2} \Delta t (\|\partial_t p^i - d_t p^i\|_\Omega + \|d_t e_p^{I,i}\|_\Omega) \right. \\ &\quad \left. + \lambda^{-1/2} \Delta t (\alpha \|\partial_t p^i - d_t p^i\|_\Omega + \alpha \|d_t e_p^{I,i}\|_\Omega + \|\partial_t p_T^i - d_t p_T^i\|_\Omega + \|d_t e_{p_T}^{I,i}\|_\Omega) \right], \\ &\leq C \mu^{1/2} h^l \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} \\ &\quad + C \max\{c_0^{1/2}, \lambda^{-1/2}\} (\Delta t \|\partial_{tt} p, \partial_{tt} p_T\|_{L^1(I; L^2(\Omega))} + h^l \|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))}). \end{aligned} \quad (4.18)$$

Next, by eq. (4.1),

$$\begin{aligned} \sum_{i=1}^n D_i &= \sum_{i=1}^n \frac{\kappa^{-1}}{2} \Delta t \|e_z^{I,i}\|_\Omega^2 \leq C \kappa^{-1} (\Delta t) h^{2l} \sum_{i=1}^n \|z(t^i)\|_{H^l(\Omega)}^2 \\ &\leq C \kappa^{-1} T h^{2l} \|z\|_{C^0(I; H^l(\Omega))}^2, \end{aligned} \quad (4.19)$$

where in the last inequality $n\Delta t \leq T$ is used. Combining eq. (4.17) with eqs. (4.18) and (4.19) and the coercivity of a_h eq. (3.4), we find:

$$\begin{aligned} c_0^{1/2} \|e_p^{h,n}\|_\Omega + \lambda^{-1/2} \|\alpha e_p^{h,n} - e_{p_T}^{h,n}\|_\Omega + \mu^{1/2} \|e_u^{h,n}\|_v + \kappa^{-1/2} \left(\sum_{i=1}^n \Delta t \|e_z^{h,i}\|_\Omega^2 \right)^{1/2} \\ \leq C \mu^{1/2} h^l \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} \\ + C \max\{c_0^{1/2}, \lambda^{-1/2}\} (\Delta t \|\partial_{tt} p, \partial_{tt} p_T\|_{L^1(I; L^2(\Omega))} + h^l \|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))}) \\ + C \kappa^{-1/2} T^{1/2} h^l \|z\|_{C^0(I; H^l(\Omega))} \\ \leq c_1 \Delta t + c_2 h^l \end{aligned} \quad (4.20)$$

where

$$\begin{aligned}
c_1 &= C \max \{c_0^{1/2}, \lambda^{-1/2}\} \|\partial_{tt} p, \partial_{tt} p_T\|_{L^1(I; L^2(\Omega))} \\
&\leq C \max \{c_0^{1/2}, \lambda^{-1/2}\} \|p, p_T\|_{W^{2,1}(I; L^2(\Omega))}, \\
c_2 &= C(\mu^{1/2} \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \max \{c_0^{1/2}, \lambda^{-1/2}\} \|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))} \\
&\quad + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))}).
\end{aligned}$$

Next, by eq. (4.4), the triangle inequality, and the approximation error eq. (4.20),

$$\begin{aligned}
&c_0^{1/2} \|p^n - p_h^n\|_{\Omega} + \lambda^{-1/2} \|\alpha(p^n - p_h^n) - (p_T^n - p_{Th}^n)\|_{\Omega} + \mu^{1/2} \|\mathbf{u}^n - \mathbf{u}_h^n\|_v \\
&\quad + \kappa^{-1/2} \left(\sum_{i=1}^n \Delta t \|z^i - z_h^i\|_{\Omega}^2 \right)^{1/2} \\
&\leq c_0^{1/2} \|e_p^{I,n}\|_{\Omega} + \lambda^{-1/2} \|\alpha e_p^{I,n} - e_{p_T}^{I,n}\|_{\Omega} + \mu^{1/2} \|\mathbf{e}_u^{I,n}\|_v + \left(2 \sum_{i=1}^n D_i \right)^{1/2} \\
&\quad + c_1 \Delta t + c_2 h^l.
\end{aligned} \tag{4.21}$$

Note that

$$\begin{aligned}
c_0^{1/2} \|e_p^{I,n}\|_{\Omega} &\leq C c_0^{1/2} h^l \|p^n\|_{l,\Omega}, \\
\lambda^{-1/2} \|\alpha e_p^{I,n} - e_{p_T}^{I,n}\|_{\Omega} &\leq C \lambda^{-1/2} h^l (\|p^n\|_{l,\Omega} + \|p_T^n\|_{l,\Omega}), \\
\mu^{1/2} \|\mathbf{e}_u^{I,n}\|_v &\leq C \mu^{1/2} h^l \|u^n\|_{l+1,\Omega}.
\end{aligned}$$

