

A STRONGLY CONSERVATIVE HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE COUPLED TIME-DEPENDENT NAVIER-STOKES AND DARCY PROBLEM

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Abstract. We present a strongly conservative and pressure-robust hybridizable discontinuous Galerkin method for the coupled time-dependent Navier–Stokes and Darcy problem. We show existence and uniqueness of a solution and present an optimal a priori error analysis for the fully discrete problem when using Backward Euler time stepping. The theoretical results are verified by numerical examples.

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1. INTRODUCTION

In this paper we present an analysis of a hybridizable discontinuous Galerkin (HDG) method for the coupled Navier–Stokes and Darcy equations that model surface/subsurface flow. While various conforming and nonconforming finite element methods have been studied for the stationary Navier–Stokes and Darcy problem, see for example [3, 13, 14, 18, 19, 22, 23], the literature on numerical methods for the time-dependent problem is limited. The first numerical methods for the time-dependent problem were studied in [10, 11]. To simplify the analysis, however, these papers included inertia effects in the balance of forces at the interface. Existence and uniqueness of a weak solution to the physically more relevant model, without inertia effects on the interface, was proven in [7], while convergence of a discontinuous Galerkin method for this model was proven in [12]. Conforming methods for the transient problem have been studied in [25, 43].

The aforementioned papers for the time-dependent Navier–Stokes and Darcy problem have in common that they consider the primal form of the Darcy problem. In contrast, we consider the mixed form of the Darcy problem as this facilitates the formulation of a *strongly conservative* discretization, i.e., a discretization that is mass conserving in the sense of $H(\operatorname{div}; \Omega)$ where the velocity is globally $H(\operatorname{div}; \Omega)$ -conforming and, in the absence of sources and sinks, pointwise divergence-free on the elements [28]. In particular, we consider an HDG method [15] that is based on the HDG method for the Navier–Stokes equations [35] and a hybridized

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formulation of the mixed form of the Darcy problem [2], although nonconforming formulations based on other strongly conservative discretizations, for example, [16, 21, 31, 41], are possible.

Previously, we proved pressure-robustness of strongly conservative HDG methods for the Stokes/Darcy [9] and stationary Navier–Stokes/Darcy [8] problems, leading to a priori error estimates for the velocity that do not depend on the best approximation of the pressure scaled by the inverse of the viscosity (see [27, 32] for a review of other pressure-robust discretizations). Using Backward Euler time stepping we now show existence and uniqueness of a solution and derive an a priori error estimate to the fully-discrete time-dependent problem. Compared to previous work on the time-dependent Navier–Stokes/Darcy problem [7, 10–12, 25, 43], the novel contributions of this work is therefore the introduction and analysis of a strongly conservative HDG discretization and an a priori error estimate for the velocity that is independent of pressure.

The remainder of this paper is organized as follows. We present the time-dependent Navier–Stokes/Darcy problem in section 2 and its HDG discretization in section 3. Consistency and well-posedness of the discrete problem are shown in section 4 while a priori error estimates are proven in section 5. We end this paper with numerical examples in section 6 and conclusions in section 7.

2. THE NAVIER–STOKES AND DARCY PROBLEM

We consider the time-dependent incompressible Navier–Stokes equations coupled to the Darcy equations on a polyhedral domain Ω in $\mathbb{R}^{\dim}$, $\dim = 2, 3$, and on the time interval $J = (0, T)$. The domain is partitioned into two non-overlapping subdomains Ω^s and Ω^d such that $\Omega = \Omega^s \cup \Omega^d$, $\Omega^s \cap \Omega^d = \emptyset$, and $\Gamma^I := \partial\Omega^s \cap \partial\Omega^d$. The boundary of the domain $\partial\Omega$ and the interface Γ^I are assumed to be Lipschitz polyhedral. We define Γ^s and Γ^d to be the exterior boundaries of Ω^s and Ω^d , respectively. We partition $\Gamma^d := \Gamma_N^d \cup \Gamma_D^d$, with $\Gamma_N^d \cap \Gamma_D^d = \emptyset$ and $|\Gamma_N^d| > 0$ and $|\Gamma_D^d| > 0$, and denote the outward unit normal on Γ^j to Ω^j ($j = s, d$) by n .

The Navier–Stokes equations are given by

$$\partial_t u^s + \nabla \cdot (u^s \otimes u^s) + \nabla p^s - \nabla \cdot (2\mu \varepsilon(u^s)) = f^s \quad \text{in } \Omega^s \times J, \quad (1a)$$

$$\nabla \cdot u^s = 0 \quad \text{in } \Omega^s \times J, \quad (1b)$$

where $u^s : \Omega^s \times J \rightarrow \mathbb{R}^{\dim}$ is the velocity in Ω^s , $p^s : \Omega^s \times J \rightarrow \mathbb{R}^{\dim}$ is the pressure in Ω^s , $\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^T)$, $\mu > 0$ is the constant fluid viscosity, and $f^s : \Omega^s \times J \rightarrow \mathbb{R}^{\dim}$ is a body force. In Ω^d the Darcy equations are given by:

$$\mu \kappa^{-1} u^d + \nabla p^d = 0 \quad \text{in } \Omega^d \times J, \quad (2a)$$

$$-\nabla \cdot u^d = f^d \quad \text{in } \Omega^d \times J, \quad (2b)$$

where $u^d : \Omega^d \times J \rightarrow \mathbb{R}^{\dim}$ is the fluid velocity in Ω^d , $p^d : \Omega^d \times J \rightarrow \mathbb{R}$ is the piezometric head in Ω^d , and $\kappa > 0$ is the permeability constant. The Navier–Stokes equations are coupled to the Darcy equations by the following interface conditions

$$u^s \cdot n = u^d \cdot n \quad \text{on } \Gamma^I \times J, \quad (3a)$$

$$-2\mu(\varepsilon(u^s)n)^t = \alpha \mu \kappa^{-1/2} (u^s)^t \quad \text{on } \Gamma^I \times J, \quad (3b)$$

$$(p^s n - 2\mu \varepsilon(u^s)n) \cdot n = p^d \quad \text{on } \Gamma^I \times J, \quad (3c)$$

where n is the unit normal vector on Γ^I pointing from Ω^s to Ω^d , $(v)^t := v - (v \cdot n)n$ is the tangential component of a vector v , and $\alpha > 0$ is an experimentally determined dimensionless constant. Note that eq. (3a) ensures continuity of the normal component of the velocity across the interface, eq. (3b) is the Beavers–Joseph–Saffman

law [4, 38], and eq. (3c) is a balance of forces. We assume the following initial and boundary conditions:

$$u^s(x, 0) = u_0(x) \quad \text{in } \Omega^s, \quad (4a)$$

$$u^s = 0 \quad \text{on } \Gamma^s \times J, \quad (4b)$$

$$u^d \cdot n = 0 \quad \text{on } \Gamma_N^d \times J, \quad (4c)$$

$$p^d = 0 \quad \text{on } \Gamma_D^d \times J, \quad (4d)$$

where $u_0 : \Omega^s \rightarrow \mathbb{R}^{\dim}$ is a solenoidal initial velocity field. We close this section by introducing $u : \Omega \times J \rightarrow \mathbb{R}^{\dim}$ and $p : \Omega \times J \rightarrow \mathbb{R}$ to be the functions such that $u|_{\Omega^j} = u^j$ and $p|_{\Omega^j} = p^j$ for $j = s, d$.

3. THE HDG METHOD

3.1. Notation

Let $j = s, d$. We denote by $\mathcal{T}_h^j = \{K\}$ a conforming triangulation of Ω^j of shape-regular simplices K . Our goal is to present a strongly conservative and pressure robust HDG discretization. A necessary requirement for our discretization to achieve this is that the velocity field is globally $H(\text{div}; \Omega)$ -conforming [35, 36]. For this reason we assume that $\mathcal{T}_h = \mathcal{T}_h^s \cup \mathcal{T}_h^d$ is a matching simplicial mesh, i.e., $\mathcal{T}_h^s$ and $\mathcal{T}_h^d$ match at the interface. Without this assumption, technical complications may arise to obtain a globally $H(\text{div}; \Omega)$ -conforming velocity field. We denote by h_K the diameter of $K \in \mathcal{T}_h$ and define the meshsize as $h := \max_{K \in \mathcal{T}_h} h_K$. A face F is an interior face if for two elements K^+ and K^- in $\mathcal{T}_h$, $F = \partial K^+ \cap \partial K^-$, and a boundary face if $F \in \partial K$ lies on the boundary $\partial\Omega$. The set of all facets in $\bar{\Omega}$ and $\bar{\Omega}^j$ are denoted by, respectively, $\mathcal{F}_h$ and $\mathcal{F}_h^j$, while the set of all facets on the interface Γ^I is denoted by $\mathcal{F}_h^I$. The set of all facets on Γ^j are denoted by $\mathcal{F}_h^{B,j}$ while the set of all facets interior to Ω^j are denoted by $\mathcal{F}_h^{int,j}$. The sets of facets on Γ_N^d and Γ_D^d are denoted by, respectively, $\mathcal{F}_h^{N,d}$ and $\mathcal{F}_h^{D,d}$. By Γ_0 and Γ_0^j we denote the union of facets in $\bar{\Omega}$ and $\bar{\Omega}^j$. The outward unit normal vector on ∂K for any element $K \in \mathcal{T}_h^j$ is denoted by n^j . On the interface Γ^I , $n = n^s = -n^d$. We will drop the superscript j if the definition of the outward unit normal vector is clear.

We partition the time interval J into N equal intervals of length $\Delta t = T/N$. We define $t^n := n\Delta t$ for $n = 0, \dots, N$ and note that $t^0 = 0$ and $t^N = T$. A function f evaluated at $t = t^n$ will be denoted by $f^n := f(t^n)$. Furthermore, we define $\delta f^{n+1} = f^{n+1} - f^n$ and $d_t f^{n+1} = \delta f^{n+1} / \Delta t = (f^{n+1} - f^n) / \Delta t$.

Denoting by $P_m(D)$ the space of polynomials of total degree m on a domain D , we define the following finite element spaces for the velocity approximation:

$$\begin{aligned} X_h &:= \{v_h \in [L^2(\Omega)]^{\dim} : v_h \in [P_k(K)]^{\dim}, \forall K \in \mathcal{T}\}, \\ X_h^j &:= \{v_h \in [L^2(\Omega^j)]^{\dim} : v_h \in [P_k(K)]^{\dim}, \forall K \in \mathcal{T}^j\}, \quad j = s, d, \\ \bar{X}_h &:= \{\bar{v}_h \in [L^2(\Gamma_0^s)]^{\dim} : \bar{v}_h \in [P_k(F)]^{\dim} \forall F \in \mathcal{F}^s, \bar{v}_h = 0 \text{ on } \Gamma^s\}. \end{aligned}$$

For notational purposes, we write $\mathbf{v}_h = (v_h, \bar{v}_h) \in \mathbf{X}_h := X_h \times \bar{X}_h$ and $\mathbf{v}_h^s = (v_h^s, \bar{v}_h) \in \mathbf{X}_h^s := X_h^s \times \bar{X}_h$. Furthermore, for the pressure approximation we define the finite element spaces

$$\begin{aligned} Q_h &:= \{q_h \in L^2(\Omega) : q_h \in P_{k-1}(K), \forall K \in \mathcal{T}\}, \\ Q_h^j &:= \{q_h \in L^2(\Omega^j) : q_h \in P_{k-1}(K), \forall K \in \mathcal{T}^j\}, \quad j = s, d, \\ \bar{Q}_h^s &:= \{\bar{q}_h^s \in L^2(\Gamma_0^s) : \bar{q}_h^s \in P_k(F) \forall F \in \mathcal{F}^s\}, \\ \bar{Q}_h^d &:= \{\bar{q}_h^d \in L^2(\Gamma_0^d) : \bar{q}_h^d \in P_k(F) \forall F \in \mathcal{F}^d, \bar{q}_h^d = 0 \text{ on } \Gamma_D^d\}. \end{aligned}$$

We write $\mathbf{q}_h = (q_h, \bar{q}_h^s, \bar{q}_h^d) \in \mathbf{Q}_h := Q_h \times \bar{Q}_h^s \times \bar{Q}_h^d$ and $\mathbf{q}_h^j = (q_h, \bar{q}_h^j) \in \mathbf{Q}_h^j := Q_h^j \times \bar{Q}_h^j$.

For scalar functions p and q , we define

$$\begin{aligned}
(p, q)_K &:= \int_K pq \, dx, \quad \forall K \in \mathcal{T}_h, & \langle p, q \rangle_{\partial K} &:= \int_{\partial K} pq \, ds, \quad \forall K \in \mathcal{T}_h, \\
\langle p, q \rangle_F &:= \int_F pq \, ds, \quad F \subset \partial K, \quad \forall K \in \mathcal{T}_h, & (p, q)_{\Omega^j} &:= \sum_{K \in \mathcal{T}_h^j} (p, q)_K, \quad j = s, d, \\
\langle p, q \rangle_{\partial \mathcal{T}_h^j} &:= \sum_{K \in \mathcal{T}_h^j} \langle p, q \rangle_{\partial K}, \quad j = s, d, & (p, q)_\Omega &:= \sum_{K \in \mathcal{T}_h} \int_K pq \, dx, \\
\langle p, q \rangle_{\partial \mathcal{T}_h} &:= \sum_{K \in \mathcal{T}_h} \langle p, q \rangle_{\partial K}, & \langle p, q \rangle_{\Gamma^I} &:= \sum_{F \in \mathcal{F}_h^I} \langle p, q \rangle_F.
\end{aligned}$$

Similar notation is used for vector- and matrix-valued functions.

3.2. The semi-discrete problem

An HDG method for the stationary Navier–Stokes and Darcy problem was proposed in [8]. Its extension to the time-dependent problem is given by: Let $u_h^{s,0} \in X_h^s \cap H(\text{div}; \Omega^s)$ be the initial condition for the velocity in Ω^s such that $\nabla \cdot u_h^{s,0} = 0$ pointwise on each $K \in \mathcal{T}_h^s$. For $t \in J$, find $(\mathbf{u}_h(t), \mathbf{p}_h(t)) \in \mathbf{X}_h \times \mathbf{Q}_h$ such that for all $(\mathbf{v}_h, \mathbf{q}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$

$$(\partial_t u_h, v_h)_{\Omega^s} + a_h(u_h; \mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_h) + b_h(\mathbf{u}_h, \mathbf{q}_h) = (f^s, v_h)_{\Omega^s} + (f^d, q_h)_{\Omega^d}. \quad (5)$$

The different forms are defined as:

$$a_h^s(\mathbf{u}, \mathbf{v}) := (2\mu\varepsilon(u), \varepsilon(v))_{\Omega^s} + \langle 2\beta\mu h_K^{-1}(u - \bar{u}), v - \bar{v} \rangle_{\partial \mathcal{T}_h^s} - \langle 2\mu\varepsilon(u)n, v - \bar{v} \rangle_{\partial \mathcal{T}_h^s} - \langle 2\mu\varepsilon(v)n, u - \bar{u} \rangle_{\partial \mathcal{T}_h^s}, \quad (6a)$$

$$a^d(u, v) := (\mu\kappa^{-1}u, v)_{\Omega^d} \quad (6b)$$

$$a^I(\bar{u}, \bar{v}) := \langle \alpha\mu\kappa^{-1/2}\bar{u}^t, \bar{v}^t \rangle_{\Gamma^I}, \quad (6c)$$

$$a_h^L(\mathbf{u}, \mathbf{v}) := a_h^s(\mathbf{u}, \mathbf{v}) + a^d(u, v) + a^I(\bar{u}, \bar{v}), \quad (6d)$$

$$t_h(w; \mathbf{u}, \mathbf{v}) := -(u \otimes w, \nabla v)_{\Omega^s} + \langle \tfrac{1}{2}w \cdot n(u + \bar{u}), v - \bar{v} \rangle_{\partial \mathcal{T}_h^s} + \langle \tfrac{1}{2}|w \cdot n|(u - \bar{u}), v - \bar{v} \rangle_{\partial \mathcal{T}_h^s} + \langle (w \cdot n)\bar{u}, \bar{v} \rangle_{\Gamma^I}, \quad (6e)$$

$$a_h(w; \mathbf{u}, \mathbf{v}) := t_h(w; \mathbf{u}, \mathbf{v}) + a_h^L(\mathbf{u}, \mathbf{v}) \quad (6f)$$

where $\beta > 0$ is a penalty parameter and where a_h^L is the linear part of a_h . For the velocity-pressure coupling we have, for $j = s, d$, the forms:

$$b_h^j(v, \mathbf{q}^j) := -(q, \nabla \cdot v)_{\Omega^j} + \langle \bar{q}^j, v \cdot n^j \rangle_{\partial \mathcal{T}_h^j},$$

$$b_h^{I,j}(\bar{v}, \bar{q}^j) := -\langle \bar{q}^j, \bar{v} \cdot n^j \rangle_{\Gamma^I},$$

$$b_h(\mathbf{v}, \mathbf{q}) := b_h^s(v, \mathbf{q}^s) + b_h^{I,s}(\bar{v}, \bar{q}^s) + b_h^d(v, \mathbf{q}^d) + b_h^{I,d}(\bar{v}, \bar{q}^d).$$

3.3. The fully-discrete problem

Using backward Euler time-stepping, and lagging the convective velocity in the nonlinear term, we obtain the following linear implicit discretization: Let $u_h^{s,0} \in X_h^s \cap H(\text{div}; \Omega^s)$ be the initial condition for the velocity

in Ω^s such that $\nabla \cdot u_h^{s,0} = 0$ pointwise on each $K \in \mathcal{T}_h^s$. Find $(\mathbf{u}_h^{n+1}, \mathbf{p}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{Q}_h$ with $n \geq 0$ such that for all $(\mathbf{v}_h, \mathbf{q}_h) \in \mathbf{X}_h \times \mathbf{Q}_h$:

$$(d_t u_h^{n+1}, v_h)_{\Omega^s} + a_h(u_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_h^{n+1}) + b_h(\mathbf{u}_h^{n+1}, \mathbf{q}_h) = (f^{s,n+1}, v_h)_{\Omega^s} + (f^{d,n+1}, q_h)_{\Omega^d}. \quad (7)$$

Remark 3.1. As observed previously in [8] for the stationary Navier–Stokes and Darcy problem, the velocity solution to eq. (7) satisfies the following properties: (i) it is exactly divergence-free on elements in Ω^s , i.e., $\nabla \cdot u_h^n = 0$ pointwise on each $K \in \mathcal{T}_h^s$; (ii) it satisfies $-\nabla \cdot u_h^n = \Pi_Q^d f^{d,n}$ pointwise on each $K \in \mathcal{T}_h^d$ (where Π_Q^d is the L^2 -projection operator into Q_h^d); (iii) the velocity solution is globally divergence-conforming, i.e., $u_h^n \in H(\text{div}; \Omega)$; and (iv) $u_h^n \cdot n = \bar{u}_h^n \cdot n$ pointwise on each $F \in \mathcal{F}^I$. Furthermore, $u_h^{d,n} \cdot n = 0$ on Γ_N^d and $u_h^{s,n} \cdot n = 0$ on Γ^s .

4. WELL-POSEDNESS

4.1. Preliminary results

Let D be a domain. Norms on $W_p^k(D)$, $L^p(D) = W_p^0(D)$, $H^k(D) = W_2^k(D)$, and $L^2(D)$ are denoted by, respectively, $\|\cdot\|_{W_p^k(D)}$, $\|\cdot\|_{L^p(D)}$, $\|\cdot\|_{k,D}$, and $\|\cdot\|_D$. Furthermore, for two real numbers a, b , and a Banach space X with norm $\|\cdot\|_X$, $L^2(a, b; X)$ is defined as the space of square integrable functions from $[a, b]$ into X with norm $\|f\|_{L^2(a,b;X)} := (\int_a^b \|f(t)\|_X^2 dt)^{1/2}$ and $L^\infty(a, b; X)$ is the space of essentially bounded functions from $[a, b]$ to X with norm $\|f\|_{L^\infty(a,b;X)} := \text{ess sup}_{[a,b]} \|f(t)\|_X$.

On Ω^s and Ω^d , we define the function spaces

$$\begin{aligned} X^s &:= \{v \in H^2(\Omega^s)^{\dim} : v = 0 \text{ on } \Gamma^s\}, & Q^s &:= H^1(\Omega^s), \\ X^d &:= \{v \in H^1(\Omega^d)^{\dim} : v \cdot n = 0 \text{ on } \Gamma_N^d\}, & Q^d &:= \{q \in H^2(\Omega^d) : q = 0 \text{ on } \Gamma_D^d\}. \end{aligned}$$

On Ω , we then define $X := \{v \in H(\text{div}; \Omega) : u^s \in X^s, u^d \in X^d\}$ and $Q := \{q \in L^2(\Omega) : q^s \in Q^s, q^d \in Q^d\}$. The trace space of X^s on facets in Γ_0^s is denoted by $\bar{X}$. If $u \in X^s$, we denote its trace by $\bar{u} := \gamma_X(u)$ where $\gamma_X : X^s \rightarrow \bar{X}$ is the trace operator restricting functions in X^s to Γ_0^s . Similarly, the trace space of Q^j on facets Γ_0^j is denoted by $\bar{Q}^j$, $\gamma_{Q^j} : Q^j \rightarrow \bar{Q}^j$ is the trace operator, and if $q \in Q^j$, then $\bar{q} := \gamma_{Q^j}(q) \in \bar{Q}^j$.

Using the notation $\mathbf{X} := X \times \bar{X}$ and $\mathbf{Q} := Q \times \bar{Q}^s \times \bar{Q}^d$, we define

$$X(h) := X_h + X, \quad X^s(h) := X_h^s + X^s, \quad \mathbf{X}(h) := \mathbf{X}_h + \mathbf{X}, \quad \mathbf{Q}(h) := \mathbf{Q}_h + \mathbf{Q}.$$

As in [8], we define the following norms on the extended function spaces:

$$\begin{aligned} \|\mathbf{v}\|_v^2 &:= \|\mathbf{v}\|_{v,s}^2 + \|\mathbf{v}\|_{v,d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2 & \mathbf{v} &\in \mathbf{X}(h), \\ \|\mathbf{v}\|_{v'}^2 &:= \|\mathbf{v}\|_v^2 + \sum_{K \in \mathcal{T}_h^s} h_K^2 |v|_{2,K}^2 = \|\mathbf{v}\|_{v',s}^2 + \|\mathbf{v}\|_{v,d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2 & \mathbf{v} &\in \mathbf{X}(h), \\ \|\mathbf{q}\|_p^2 &:= \|\mathbf{q}^s\|_{p,s}^2 + \|\mathbf{q}^d\|_{p,d}^2 & \mathbf{q} &\in \mathbf{Q}(h), \end{aligned}$$

where

$$\begin{aligned}
\|\mathbf{v}\|_{v,s}^2 &:= \sum_{K \in \mathcal{T}_h^s} (\|\nabla v\|_K^2 + h_K^{-1} \|v - \bar{v}\|_{\partial K}^2), \\
\|\mathbf{v}\|_{v',s}^2 &:= \|\mathbf{v}\|_{v,s}^2 + \sum_{K \in \mathcal{T}_h^s} h_K^2 |v|_{2,K}^2, \\
\|\mathbf{v}\|_{v,d}^2 &:= \|v\|_{\text{div};\Omega^d}^2 + \sum_{F \in \mathcal{F}_h^d \setminus (\mathcal{F}_h^I \cup \mathcal{F}_h^{D,d})} h_F^{-1} \|\llbracket v \cdot n \rrbracket\|_F^2 + \sum_{K \in \mathcal{T}_h^d} h_K^{-1} \|(v - \bar{v}) \cdot n\|_{\partial K \cap \Gamma^I}^2, \\
\|\mathbf{q}^j\|_{p,j}^2 &:= \|q\|_{\Omega^j}^2 + \sum_{K \in \mathcal{T}_h^j} h_K \|\bar{q}^j\|_{\partial K}^2, \quad j = s, d.
\end{aligned}$$

Here $\llbracket v \cdot n \rrbracket$ is the usual jump operator and $\|v\|_{\text{div};\Omega^d}^2 := \|v\|_{\Omega^d}^2 + \|\nabla \cdot v\|_{\Omega^d}^2$. Let us furthermore note that $\|v_h\|_{1,h,\Omega^s} := \| (v_h, \llbracket v_h \rrbracket) \|_{v,s}$, where $\|v_h\|_{1,h,\Omega^s}$ is the standard discrete H^1 -norm of v_h in Ω^s [6]. Finally, we will also require the following two norms on the pressure in Ω^d :

$$\begin{aligned}
\|q_h\|_{1,h,\Omega^d}^2 &:= \sum_{K \in \mathcal{T}_h^d} \|\nabla q_h\|_K^2 + \sum_{F \in \mathcal{F}_h^{int,d} \cup \mathcal{F}_h^{D,d}} h_F^{-1} \|\llbracket q_h \rrbracket\|_F^2 & \forall q_h \in Q_h^d, \\
\|\mathbf{q}_h\|_{1,h,d}^2 &:= \sum_{K \in \mathcal{T}_h^d} (\|\nabla q_h\|_K^2 + h_K^{-1} \|q_h - \bar{q}_h\|_{\partial K}^2) & \forall \mathbf{q}_h \in \mathbf{Q}_h^d.
\end{aligned}$$

That $\|q_h\|_{1,h,\Omega^d}$ is a norm on Q_h^d follows because $|\Gamma_D^d| > 0$.

The following inequalities will be used in the remainder of this paper (see [42, eq. (5.5)], [23, Theorem 4.4 and Proposition 4.5], and [17, Lemma 1.46]):

$$\begin{aligned}
\|\mathbf{v}_h\|_v &\leq \|\mathbf{v}_h\|_{v'} \leq c_e \|\mathbf{v}_h\|_v & \forall \mathbf{v}_h \in \mathbf{X}_h, & (8a) \\
\|v_h\|_{\Omega^s} &\leq c_p \|v_h\|_{1,h,\Omega^s} \leq c_p \|\mathbf{v}_h\|_{v,s} & \forall \mathbf{v}_h \in \mathbf{X}_h^s, & (8b) \\
\|q_h\|_{\Omega^d} &\leq c_{pp} \|q_h\|_{1,h,\Omega^d} \leq c_{pp} \|\mathbf{q}_h\|_{1,h,d} & \forall \mathbf{q}_h \in \mathbf{Q}_h^d, & (8c) \\
\|v_h^s\|_{L^r(\Gamma^I)} &\leq c_{si,r} \|v_h\|_{1,h,\Omega^s} \leq c_{si,r} \|\mathbf{v}_h\|_{v,s} & \forall \mathbf{v}_h \in \mathbf{X}_h^s, \quad r \geq 2, & (8d) \\
\|v\|_{\partial K} &\leq c_{tr} h_K^{-1/2} \|v\|_K & \forall v \in P_k(K), \quad K \in \mathcal{T}_h, & (8e)
\end{aligned}$$

where c_e , c_p , $c_{si,r}$, and c_{tr} are positive constants independent of h and Δt .

For b_h we have:

$$c_{bb} \|\mathbf{q}_h\|_p \leq \sup_{0 \neq \mathbf{v}_h \in \mathbf{X}_h} \frac{b_h(\mathbf{v}_h, \mathbf{q}_h)}{\|\mathbf{v}_h\|_v} \quad \forall \mathbf{q}_h \in \mathbf{Q}_h, \quad (9a)$$

$$|b_h(\mathbf{v}, \mathbf{q})| \leq c_{bc} \|\mathbf{v}\|_v \|\mathbf{q}\|_p \quad \forall (\mathbf{v}, \mathbf{q}) \in \mathbf{X}(h) \times \mathbf{Q}_h. \quad (9b)$$

Due to the use of different function spaces, the inf-sup condition eq. (9a) is different from that proven in [8]. We therefore prove eq. (9a) in appendix A. Equation (9b) is proven in [8, Lemma 3]. For a_h^s , a^d , and a^I , we have that for all $\mathbf{u}, \mathbf{v} \in \mathbf{X}(h)$,

$$|a_h^s(\mathbf{u}, \mathbf{v})| \leq \mu c_{ac}^s \|\mathbf{u}\|_{v',s} \|\mathbf{v}\|_{v',s}, \quad |a^d(u, v)| \leq \mu \kappa^{-1} \|u\|_{\Omega^d} \|v\|_{\Omega^d}, \quad |a^I(\bar{u}, \bar{v})| \leq \alpha \mu \kappa^{-1/2} \|\bar{u}^t\|_{\Gamma^I} \|\bar{v}^t\|_{\Gamma^I}, \quad (10)$$

where $c_{ac}^s > 0$ is a constant independent of h and Δt . For $\mathbf{v}_h \in \mathbf{X}_h$ we have

$$a_h^s(\mathbf{v}_h, \mathbf{v}_h) \geq \mu c_{ae}^s \|\mathbf{v}_h\|_{v,s}^2, \quad a^d(v_h, v_h) \geq \mu \kappa^{-1} \|v_h\|_{\Omega^d}^2, \quad a^I(\bar{v}_h, \bar{v}_h) \geq \alpha \mu \kappa^{-1/2} \|\bar{v}_h^t\|_{\Gamma^I}^2, \quad (11)$$

where the first inequality holds for β large enough and where $c_{ae}^s > 0$ is a constant independent of h and Δt . A direct consequence of eqs. (10) and (11) is that

$$|a_h^L(\mathbf{u}, \mathbf{v})| \leq \mu c_{ac}^L \|\mathbf{u}\|_{v'} \|\mathbf{v}\|_{v'} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}(h), \quad (12a)$$

$$|a_h^L(\mathbf{v}_h, \mathbf{v}_h)| \geq \mu c_{ae}^L \|\mathbf{v}_h\|_v^2 \quad \forall \mathbf{v}_h \in \mathbf{X}_h, \quad (12b)$$

where $c_{ac}^L := \max(c_{ac}^s, \kappa^{-1}, \alpha \kappa^{-1/2}) > 0$ and $c_{ae}^L := \min(c_{ae}^s, \kappa^{-1}, \alpha \kappa^{-1/2}) > 0$ are constants independent of h and Δt , and where eq. (12b) holds for β large enough.

We also recall the following inequality from [8, Lemma 4], [6, Proposition 3.4] related to the form t_h . Assuming that $w_1, w_2 \in X^s(h) \cap H(\text{div}; \Omega^s)$ are such that $\nabla \cdot w_1 = \nabla \cdot w_2 = 0$ on each $K \in \mathcal{T}^s$ it holds for any $\mathbf{u} \in \mathbf{X}^s(h)$, $\mathbf{v} \in \mathbf{X}_h^s$ that

$$|t_h(w_1; \mathbf{u}, \mathbf{v}) - t_h(w_2; \mathbf{u}, \mathbf{v})| \leq c_w \|w_1 - w_2\|_{1,h,\Omega^s} \|\mathbf{u}\|_{v,s} \|\mathbf{v}\|_{v,s}, \quad (13)$$

where $c_w > 0$ is a constant independent of h and Δt .

Assuming $w \in X^s(h) \cap H(\text{div}; \Omega^s)$ is such that $\nabla \cdot w = 0$ on each $K \in \mathcal{T}_h^s$, then [8, Lemma 5]

$$|a_h(w; \mathbf{u}, \mathbf{v})| \leq c_{ac} \mu \|\mathbf{u}\|_{v'} \|\mathbf{v}\|_{v'} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}(h), \quad (14a)$$

$$|a_h(w; \mathbf{u}_h, \mathbf{v}_h)| \leq c_{ac} \mu \|\mathbf{u}_h\|_v \|\mathbf{v}_h\|_v \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h, \quad (14b)$$

where $c_{ac} = 2c_e^2 \max(c_w \mu^{-1} \|w\|_{1,h,\Omega^s} + c_{ac}^s, \kappa^{-1}, \alpha \kappa^{-1/2})$. Let us now define

$$\begin{aligned} \mathbf{Z}_h^s &:= \{\mathbf{v}_h \in \mathbf{X}_h^s : b_h^s(v_h, \mathbf{q}_h^s) + b_h^{I,s}(\bar{v}_h, \bar{\mathbf{q}}_h^s) = 0 \ \forall \mathbf{q}_h^s \in \mathbf{Q}_h^s\}, \\ \mathbf{Z}_h &:= \{\mathbf{v}_h \in \mathbf{X}_h : \sum_{j=s,d} (b_h^j(v_h, \mathbf{q}_h^j) + b_h^{I,j}(\bar{v}_h, \bar{\mathbf{q}}_h^j)) = 0 \ \forall \mathbf{q}_h \in \mathbf{Q}_h\}. \end{aligned}$$

If $w \in X^s(h) \cap H(\text{div}; \Omega^s)$ such that $\nabla \cdot w = 0$ on each $K \in \mathcal{T}_h^s$ and $\|w \cdot n\|_{\Gamma_I} \leq \frac{1}{2} \mu c_{ae}^s / (c_{pq}^2 + c_{si,4}^2)$ on the interface, and if β is large enough that the first inequality in eq. (11) holds, then it was shown in [8, Lemma 6] that,

$$a_h(w; \mathbf{v}_h, \mathbf{v}_h) \geq c_{ae} \mu \|\mathbf{v}_h\|_v^2 \quad \forall \mathbf{v}_h \in \mathbf{Z}_h, \quad (15)$$

where $c_{ae} = \min(\frac{1}{2} c_{ae}^s, \kappa^{-1}, \alpha \kappa^{-1/2})$.

Using a proof similar to [8, Lemma 1], it is straightforward to obtain the following result.

Lemma 4.1 (Consistency). *Suppose that (u, p) is the solution to eqs. (1) to (4) that satisfies $u \in L^2(J; X)$, $p \in L^2(J; Q)$, and $\partial_t u \in L^2(J; L^2(\Omega^s))$. Let $\mathbf{u} = (u, \bar{u})$ and $\mathbf{p} = (p, \bar{p}^s, \bar{p}^d)$ and assume that $f^s \in C^0(J; L^2(\Omega^s)^{\dim})$ and $f^d \in C^0(J; L^2(\Omega^d))$. Then $(\mathbf{u}, \mathbf{p})$ satisfies eq. (5) for all $t > 0$.*

4.2. Existence and uniqueness

We start this section with some auxiliary results.

Lemma 4.2. *For $\mathbf{p}_h^{d,n}$ and $u_h^{d,n}$ that satisfy eq. (7), there exists a $c_{pd} > 0$, independent of h and Δt , such that*

$$\|\mathbf{p}_h^{d,n}\|_{1,h,d} \leq c_{pd} \mu \kappa^{-1} \|u_h^{d,n}\|_{\Omega^d}. \quad (16)$$

Proof. We will prove eq. (16) in three dimensions only noting that the proof in two dimensions is similar. To ease notation we will drop the “time” superscript n . The proof follows the proof of [34, Lemma 2.1] with modifications made to take into account Brezzi–Douglas–Marini (BDM) elements and HDG facet functions.