When combined with eqs. (4.19) and (4.21),

$$\begin{aligned}
&c_0^{1/2} \|p^n - p_h^n\|_{\Omega} + \lambda^{-1/2} \|\alpha(p^n - p_h^n) - (p_T^n - p_{Th}^n)\|_{\Omega} + \mu^{1/2} \|\mathbf{u}^n - \mathbf{u}_h^n\|_v \\
&\quad + \kappa^{-1/2} \left(\sum_{i=1}^n \Delta t \|z^i - z_h^i\|_{\Omega}^2 \right)^{1/2} \leq c_1 \Delta t + (c_2 + c_3) h^l,
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
c_3 &= C \left(c_0^{1/2} \|p^n\|_{l,\Omega} + \lambda^{-1/2} (\|p^n\|_{l,\Omega} + \|p_T^n\|_{l,\Omega}) + \mu^{1/2} \|u^n\|_{l+1,\Omega} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right) \\
&= C \left((c_0^{1/2} + \lambda^{-1/2}) \|p^n\|_{l,\Omega} + \lambda^{-1/2} \|p_T^n\|_{l,\Omega} + \mu^{1/2} \|u^n\|_{l+1,\Omega} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right).
\end{aligned} \tag{4.23}$$

Let us have a closer look at $c_2 + c_3$. Using the Sobolev embedding $W^{s,q}(I; H^l(\Omega)) \hookrightarrow C^0(I; H^l(\Omega))$ for $(s, q) = (1, 1)$ and $(s, q) = (2, 1)$,

$$\begin{aligned}
c_2 + c_3 &\leq C(\mu^{1/2} \{\|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \|u^n\|_{l+1, \Omega}\} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \\
&\quad + \max\{c_0^{1/2}, \lambda^{-1/2}\} \{\|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))} + \|p^n\|_{l, \Omega} + \|p_T^n\|_{l, \Omega}\}) \\
&\leq C(\mu^{1/2} \{\|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \|u\|_{C^0(I; H^{l+1}(\Omega))}\} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \\
&\quad + \max\{c_0^{1/2}, \lambda^{-1/2}\} \{\|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))} + \|p, p_T\|_{C^0(I; H^l(\Omega))}\}) \\
&\leq C(\mu^{1/2} \|u\|_{W^{1,1}(I; H^{l+1}(\Omega))} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \\
&\quad + \max\{c_0^{1/2}, \lambda^{-1/2}\} \|p, p_T\|_{W^{1,1}(I; H^l(\Omega))}),
\end{aligned}$$

proving eq. (4.8a).

Finally, eq. (4.8b) follows from eq. (4.4) and the triangle inequality, eq. (4.12), eq. (4.20), and usual interpolation estimates for the L^2 -projection:

$$\begin{aligned}
\mu^{-1/2} \|\mathbf{p}_T^n - \mathbf{p}_{Th}^n\|_q &\leq \mu^{-1/2} \|\mathbf{e}_{p_T}^{I,n}\|_q + C A_i \\
&\leq c_1 \Delta t + (C \mu^{-1/2} \|p_T^n\|_{l, \Omega} + c_2) h^l \\
&\leq c_1 \Delta t + (C \mu^{-1/2} \|p_T\|_{C^0(I; H^l(\Omega))} + c_2) h^l
\end{aligned} \tag{4.24}$$

Let us have a closer look at the constant in front of the second term:

$$\begin{aligned}
&C \mu^{-1/2} \|p_T\|_{C^0(I; H^l(\Omega))} + c_2 \\
&\leq C(\mu^{1/2} \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \max\{c_0^{1/2}, \lambda^{-1/2}\} \|\partial_t p, \partial_t p_T\|_{L^1(I; H^l(\Omega))} \\
&\quad + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} + C \mu^{-1/2} \|p_T\|_{C^0(I; H^l(\Omega))}) \\
&\leq C(\mu^{1/2} \|\partial_t u\|_{L^1(I; H^{l+1}(\Omega))} + \max\{c_0^{1/2}, \mu^{-1/2}\} \|p, p_T\|_{W^{1,1}(I; H^l(\Omega))} \\
&\quad + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))}),
\end{aligned}$$

where in the last step we used that $C^* \lambda^{-1/2} \leq \mu^{1/2}$. This proves eq. (4.8b). $\square$

We end this section by noting that the estimates in Theorem 4.1 for the displacement, Darcy velocity, and total pressure are unconditionally robust in the incompressible limit $c_0 \rightarrow 0$ and $\lambda \rightarrow \infty$.

5 Numerical examples

We now validate our theoretical analysis. As stated previously in remark 3.1, the analysis in this paper holds both for HDG and EDG-HDG. As such, both methods are implemented using the Netgen/NGSolve finite element library [41, 42]. Numerical results are compared to analytical solutions and some benchmark problems.