The local degrees of freedom for the BDM element are [5, Proposition 2.3.2]:

$$\langle v_h \cdot n, \bar{r}_h \rangle_{\partial K}, \ \forall \bar{r}_h \in R_k(\partial K) \quad \text{and} \quad (v_h, z_h)_K, \ \forall z_h \in \mathcal{N}_{k-2}(K), \quad (17)$$

where $R_k(\partial K) := \{\bar{r} \in L^2(\partial K) : \bar{r}|_F \in P_k(F), \forall F \subset \partial K\}$ and $\mathcal{N}_{k-2}(K)$ is the Nédélec space. Therefore, given $\mathbf{p}_h^d \in \mathbf{Q}_h^d$, we define $w_h \in V_h^d \cap H(\text{div}; \Omega^d)$ such that

$$\langle w_h \cdot n, \bar{r}_h \rangle_{\partial K} = h_K^{-1} \langle p_h^d - \bar{p}_h^d, \bar{r}_h \rangle_{\partial K} \quad \forall \bar{r}_h \in R_k(\partial K), \forall K \in \mathcal{T}_h^d, \quad (18a)$$

$$(w_h, z_h)_K = -(\nabla p_h^d, z_h)_K \quad \forall z_h \in \mathcal{N}_{k-2}(K), \forall K \in \mathcal{T}_h^d. \quad (18b)$$

Since $\nabla p_h^d \in \nabla P_{k-1}(K) \subset [P_{k-2}]^3 \subset \mathcal{N}_{k-2}(K)$ and since $p_h^d - \bar{p}_h^d \in R_k(\partial K)$, we obtain from eq. (18) that

$$\langle w_h \cdot n, p_h^d - \bar{p}_h^d \rangle_{\partial K} = h_K^{-1} \|p_h^d - \bar{p}_h^d\|_{\partial K}^2 \quad \forall K \in \mathcal{T}_h^d, \quad (19a)$$

$$(w_h, \nabla p_h^d)_K = -\|\nabla p_h^d\|_K^2 \quad \forall K \in \mathcal{T}_h^d. \quad (19b)$$

Setting now $\mathbf{v}_h^s = 0$ and $\mathbf{q}_h = 0$ in eq. (7), and after integration by parts, we find for all $v_h \in V_h^d$ that:

$$\begin{aligned} 0 &= (\mu\kappa^{-1}u_h^d, v_h)_{\Omega^d} - (p_h^d, \nabla \cdot v_h)_{\Omega^d} + \langle \bar{p}_h^d, v_h \cdot n^d \rangle_{\partial \mathcal{T}_h^d} \\ &= (\mu\kappa^{-1}u_h^d, v_h)_{\Omega^d} + (\nabla p_h^d, v_h)_{\Omega^d} - \langle p_h^d - \bar{p}_h^d, v_h \cdot n^d \rangle_{\partial \mathcal{T}_h^d}. \end{aligned} \quad (20)$$

Choose $v_h = w_h$, with w_h defined in eq. (18). By eq. (19), eq. (20), and the definition of $\|\mathbf{p}_h^d\|_{1,h,d}^2$, we find

$$\|\mathbf{p}_h^d\|_{1,h,d}^2 = (\mu\kappa^{-1}u_h^d, w_h)_{\Omega^d} \leq \mu\kappa^{-1} \|u_h^d\|_{\Omega^d} \|w_h\|_{\Omega^d}. \quad (21)$$

To find out more about $\|w_h\|_{\Omega^d}$, let us define the norm $\|\cdot\|_{0,h}$ for functions in $V_h^d \cap H(\text{div}; \Omega^d)$:

$$\|w_h\|_{0,h}^2 := \|w_h\|_{\Omega^d}^2 + \sum_{F \in \mathcal{F}_h^d} h_F \|w_h \cdot n\|_F^2. \quad (22)$$

Consider now a single element K and denote by $\mathcal{F}_K$ the set of faces of K . In an approach similar to that used in the proof of [33, Lemma 4.4], we have:

$$\begin{aligned} \|w_h\|_K^2 + \sum_{F \in \mathcal{F}_K} h_F \|w_h \cdot n\|_F^2 &\lesssim \sup_{\substack{z_h \in \mathcal{N}_{k-2}(K)^3 \\ \|z_h\|_K=1}} |(w_h, z_h)_K|^2 + \sup_{\substack{\bar{r}_h \in R_k(\partial K) \\ \|\bar{r}_h\|_{\partial K}=1}} h_K |\langle w_h \cdot n, \bar{r}_h \rangle_F|^2 \\ &= \sup_{\substack{z_h \in \mathcal{N}_{k-2}(K)^3 \\ \|z_h\|_K=1}} |(\nabla p_h^d, z_h)_K|^2 + \sup_{\substack{\bar{r}_h \in R_k(\partial K) \\ \|\bar{r}_h\|_{\partial K}=1}} h_K h_K^{-2} |\langle p_h^d - \bar{p}_h^d, \bar{r}_h \rangle_{\partial K}|^2 \\ &\leq \|\nabla p_h^d\|_K^2 + h_K^{-1} \|p_h^d - \bar{p}_h^d\|_{\partial K}^2, \end{aligned} \quad (23)$$

where the first line on the right hand side is by using the degrees of freedom eq. (17), the second by definition of w_h given by eq. (18), and the last is by the Cauchy–Schwarz inequality. Therefore, after summing eq. (23) over all K in $\mathcal{T}_h^d$:

$$\|w_h\|_{\Omega^d}^2 \leq \|w_h\|_{0,h}^2 \lesssim \sum_{K \in \mathcal{T}_h^d} (\|\nabla p_h^d\|_K^2 + h_K^{-1} \|p_h^d - \bar{p}_h^d\|_{\partial K}^2) = \|\mathbf{p}_h^d\|_{1,h,d}^2.$$

The result follows after combining this with eq. (21). $\square$

An immediate consequence of eq. (8c) and Lemma 4.2 is that if $\mathbf{p}_h^{d,n}$ and $u_h^{d,n}$ satisfy eq. (7), then for $1 \leq n \leq N$:

$$\|p_h^{d,n}\|_{\Omega^d} \leq c_{pp} \|p_h^{d,n}\|_{1,h,\Omega^d} \leq c_{pp} \|\mathbf{p}_h^{d,n}\|_{1,h,d} \leq c_{td} \mu \kappa^{-1} \|u_h^{d,n}\|_{\Omega^d}, \quad (24)$$

where $c_{td} = c_{pp}c_{pd}$.

The following result, which was shown in [12, Theorem 5.2], will be used to prove the next lemma: there exists a constant $c > 0$, independent of h and Δt , such that

$$|\langle q_h, v_h \cdot n \rangle_{\Gamma^I}| \leq c \|q_h\|_{1,h,\Omega^d} \|v_h\|_{\Omega^s} \quad \forall v_h \in \tilde{V}_h^s, \quad \forall q_h \in Q_h^d, \quad (25)$$

where $\tilde{V}_h^s := \{v_h \in X_h^s : b_s(v_h, q_h) = 0 \quad \forall q_h \in Q_h^s\}$ with $b_s(v, q) := -(q, \nabla \cdot v)_{\Omega^s} + \sum_{F \in \mathcal{F}_h^{int,s} \cup \mathcal{F}_h^{B,s}} \langle \llbracket q \rrbracket, \llbracket v \rrbracket \cdot n \rangle_F$.

Lemma 4.3. *Let $u_h^{s,n}$, $u_h^{d,n}$ and $\bar{p}_h^{d,n}$ be (part of) the solution to eq. (7). There exists a constant $c_{sdi} > 0$, independent of h and Δt , such that for all $n \geq 1$*

$$|\langle \bar{p}_h^{d,n}, u_h^{s,n} \cdot n \rangle_{\Gamma^I}| \leq c_{sdi} \mu \kappa^{-1} \|u_h^{d,n}\|_{\Omega^d} \|u_h^{s,n}\|_{\Omega^s}. \quad (26)$$

Proof. For ease of notation we will drop the “time” superscript n . Then, note that

$$|\langle \bar{p}_h^d, \bar{u}_h^s \cdot n \rangle_{\Gamma^I}| \leq |\langle \bar{p}_h^d - p_h^d, \bar{u}_h^s \cdot n \rangle_{\Gamma^I}| + |\langle p_h^d, \bar{u}_h^s \cdot n \rangle_{\Gamma^I}| \leq |\langle \bar{p}_h^d - p_h^d, u_h^s \cdot n \rangle_{\Gamma^I}| + |\langle p_h^d, u_h^s \cdot n \rangle_{\Gamma^I}|. \quad (27)$$

Since u_h^s is a solution to eq. (7), by Remark 3.1 we know that $\nabla \cdot u_h^s = 0$ and $\llbracket u_h^s \rrbracket \cdot n = 0$ on $F \in \mathcal{F}_h^{int,s} \cup \mathcal{F}_h^{B,s}$ so that $u_h^s \in \tilde{V}_h^s$. Therefore, using eq. (25),

$$|\langle p_h^d, u_h^s \cdot n \rangle_{\Gamma^I}| \leq C \|u_h^s\|_{\Omega^s} \|p_h^d\|_{1,h,\Omega^d}. \quad (28)$$

Next, using eq. (8e) and Lemma 4.2, we note that

$$\begin{aligned} |\langle \bar{p}_h^d - p_h^d, u_h^s \cdot n \rangle_{\Gamma^I}| &\leq \left(\sum_{K \in \mathcal{T}_h^s} h_K \|u_h^s \cdot n\|_{\partial K}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h^d} h_K^{-1} \|\bar{p}_h^d - p_h^d\|_{\partial K}^2 \right)^{1/2} \\ &\leq C \|u_h^s\|_{\Omega^s} \left(\sum_{K \in \mathcal{T}_h^d} h_K^{-1} \|\bar{p}_h^d - p_h^d\|_{\partial K}^2 \right)^{1/2} \\ &\leq C \|u_h^s\|_{\Omega^s} \|\mathbf{p}_h^d\|_{1,h,d} \leq C \mu \kappa^{-1} \|u_h^s\|_{\Omega^s} \|u_h^d\|_{\Omega^d}. \end{aligned} \quad (29)$$

The result follows by combining eqs. (24) and (27) to (29). $\square$

For the remainder of this section we define

$$\mathbf{B}_h^s := \{\mathbf{v}_h^s \in \mathbf{Z}_h^s : \|\mathbf{v}_h^s\|_{v,s} \leq \frac{1}{2} \mu \min(c_{ae}^s c_{si,2}^{-1} (c_{pq}^2 + c_{si,4}^2)^{-1}, c_{ae} c_w^{-1})\}.$$

Let us remark that by eq. (8d) with $r = 2$, $\|v_h^s \cdot n\|_{L^2(\Gamma^I)} \leq \frac{1}{2} \mu c_{ae}^s (c_{pq}^2 + c_{si,4}^2)^{-1}$ for all $\mathbf{v}_h^s \in \mathbf{B}_h^s$. This result, in combination with eq. (15), is used in the following lemma to prove a well-posedness result.

Lemma 4.4. *For $0 \leq n \leq N - 1$, let $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$. Then eq. (7) has a unique solution $(\mathbf{u}_h^{n+1}, \mathbf{p}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{Q}_h$.*

Proof. Consider eq. (7) for the solution at time level t^{n+1} which we write here as:

$$\begin{aligned} \frac{1}{\Delta t} (u_h^{s,n+1}, v_h)_{\Omega^s} + a_h(u_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}_h) + b_h(v_h, \mathbf{p}_h^{n+1}) + b_h(\mathbf{u}_h^{n+1}, q_h) \\ = \frac{1}{\Delta t} (u_h^{s,n}, v_h)_{\Omega^s} + (f^{s,n+1}, v_h)_{\Omega^s} + (f^{d,n+1}, q_h)_{\Omega^d}. \end{aligned} \quad (30)$$

Given $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$ we remark that, by eq. (8d) with $r = 2$ and eq. (15),

$$\frac{1}{\Delta t} (v_h, v_h)_{\Omega^s} + a_h(u_h^n; \mathbf{v}_h, \mathbf{v}_h) \geq c_{ae} \mu \|\mathbf{v}_h\|_v^2 \quad \forall \mathbf{v}_h \in \mathbf{Z}_h, \quad (31)$$

Furthermore, by eq. (14b) and eq. (8b), we obtain the following boundedness result:

$$\frac{1}{\Delta t}(u_h^s, v_h)_{\Omega^s} + |a_h(u_h^{s,n}; \mathbf{u}_h, \mathbf{v}_h)| \leq (\frac{1}{\Delta t}c_p^2 + c_f\mu) \|\mathbf{u}_h\|_v \|\mathbf{v}_h\|_v \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h, \quad (32)$$

where

$$c_f = 2c_e^2 \max(\frac{1}{2} \min(c_w c_{ae}^s c_{si,2}^{-1} (c_{pq}^2 + c_{si,4}^2)^{-1}, c_{ae}) + c_{ac}^s, \kappa^{-1}, \alpha \kappa^{-1/2}).$$

Here c_f is an upper bound for c_{ac} using that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$. Since $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$, boundedness of the right hand side of eq. (30) follows from the Cauchy–Schwarz inequality. Existence of a unique solution $(\mathbf{u}_h^{n+1}, \mathbf{p}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{Q}_h$ to eq. (7) is now a consequence of eq. (31), eq. (32), eq. (9) and [5, Theorem 3.4.3]. $\square$

Lemma 4.4 guarantees existence and uniqueness of a solution $(\mathbf{u}_h^{n+1}, \mathbf{p}_h^{n+1}) \in \mathbf{X}_h \times \mathbf{Q}_h$ at time level $n+1$ provided that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$. However, Lemma 4.4 does not guarantee that $\mathbf{u}_h^{s,n+1} \in \mathbf{B}_h^s$. Therefore, the remainder of this section is dedicated to showing that $\mathbf{u}_h^{s,n+1} \in \mathbf{B}_h^s$ under a smallness assumption on the data. First we obtain bounds on $\|d_t u_h^{s,k}\|_{\Omega^s}$ and $\Delta t^{-1/2} \|\mathbf{u}_h^{s,1}\|_{v,s}$ (which are proven in Lemma 4.5) after which we prove a bound on $\|\mathbf{u}_h^k\|_v$ (see Lemma 4.6). The steps used to obtain these results are similar to [12]. In Lemma 4.7 we then impose a smallness assumption on the data to show existence and uniqueness of the solution $(\mathbf{u}_h^n, \mathbf{p}_h^n) \in \mathbf{X}_h \times \mathbf{Q}_h$ for all time levels $1 \leq n \leq N$.

The following lemmas will be proven in three dimensions with similar proofs holding for two dimensions. We assume that $f^s \in C^0(J; L^2(\Omega^s)^3)$ and $f^d \in C^0(J; L^2(\Omega^d))$. It will furthermore be useful to define

$$F^m := \frac{c_p^2}{c_{ae}\mu} \Delta t \sum_{k=1}^m \|d_t f^{s,k+1}\|_{\Omega^s}^2 + \frac{c_{td}^2 \mu}{\kappa^2 c_{ae}} \Delta t \sum_{k=1}^m \|d_t f^{d,k+1}\|_{\Omega^d}^2, \quad (33)$$

$$(M^m)^2 := (M^0)^2 + \frac{1}{2} c_{ae} \mu G^2 + F^m, \quad (34)$$

where

$$M^0 := (1 + c_{sdi}(\frac{\mu \Delta t}{2\kappa})^{1/2}) \|f^{s,1}\|_{L^2(\Omega^s)^3} + c_{sdi} \mu \kappa^{-1} c_{td} \|f^{d,1}\|_{L^2(\Omega^d)} \quad (35)$$

$$G^2 := \frac{1}{2c_{ae}^s} ((\frac{1}{\mu} + \frac{c_{sdi}^2}{2\kappa} \Delta t) \|f^{s,1}\|_{L^2(\Omega^s)^3}^2 + \frac{c_{sdi}^2 c_{td}^2 \mu}{\kappa^2} \|f^{d,1}\|_{L^2(\Omega^d)}^2). \quad (36)$$

Lemma 4.5. *Let $\mathbf{u}_h^{s,0} = \mathbf{0}$ and let M^0 , G , F^m and G^m be defined as in eqs. (33) to (36). Suppose that eq. (7) has a solution $(\mathbf{u}_h^k, \mathbf{p}_h^k)$ for all $1 \leq k \leq n$. For $k = 1$,*

$$\|d_t u_h^{s,1}\|_{\Omega^s} \leq M^0, \quad (37a)$$

$$\frac{1}{(\Delta t)^{1/2}} \|\mathbf{u}_h^{s,1}\|_{v,s} \leq G. \quad (37b)$$

Furthermore, if $\mathbf{u}_h^{s,k} \in \mathbf{B}_h^s$ for all $0 \leq k \leq n$, with $1 \leq n \leq N-1$, then

$$\|d_t u_h^{s,n+1}\|_{\Omega^s} \leq M^n. \quad (38)$$

Proof. We first prove eq. (37). Choose $\mathbf{v}_h^s = \mathbf{u}_h^{s,1}$, $\mathbf{q}_h^s = -\mathbf{p}_h^{s,1}$, $v_h^d = 0$, $q_h^d = 0$, and $\bar{q}_h^d = -\bar{p}_h^{d,1}$ in eq. (7). At $n = 0$, since $u_h^{s,0} = 0$, this reduces to:

$$\frac{1}{\Delta t} \|u_h^{s,1}\|_{\Omega^s}^2 + a_h^s(\mathbf{u}_h^{s,1}, \mathbf{u}_h^{s,1}) + a^I(\bar{u}_h^{s,1}, \bar{u}_h^{s,1}) = (f^{s,1}, u_h^{s,1})_{\Omega^s} + b_h^d(u_h^{d,1}, (0, \bar{p}_h^{d,1})). \quad (39)$$

We bound the second term on the right hand side:

$$\begin{aligned} |b_h^d(u_h^{d,1}, (0, \bar{p}_h^{d,1}))| &= |\langle \bar{p}_h^{d,1}, u_h^{d,1} \cdot n^d \rangle_{\partial \mathcal{T}_h^d}| = |\langle \bar{p}_h^{d,1}, u_h^{d,1} \cdot n^d \rangle_{\Gamma^I}| \\ &= |\langle \bar{p}_h^{d,1}, u_h^{s,1} \cdot n \rangle_{\Gamma^I}| \leq c_{sdi} \mu \kappa^{-1} \|u_h^{d,1}\|_{\Omega^d} \|u_h^{s,1}\|_{\Omega^s}, \end{aligned}$$

where the first equality is by definition, the second equality is because $\bar{p}_h^{d,1}$ and $u_h^{d,1} \cdot n^d$ are single-valued on $F \in \mathcal{F}_h^{int,d}$, $\bar{p}_h^{d,1} = 0$ on Γ_D^d and $u_h^{d,1} \cdot n^d = 0$ on Γ_N^d , and the third equality is because $u_h^1 \in H(\text{div}; \Omega)$ (see Remark 3.1). Finally, the inequality is by Lemma 4.3. Combining this with eq. (39), the coercivity of a_h^s and a^I eq. (11), and the Cauchy–Schwarz inequality,

$$\frac{1}{\Delta t} \|u_h^{s,1}\|_{\Omega^s}^2 + \mu c_{ae}^s \|u_h^{s,1}\|_{v,s}^2 + \alpha \mu \kappa^{-1/2} \|(\bar{u}_h^{s,1})^t\|_{\Gamma^I}^2 \leq \|f^{s,1}\|_{\Omega^s} \|u_h^{s,1}\|_{\Omega^s} + c_{sdi} \mu \kappa^{-1} \|u_h^{d,1}\|_{\Omega^d} \|u_h^{s,1}\|_{\Omega^s}, \quad (40)$$

directly implying, after ignoring the non-negative second and third terms on the left hand side, and canceling $\|u_h^{s,1}\|_{\Omega^s}$,

$$\|d_t u_h^{s,1}\|_{\Omega^s} \leq \|f^{s,1}\|_{\Omega^s} + c_{sdi} \mu \kappa^{-1} \|u_h^{d,1}\|_{\Omega^d}. \quad (41)$$

Furthermore, applying Young's inequality to both terms on the right hand side of eq. (40) we also find:

$$\frac{1}{\Delta t} \|u_h^{s,1}\|_{\Omega^s}^2 + \mu c_{ae}^s \|u_h^{s,1}\|_{v,s}^2 + \alpha \mu \kappa^{-1/2} \|(\bar{u}_h^{s,1})^t\|_{\Gamma^I}^2 \leq \frac{1}{2\psi} \|f^{s,1}\|_{\Omega^s}^2 + \frac{c_{sdi}^2 \mu^2}{2\kappa^2 \psi} \|u_h^{d,1}\|_{\Omega^d}^2 + \psi \|u_h^{s,1}\|_{\Omega^s}^2. \quad (42)$$

Choosing $\psi = 1/\Delta t$ and reordering,

$$\frac{1}{\Delta t} \|u_h^{s,1}\|_{v,s}^2 \leq \frac{1}{2\mu c_{ae}^s} \|f^{s,1}\|_{\Omega^s}^2 + \frac{c_{sdi}^2 \mu^2}{2\kappa^2 c_{ae}^s} \|u_h^{d,1}\|_{\Omega^d}^2. \quad (43)$$

To further bound eqs. (41) and (43) we require a bound on $\|u_h^{d,1}\|_{\Omega^d}^2$. To obtain this bound, we set $n = 0$ and choose $(\mathbf{v}_h, \mathbf{q}_h) = (\mathbf{u}_h^1, -\mathbf{p}_h^1)$ in eq. (7) and recall that $u_h^{s,0} = 0$ to find

$$(d_t u_h^{s,1}, u_h^{s,1})_{\Omega^s} + a_h^L(\mathbf{u}_h^1, \mathbf{u}_h^1) = (f^{s,1}, u_h^{s,1})_{\Omega^s} - (f^{d,1}, p_h^{d,1})_{\Omega^d}.$$

Using that $(d_t u_h^{s,1}, u_h^{s,1})_{\Omega^s} = \Delta t^{-1} \|u_h^{s,1}\|_{\Omega^s}^2$, the Cauchy–Schwarz inequality, eqs. (11) and (24), and Young's inequality,

$$\begin{aligned} \frac{1}{\Delta t} \|u_h^{s,1}\|_{\Omega^s}^2 + c_{ae}^s \mu \|u_h^{s,1}\|_{v,s}^2 + \mu \kappa^{-1} \|u_h^{d,1}\|_{\Omega^d}^2 &\leq \|f^{s,1}\|_{\Omega^s} \|u_h^{s,1}\|_{\Omega^s} + \|f^{d,1}\|_{\Omega^d} c_{td} \mu \kappa^{-1} \|u_h^{d,1}\|_{\Omega^d} \\ &\leq \frac{1}{2\psi} \|f^{s,1}\|_{\Omega^s}^2 + \frac{\psi}{2} \|u_h^{s,1}\|_{\Omega^s}^2 + \frac{c_{td}^2 \mu^2}{2\kappa^2 \phi} \|f^{d,1}\|_{\Omega^d}^2 + \frac{\phi}{2} \|u_h^{d,1}\|_{\Omega^d}^2. \end{aligned} \quad (44)$$

Choosing $\psi = 2/\Delta t$ and $\phi = \mu \kappa^{-1}$, we find from eq. (44), after reordering, that

$$\|u_h^{d,1}\|_{\Omega^d}^2 \leq \frac{\kappa \Delta t}{2\mu} \|f^{s,1}\|_{L^2(\Omega^s)}^2 + c_{td}^2 \|f^{d,1}\|_{L^2(\Omega^d)}^2. \quad (45)$$

Equation (37a) follows from eq. (45) and eq. (41). Equation (37b) follows from eq. (45) and eq. (43).

We proceed with proving eq. (38). Let $1 \leq n \leq N - 1$. Consider eq. (7) at time levels $n + 1$ and n :

$$\begin{aligned} (d_t u_h^{s,n+1}, v_h^s)_{\Omega^s} + a_h(u_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_h^{n+1}) + b_h(\mathbf{u}_h^{n+1}, \mathbf{q}_h) &= (f^{s,n+1}, v_h)_{\Omega^s} + (f^{d,n+1}, q_h)_{\Omega^d}, \\ (d_t u_h^{s,n}, v_h^s)_{\Omega^s} + a_h(u_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{p}_h^n) + b_h(\mathbf{u}_h^n, \mathbf{q}_h) &= (f^{s,n}, v_h)_{\Omega^s} + (f^{d,n}, q_h)_{\Omega^d}. \end{aligned}$$

Subtracting the latter from the former, choosing $(\mathbf{v}_h, \mathbf{q}_h) = (\delta \mathbf{u}_h^{n+1}, -\delta \mathbf{p}_h^{n+1})$, and noting that

$$\begin{aligned} t_h(u_h^{s,n}; \mathbf{u}_h^{n+1}, \delta \mathbf{u}_h^{n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) \\ = t_h(u_h^{s,n}; \delta \mathbf{u}_h^{n+1}, \delta \mathbf{u}_h^{n+1}) + t_h(u_h^{s,n}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}), \end{aligned}$$

we find

$$\begin{aligned} \frac{1}{\Delta t} (\delta u_h^{s,n+1} - \delta u_h^{s,n}, \delta u_h^{s,n+1})_{\Omega^s} + t_h(u_h^{s,n}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) \\ + a_h(u_h^{s,n}; \delta \mathbf{u}_h^{n+1}, \delta \mathbf{u}_h^{n+1}) = (\delta f^{s,n+1}, \delta u_h^{s,n+1})_{\Omega^s} + (\delta f^{d,n+1}, \delta p_h^{d,n+1})_{\Omega^d}. \quad (46) \end{aligned}$$

Equation (8d) and $\mathbf{u}_h^{s,k} \in \mathbf{B}_h^s$ imply $\|u_h^k \cdot n\|_{\Gamma^I} \leq \frac{1}{2} \mu c_{ae}^s (c_{pq}^2 + c_{si,4}^2)^{-1}$ for $0 \leq k \leq n$. Therefore, coercivity of a_h eq. (15) holds. Also using the Cauchy-Schwarz inequality and eq. (8b):

$$\begin{aligned} \frac{1}{\Delta t} (\delta u_h^{s,n+1} - \delta u_h^{s,n}, \delta u_h^{s,n+1})_{\Omega^s} + t_h(u_h^{s,n}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{n+1}) \\ + c_{ae} \mu \|\delta \mathbf{u}_h^{n+1}\|_v^2 \leq c_p \|\delta f^{s,n+1}\|_{\Omega^s} \|\delta \mathbf{u}_h^{s,n+1}\|_{v,s} + \|\delta f^{d,n+1}\|_{\Omega^d} \|\delta p_h^{d,n+1}\|_{\Omega^d}. \quad (47) \end{aligned}$$

A simple modification of the proof of Lemma 4.2 allows us to show that $\|\delta p_h^{d,n+1}\|_{1,h,\Omega^d} \leq C \mu \kappa^{-1} \|\delta u_h^{d,n+1}\|_{\Omega^d}$. Then, following the same steps used to find eq. (24), $\|\delta p_h^{d,n+1}\|_{\Omega^d} \leq c_{td} \mu \kappa^{-1} \|\delta u_h^{d,n+1}\|_{\Omega^d}$ so that

$$\begin{aligned} \frac{1}{\Delta t} (\delta u_h^{s,n+1} - \delta u_h^{s,n}, \delta u_h^{s,n+1})_{\Omega^s} + c_{ae} \mu \|\delta \mathbf{u}_h^{n+1}\|_v^2 \\ \leq c_p \|\delta f^{s,n+1}\|_{\Omega^s} \|\delta \mathbf{u}_h^{s,n+1}\|_{v,s} + c_{td} \mu \kappa^{-1} \|\delta f^{d,n+1}\|_{\Omega^d} \|\delta u_h^{d,n+1}\|_{\Omega^d} \\ + |t_h(u_h^{s,n}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{s,n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{s,n+1})|. \quad (48) \end{aligned}$$

To bound the convective terms we use eq. (13), eq. (8b), and Young's inequality:

$$\begin{aligned} |t_h(u_h^{s,n}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{s,n+1}) - t_h(u_h^{s,n-1}; \mathbf{u}_h^n, \delta \mathbf{u}_h^{s,n+1})| &\leq c_w \|\delta \mathbf{u}_h^{s,n}\|_{v,s} \|\mathbf{u}_h^{s,n}\|_{v,s} \|\delta \mathbf{u}_h^{s,n+1}\|_{v,s} \\ &\leq \frac{c_w}{2} \|\mathbf{u}_h^{s,n}\|_{v,s} \|\delta \mathbf{u}_h^{s,n}\|_{v,s}^2 + \frac{c_w}{2} \|\mathbf{u}_h^{s,n}\|_{v,s} \|\delta \mathbf{u}_h^{s,n+1}\|_{v,s}^2. \end{aligned}$$

Applying Young's inequality to the first two terms on the right hand side of eq. (48),

$$\begin{aligned} c_p \|\delta f^{s,n+1}\|_{\Omega^s} \|\delta \mathbf{u}_h^{s,n+1}\|_{v,s} + c_{td} \mu \kappa^{-1} \|\delta f^{d,n+1}\|_{\Omega^d} \|\delta u_h^{d,n+1}\|_{\Omega^d} \\ \leq \frac{c_p^2}{2\phi} \|\delta f^{s,n+1}\|_{\Omega^s}^2 + \frac{c_{td}^2 \mu^2}{2\kappa^2 \phi} \|\delta f^{d,n+1}\|_{\Omega^d}^2 + \frac{\phi}{2} (\|\delta \mathbf{u}_h^{s,n+1}\|_{v,s}^2 + \|\delta \mathbf{u}_h^{d,n+1}\|_{v,d}^2), \end{aligned}$$

and choosing $\phi = c_{ae} \mu$, we find after combining with eq. (48) that

$$\begin{aligned} \frac{1}{\Delta t} (\delta u_h^{s,n+1} - \delta u_h^{s,n}, \delta u_h^{s,n+1})_{\Omega^s} + (\frac{1}{2} c_{ae} \mu - \frac{c_w}{2} \|\mathbf{u}_h^{s,n}\|_{v,s}) \|\delta \mathbf{u}_h^{n+1}\|_v^2 \\ \leq \frac{c_p^2}{2c_{ae} \mu} \|\delta f^{s,n+1}\|_{\Omega^s}^2 + \frac{c_{td}^2 \mu}{2\kappa^2 c_{ae}} \|\delta f^{d,n+1}\|_{\Omega^d}^2 + \frac{c_w}{2} \|\mathbf{u}_h^{s,n}\|_{v,s} \|\delta \mathbf{u}_h^{s,n}\|_{v,s}^2. \quad (49) \end{aligned}$$

Multiplying both sides by 2, using the assumption that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$, that $a(a-b) \geq \frac{1}{2}(a^2 - b^2)$, and that $\|\delta \mathbf{u}_h^{n+1}\|_{v,s} \leq \|\delta \mathbf{u}_h^{n+1}\|_v$,

$$\begin{aligned} \frac{1}{\Delta t} \|\delta u_h^{s,n+1}\|_{\Omega^s}^2 - \frac{1}{\Delta t} \|\delta u_h^{s,n}\|_{\Omega^s}^2 + \frac{1}{2} c_{ae} \mu \|\delta \mathbf{u}_h^{n+1}\|_{v,s}^2 \\ \leq \frac{c_p^2}{c_{ae} \mu} \|\delta f^{s,n+1}\|_{\Omega^s}^2 + \frac{c_{td}^2 \mu}{\kappa^2 c_{ae}} \|\delta f^{d,n+1}\|_{\Omega^d}^2 + \frac{1}{2} c_{ae} \mu \|\delta \mathbf{u}_h^{s,n}\|_{v,s}^2. \end{aligned} \quad (50)$$

Replacing n by k , summing for $k = 1$ to $k = n$, using that $d_t u_h^{s,n+1} = \Delta t^{-1} \delta u_h^{s,n+1}$ and that $\delta \mathbf{u}_h^{s,1} = \mathbf{u}_h^{s,1}$ (because $\mathbf{u}_h^{s,0} = \mathbf{0}$), and the definition of F^n (see eq. (33)):

$$\|d_t u_h^{s,n+1}\|_{\Omega^s}^2 \leq \|d_t u_h^{s,1}\|_{\Omega^s}^2 + \frac{c_{ae} \mu}{2 \Delta t} \|\mathbf{u}_h^{s,1}\|_{v,s}^2 + F^n. \quad (51)$$

Equation (38) now follows by inserting eqs. (37a) and (37b) into the above inequality. $\square$

We are now ready to prove a bound on $\mathbf{u}_h^{n+1}$.

Lemma 4.6. *Let $\mathbf{u}_h^{s,0} = \mathbf{0}$, and let M^n be defined as in eqs. (34) and (35) for $0 \leq n \leq N-1$. If eq. (7) has a solution $(\mathbf{u}_h^k, \mathbf{p}_h^k)$ for all $0 \leq k \leq n$ such that $\mathbf{u}_h^{s,k} \in \mathbf{B}_h^s$, then*

$$\|\mathbf{u}_h^{n+1}\|_v^2 \leq \frac{2}{c_{ae} \mu} \left(\frac{c_p^2}{c_{ae} \mu} (M^n)^2 + \frac{c_p^2}{c_{ae} \mu} \|f^s\|_{L^\infty(J; L^2(\Omega^s))}^2 + \frac{c_{td}^2 \mu}{2 \kappa^2 c_{ae}} \|f^d\|_{L^\infty(J; L^2(\Omega^d))}^2 \right). \quad (52)$$

Proof. Choose $(\mathbf{v}_h, \mathbf{q}_h) = (\mathbf{u}_h^{n+1}, -\mathbf{p}_h^{n+1})$ in eq. (7). Coercivity of a_h eq. (15) (which holds by eq. (8d) and the assumption that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$) then implies:

$$(d_t u_h^{s,n+1}, u_h^{s,n+1})_{\Omega^s} + c_{ae} \mu \|\mathbf{u}_h^{n+1}\|_v^2 \leq (f^{s,n+1}, u_h^{s,n+1})_{\Omega^s} - (f^{d,n+1}, p_h^{d,n+1})_{\Omega^d}. \quad (53)$$

Using the Cauchy-Schwarz inequality, eqs. (8b) and (24), Young's inequality and eq. (38), we obtain:

$$\begin{aligned} c_{ae} \mu \|\mathbf{u}_h^{n+1}\|_v^2 &\leq |(f^{s,n+1}, u_h^{s,n+1})_{\Omega^s} - (f^{d,n+1}, p_h^{d,n+1})_{\Omega^d} - (d_t u_h^{s,n+1}, u_h^{s,n+1})_{\Omega^s}| \\ &\leq c_p \|f^{s,n+1}\|_{\Omega^s} \|\mathbf{u}_h^{s,n+1}\|_{v,s} + c_{td} \mu \kappa^{-1} \|f^{d,n+1}\|_{\Omega^d} \|u_h^{d,n+1}\|_{\Omega^d} + c_p \|d_t u_h^{s,n+1}\|_{\Omega^s} \|\mathbf{u}_h^{s,n+1}\|_{v,s} \\ &\leq \frac{c_p^2}{2\chi} (M^n)^2 + \frac{c_p^2}{2\chi} \|f^{s,n+1}\|_{\Omega^s}^2 + \chi \|\mathbf{u}_h^{s,n+1}\|_{v,s}^2 + \frac{c_{td}^2 \mu^2}{2 \kappa^2 \phi} \|f^{d,n+1}\|_{\Omega^d}^2 + \frac{\phi}{2} \|u_h^{d,n+1}\|_{\Omega^d}^2. \end{aligned}$$

The result follows by choosing $\chi = \frac{1}{2} c_{ae} \mu$, and $\phi = c_{ae} \mu$, and using the definition of $\|\cdot\|_v$. $\square$

We end this section by proving existence and uniqueness for all time levels under a suitable data assumption.

Lemma 4.7. *Let M^n be defined as in eq. (34). Assume the data satisfy for $1 \leq n \leq N-1$,*

$$\begin{aligned} \frac{2}{c_{ae} \mu} \left(\frac{c_p^2}{c_{ae} \mu} (M^n)^2 + \frac{c_p^2}{c_{ae} \mu} \|f^s\|_{L^\infty(J; L^2(\Omega^s))}^2 + \frac{c_{td}^2 \mu}{2 \kappa^2 c_{ae}} \|f^d\|_{L^\infty(J; L^2(\Omega^d))}^2 \right) \\ \leq \left[\min \left(\frac{\mu c_{ae}^s}{2 c_{si,2} (c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae} \mu}{2 c_w} \right) \right]^2. \end{aligned} \quad (54)$$

Then, starting with $\mathbf{u}_h^{s,0} = \mathbf{0}$, eq. (7) has a unique solution. Furthermore, for $1 \leq n \leq N$, the velocity solution is such that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$ and the pressure solution satisfies,

$$\|\mathbf{p}_h^n\|_p^2 \leq \left(\frac{1}{2} c_{ae}^2 + c_{ac}^2 \right) \frac{3\mu^2}{c_{bb}^2} \left[\min \left(\frac{\mu c_{ae}^s}{2 c_{si,2} (c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae} \mu}{2 c_w} \right) \right]^2. \quad (55)$$

Proof. Existence and uniqueness of $(\mathbf{u}_h^{n+1}, \mathbf{p}_h^{n+1})$ under the assumption that $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$ for $0 \leq n \leq N$ is established by Lemma 4.4. That $\mathbf{u}_h^{s,n} \in \mathbf{B}_h^s$ for $1 \leq n \leq N$ is due to eqs. (52) and (54).