5.1 Convergence rates for a static problem

We consider a test case proposed in [31, Example 1]. Consider the static Biot problem eq. (2.2) on Ω where eq. (2.2c) is replaced by

$$c_0 p + \lambda^{-1} \alpha (\alpha p - p_T) + \nabla \cdot z = g \quad \text{in } \Omega. \quad (5.1)$$

We consider a domain Ω with four curved boundaries parametrized as

$$\begin{aligned} \Gamma_1 &= \{\omega \in [0, 1] : x_1 = \omega + \gamma \cos(\pi\omega) \sin(\pi\omega), x_2 = -\gamma \cos(\pi\omega) \sin(\pi\omega)\}, \\ \Gamma_2 &= \{\omega \in [0, 1] : x_1 = 1 + \gamma \cos(\pi\omega) \sin(\pi\omega), x_2 = \omega - \gamma \cos(\pi\omega) \sin(\pi\omega)\}, \\ \Gamma_3 &= \{\omega \in [1, 0] : x_1 = \omega + \gamma \cos(\pi\omega) \sin(\pi\omega), x_2 = 1 - \gamma \cos(\pi\omega) \sin(\pi\omega)\}, \\ \Gamma_4 &= \{\omega \in [1, 0] : x_1 = \gamma \cos(\pi\omega) \sin(\pi\omega), x_2 = \omega - \gamma \cos(\pi\omega) \sin(\pi\omega)\}, \end{aligned}$$

with $\gamma = -0.08$. We then define $\Gamma_D = \Gamma_1 \cup \Gamma_3 \cup \Gamma_4$, $\Gamma_P = \Gamma_1 \cup \Gamma_2$, $\Gamma_T = \Gamma_2$, and $\Gamma_F = \Gamma_3 \cup \Gamma_4$. The solution to the Biot problem is taken as

$$u = a \begin{bmatrix} \sin(\pi x_1) \cos(\pi x_2) + x_1^2/(2\lambda) \\ -\cos(\pi x_1) \sin(\pi x_2) + x_2^2/(2\lambda) \end{bmatrix}, \quad p = b \sin(\pi x_1) \sin(\pi x_2). \quad (5.2)$$

This solution eq. (5.2) is used to set the body force f , the source/sink term g , and *inhomogeneous* boundary conditions. As parameters we set $a = 10^{-4}$, $b = \pi$, $\kappa = 10^{-7}$, $\alpha = 0.1$ and $c_0 = 10^{-5}$. We consider both mild incompressibility ($\nu = 0.4$) and quasi-incompressibility ($\nu = 0.49999$) and consider two values for E , namely $E = 10^4$ and $E = 1$. Furthermore, we consider the rates of convergence for the lowest order ($k = 1$) and a higher order ($k = 3$) approximation (with k the polynomial approximation in eq. (3.1)).

Theorem 4.1 does not present an estimate in the L^2 -norm for the displacement and only a suboptimal estimate for the Darcy velocity. Nevertheless, since our main objective here is to show the robustness of the discretization in the incompressible limit, we present in Tables 5.1 and 5.2, for the HDG and EDG-HDG schemes, respectively, the errors and rates of convergence of all unknowns in the L^2 -norm. Let us first observe that the velocity and displacement converge at rate $k + 1$ and that the pressures converge at rate k . These are optimal rates of convergence. We furthermore observe that the errors for all unknowns are independent of the value of Poisson's ratio ν and for the modulus of elasticity E . It is particularly interesting to note that the choice $E = 10^4$ and $\nu = 0.49999$ (corresponding to $\lambda \approx 1.7 \cdot 10^8$) does not affect the quality of the approximation. This confirms the robustness of the error estimates in Theorem 4.1 in the incompressible limit.

5.2 Convergence rates for the quasi-static problem

We now consider a manufactured solution for the quasi-static problem eq. (2.2) on the unit square. We divide the boundary of our domain into

$$\begin{aligned} \Gamma_1 &= \{(x_1, x_2) \in \partial\Omega : x_2 = 0\}, & \Gamma_2 &= \{(x_1, x_2) \in \partial\Omega : x_1 = 1\}, \\ \Gamma_3 &= \{(x_1, x_2) \in \partial\Omega : x_2 = 1\}, & \Gamma_4 &= \{(x_1, x_2) \in \partial\Omega : x_1 = 0\}, \end{aligned}$$

Table 5.1 Rates of convergence for HDG the test case described in Section 5.1 for $E = 1$ or $E = 10^4$, $\nu = 0.4$ or $\nu = 0.49999$, and for $k = 1$ or $k = 3$. Here r is the rate of convergence.

Cells	$\ u_h - u\ _\Omega$	r	$\ p_{Th} - p_T\ _\Omega$	r	$\ z_h - z\ _\Omega$	r	$\ p_h - p\ _\Omega$	r
$k = 1, E = 10^4, \nu = 0.4$								
384	4.2e-07	2.0	4.7e-02	1.1	5.6e-09	2.1	1.3e-01	1.0
1536	1.1e-07	2.0	2.2e-02	1.1	1.3e-09	2.1	6.3e-02	1.0
6144	2.6e-08	2.0	1.0e-02	1.1	3.3e-10	2.0	3.2e-02	1.0
24576	6.6e-09	2.0	5.0e-03	1.0	8.1e-11	2.0	1.6e-02	1.0
$k = 1, E = 10^4, \nu = 0.49999$								
384	4.3e-07	2.0	6.7e-02	1.2	3.9e-09	2.1	1.3e-01	1.0
1536	1.1e-07	2.0	2.9e-02	1.2	9.6e-10	2.0	6.3e-02	1.0
6144	2.7e-08	2.0	1.3e-02	1.1	2.4e-10	2.0	3.2e-02	1.0
24576	6.7e-09	2.0	6.3e-03	1.1	5.9e-11	2.0	1.6e-02	1.0
$k = 3, E = 10^4, \nu = 0.4$								
384	3.2e-10	4.0	9.1e-05	3.0	5.0e-12	4.0	1.8e-04	3.0
1536	2.0e-11	4.0	1.1e-05	3.1	3.0e-13	4.0	2.3e-05	3.0
6144	1.2e-12	4.0	1.3e-06	3.0	1.8e-14	4.0	2.8e-06	3.0
24576	7.6e-14	4.0	1.7e-07	3.0	1.1e-15	4.0	3.5e-07	3.0
$k = 3, E = 10^4, \nu = 0.49999$								
384	3.6e-10	4.0	1.7e-04	3.1	2.6e-12	4.0	1.8e-04	3.0
1536	2.2e-11	4.0	2.0e-05	3.1	1.6e-13	4.0	2.3e-05	3.0
6144	1.4e-12	4.0	2.4e-06	3.1	1.0e-14	4.0	2.8e-06	3.0
24576	8.5e-14	4.0	2.9e-07	3.0	6.4e-16	4.0	3.5e-07	3.0
$k = 1, E = 1, \nu = 0.4$								
384	6.2e-07	3.6	1.3e-02	1.0	3.7e-09	2.1	1.3e-01	1.0
1536	1.1e-07	2.5	6.3e-03	1.0	9.6e-10	2.0	6.3e-02	1.0
6144	2.6e-08	2.1	3.2e-03	1.0	2.6e-10	1.9	3.2e-02	1.0
24576	6.6e-09	2.0	1.6e-03	1.0	7.2e-11	1.9	1.6e-02	1.0
$k = 1, E = 1, \nu = 0.49999$								
384	6.4e-07	3.7	1.3e-02	1.0	3.9e-09	2.1	1.3e-01	1.0
1536	1.1e-07	2.5	6.3e-03	1.0	9.5e-10	2.0	6.3e-02	1.0
6144	2.7e-08	2.0	3.2e-03	1.0	2.4e-10	2.0	3.2e-02	1.0
24576	6.7e-09	2.0	1.6e-03	1.0	5.9e-11	2.0	1.6e-02	1.0
$k = 3, E = 1, \nu = 0.4$								
384	3.4e-10	4.1	1.8e-05	3.0	2.2e-12	3.7	1.8e-04	3.0
1536	2.0e-11	4.1	2.3e-06	3.0	2.0e-13	3.5	2.3e-05	3.0
6144	1.2e-12	4.0	2.8e-07	3.0	1.6e-14	3.7	2.8e-06	3.0
24576	7.6e-14	4.0	3.5e-08	3.0	1.1e-15	3.9	3.5e-07	3.0
$k = 3, E = 1, \nu = 0.49999$								
384	3.6e-10	4.0	1.8e-05	3.0	2.6e-12	4.0	1.8e-04	3.0
1536	2.2e-11	4.0	2.3e-06	3.0	1.6e-13	4.0	2.3e-05	3.0
6144	1.4e-12	4.0	2.8e-07	3.0	1.0e-14	4.0	2.8e-06	3.0
24576	8.5e-14	4.0	3.5e-08	3.0	6.4e-16	4.0	3.5e-07	3.0

and set $\Gamma_D = \Gamma_1 \cup \Gamma_3 \cup \Gamma_4$, $\Gamma_P = \Gamma_1 \cup \Gamma_2$, $\Gamma_T = \Gamma_2$, and $\Gamma_F = \Gamma_3 \cup \Gamma_4$. As exact solution we take

$$u = \begin{bmatrix} \sin(\pi t) \sin(\pi x_1) \sin(\pi x_2) \\ \sin(\pi t) \sin(\pi x_1) \cos(\pi x_2) \end{bmatrix}, \quad p = \sin(\pi(x_1 - x_2 - t)), \quad (5.3)$$

and set body force terms, source/sink terms, initial and boundary conditions accordingly. As parameters we set $E = 10^4$, $\kappa = 10^{-2}$, $\alpha = 0.1$, $c_0 = 0.1$, $\nu = 0.2$. We consider the solution over the time interval $I = (0, 0.1]$ and show the rates of convergence at $t = 0.1$ in Table 5.3 for HDG and EDG-HDG using $k = 1$ and $k = 2$. We are interested in the spatial rates of convergence and so we implement

Table 5.2 Rates of convergence for EDG-HDG the test case described in Section 5.1 for $E = 1$ or $E = 10^4$, $\nu = 0.4$ or $\nu = 0.49999$, and for $k = 1$ or $k = 3$. Here r is the rate of convergence.