We now prove the pressure bound eq. (55). By the inf-sup condition eq. (9a) and the HDG method eq. (7), with $\mathbf{q}_h = \mathbf{0}$, we find for $0 \leq n \leq N-1$:

$$c_{bb} \|\mathbf{p}_h^{n+1}\|_p \leq \sup_{0 \neq \mathbf{v}_h \in \mathbf{X}_h} \frac{|b_h(\mathbf{v}_h, \mathbf{p}_h^{n+1})|}{\|\mathbf{v}_h\|_v} = \sup_{0 \neq \mathbf{v}_h \in \mathbf{X}_h} \frac{|(f^{s,n+1}, v_h)_{\Omega^s} - (d_t u_h^{n+1}, v_h)_{\Omega^s} - a_h(u_h^n; \mathbf{u}_h^{n+1}, v_h)|}{\|\mathbf{v}_h\|_v}.$$

By the Cauchy–Schwarz inequality, eqs. (8b) and (14b), squaring and using Hölder’s inequality for sums,

$$\|\mathbf{p}_h^{n+1}\|_p^2 \leq 3(c_p^2 c_{bb}^{-2} \|f^{s,n+1}\|_{\Omega^s}^2 + c_p^2 c_{bb}^{-2} \|d_t u_h^{n+1}\|_{\Omega^s}^2 + c_{ac}^2 c_{bb}^{-2} \mu^2 \|\mathbf{u}_h^{n+1}\|_v^2).$$

A bound for $\|\mathbf{u}_h^{n+1}\|_v$ is given by Lemma 4.6 and the data assumption eq. (54). Together with eq. (38) we obtain

$$\|\mathbf{p}_h^{n+1}\|_p^2 \leq 3(c_p^2 c_{bb}^{-2} (\|f^s\|_{L^\infty(J; L^2(\Omega^s))}^2 + (M^n)^2) + c_{ac}^2 c_{bb}^{-2} \mu^2 [\min(\frac{\mu c_{ae}^s}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae}\mu}{2c_w})]^2). \quad (56)$$

Note that the data assumption eq. (54) implies that

$$c_p^2 c_{bb}^{-2} (\|f^s\|_{L^\infty(J; L^2(\Omega^s))}^2 + (M^n)^2) \leq \frac{1}{2} c_{ae}^2 c_{bb}^{-2} \mu^2 [\min(\frac{\mu c_{ae}^s}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae}\mu}{2c_w})]^2. \quad (57)$$

The result follows from eqs. (56) and (57). $\square$

5. A PRIORI ERROR ESTIMATES

Let Π_Q be the L^2 -projection into Q_h and let $\bar{\Pi}_V$ and $\bar{\Pi}_Q^j$, $j = s, d$, be the L^2 -projections into the facet spaces $\bar{V}_h$ and $\bar{Q}_h^j$, $j = s, d$, respectively. Let $\Pi_V : H(\text{div}; \Omega) \cap [L^r(\Omega)]^{\text{dim}} \rightarrow X_h \cap H(\text{div}; \Omega)$, where $r > 2$, be an interpolant such that

$$(q_h, \nabla \cdot \Pi_V u)_K = (q_h, \nabla \cdot u)_K \quad \forall q_h \in P_{k-1}(K), \quad (58)$$

$$\langle \bar{q}_h, n \cdot \Pi_V u \rangle_F = \langle \bar{q}_h, n \cdot u \rangle_F \quad \forall \bar{q}_h \in P_k(F), \quad \forall \text{ faces } F \text{ of } K, \quad (59)$$

and with the properties that for any $u \in [H^{k+1}(K)]^{\text{dim}}$,

$$\|u - \Pi_V u\|_{m,K} \leq C h_K^{\ell-m} \|u\|_{\ell,K} \quad m = 0, 1, 2, \quad \max(1, m) \leq \ell \leq k+1, \quad (60)$$

and for any $u \in [W_\infty^1(K)]^{\text{dim}}$,

$$\|u - \Pi_V u\|_{L^\infty(K)} \leq C h_K |u|_{W_\infty^1(K)}. \quad (61)$$

Examples of such operators are the Brezzi–Douglas–Marini (BDM) and Raviart–Thomas (RT) interpolation operators [5].

We partition the errors into their interpolation and approximation parts as $\zeta - \zeta_h = e_\zeta^I - e_\zeta^h$ for $\zeta = u, \bar{u}, p, \bar{p}^j$ and for $j = s, d$, where

$$\begin{aligned} e_u^I &= u - \Pi_V u, & e_u^h &= u_h - \Pi_V u, & e_p^I &= p - \Pi_Q p, & e_p^h &= p_h - \Pi_Q p, \\ \bar{e}_u^I &= \gamma(u) - \bar{\Pi}_V u, & \bar{e}_u^h &= \bar{u}_h - \bar{\Pi}_V u, & \bar{e}_{p^j}^I &= \gamma(p^j) - \bar{\Pi}_Q^j p, & \bar{e}_{p^j}^h &= \bar{p}_h^j - \bar{\Pi}_Q^j p. \end{aligned}$$

To be consistent with the notation for elements in $\mathbf{X}_h, \mathbf{Q}_h, \mathbf{Q}_h^j$, $j = s, d$, we also define

$$\mathbf{e}_u^\zeta = (e_u^\zeta, \bar{e}_u^\zeta), \quad \mathbf{e}_p^\zeta = (e_p^\zeta, \bar{e}_{p^s}^\zeta, \bar{e}_{p^d}^\zeta), \quad \mathbf{e}_{p^j}^\zeta = (e_{p^j}^\zeta, \bar{e}_{p^j}^\zeta), \quad \zeta = I, h.$$

In the following we will use that the initial condition is given by $u_h^{s,0} = \Pi_V u_0$ and so $e_u^{h,0} = 0$.

To determine the error equation we first note that by Lemma 4.1, the exact solution $(\mathbf{u}, \mathbf{p})$ satisfies eq. (5). Therefore, subtracting eq. (5) at time level $t = t^{n+1}$, with $(\mathbf{u}_h, \mathbf{p}_h)$ replaced by $(\mathbf{u}, \mathbf{p})$, from eq. (7), splitting the errors into their interpolation and approximation parts, using that $b_h(\mathbf{v}_h, \mathbf{e}_p^{I,n+1}) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_h$ (since $\Pi_Q, \bar{\Pi}_Q$ are L^2 -projections onto Q_h and $\bar{Q}_h$, respectively, and $\nabla \cdot \mathbf{V}_h = Q_h$) and that $b_h(\mathbf{e}_u^{I,n+1}, \mathbf{q}_h) = 0$ for all $\mathbf{q}_h \in \mathbf{Q}_h$ (by eqs. (58) and (59) and properties of the L^2 -projection $\bar{\Pi}_V$) we obtain:

$$\begin{aligned} (d_t e_u^{h,n+1}, v_h)_{\Omega^s} + t_h(u_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}_h) - t_h(u^{n+1}; \mathbf{u}^{n+1}, \mathbf{v}_h) + a_h^L(\mathbf{e}_u^{h,n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{e}_p^{h,n+1}) + b_h(\mathbf{e}_u^{h,n+1}, \mathbf{q}_h) \\ = (d_t e_u^{I,n+1}, v_h)_{\Omega^s} + (\partial_t u^{n+1} - d_t u^{n+1}, v_h)_{\Omega^s} + a_h^L(\mathbf{e}_u^{I,n+1}, \mathbf{v}_h). \end{aligned} \quad (62)$$

The following theorem now determines an upper bound for the approximation error $\mathbf{e}_u^{h,n}$.

Theorem 5.1. *Suppose that $u \in L^\infty(0, T; [H^{k+1}(\Omega)]^{\dim})$ such that $u^s \in L^2(0, T; [W_3^1(\Omega^s)]^{\dim})$, $\partial_t u \in L^2(0, T; [H^k(\Omega^s)]^{\dim})$, and $\partial_{tt} u \in L^2(0, T; [L^2(\Omega^s)]^{\dim})$. Suppose also that the data satisfies the assumptions of Lemma 4.7. Then, for $1 \leq m \leq N$,*

$$\begin{aligned} \|\mathbf{e}_u^{h,m}\|_{\Omega^s}^2 + \Delta t^2 \sum_{n=0}^{m-1} \|d_t \mathbf{e}_u^{h,n+1}\|_{\Omega^s}^2 + c_{ae} \mu \Delta t \sum_{n=0}^{m-1} \|\mathbf{e}_u^{h,n+1}\|_v^2 \\ \leq C C_G [h^{2k} \{\mu^{-1} \|\partial_t u\|_{L^2(J; H^k(\Omega^s))}^2 + T((\mu + \mu^{-1}) \|u\|_{L^\infty(J; H^{k+1}(\Omega^s))}^2 \|u\|_{L^\infty(J; H^{k+1}(\Omega))}^2) \\ + (\Delta t)^2 \mu^{-1} \{\|\partial_{tt} u\|_{L^2(J; L^2(\Omega^s))}^2 + \|\partial_t u\|_{L^2(J; L^2(\Omega^s))}^2 \|u\|_{L^\infty(J; H^1(\Omega^s))}^2\}], \end{aligned} \quad (63)$$

where $C_G = \exp(\Delta t \sum_{n=0}^{m-1} C \mu^{-1} \|u^{n+1}\|_{W_3^1(\Omega^s)}^2)$.

Proof. Consider the convective terms in eq. (62). We note that

$$\begin{aligned} t_h(u_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}_h) - t_h(u^{n+1}; \mathbf{u}^{n+1}, \mathbf{v}_h) = t_h(u_h^n; \mathbf{e}_u^{h,n+1}, \mathbf{v}_h) - t_h(u^{n+1}; \mathbf{e}_u^{I,n+1}, \mathbf{v}_h) \\ + [t_h(u_h^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h) - t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h)] \\ + [t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h) - t_h(u^{n+1}; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h)]. \end{aligned}$$

We furthermore note that

$$\begin{aligned} a_h(u_h^n; \mathbf{e}_u^{h,n+1}, \mathbf{v}_h) &= t_h(u_h^n; \mathbf{e}_u^{h,n+1}, \mathbf{v}_h) + a_h^L(\mathbf{e}_u^{h,n+1}, \mathbf{v}_h), \\ a_h(u^{n+1}; \mathbf{e}_u^{I,n+1}, \mathbf{v}_h) &= t_h(u^{n+1}; \mathbf{e}_u^{I,n+1}, \mathbf{v}_h) + a_h^L(\mathbf{e}_u^{I,n+1}, \mathbf{v}_h), \end{aligned}$$

so that we can write eq. (62) as

$$\begin{aligned} (d_t e_u^{h,n+1}, v_h)_{\Omega^s} + a_h(u_h^n; \mathbf{e}_u^{h,n+1}, \mathbf{v}_h) + b_h(\mathbf{v}_h, \mathbf{e}_p^{h,n+1}) + b_h(\mathbf{e}_u^{h,n+1}, \mathbf{q}_h) \\ = (d_t e_u^{I,n+1}, v_h)_{\Omega^s} + (\partial_t u^{n+1} - d_t u^{n+1}, v_h)_{\Omega^s} + a_h(u^{n+1}; \mathbf{e}_u^{I,n+1}, \mathbf{v}_h) \\ + [t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h) - t_h(u_h^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h)] + [t_h(u^{n+1}; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h) - t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{v}_h)]. \end{aligned} \quad (64)$$

Let us now choose $(\mathbf{v}_h, \mathbf{q}_h) = (\mathbf{e}_u^{h,n+1}, -\mathbf{e}_p^{h,n+1})$ in eq. (64). By the assumption on the data we have coercivity of a_h eq. (15) so that:

$$\begin{aligned}
(d_t \mathbf{e}_u^{h,n+1}, \mathbf{e}_u^{h,n+1})_{\Omega^s} + c_{ae} \mu \|\mathbf{e}_u^{h,n+1}\|_v^2 &\leq (d_t \mathbf{e}_u^{I,n+1}, \mathbf{e}_u^{h,n+1})_{\Omega^s} + (\partial_t u^{n+1} - d_t u^{n+1}, \mathbf{e}_u^{h,n+1})_{\Omega^s} \\
&\quad + a_h(u^{n+1}; \mathbf{e}_u^{I,n+1}, \mathbf{e}_u^{h,n+1}) \\
&\quad + [t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u_h^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\
&\quad + [t_h(u^{n+1}; \mathbf{\Pi}_V u^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u^n; \mathbf{\Pi}_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\
&=: \sum_{j=1}^5 I_j.
\end{aligned} \tag{65}$$

Using eq. (74), eq. (60), eq. (8b), and Young's inequality we find:

$$\begin{aligned}
I_1 &\leq \|d_t \mathbf{e}_u^{I,n+1}\|_{\Omega^s} \|\mathbf{e}_u^{h,n+1}\|_{\Omega^s} \leq C h^k (\Delta t)^{-1/2} \|\partial_t u\|_{L^2(t^n, t^{n+1}; H^k(\Omega^s))} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\
&\leq \gamma \|\mathbf{e}_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{2k} (\Delta t)^{-1} \|\partial_t u\|_{L^2(t^n, t^{n+1}; H^k(\Omega^s))}^2,
\end{aligned} \tag{66}$$

where $\gamma > 0$ will be chosen later. By eq. (72), eq. (8b), and Young's inequality,

$$\begin{aligned}
I_2 &\leq \|\partial_t u^{n+1} - d_t u^{n+1}\|_{\Omega^s} \|\mathbf{e}_u^{h,n+1}\|_{\Omega^s} \\
&\leq C (\Delta t)^{1/2} \|\partial_{tt} u\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\
&\leq \gamma \|\mathbf{e}_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} \Delta t \|\partial_{tt} u\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}^2.
\end{aligned} \tag{67}$$

Observe that by eq. (14a), eq. (8a), [8, Lemma 7], and Young's inequality,

$$\begin{aligned}
I_3 &\leq c_{ac} \mu \|\mathbf{e}_u^{I,n+1}\|_{v'} \|\mathbf{e}_u^{h,n+1}\|_{v'} \leq C \mu h^k \|u^{n+1}\|_{k+1, \Omega} \|\mathbf{e}_u^{h,n+1}\|_v \\
&\leq \gamma \|\mathbf{e}_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} \mu^2 h^{2k} \|u^{n+1}\|_{k+1, \Omega}^2.
\end{aligned}$$

For I_4 we have

$$I_4 \leq 2\gamma \|\mathbf{e}_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{2k} \|u^{n+1}\|_{k+1, \Omega^s}^2 \|u^n\|_{k+1, \Omega^s}^2 + \frac{C}{\gamma} \|\mathbf{e}_u^{h,n}\|_{\Omega^s}^2 \|u^{n+1}\|_{W_3^1(\Omega^s)}^2, \tag{68}$$

the proof of which, due to its length, is given in appendix C.

By eq. (13), eq. (74), properties of Π_V and $\bar{\Pi}_V$ so that $\|\mathbf{\Pi}_V u^{n+1}\|_{v,s} \leq c \|u^{n+1}\|_{1, \Omega^s}$ (see [36, Eq. (28)]) and Young's inequality,

$$\begin{aligned}
I_5 &\leq c_w \|\nabla(u^{n+1} - u^n)\|_{\Omega^s} \|\mathbf{\Pi}_V u^{n+1}\|_{v,s} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\
&\leq C (\Delta t)^{1/2} \|\partial_t u\|_{L^2(t^n, t^{n+1}; H^1(\Omega^s))} \|u^{n+1}\|_{1, \Omega^s} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\
&\leq \gamma \|\mathbf{e}_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} \Delta t \|\partial_t u\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}^2 \|u^{n+1}\|_{1, \Omega^s}^2.
\end{aligned}$$

Collecting the above estimates for $I_1, \dots, I_5$, combining with eq. (65), using that $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$, choosing $\gamma = \frac{1}{12}c_{ae}\mu$, and multiplying by $2\Delta t$:

$$\begin{aligned} & (\|e_u^{h,n+1}\|_{\Omega^s}^2 - \|e_u^{h,n}\|_{\Omega^s}^2) + \|e_u^{h,n+1} - e_u^{h,n}\|_{\Omega^s}^2 + c_{ae}\mu\Delta t \|e_u^{h,n+1}\|_v^2 \\ & \leq C[h^{2k}\{\mu^{-1}\|\partial_t u\|_{L^2(t^n, t^{n+1}; H^k(\Omega^s))}^2 + \Delta t\mu\|u^{n+1}\|_{k+1, \Omega}^2 + \Delta t\mu^{-1}\|u^n\|_{k+1, \Omega^s}^2\|u^{n+1}\|_{k+1, \Omega^s}^2\} \\ & \quad + (\Delta t)^2\mu^{-1}\{\|\partial_{tt}u\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}^2 + \|\partial_t u\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}^2\|u^{n+1}\|_{1, \Omega^s}^2\} \\ & \quad + \Delta t\mu^{-1}\|u^{n+1}\|_{W_3^1(\Omega^s)}^2\|e_u^{h,n}\|_{\Omega^s}^2]. \end{aligned}$$

Summing from $n = 0$ to $n = m - 1$ and noting that $e_u^{h,0} = 0$ gives

$$\begin{aligned} & \|e_u^{h,m}\|_{\Omega^s}^2 + \Delta t^2 \sum_{n=0}^{m-1} \|d_t e_u^{h,n+1}\|_{\Omega^s}^2 + c_{ae}\mu\Delta t \sum_{n=0}^{m-1} \|e_u^{h,n+1}\|_v^2 \\ & \leq Ch^{2k}\{\mu^{-1}\|\partial_t u\|_{L^2(J; H^k(\Omega^s))}^2 + T((\mu + \mu^{-1})\|u\|_{L^\infty(J; H^{k+1}(\Omega^s))}^2)\|u\|_{L^\infty(J; H^{k+1}(\Omega))}^2\} \\ & \quad + C\Delta t^2\mu^{-1}\{\|\partial_{tt}u\|_{L^2(J; L^2(\Omega^s))}^2 + \|\partial_t u\|_{L^2(J; L^2(\Omega^s))}^2\|u\|_{L^\infty(J; H^1(\Omega^s))}^2\} \\ & \quad + C\Delta t \sum_{n=0}^{m-1} \mu^{-1}\|u^{n+1}\|_{W_3^1(\Omega^s)}^2\|e_u^{h,n}\|_{\Omega^s}^2. \end{aligned}$$

The result now follows by Grönwall's inequality [29, Lemma 28] for all $\Delta t > 0$. $\square$

By a triangle inequality and properties of the interpolant Π_V and projection $\bar{\Pi}_V$, we obtain the following velocity error estimate that is independent of the pressure.