Cells	$\ u_h - u\ _\Omega$	r	$\ p_{Th} - p_T\ _\Omega$	r	$\ z_h - z\ _\Omega$	r	$\ p_h - p\ _\Omega$	r
$k = 1, E = 10^4, \nu = 0.4$								
384	5.4e-07	2.1	7.1e-02	1.3	7.7e-09	2.2	1.3e-01	1.0
1536	1.3e-07	2.0	2.9e-02	1.3	1.7e-09	2.2	6.3e-02	1.0
6144	3.2e-08	2.0	1.3e-02	1.2	3.7e-10	2.2	3.2e-02	1.0
24576	8.1e-09	2.0	5.7e-03	1.1	8.7e-11	2.1	1.6e-02	1.0
$k = 1, E = 10^4, \nu = 0.49999$								
384	5.4e-07	2.1	1.6e-01	1.4	3.9e-09	2.1	1.3e-01	1.0
1536	1.3e-07	2.0	6.0e-02	1.4	9.6e-10	2.0	6.3e-02	1.0
6144	3.3e-08	2.0	2.3e-02	1.4	2.4e-10	2.0	3.2e-02	1.0
24576	8.1e-09	2.0	9.5e-03	1.3	5.9e-11	2.0	1.6e-02	1.0
$k = 3, E = 10^4, \nu = 0.4$								
384	3.4e-10	4.0	9.8e-05	3.1	5.3e-12	4.0	1.8e-04	3.0
1536	2.1e-11	4.0	1.2e-05	3.1	3.2e-13	4.1	2.3e-05	3.0
6144	1.3e-12	4.0	1.4e-06	3.0	1.9e-14	4.0	2.8e-06	3.0
24576	8.1e-14	4.0	1.7e-07	3.0	1.2e-15	4.0	3.5e-07	3.0
$k = 3, E = 10^4, \nu = 0.49999$								
384	4.0e-10	4.0	2.0e-04	3.2	2.6e-12	4.0	1.8e-04	3.0
1536	2.4e-11	4.0	2.3e-05	3.1	1.6e-13	4.0	2.3e-05	3.0
6144	1.5e-12	4.0	2.6e-06	3.1	1.0e-14	4.0	2.8e-06	3.0
24576	9.2e-14	4.0	3.1e-07	3.1	6.4e-16	4.0	3.5e-07	3.0
$k = 1, E = 1, \nu = 0.4$								
384	7.0e-07	3.3	1.3e-02	1.0	3.9e-09	2.0	1.3e-01	1.0
1536	1.3e-07	2.4	6.3e-03	1.0	1.1e-09	1.9	6.3e-02	1.0
6144	3.2e-08	2.1	3.2e-03	1.0	3.0e-10	1.8	3.2e-02	1.0
24576	8.0e-09	2.0	1.6e-03	1.0	7.8e-11	1.9	1.6e-02	1.0
$k = 1, E = 1, \nu = 0.49999$								
384	7.2e-07	3.4	1.3e-02	1.0	3.9e-09	2.1	1.3e-01	1.0
1536	1.4e-07	2.4	6.3e-03	1.0	9.5e-10	2.0	6.3e-02	1.0
6144	3.3e-08	2.1	3.2e-03	1.0	2.4e-10	2.0	3.2e-02	1.0
24576	8.1e-09	2.0	1.6e-03	1.0	5.9e-11	2.0	1.6e-02	1.0
$k = 3, E = 1, \nu = 0.4$								
384	3.7e-10	4.1	1.8e-05	3.0	2.3e-12	3.6	1.8e-04	3.0
1536	2.2e-11	4.1	2.3e-06	3.0	2.1e-13	3.5	2.3e-05	3.0
6144	1.3e-12	4.0	2.8e-07	3.0	1.6e-14	3.7	2.8e-06	3.0
24576	8.1e-14	4.0	3.5e-08	3.0	1.1e-15	3.9	3.5e-07	3.0
$k = 3, E = 1, \nu = 0.49999$								
384	4.0e-10	4.0	1.8e-05	3.0	2.6e-12	4.0	1.8e-04	3.0
1536	2.4e-11	4.0	2.3e-06	3.0	1.6e-13	4.0	2.3e-05	3.0
6144	1.5e-12	4.0	2.8e-07	3.0	1.0e-14	4.0	2.8e-06	3.0
24576	9.2e-14	4.0	3.5e-08	3.0	6.4e-16	4.0	3.5e-07	3.0

a second order backward differentiation formulae (BDF2) time stepping scheme and take a time step of $\Delta t = 10^{-3}$ so that spatial errors dominate over temporal errors. In Table 5.3, we observe optimal rates of convergence for all unknowns, both for the HDG and EDG-HDG schemes. Furthermore, note that although the error in the total pressure is relatively large, this has no effect on the errors in the displacement, velocity and pore pressure of the fluid, which are all magnitudes smaller.

Table 5.3 Rates of convergence for HDG and EDG-HDG for the test case described in Section 5.2 for $k = 1$ and $k = 2$. Here dofs are the total number of degrees-of-freedom and r is the rate of convergence.

Dofs	$\ u_h - u\ _\Omega$	r	$\ p_{Th} - p_T\ _\Omega$	r	$\ z_h - z\ _\Omega$	r	$\ p_h - p\ _\Omega$	r
HDG								
$k = 1$								
896	1.7e-02	2.5	7.2e+02	1.1	4.1e-03	0.9	1.9e-01	0.7
3456	4.1e-03	2.1	3.6e+02	1.0	1.1e-03	1.9	9.3e-02	1.0
13568	1.0e-03	2.0	1.8e+02	1.0	3.0e-04	1.9	4.6e-02	1.0
53760	2.5e-04	2.0	9.0e+01	1.0	7.6e-05	2.0	2.3e-02	1.0
$k = 2$								
1632	1.7e-03	1.6	1.1e+02	1.0	3.5e-04	3.3	2.8e-02	2.3
6336	2.1e-04	3.0	2.7e+01	2.0	4.3e-05	3.0	7.0e-03	2.0
24960	2.7e-05	3.0	6.8e+00	2.0	5.3e-06	3.0	1.8e-03	2.0
99072	3.4e-06	3.0	1.7e+00	2.0	6.8e-07	3.0	4.4e-04	2.0
EDG-HDG								
$k = 1$								
722	2.2e-02	2.3	7.4e+02	1.1	4.9e-03	0.4	1.8e-01	0.6
2786	5.5e-03	2.0	3.7e+02	1.0	1.2e-03	2.0	9.2e-02	1.0
10946	1.3e-03	2.0	1.8e+02	1.0	3.0e-04	2.0	4.6e-02	1.0
43394	3.3e-04	2.0	9.2e+01	1.0	7.5e-05	2.0	2.3e-02	1.0
$k = 2$								
1458	1.8e-03	1.6	1.1e+02	1.1	4.4e-04	3.0	2.8e-02	2.3
5666	2.4e-04	2.9	2.8e+01	2.0	5.5e-05	3.0	7.0e-03	2.0
22338	3.0e-05	3.0	6.9e+00	2.0	6.7e-06	3.0	1.8e-03	2.0
88706	3.7e-06	3.0	1.7e+00	2.0	8.5e-07	3.0	4.4e-04	2.0

5.3 The footing problem

The two-dimensional footing problem has been proposed in the literature to study the locking-free properties of numerical methods for the Biot equations [16, 31]. We follow here the setup of [2] and consider the domain $\Omega = (-50, 50) \times (0, 75)$ and model parameters $\kappa = 10^{-4}$, $c_0 = 10^{-3}$, $\alpha = 0.1$, $E = 3 \cdot 10^4$, and $\nu = 0.4995$ (so that $\lambda \approx 10^7$). We define the boundaries $\Gamma_1 = \{(x_1, x_2) \in \partial\Omega, |x_1| \leq 50/3, x_2 = 75\}$, $\Gamma_2 = \{(x_1, x_2) \in \partial\Omega, |x_1| > 50/3, x_2 = 75\}$, and $\Gamma_3 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$ and impose the following boundary conditions:

$$\sigma n = (0, -\sigma_0)^T \text{ on } \Gamma_1, \quad \sigma n = 0 \text{ on } \Gamma_2, \quad u = 0 \text{ on } \Gamma_3, \quad p = 0 \text{ on } \partial\Omega,$$

where $\sigma_0 = 10^4$. As initial conditions we impose $u(x, 0) = 0$ and $p(x, 0) = 0$. We solve this problem with HDG until $T = 50$ using BDF2 time stepping. We choose a time step of size $\Delta t = 1$, take $k = 2$ in our finite element spaces, and compute the solution on an unstructured mesh consisting of 169984 simplices.

We show the solution to this problem at time $t = 50$ in Figure 5.1. In this incompressible limit we observe that the discretization results in pressure and displacement solutions are free of, respectively, spurious oscillations and locking effects.

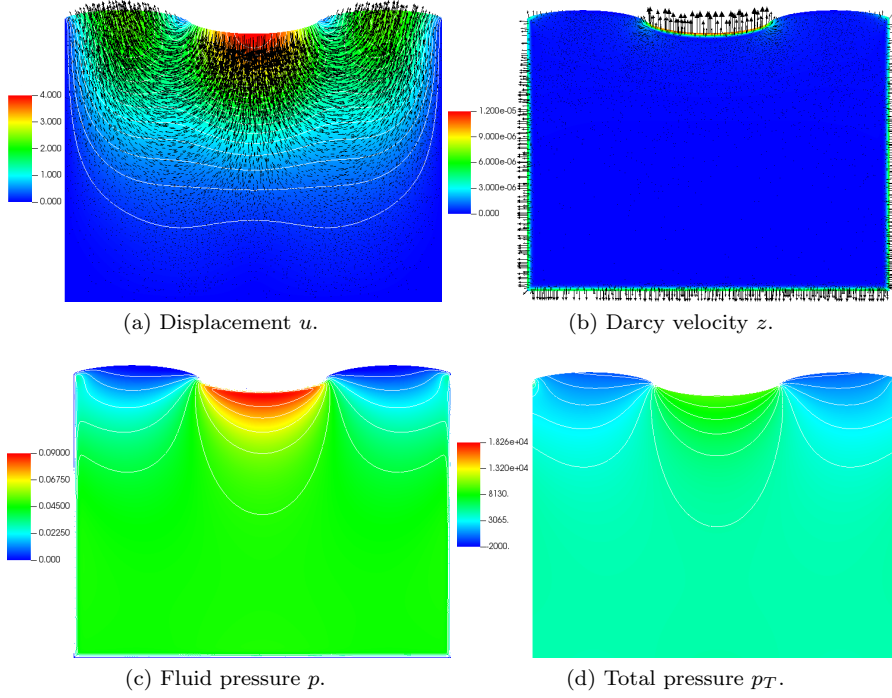


Fig. 5.1 The solution to the footing problem of Section 5.3 in the deformed domain at $t = 50$.