Corollary 5.2. *Suppose that u , $\mathbf{u}_h$, and the data satisfy the assumptions of Theorem 5.1. Then, for $1 \leq m \leq N$,*

$$\begin{aligned} & \|u^m - u_h^m\|_{\Omega^s}^2 + c_{ae}\mu\Delta t \sum_{n=0}^{m-1} \|\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}\|_v^2 \\ & \leq C[h^{2k}\{\mu^{-1}\|\partial_t u\|_{L^2(J; H^k(\Omega^s))}^2 + (1 + T(\mu + \mu^{-1})\|u\|_{L^\infty(J; H^{k+1}(\Omega^s))}^2)\|u\|_{L^\infty(J; H^{k+1}(\Omega))}^2\} \\ & \quad + (\Delta t)^2\mu^{-1}\{\|\partial_{tt}u\|_{L^2(J; L^2(\Omega^s))}^2 + \|\partial_t u\|_{L^2(J; L^2(\Omega^s))}^2\|u\|_{L^\infty(J; H^1(\Omega^s))}^2\}]. \end{aligned}$$

6. NUMERICAL EXAMPLES

We implement the fully discrete HDG method eq. (7) in Netgen/NGSolve [39, 40]. For all examples we choose the penalty parameter as $\beta = 8k^2$ (see [1, 37]), where k is the polynomial degree in the approximation spaces.

6.1. Rates of convergence

In this section we verify the rates of convergence by the method of manufactured solutions. For this we consider the domains $\Omega^s = (0, 1) \times (0, 0.5)$ and $\Omega^d = (0, 1) \times (-0.5, 0)$. The interface is given by $\Gamma^I = \bar{\Omega}^s \cap \bar{\Omega}^d$ while $\Gamma_D^s = \{x \in \Gamma^s : x_1 = 0 \text{ or } x_2 = 0.5\}$, $\Gamma_N^s = \Gamma^s \setminus \Gamma_D^s$, $\Gamma_D^d = \{x \in \Gamma^d : x_2 = -0.5\}$, and $\Gamma_N^d = \Gamma^d \setminus \Gamma_D^d$. To construct a manufactured solution, we consider the following inhomogeneous boundary conditions and modified

interface conditions:

$$\begin{aligned}
u^s &= U^s && \text{on } \Gamma_D^s \times J, \\
\sigma_d^s n &= S^s && \text{on } \Gamma_N^s \times J, \\
u^d \cdot n &= U^d && \text{on } \Gamma_N^d \times J, \\
p^d &= P^d && \text{on } \Gamma_D^d \times J, \\
u^s \cdot n &= u^d \cdot n + M^u && \text{on } \Gamma^I \times J, \\
-2\mu(\varepsilon(u^s)n)^t &= \alpha\mu\kappa^{-1/2}(u^s)^t + (M^e)^t && \text{on } \Gamma^I \times J, \\
(\sigma_d^s n) \cdot n &= p^d + M^p && \text{on } \Gamma^I \times J,
\end{aligned}$$

where U^s , S^s , U^d , P^d , M^u , M^e , and M^p , and the functions f^s and f^d in eqs. (1a) and (2b) are chosen such that the exact solution is given by:

$$\begin{aligned}
p^s &= \sin(3x_1 - t) \cos(4x_2) + \sin(2\pi x_1 x_2), & u^s &= \begin{bmatrix} \pi x_1 \cos(\pi x_1 x_2 - t) + 1 \\ -\pi x_2 \cos(\pi x_1 x_2 - t) + 2x_1 \end{bmatrix}, \\
p^d &= \cos(3x_1 x_2 - t/10), & u^d &= -(\kappa/\mu) \nabla p^d.
\end{aligned}$$

The initial condition for the velocity is set by first solving the stationary Stokes–Darcy problem with the above boundary/interface conditions and functions f^s and f^d . In our simulations we choose $\kappa = 10^{-4}$ and $\alpha = 1$. We consider polynomial degrees $k = 1$ (corresponding to approximating the cell pressure by piecewise constants and the other unknowns by piecewise linear polynomials) and $k = 2$ (in which the cell pressure is approximated by piecewise linears and the other unknowns by piecewise quadratic polynomials). We compare results obtained by choosing $\mu = 10^{-1}$, $\mu = 10^{-3}$, and $\mu = 10^{-5}$.

Let us define $e_u := u - u_h$ and, similar to [23], $\|e_u\|_E^2 := (\sum_{K \in \mathcal{T}^s} |e_u|_{1,K}^2 + \|e_u\|_{\Omega^d}^2)$. From Corollary 5.2 we expect that, for smooth enough solutions, $\|e_u\|_E = \mathcal{O}(h^k + \Delta t)$. The spatial rate of convergence is indeed observed in table 1 (to obtain these results we chose our time step as $\Delta t = 0.8h^{k+1}$ and set $J = (0, 0.1)$). Table 1 also lists the L^2 -norm of e_u and $e_p := p - p_h$. For the velocity we observe that $\|e_u\|_\Omega = \mathcal{O}(h^{k+1})$ for $\mu = 10^{-1}$ and $\|e_u\|_\Omega \approx \mathcal{O}(h^{k+1/2})$ for $\mu = 10^{-5}$. For $\mu = 10^{-3}$ we have that $\|e_u\|_\Omega$ lies between $\mathcal{O}(h^{k+1/2})$ and $\mathcal{O}(h^{k+1})$, depending on whether $k = 1$ or $k = 2$. The slower convergence in the L^2 -norm for $\mu = 10^{-5}$ is not surprising; the flow problem is advection dominated and analysis of HDG methods for the scalar advection equation reveals a priori error estimates for the solution to be $\mathcal{O}(h^{k+1/2})$, see [42, Lemma 4.8]. We furthermore observe optimal rates of convergence for the pressure: $\|e_p\|_\Omega = \mathcal{O}(h^k)$.

We next consider the temporal rates of convergence. For this we consider a fine mesh with 9508 cells and set $k = 2$ and $J = (0, 1)$. In table 2 we vary the time step and present the errors and rates of convergence. All errors are $\mathcal{O}(\Delta t)$.

Finally, let us remark that despite our analysis holding only under the small data assumption (see eq. (54)), we are nevertheless able to compute the solution for very small values of viscosity. From tables 1 and 2 we even observe that the variation in $\|e_u\|_E$ for the different values of μ is small, despite the upper bound in Corollary 5.2 depending on μ and μ^{-1} .

6.2. Surface/subsurface flow with nonuniform permeability field

In this example we consider surface/subsurface flow. For this example we divide the domain $\Omega = (0, 1) \times (-0.5, 0.5)$ into two subdomains Ω^s and Ω^d . We consider a case where the interface $\Gamma^I = \overline{\Omega^s} \cap \overline{\Omega^d}$ is not horizontal (see fig. 1a). Furthermore, let $\Gamma_D^d = \{x \in \Gamma^d : x_2 = -0.5\}$, and $\Gamma_N^d = \Gamma^d \setminus \Gamma_D^d$. We then impose the

TABLE 1. Errors and spatial rates of convergence for a manufactured solution (see section 6.1). Results are for $k = 1$ and $k = 2$ with parameters $\kappa = 10^{-4}$, $\alpha = 1$, and $\mu \in \{10^{-1}, 10^{-3}, 10^{-5}\}$. Here $e_u = u - u_h$ and $e_p = p - p_h$. The rate of convergence is denoted by r .

Cells	$\ e_u\ _E$	r	$\ e_u\ _\Omega$	r	$\ e_p\ _\Omega$	r
$k = 1, \mu = 10^{-1}$						
152	4.8e-01	0.9	9.4e-03	1.9	1.1e-01	0.7
580	2.1e-01	1.2	1.8e-03	2.4	4.5e-02	1.3
2362	1.0e-01	1.0	4.2e-04	2.1	2.2e-02	1.0
9508	5.1e-02	1.0	9.8e-05	2.1	1.0e-02	1.1
$k = 1, \mu = 10^{-3}$						
152	5.5e-01	0.9	1.4e-02	2.0	6.7e-02	1.1
580	2.5e-01	1.1	3.9e-03	1.9	3.3e-02	1.0
2362	1.2e-01	1.1	1.2e-03	1.7	1.6e-02	1.0
9508	5.6e-02	1.1	3.6e-04	1.7	7.9e-03	1.0
$k = 1, \mu = 10^{-5}$						
152	2.4e+00	3.8	1.5e-01	4.3	7.8e-02	6.5
580	3.2e-01	2.9	2.5e-02	2.6	3.3e-02	1.3
2362	1.4e-01	1.2	5.6e-03	2.2	1.6e-02	1.0
9508	7.2e-02	0.9	1.6e-03	1.8	7.9e-03	1.0
$k = 2, \mu = 10^{-1}$						
152	3.7e-02	2.1	5.9e-04	3.1	8.7e-03	2.5
580	7.6e-03	2.3	5.6e-05	3.4	1.9e-03	2.2
2362	1.7e-03	2.2	5.6e-06	3.3	4.8e-04	2.0
9508	4.0e-04	2.1	6.4e-07	3.1	1.2e-04	2.1
$k = 2, \mu = 10^{-3}$						
152	4.7e-02	2.0	1.0e-03	2.7	5.5e-03	2.2
580	9.2e-03	2.3	1.3e-04	3.0	1.3e-03	2.1
2362	2.0e-03	2.2	1.6e-05	2.9	3.0e-04	2.1
9508	4.9e-04	2.0	2.2e-06	2.9	7.6e-05	2.0
$k = 2, \mu = 10^{-5}$						
152	5.9e-02	4.0	3.6e-03	4.2	5.5e-03	3.1
580	1.0e-02	2.5	4.3e-04	3.1	1.3e-03	2.1
2362	2.4e-03	2.1	6.1e-05	2.8	3.0e-04	2.1
9508	5.3e-04	2.1	1.1e-05	2.5	7.6e-05	2.0

following boundary conditions:

$$\begin{aligned}
 u^s &= (\frac{5}{42}(10x_2 + 1)(1 - x_1/5)(\cos(\pi t/5) + \frac{11}{10}), 0) && \text{on } \Gamma^s \times J, \\
 u^d \cdot n &= 0 && \text{on } \Gamma_N^d \times J, \\
 p^d &= 0 && \text{on } \Gamma_D^d \times J,
 \end{aligned}$$

and set $f^s = 0$ and $f^d = 0$. We consider both $\mu = 10^{-1}$ and $\mu = 10^{-3}$ together with $\alpha = 0.5$, and choose the permeability to be piecewise constant such that $\mu^{-1}\kappa = 10^{-r}$ with $r \in [2, 6]$ a random number that is chosen differently in each element of the mesh in Ω^d . (The analysis presented in this paper assumes a constant permeability, but noting that $0 < \kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}$ the analysis is easily extended to this situation.) A plot of the permeability is given in fig. 1b. To set the initial condition for the velocity in Ω^s we solve the stationary Stokes–Darcy problem.

TABLE 2. Errors and temporal rates of convergence for a manufactured solution (see section 6.1). Parameters: $k = 2$, $\kappa = 10^{-4}$, $\alpha = 1$, and $\mu \in \{10^{-1}, 10^{-3}, 10^{-5}\}$. Here $e_u = u - u_h$ and $e_p = p - p_h$. The rate of convergence is denoted by r .

Δt	$\ e_u\ _E$	r	$\ e_u\ _\Omega$	r	$\ e_p\ _\Omega$	r
$\mu = 10^{-1}$						
1/8	3.5e-02	1.1	2.4e-03	1.1	8.0e-02	0.9
1/16	1.7e-02	1.1	1.2e-03	1.0	4.2e-02	0.9
1/32	8.2e-03	1.0	5.8e-04	1.0	2.1e-02	1.0
1/64	4.1e-03	1.0	2.9e-04	1.0	1.1e-02	1.0
$\mu = 10^{-3}$						
1/8	1.5e-01	1.0	2.1e-02	0.9	3.0e-02	0.8
1/16	7.7e-02	0.9	1.1e-02	0.9	1.6e-02	0.9
1/32	4.0e-02	1.0	5.5e-03	1.0	8.3e-03	1.0
1/64	2.0e-02	1.0	2.8e-03	1.0	4.2e-03	1.0
$\mu = 10^{-5}$						
1/8	1.5e-01	0.9	2.8e-02	0.9	2.2e-02	0.7
1/16	7.8e-02	0.9	1.5e-02	0.9	1.2e-02	0.9
1/32	4.0e-02	1.0	7.4e-03	1.0	6.2e-03	0.9
1/64	2.0e-02	1.0	3.8e-03	1.0	3.2e-03	1.0

We compute the solution on a mesh consisting of 91720 elements, using $k = 2$, a time step of $\Delta t = 0.01$, and on the time interval $J = (0, 10)$. Plots of the velocity and pressure fields at different time levels are shown in figs. 2 and 3, both for $\mu = 10^{-1}$ and $\mu = 10^{-3}$. The velocity fields at $t = 0$ and $t = 10$ for both values of viscosity are similar: flow in Ω^s away from the interface is more or less horizontal while in Ω^d flow finds its way through the permeability maze in the direction of negative pressure gradient. At $t = 5.2$ (when the inflow magnitude of the velocity is close to its minimum), the behavior of the velocity fields when $\mu = 10^{-1}$ and $\mu = 10^{-3}$ are significantly different: when $\mu = 10^{-1}$ the velocity field is similar to that at $t = 0$ and $t = 10$, but when $\mu = 10^{-3}$ we obtain a large area of circulation. The pressure fields are similar for the two values of viscosity and follow a more or less linear profile in Ω^d . Pressure variations in Ω^s are small.

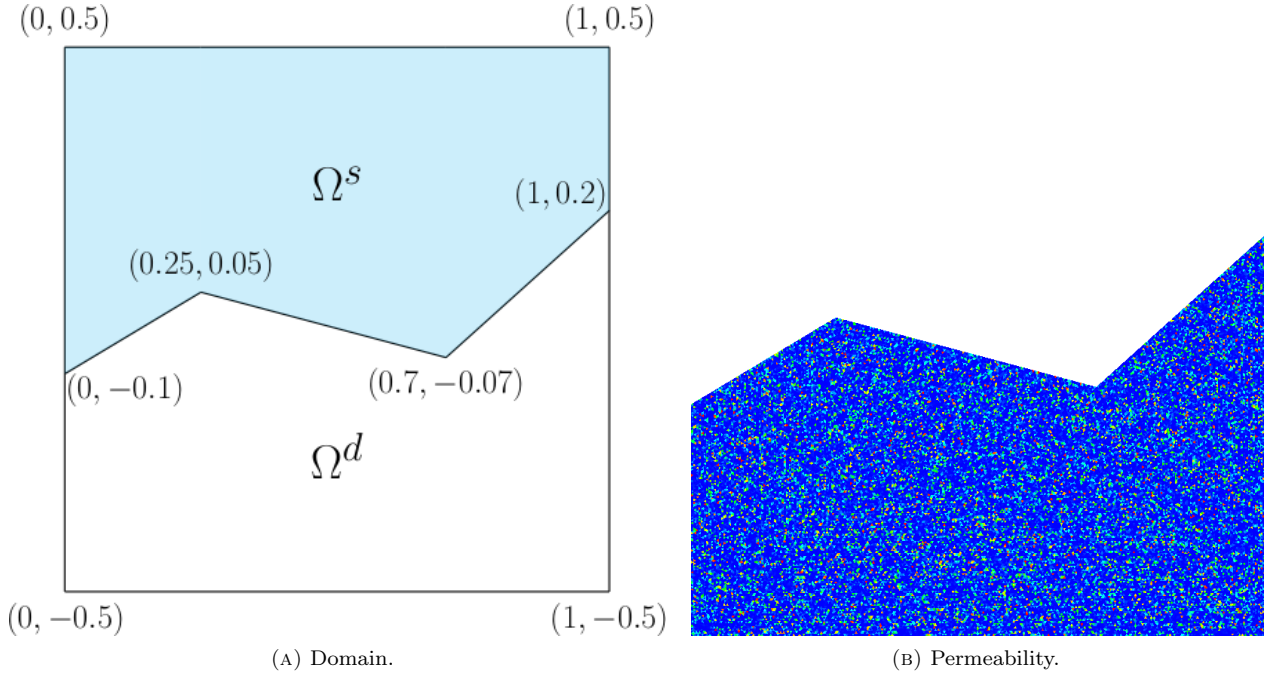
7. CONCLUSIONS

We presented a strongly conservative HDG method for the coupled time-dependent Navier–Stokes and Darcy problem. Existence and uniqueness of a solution to the fully discrete problem were proven assuming a small data assumption. We furthermore determined a pressure-independent a priori error estimate for the discrete velocity. This estimate is optimal in space in the combined discrete H^1 -norm on Ω^s and $H(\text{div})$ -norm on Ω^d , and optimal in time. Our analysis is supported by numerical examples.

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FIGURE 1. The surface/subsurface domain Ω used in section 6.2.

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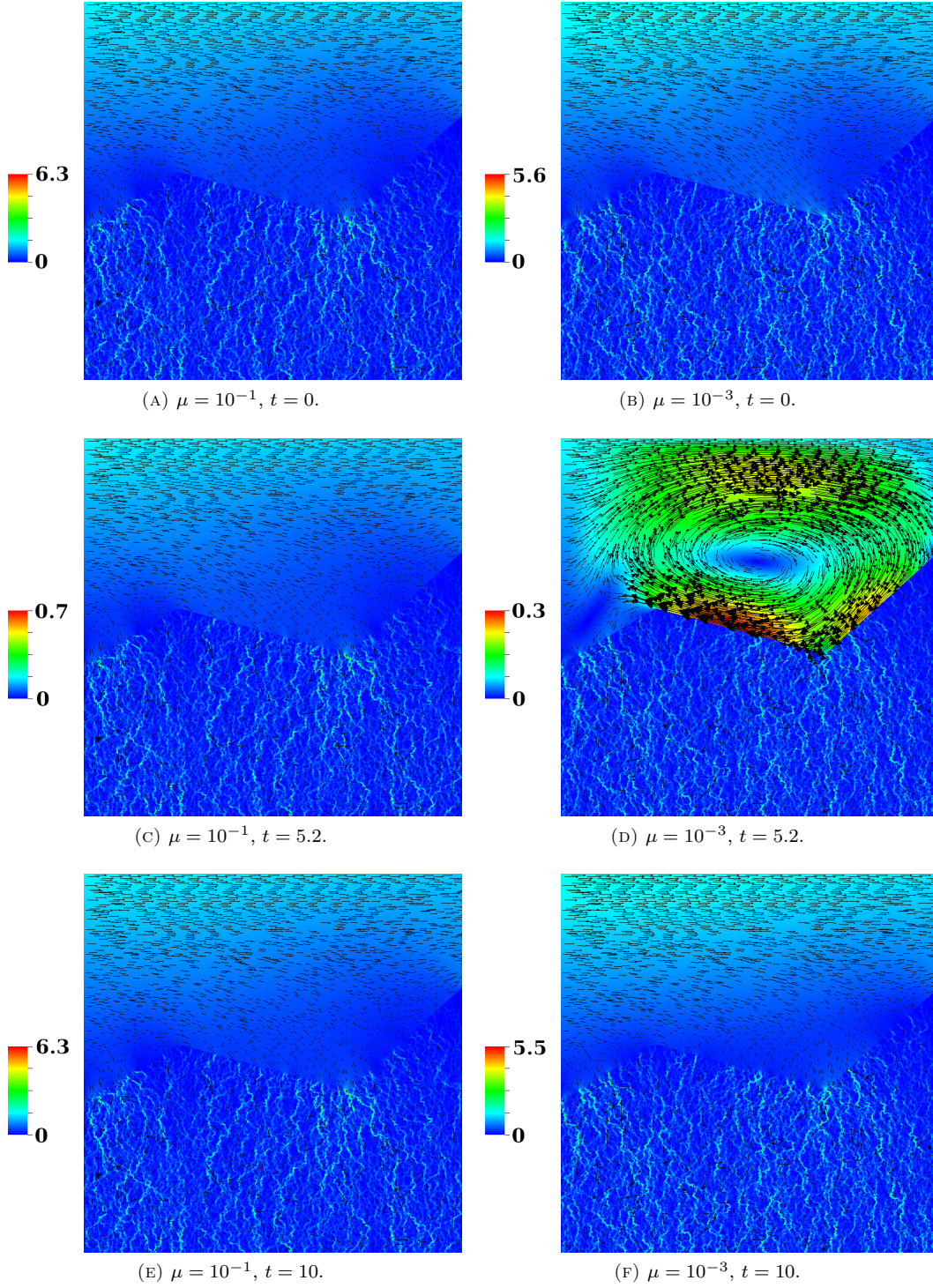


FIGURE 2. Velocity magnitude and velocity vector field at time levels $t = 0$, $t = 5.2$, and $t = 10$. Left column: $\mu = 10^{-1}$. Right column: $\mu = 10^{-3}$. See also section 6.2.

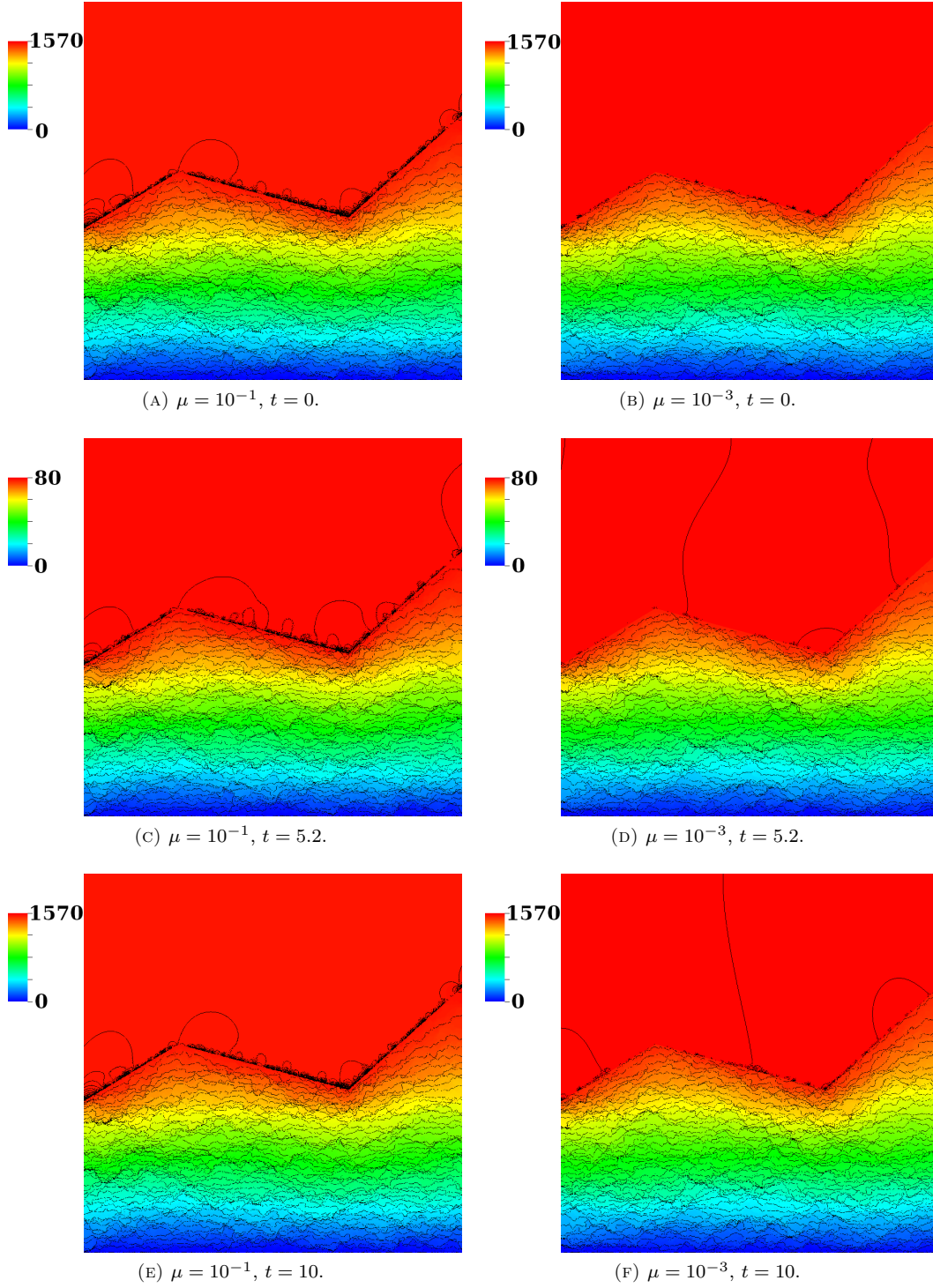


FIGURE 3. Pressure magnitude and contour plots at time levels $t = 0$, $t = 5.2$, and $t = 10$. Left column: $\mu = 10^{-1}$. Right column: $\mu = 10^{-3}$. For visualization purposes at $t = 0$ and $t = 10$ we consider in Ω^d the pressure interval $[0, 1570]$ divided into 50 contour lines using a linear scale while in Ω^s we consider the pressure interval $[1500, 1570]$ divided into 100 contour lines using a log scale. At $t = 5.2$ we consider in Ω^d and Ω^s the pressure intervals $[0, 80]$ and $[75, 80]$, respectively. See also section 6.2.

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APPENDIX A. PROOF OF THE INF-SUP CONDITION EQ. (9a)

An inf-sup condition of the form eq. (9a) was proven in [8, Lemma 2] assuming that $u = 0$ on Γ^s and $u \cdot n = 0$ on Γ^d . We modify this proof to take into account the boundary conditions eqs. (4b) to (4d). The proof requires the BDM interpolation operator $\Pi_V : H(\text{div}; \Omega) \cap [L^r(\Omega)]^{\dim} \rightarrow X_h \cap H(\text{div}; \Omega)$, $r > 2$, which satisfies eqs. (58)

to (60) for all $u \in [H^{k+1}(K)]^{\dim}$. We will also require the following function space:

$$[H_{0,\Gamma^s \cup \Gamma_N^d}^1(\Omega)]^{\dim} := \{w \in [H^1(\Omega)]^{\dim} : w|_{\Gamma^s \cup \Gamma_N^d} = 0\}.$$

Defining

$$\begin{aligned} \tilde{b}_h(\mathbf{v}_h, (\bar{q}_h^s, \bar{q}_h^d)) &:= \sum_{j=s,d} (\langle \bar{q}_h^j, v_h \cdot n^j \rangle_{\partial \mathcal{T}_h^j} - \langle \bar{q}_h^j, \bar{v}_h \cdot n^j \rangle_{\Gamma^I}), \\ \text{Ker}(\tilde{b}_h) &:= \{\mathbf{v}_h \in \mathbf{X}_h : \tilde{b}_h(\mathbf{v}_h, (\bar{q}_h^s, \bar{q}_h^d)) = 0 \ \forall (\bar{q}_h^s, \bar{q}_h^d) \in \bar{Q}_h^s \times \bar{Q}_h^d\}, \end{aligned}$$

and noting that $b_h(\mathbf{v}_h, \mathbf{q}_h) = -(q_h, \nabla \cdot v_h)_\Omega + \tilde{b}_h(\mathbf{v}_h, (\bar{q}_h^s, \bar{q}_h^d))$, by [24, Theorem 3.1] the inf-sup condition eq. (9a) holds for all $\mathbf{q}_h \in \mathbf{Q}_h$ if there exist constants $c_{b1} > 0$ and $c_{b2} > 0$, independent of h and Δt , such that

$$c_{b1} \|q_h\|_\Omega \leq \sup_{\substack{\mathbf{v}_h \in \text{Ker}(\tilde{b}_h) \\ \mathbf{v}_h \neq 0}} \frac{-(q_h, \nabla \cdot v_h)_\Omega}{\|\mathbf{v}_h\|_v} \quad \forall q_h \in Q_h, \quad (69a)$$

$$(c_{b2} \sum_{j=s,d} \sum_{K \in \mathcal{T}_h^j} h_K \|\bar{q}_h^j\|_{\partial K}^2)^{1/2} \leq \sup_{\substack{\mathbf{v}_h \in \mathbf{X}_h \\ \mathbf{v}_h \neq 0}} \frac{\tilde{b}_h(\mathbf{v}_h, (\bar{q}_h^s, \bar{q}_h^d))}{\|\mathbf{v}_h\|_v} \quad \forall (\bar{q}_h^s, \bar{q}_h^d) \in \bar{Q}_h^s \times \bar{Q}_h^d. \quad (69b)$$

Compared to [8, Lemma 2], only the proof for eq. (69a) needs to be modified.

We first seek a suitable $\mathbf{v}_h \in \text{Ker}(\tilde{b}_h)$. Let $q_h \in Q_h$. By [30, Remark 3.3] there exists $v \in [H_{0,\Gamma^s \cup \Gamma_N^d}^1(\Omega)]^{\dim}$ such that

$$-(\nabla \cdot v, q_h)_\Omega = \|q_h\|_\Omega^2, \quad c_{vq} \|v\|_{1,\Omega} \leq \|q_h\|_\Omega, \quad (70)$$

where $c_{vq} > 0$ is a constant independent of h and Δt . Let $\bar{\Pi}_V : [H^1(\Omega^s)]^{\dim} \rightarrow \bar{X}_h$ be the L^2 -projection into the facet velocity space and note that the pair $\mathbf{v}_h = (\Pi_V v, \bar{\Pi}_V v)$ lies in $\text{Ker}(\tilde{b}_h)$:

$$\begin{aligned} \tilde{b}_h((\Pi_V v, \bar{\Pi}_V v), (\bar{q}_h^s, \bar{q}_h^d)) &= \langle \bar{q}_h^s, (\Pi_V v - \bar{\Pi}_V v) \cdot n^s \rangle_{\Gamma^I} + \langle \bar{q}_h^s, \Pi_V v \cdot n^s \rangle_{\Gamma^s} + \langle \bar{q}_h^d, (\Pi_V v - \bar{\Pi}_V v) \cdot n^d \rangle_{\Gamma^I} + \langle \bar{q}_h^d, \Pi_V v \cdot n^d \rangle_{\Gamma^d} \\ &= \langle \bar{q}_h^s, (v - v) \cdot n^s \rangle_{\Gamma^I} + \langle \bar{q}_h^d, (v - v) \cdot n^d \rangle_{\Gamma^I} = 0, \end{aligned}$$

where the first equality is because $\Pi_V v \cdot n^j$ is continuous on element boundaries and $\bar{q}_h^j$ is single-valued. The second equality is by properties of Π_V and $\bar{\Pi}_V$, $v \cdot n^j = 0$ on $\Gamma^s \cup \Gamma_N^d$, and $\bar{q}_h^d = 0$ on Γ_D^d . Therefore, $(\Pi_V v, \bar{\Pi}_V v) \in \text{Ker}(\tilde{b}_h)$.

We now proceed to find a bound for $\|(\Pi_V v, \bar{\Pi}_V v)\|_v$ in terms of $\|v\|_{1,\Omega}$. First, note that by definition,

$$\begin{aligned} \|(\Pi_V v, \bar{\Pi}_V v)\|_{v,d}^2 &= \|\Pi_V v\|_{\text{div};\Omega^d}^2 + \sum_{F \in \mathcal{F}_h^d \setminus (\mathcal{F}_h^I \cup \mathcal{F}_h^{D,d})} h_F^{-1} \|[\Pi_V v \cdot n]\|_F^2 \\ &\quad + \sum_{K \in \mathcal{T}_h^d} h_K^{-1} \|(\Pi_V v - \bar{\Pi}_V v) \cdot n\|_{\partial K \cap \Gamma^I}^2 =: I_1 + I_2 + I_3. \end{aligned}$$

In [8, Lemma 2] it was shown that $I_1 + I_3 \leq C \|v\|_{1,\Omega^d}^2$. Furthermore, $I_2 = 0$ because $\Pi_V v \in H(\text{div}; \Omega^d)$ and $v = 0$ on Γ_N^d . Therefore, $\|(\Pi_V v, \bar{\Pi}_V v)\|_{v,d} \leq C \|v\|_{1,\Omega^d}$. In the proof of [8, Lemma 2] it was also shown that

$$\|(\Pi_V v, \bar{\Pi}_V v)\|_{v,s} \leq C \|v\|_{1,\Omega^s}, \quad \|(\bar{\Pi}_V v)^t\|_{\Gamma^I} \leq C \|v\|_{1,\Omega^s}. \quad (71)$$

By definition of $\|\cdot\|_v$ and using the preceding bounds on $\|(\Pi_V v, \bar{\Pi}_V v)\|_{v,s}$, $\|(\bar{\Pi}_V v)^t\|_{\Gamma^I}$, and $\|(\Pi_V v, \bar{\Pi}_V v)\|_{v,d}$, we find

$$\|(\Pi_V v, \bar{\Pi}_V v)\|_v \leq C \|v\|_{1,\Omega}.$$

Equation (69a) now follows from this and eq. (70):

$$\sup_{\substack{\mathbf{v}_h \in \text{Ker}(\tilde{b}_h) \\ \mathbf{v}_h \neq 0}} \frac{-(q_h, \nabla \cdot \mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|_v} \geq \frac{-(q_h, \nabla \cdot \Pi_V \mathbf{v})_\Omega}{\|(\Pi_V \mathbf{v}, \bar{\Pi}_V \mathbf{v})\|_v} \geq \frac{\|q_h\|_\Omega^2}{C \|\mathbf{v}\|_{1,\Omega}} \geq \frac{c_{vq}}{C} \|q_h\|_\Omega.$$

APPENDIX B. USEFUL INEQUALITIES

Let g be a sufficiently smooth function. Using Taylor's theorem in integral form, it is shown in [26, Lemma 7.67]) that

$$\|\partial_t g^{n+1} - d_t g^{n+1}\|_{\Omega^s} \leq C \sqrt{\Delta t} \|\partial_{tt} g\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}. \quad (72)$$

A minor modification of the proof of eq. (72) leads to:

$$\|g^{n+1} - g^n\|_{\Omega^s} \leq C \Delta t (\|\partial_t g^n\|_{\Omega^s} + \|\partial_{tt} g\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}). \quad (73)$$

We also have, by the fundamental theorem of Calculus and the Cauchy-Schwarz inequality, that

$$\|g^{n+1} - g^n\|_{\Omega^s} = \left\| \int_{t^n}^{t^{n+1}} \partial_t g \, dt \right\| \leq \left(\int_{t^n}^{t^{n+1}} dt \right)^{1/2} \left(\int_{t^n}^{t^{n+1}} \|\partial_t g\|_{\Omega^s}^2 \, dt \right)^{1/2} \leq \sqrt{\Delta t} \|\partial_t g\|_{L^2(t^n, t^{n+1}; L^2(\Omega^s))}. \quad (74)$$

APPENDIX C. PROOF OF EQ. (68)

To prove eq. (68) we will use the following result, which is due to a discrete Sobolev embedding [17, Theorem 5.3] and eq. (8b):

$$\left(\sum_{K \in \mathcal{T}_h^s} \|v_h\|_{L^6(K)}^6 \right)^{1/6} \leq C \|\mathbf{v}_h\|_{v,s} \quad \forall \mathbf{v}_h \in \mathbf{X}_h. \quad (75)$$

Let us first write I_4 as:

$$\begin{aligned} I_4 &= [t_h(u^n; \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u_h^n; \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\ &= [t_h(u^n; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u^n; \mathbf{u}^{n+1} - \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\ &\quad - [t_h(u_h^n; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u_h^n; \mathbf{u}^{n+1} - \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\ &= [t_h(u^n; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u_h^n; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1})] \\ &\quad + [t_h(u_h^n; \mathbf{u}^{n+1} - \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(u^n; \mathbf{u}^{n+1} - \Pi_V u^{n+1}, \mathbf{e}_u^{h,n+1})] \\ &=: I_{41} + I_{42}. \end{aligned}$$

For I_{41} we note that since the second argument of t_h is continuous almost everywhere:

$$I_{41} = t_h(u^n - u_h^n; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) = t_h(e_u^{I,n}; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) - t_h(e_u^{h,n}; \mathbf{u}^{n+1}, \mathbf{e}_u^{h,n+1}) =: I_{411} + I_{412}.$$