5.4 The cantilever bracket problem

The cantilever bracket problem was used in [2,29,33] to study locking phenomena at low permeability and when the storage coefficient is zero. Consider the domain $\Omega = (0, 1)^2$ and define

$$\begin{aligned} \Gamma_1 &= \{(x_1, x_2) \in \partial\Omega, x_2 = 0\}, & \Gamma_2 &= \{(x_1, x_2) \in \partial\Omega, x_1 = 1\}, \\ \Gamma_3 &= \{(x_1, x_2) \in \partial\Omega, x_2 = 1\}, & \Gamma_4 &= \{(x_1, x_2) \in \partial\Omega, x_1 = 0\}. \end{aligned}$$

We impose the boundary conditions

$$z \cdot n = 0 \text{ on } \partial\Omega, \quad \sigma n = (0, -1)^T \text{ on } \Gamma_3, \quad \sigma n = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad u = 0 \text{ on } \Gamma_4.$$

At $t = 0$ we set $u = 0$ and $p = 0$. The model parameters are chosen as $E = 10^5$, $\nu = 0.4$, $\alpha = 0.93$, $c_0 = 0$, and $\kappa = 10^{-7}$ [33]. As shown in [33], with these parameters continuous Galerkin numerical methods show spurious oscillations in the pressure on a very short time interval. Therefore, we consider here the time interval $I = [0, 0.005]$. In our discretization we combine the EDG-HDG discretization with BDF2 time stepping, choose a time step of $\Delta t = 0.001$, set $k = 2$ in our finite element spaces, and compute the solution on a mesh consisting of 128 simplices.

We plot the solution in Figure 5.2. In Figure 5.2a we observe that the pressure field at $t = 0.001$ is free from spurious oscillations, similar to the discontinuous Galerkin solutions obtained in [33]. We further show in Figure 5.2b that the pressure solution along the lines $x = 0.26$, $x = 0.33$, $x = 0.4$, and $x = 0.46$ at $t = 0.005$

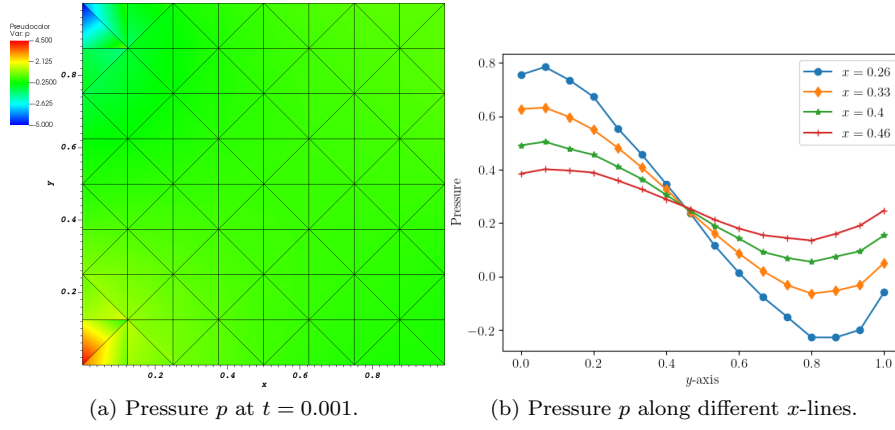


Fig. 5.2 The solution to the cantilever bracket problem Section 5.4 using EDG-HDG. Left: the pressure solution at $t = 0.001$. Right: the pressure solution along different x -lines at time $t = 0.005$.

is free of oscillations, agreeing with other stable finite element methods for this problem [2, 29, 33].

6 Conclusions

An HDG and an EDG-HDG method have been presented and analyzed for the total pressure formulation of the quasi-static poroelasticity model. Both discretization methods are shown to be well-posed and space-time a priori error estimates show robustness of the proposed methods when $\lambda \rightarrow \infty$ and $c_0 \rightarrow 0$; both methods are free of volumetric locking. Numerical examples confirm our theory and further show optimal spatial rates of convergence in the L^2 -norm.

Statements and Declarations

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Data Availability Enquiries about data availability should be directed to the authors.

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A The inf-sup condition for $b_h((w_h, 0), q_h)$

By definition of b_h eq. (3.3b),

$$b_h((w_h, 0), q_h) := \underbrace{-(q_h, \nabla \cdot w_h)_\Omega}_{=: b_h^1(w_h, q_h)} + \underbrace{\langle \bar{q}_h, w_h \cdot n \rangle_{\partial \mathcal{T}}}_{=: b_h^2(w_h, \bar{q}_h)} \quad \forall w_h \in V_h, \forall q_h \in Q_h^0. \quad (\text{A.1})$$