We have by eq. (13) and Young's inequality,

$$\begin{aligned} I_{411} &\leq c_w \|e_u^{I,n}\|_{1,h,\Omega^s} \|\mathbf{u}^{n+1}\|_{v,s} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\ &\leq C h^k \|u^n\|_{k+1,\Omega^s} \|\nabla u^{n+1}\|_{\Omega^s} \|\mathbf{e}_u^{h,n+1}\|_{v,s} \\ &\leq \frac{1}{2} \gamma \|\mathbf{e}_u^{h,n+1}\|_{v,s}^2 + \frac{C}{\gamma} h^{2k} \|u^n\|_{k+1,\Omega^s}^2 \|\nabla u^{n+1}\|_{\Omega^s}^2. \end{aligned} \quad (76)$$

Next, using that $u^{n+1} = \bar{u}^{n+1}$ on facets,

$$I_{412} = -(u^{n+1} \otimes e_u^{h,n}, \nabla e_u^{h,n+1})_{\Omega^s} + \langle e_u^{h,n} \cdot n, (e_u^{h,n+1} - \bar{e}_u^{h,n+1}) \cdot u^{n+1} \rangle_{\partial \mathcal{T}_h^s} + \langle e_u^{h,n} \cdot n, \bar{e}_u^{h,n+1} \cdot u^{n+1} \rangle_{\Gamma^I}.$$

At this point we note that since $e_u^{h,n} \cdot n$, $\bar{e}_u^{h,n+1}$, and u^{n+1} are single-valued on facets, and because $u = 0$ on Γ^s , we have that $\langle e_u^{h,n} \cdot n, \bar{e}_u^{h,n+1} \cdot u^{n+1} \rangle_{\partial \mathcal{T}_h^s} = \langle e_u^{h,n} \cdot n, \bar{e}_u^{h,n+1} \cdot u^{n+1} \rangle_{\Gamma^I}$. Therefore,

$$I_{412} = -(u^{n+1} \otimes e_u^{h,n}, \nabla e_u^{h,n+1})_{\Omega^s} + \langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot u^{n+1} \rangle_{\partial \mathcal{T}_h^s}.$$

Integrating by parts, using that $\nabla \cdot e_u^{h,n} = 0$ on each $K \in \mathcal{T}_h^s$, the generalized Hölder's inequality, eq. (75), and Young's inequality:

$$\begin{aligned} I_{412} &= (\nabla \cdot (u^{n+1} \otimes e_u^{h,n}), e_u^{h,n+1})_{\Omega^s} = (e_u^{h,n} \cdot \nabla u^{n+1}, e_u^{h,n+1})_{\Omega^s} \\ &\leq \|e_u^{h,n}\|_{\Omega^s} |u^{n+1}|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{L^6(\Omega^s)} \\ &\leq C \|e_u^{h,n}\|_{\Omega^s} |u^{n+1}|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{v,s} \\ &\leq \frac{1}{2} \gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} \|e_u^{h,n}\|_{\Omega^s}^2 |u^{n+1}|_{W_3^1(\Omega^s)}^2. \end{aligned} \tag{77}$$

Combining eqs. (76) and (77) we find

$$I_{41} \leq \gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{2k} \|u^n\|_{k+1,\Omega^s}^2 \|\nabla u^{n+1}\|_{\Omega^s}^2 + \frac{C}{\gamma} \|e_u^{h,n}\|_{\Omega^s}^2 |u^{n+1}|_{W_3^1(\Omega^s)}^2. \tag{78}$$

We next consider I_{42} which we first write as:

$$\begin{aligned} I_{42} &= [t_h(u_h^n; u^{n+1} - \Pi_V u^{n+1}, e_u^{h,n+1}) - t_h(\Pi_V u^n; u^{n+1} - \Pi_V u^{n+1}, e_u^{h,n+1})] \\ &\quad + [t_h(\Pi_V u^n; u^{n+1} - \Pi_V u^{n+1}, e_u^{h,n+1}) - t_h(u^n; u^{n+1} - \Pi_V u^{n+1}, e_u^{h,n+1})] \\ &= [t_h(\Pi_V u^n; e_u^{I,n+1}, e_u^{h,n+1}) - t_h(u^n; e_u^{I,n+1}, e_u^{h,n+1})] \\ &\quad + [t_h(u_h^n; e_u^{I,n+1}, e_u^{h,n+1}) - t_h(\Pi_V u^n; e_u^{I,n+1}, e_u^{h,n+1})] \\ &=: I_{421} + I_{422}. \end{aligned}$$

For I_{421} we have by eq. (13), [8, Lemma 7], properties of Π_V and $\bar{\Pi}_V$, and Young's inequality,

$$\begin{aligned} I_{421} &\leq c_w \|\Pi_V u^n - u^n\|_{1,h,\Omega^s} \|e_u^{I,n+1}\|_{v,s} \|e_u^{h,n+1}\|_{v,s} \\ &= c_w \|e_u^{I,n}\|_{1,h,\Omega^s} \|e_u^{I,n+1}\|_{v,s} \|e_u^{h,n+1}\|_{v,s} \\ &\leq C h^k \|u^{n+1}\|_{k+1,\Omega^s} \|e_u^{I,n}\|_{1,h,\Omega^s} \|e_u^{h,n+1}\|_{v,s} \\ &\leq C h^{2k} \|u^{n+1}\|_{k+1,\Omega^s} \|u^n\|_{k+1,\Omega^s} \|e_u^{h,n+1}\|_{v,s} \\ &\leq \frac{1}{2} \gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{4k} \|u^{n+1}\|_{k+1,\Omega^s}^2 \|u^n\|_{k+1,\Omega^s}^2. \end{aligned} \tag{79}$$

For I_{422} we find, after integrating by parts,

$$\begin{aligned}
I_{422} &= t_h(u_h^n; e_u^{I,n+1}, e_u^{h,n+1}) - t_h(\Pi_V u^n; e_u^{I,n+1}, e_u^{h,n+1}) \\
&= (\nabla e_u^{I,n+1}, e_u^{h,n+1} \otimes (u_h^n - \Pi_V u^n))_{\Omega^s} - \langle ((e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \otimes (u_h^n - \Pi_V u^n))n, e_u^{h,n+1} \rangle_{\partial\mathcal{T}^s} \\
&\quad + \langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))(e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_{\partial\mathcal{T}^s} \\
&= (\nabla e_u^{I,n+1}, e_u^{h,n+1} \otimes e_u^{h,n})_{\Omega^s} - \langle ((e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \otimes e_u^{h,n})n, e_u^{h,n+1} \rangle_{\partial\mathcal{T}^s} \\
&\quad + \langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))(e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_{\partial\mathcal{T}^s} \\
&= (e_u^{h,n} \cdot \nabla e_u^{I,n+1}, e_u^{h,n+1})_{\Omega^s} - \langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot (e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \rangle_{\partial\mathcal{T}^s} \\
&\quad + \langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))(e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_{\partial\mathcal{T}^s} \\
&= I_{422a} + I_{422b} + I_{422c}.
\end{aligned}$$

For I_{422a} , using generalized Hölder's inequality, eq. (75), that $|u^{n+1} - \Pi_V u^{n+1}|_{W_3^1(\Omega^s)} \leq c|u^{n+1}|_{W_3^1(\Omega^s)}$ (see [20, Theorem 16.4]) we have:

$$\begin{aligned}
I_{422a} &= (e_u^{h,n} \cdot \nabla e_u^{I,n+1}, e_u^{h,n+1})_{\Omega^s} \\
&\leq \|e_u^{h,n}\|_{\Omega^s} \|\nabla e_u^{I,n+1}\|_{L^3(\Omega^s)} \|e_u^{h,n+1}\|_{L^6(\Omega^s)} \\
&\leq C \|e_u^{h,n}\|_{\Omega^s} |e_u^{I,n+1}|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{v,s} \\
&= C \|e_u^{h,n}\|_{\Omega^s} |u^{n+1} - \Pi_V u^{n+1}|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{v,s} \\
&\leq C \|e_u^{h,n}\|_{\Omega^s} |u^{n+1}|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{v,s}.
\end{aligned} \tag{80}$$

To bound I_{422b} let us first consider a single facet $F \subset \partial K$. By Hölder's inequality,

$$|\langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot (e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \rangle_F| \leq \|e_u^{h,n}\|_{L^{3/2}(F)} \|e_u^{I,n+1} - \bar{e}_u^{I,n+1}\|_{L^3(F)} \|e_u^{h,n+1}\|_{L^\infty(F)}. \tag{81}$$

Noting that $\bar{\Pi}_V \Pi_V u = \Pi_V u$ on F , we have:

$$\begin{aligned}
\|e_u^{I,n+1} - \bar{e}_u^{I,n+1}\|_{L^3(F)} &= \|u^{n+1} - \Pi_V u^{n+1} - \gamma(u^{n+1}) + \bar{\Pi}_V u^{n+1}\|_{L^3(F)} \\
&= \|\bar{\Pi}_V u^{n+1} - \Pi_V u^{n+1}\|_{L^3(F)} \\
&= \|\bar{\Pi}_V(u^{n+1} - \Pi_V u^{n+1})\|_{L^3(F)} \\
&\leq C \|u^{n+1} - \Pi_V u^{n+1}\|_{L^3(F)},
\end{aligned} \tag{82}$$

where the inequality is by [20, Lemma 11.18]. By a multiplicative trace inequality [20, Lemma 12.15], we have that

$$\begin{aligned}
\|u^{n+1} - \Pi_V u^{n+1}\|_{L^3(F)} &\leq c \|u^{n+1} - \Pi_V u^{n+1}\|_{L^3(K)}^{2/3} (h_K^{-1/3} \|u^{n+1} - \Pi_V u^{n+1}\|_{L^3(K)}^{1/3} + \|\nabla(u^{n+1} - \Pi_V u^{n+1})\|_{L^3(K)}^{1/3}), \tag{83}
\end{aligned}$$

and by [20, Theorem 16.4] we have

$$\begin{aligned}
\|u^{n+1} - \Pi_V u^{n+1}\|_{L^3(K)} &\leq c h_K |\nabla u^{n+1}|_{W_3^1(K)}, \\
\|\nabla(u^{n+1} - \Pi_V u^{n+1})\|_{L^3(K)} &\leq c \|u^{n+1}\|_{W_3^1(K)}.
\end{aligned} \tag{84}$$

Combining eqs. (82) to (84),

$$\begin{aligned} \|e_u^{I,n+1} - \bar{e}_u^{I,n+1}\|_{L^3(F)} &\leq ch_K^{2/3} |\nabla u^{n+1}|_{W_3^1(K)}^{2/3} (h_K^{-1/3} h_K^{1/3} |\nabla u^{n+1}|_{W_3^1(K)}^{1/3} + \|u^{n+1}\|_{W_3^1(K)}^{1/3}) \\ &\leq ch_K^{2/3} \|u^{n+1}\|_{W_3^1(K)}. \end{aligned} \quad (85)$$

We also have, by a discrete trace inequality [17, Lemma 1.52], that

$$\|e_u^{h,n}\|_{L^{3/2}(F)} \leq Ch_K^{-2/3} \|e_u^{h,n}\|_{L^{3/2}(K)}, \quad \|e_u^{h,n+1}\|_{L^\infty(F)} \leq C \|e_u^{h,n+1}\|_{L^\infty(K)}. \quad (86)$$

Combining eq. (81) with eqs. (85) and (86)

$$\begin{aligned} |\langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot (e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \rangle_F| &\leq Ch_K^{-2/3} \|e_u^{h,n}\|_{L^{3/2}(K)} h_K^{2/3} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1}\|_{L^\infty(K)} \\ &= C \|e_u^{h,n}\|_{L^{3/2}(K)} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1}\|_{L^\infty(K)}. \end{aligned}$$

By [17, Lemma 1.50], for $\dim = 2, 3$,

$$\|e_u^{h,n}\|_{L^{3/2}(K)} \leq Ch_K^{\dim/6} \|e_u^{h,n}\|_{L^2(K)}, \quad (87a)$$

$$\|e_u^{h,n+1}\|_{L^\infty(K)} \leq Ch_K^{-\dim/6} \|e_u^{h,n+1}\|_{L^6(K)}, \quad (87b)$$

so that

$$|\langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot (e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \rangle_F| \leq C \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1}\|_{L^6(K)}.$$

Since we assumed $F \subset \partial K$ it follows that

$$|\langle e_u^{h,n} \cdot n, e_u^{h,n+1} \cdot (e_u^{I,n+1} - \bar{e}_u^{I,n+1}) \rangle_{\partial K}| \leq C \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1}\|_{L^6(K)}.$$

Summing over all elements in $\mathcal{T}_h^s$, using a generalized Hölder's inequality for the summation over the elements, and eq. (75),

$$\begin{aligned} I_{422b} &\leq C \sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1}\|_{L^6(K)} \\ &\leq C \left(\sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n}\|_{L^2(K)}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h^s} \|u^{n+1}\|_{W_3^1(K)}^3 \right)^{1/3} \left(\sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n+1}\|_{L^6(K)}^6 \right)^{1/6} \\ &\leq C \|e_u^{h,n}\|_{L^2(\Omega^s)} \|u^{n+1}\|_{W_3^1(\Omega^s)} \|e_u^{h,n+1}\|_{v,s}. \end{aligned} \quad (88)$$

Let us now consider I_{422c} . Starting again with a single facet $F \subset \partial K$, we find using Hölder's inequality,

$$\begin{aligned} &|\langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0)) (e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_F| \\ &\leq \|(\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))\|_{L^{3/2}(F)} \|e_u^{I,n+1} - \bar{e}_u^{I,n+1}\|_{L^3(F)} \times \\ &\quad \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^\infty(F)}. \end{aligned} \quad (89)$$

Since $a \mapsto \max(a, 0)$ is Lipschitz ([6, Appendix A.3.1]), and using eq. (86):

$$\begin{aligned} \|\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0)\|_{L^{3/2}(F)} \\ \leq C \|u_h^n - \Pi_V u^n\|_{L^{3/2}(F)} = C \|e_u^{h,n}\|_{L^{3/2}(F)} \leq Ch_K^{-2/3} \|e_u^{h,n}\|_{L^{3/2}(K)}. \end{aligned} \quad (90)$$

Furthermore, by [17, Lemma 1.50],

$$\|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^\infty(F)} \leq ch_K^{(1-\dim)/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(F)}, \quad (91)$$

From eq. (89), eq. (90), eq. (85), eq. (87a), and eq. (91) we therefore find that

$$\begin{aligned} & |\langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))(e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_F | \\ & \leq Ch_K^{-2/3} \|e_u^{h,n}\|_{L^{3/2}(K)} \|e_u^{I,n+1} - \bar{e}_u^{I,n+1}\|_{L^3(F)} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^\infty(F)} \quad (\text{by eq. (90)}) \\ & \leq Ch_K^{-2/3} \|e_u^{h,n}\|_{L^{3/2}(K)} h_K^{2/3} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^\infty(F)} \quad (\text{by eq. (85)}) \\ & \leq Ch_K^{\dim/6} \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^\infty(F)} \quad (\text{by eq. (87a)}) \\ & \leq Ch_K^{\dim/6} \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} h_K^{(1-\dim)/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(F)} \quad (\text{by eq. (91)}) \\ & \leq C \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} (h_K^{-1/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(F)}), \end{aligned}$$

where the last inequality is because $h_K^{\dim/6} h_K^{(1-\dim)/2} \leq h_K^{-1/2}$ for $\dim = 2, 3$. Since $F \subset \partial K$ it follows that

$$\begin{aligned} & |\langle (\max(u_h^n \cdot n, 0) - \max(\Pi_V u^n \cdot n, 0))(e_u^{I,n+1} - \bar{e}_u^{I,n+1}), e_u^{h,n+1} - \bar{e}_u^{h,n+1} \rangle_{\partial K} | \\ & \leq C \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} (h_K^{-1/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(\partial K)}). \end{aligned}$$

Summing over all elements in $\mathcal{T}_h^s$ and by the Cauchy–Schwarz inequality,

$$\begin{aligned} I_{422c} & \leq C \sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n}\|_{L^2(K)} \|u^{n+1}\|_{W_3^1(K)} (h_K^{-1/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(\partial K)}) \\ & \leq C \max_{K \in \mathcal{T}_h^s} \|u^{n+1}\|_{W_3^1(K)} \sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n}\|_{L^2(K)} (h_K^{-1/2} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(\partial K)}) \\ & \leq C \max_{K \in \mathcal{T}_h^s} \|u^{n+1}\|_{W_3^1(K)} \left(\sum_{K \in \mathcal{T}_h^s} \|e_u^{h,n}\|_{L^2(K)}^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}_h^s} h_K^{-1} \|e_u^{h,n+1} - \bar{e}_u^{h,n+1}\|_{L^2(\partial K)}^2 \right)^{1/2} \\ & \leq C \|u^{n+1}\|_{W_3^1(\Omega^s)} \|e_u^{h,n}\|_{\Omega^s} \|e_u^{h,n+1}\|_{v,s}. \end{aligned} \quad (92)$$

Combining eqs. (80), (88) and (92), and applying Young's inequality, we find the following bound for I_{422} :

$$I_{422} \leq \frac{1}{2} \gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} \|e_u^{h,n}\|_{\Omega^s}^2 \|u^{n+1}\|_{W_3^1(\Omega^s)}^2. \quad (93)$$

Combining now eqs. (79) and (93) we find that

$$I_{42} \leq \gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{4k} \|u^{n+1}\|_{k+1,\Omega^s}^2 \|u^n\|_{k+1,\Omega^s}^2 + \frac{C}{\gamma} \|e_u^{h,n}\|_{\Omega^s}^2 \|u^{n+1}\|_{W_3^1(\Omega^s)}^2,$$

which, when combined with eq. (78), gives us:

$$I_4 \leq 2\gamma \|e_u^{h,n+1}\|_v^2 + \frac{C}{\gamma} h^{2k} \|u^{n+1}\|_{k+1,\Omega^s}^2 \|u^n\|_{k+1,\Omega^s}^2 + \frac{C}{\gamma} \|e_u^{h,n}\|_{\Omega^s}^2 \|u^{n+1}\|_{W_3^1(\Omega^s)}^2,$$

which is the desired result.