Let $q_h \in Q_h$. It is known (see, for example, [13, Section 4.1.4] or [28, Remark 3.3]) that there exists $w \in [H_{0, \Gamma_F}^1(\Omega)]^d := \{v \in [H^1(\Omega)]^d : v|_{\Gamma_F} = 0\}$ such that

$$-(\nabla \cdot w, q_h)_\Omega = \|q_h\|_\Omega^2, \quad \|w\|_{1, \Omega} \leq C \|q_h\|_\Omega, \quad (\text{A.2})$$

for some positive constant C that only depends on Ω . Let $\Pi_V : [H^1(\Omega)]^d \rightarrow V_h$ be the BDM interpolation operator [8, Section III.3] and observe that by the single-valuedness of $\bar{q}_h$, continuity of $\Pi_V w \cdot n$ across interior faces, and since $\bar{q}_h = 0$ on Γ_P and $w = 0$ on Γ_F ,

$$b_h^2(\Pi_V w, \bar{q}_h) = \langle \bar{q}_h, \Pi_V w \cdot n \rangle_{\Gamma_F} = 0,$$

i.e., $\Pi_V w \in \text{Ker } b_2 := \{w_h \in V_h : b_h^2(w_h, \bar{q}_h) = 0 \quad \forall \bar{q}_h \in \bar{Q}_h\}$. Recall also that $(q_h, \nabla \cdot \Pi_V w)_\Omega = (q_h, \nabla \cdot w)_\Omega$ and $\|\Pi_V w\|_\Omega \leq C \|w\|_{1, \Omega}$. Then, by eq. (A.2),

$$\sup_{0 \neq w_h \in \text{Ker } b_2} \frac{b_h^1(w_h, q_h)}{\|w_h\|_\Omega} \geq \frac{-(q_h, \nabla \cdot \Pi_V w)_\Omega}{\|\Pi_V w\|_\Omega} \geq \frac{\|q_h\|_\Omega^2}{C \|q_h\|_\Omega} = C \|q_h\|_\Omega.$$

Next, let $w_h := L\bar{q}_h \in P_k(K)^d$ where L is the local BDM interpolation operator [8] such that

$$(L\bar{q}_h) \cdot n = \bar{q}_h, \quad \|L\bar{q}_h\|_K \leq Ch_K^{1/2} \|\bar{q}_h\|_{\partial K}, \quad K \in \mathcal{T}. \quad (\text{A.3})$$

Then

$$\begin{aligned} \sup_{0 \neq w_h \in V_h} \frac{b_h^2(w_h, \bar{q}_h)}{\|w_h\|_\Omega} &\geq \frac{\|\bar{q}_h\|_{\partial \mathcal{T}}^2}{\|w_h\|_\Omega} \geq \frac{\|\bar{q}_h\|_{\partial \mathcal{T}}^2}{C \sum_{K \in \mathcal{T}} h_K^{-1} \|w_h\|_\Omega} \\ &\geq Ch_{\min} h_{\max}^{-1} \left(\sum_{K \in \mathcal{T}} h_K \|\bar{q}_h\|_{\partial K}^2 \right)^{1/2}. \end{aligned}$$

Therefore, by [20, Theorem 3.1],

$$\begin{aligned} \sup_{0 \neq w_h \in V_h} \frac{b_h((w_h, 0), q_h)}{\|w_h\|_\Omega} &= \sup_{0 \neq w_h \in V_h} \frac{b_h^1(w_h, q_h) + b_h^2(w_h, \bar{q}_h)}{\|w_h\|_\Omega} \\ &\geq C \left(\|q_h\|_\Omega + h_{\min} h_{\max}^{-1} \left(\sum_{K \in \mathcal{T}} h_K \|\bar{q}_h\|_{\partial K}^2 \right) \right) \geq C \|q_h\|_q. \end{aligned}$$

B Error estimates following from Taylor's theorem

We prove here eqs. (4.5a) and (4.5b). Let $D \subset \mathbb{R}^d$. Then for ψ a regular enough function defined on $[0, T] \times D$, using Taylor's theorem,

$$\begin{aligned} \Delta t \|\partial_t \psi^m - d_t \psi^m\|_{0,D} &= \|\Delta t \partial_t \psi^m - (\psi^m - \psi^{m-1})\|_{0,D} \\ &= \left\| \int_{t_{m-1}}^{t_m} \partial_{tt} \psi(t) \underbrace{(t - t_{m-1})}_{\leq \Delta t} dt \right\|_{0,D} \\ &\leq \Delta t \int_{t_{m-1}}^{t_m} \|\partial_{tt} \psi(t)\|_{0,D} dt \\ &= \Delta t \|\partial_{tt} \psi\|_{L^1(t_{m-1}, t_m; L^2(D))}, \end{aligned}$$

from which eq. (4.5a) follows. Next, to show eq. (4.5b) we use the identity

$$\Delta t d_t e_\psi^{I,m} = \int_{t_{m-1}}^{t_m} (\partial_t \psi(s) - \Pi_Q \partial_t \psi(s)) ds,$$

and eq. (4.1). Then, by the approximation property of the L^2 -projection,

$$\begin{aligned} \Delta t \sum_{m=1}^n \|d_t e_\psi^{I,m}\|_{0,D} &\leq \sum_{m=1}^n \int_{t_{m-1}}^{t_m} \|\partial_t \psi(s) - \Pi_Q \partial_t \psi(s)\|_{0,D} ds \\ &\leq Ch^l \|\partial_t \psi\|_{L^1(I; H^l(D))}. \end{aligned}$$