Regularity of the Level Set Flow

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Abstract

We showed earlier that the level set function of a monotonic advancing front is twice differentiable everywhere with bounded second derivative and satisfies the equation classically. We show here that the second derivative is continuous if and only if the flow has a single singular time where it becomes extinct and the singular set consists of a closed C^1 manifold with cylindrical singularities.

1 Introduction

The level set method has been used with great success the last thirty years in both pure and applied mathematics to describe evolutions of various physical situations. In mean curvature flow, the evolving hypersurface (front) is thought of as the level set of a function that satisfies a nonlinear degenerate parabolic equation. Solutions are defined weakly in the viscosity sense; in general, they may not even be differentiable (let alone twice differentiable).

For a monotonically advancing front, we showed in [9] (cf. [10]) that viscosity solutions are in fact twice differentiable and satisfy the equation in the classical sense. Here we characterize when they are C^2. As we will see, the situation becomes very rigid when the second derivative is continuous.

When $v : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$ is a function and for each s the level set $t \to \{x \mid v(x,t) = s\}$ evolves by the mean curvature flow, then v satisfies the level set equation

$$\partial_t v = |\nabla v| \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right). \tag{1.1}$$

This equation has been studied extensively. Whereas the work of Osher and Sethian [22] was numerical, Evans and Spruck [13] and, independently, Chen, Giga, and Goto [4] provided the theoretical justification. This is analytically subtle, principally because the mean curvature evolution equation is nonlinear, degenerate, and indeed defined only weakly at points where $\nabla v = 0$. Moreover, v is a priori not even differentiable, let alone twice differentiable. They resolved these problems by

Communications on Pure and Applied Mathematics, Vol. LXXI, 0814–0824 (2018)

introducing an appropriate definition of a weak solution, inspired by the notion of viscosity solutions, and showed existence and uniqueness.

When the initial hypersurface is mean convex (the mean curvature is nonnegative), so are all future ones and the front advances monotonically. In this case, Evans and Spruck [13] showed that $v(x,t) = u(x) - t$, where u is Lipschitz and satisfies (in the viscosity sense)

$$-1 = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{1.2}$$

As the front moves monotonically inwards, it sweeps out the entire domain inside the initial hypersurface. The function u is the *arrival time* since $u(x)$ is the time when the front passes through x. It is defined on the entire compact domain bounded by the initial hypersurface. Singular points for the flow correspond to critical points for u: the flow has a singularity at x at time $u(x)$ if and only if $\nabla u(x) = 0$.

When the initial hypersurface is convex, the flow is smooth except at the point it becomes extinct, and Huisken showed that the arrival time is C^2 [15, 16]. In [19, 20], Ilmanen gave an example of a rotationally symmetric mean convex dumbbell in $\mathbb{R}^3$ for which the arrival time was not C^2. There is even more regularity in the plane, where Kohn and Serfaty showed that it is at least C^3 [21]. For $n > 1$, Sesum [23] showed that Huisken's result is optimal; namely, she gave examples of convex initial hypersurfaces where the arrival time is not three times differentiable.

In the next two theorems and corollary, u is the arrival time of a mean convex flow in $\mathbb{R}^{n+1}$ starting from a smooth closed connected hypersurface.

THEOREM 1.1. *u is C^2 if and only if both* (1) *and* (2) *hold*:

(1) *There is exactly one singular time T* (*where the flow becomes extinct*).
(2) *The singular set $\mathcal{S}$ is a k-dimensional closed, connected, embedded C^1 submanifold of singularities where the blowup is a cylinder $\mathbb{S}^{n-k} \times \mathbb{R}^k$ at each point.*

Moreover, $\mathcal{S}$ is tangent to the $\mathbb{R}^k$ factor in (2).

In general, even if u is not C^2, it follows from [11] that $\mathcal{S}$ is contained in a union of C^1 submanifolds with each submanifold tangent to the axis of the corresponding cylinder at each singular point.[1] There are finitely many $(n-1)$-dimensional submanifolds and at most countably many in each lower dimension. Theorem 1.1 gives a much stronger statement when u is C^2: there is only one submanifold;, it is closed, connected, and embedded, it lies in one singular time, and $\mathcal{S}$ fills out the entire submanifold (rather than being a subset of it).

[1] The main theorem of [11] states the submanifolds are Lipschitz, but the proof shows that they are C^1.

A convex mean curvature flow gives an example where u is C^2 and $\mathcal{S}$ is a point (i.e., $k = 0$), while the marriage ring[2] gives an example where u is C^2 and $\mathcal{S}$ is a circle of cylindrical singularities. In contrast, any of the examples of rotationally symmetric surfaces studied in [1] has isolated cylindrical singular points and, thus, is not C^2.

We can restate the theorem in terms of the function u as follows:

COROLLARY 1.2. *u is C^2 if and only if both* (1) *and* (2) *hold*:

(1) *There is exactly one critical value $T = \max u$.*
(2) *The critical set $\mathcal{S}$ is a k-dimensional closed, connected, embedded C^1 submanifold. At each critical point,* Hess_u *has a k-dimensional kernel tangent to the critical set and is $-\frac{1}{n-k}$ times the identity on the orthogonal complement.*

The Hessian is always continuous where the flow is smooth. Thus, discontinuity of Hess_u only occurs at critical points of u. The next proposition shows that Hess_u is still continuous at a critical point if we approach it transversely to the kernel K of Hess_u at the critical point: u is C^2 where the projection Π_{axis} onto K is bounded by the projection Π onto $K^{\perp}$.

THEOREM 1.3. *Suppose that $\nabla u(0) = 0$. Given any C, there exists $\delta > 0$ so that u is C^2 in the region*

$$B_\delta \cap \{x \mid |\Pi_{\mathrm{axis}}(x)| \le C|\Pi(x)|\}. \tag{1.3}$$

Thus, any lack of continuity only occurs along paths tangent to the kernel of Hess_u.

2 C^2 Arrival Times

In this section, we will prove one direction of the main theorem: If the arrival time is C^2, then the flow has the one singular time and the singular set is a closed, connected, embedded C^1 submanifold.

Throughout this section, u is the arrival time of a mean convex flow in $\mathbb{R}^{n+1}$ starting from a smooth, closed, connected hypersurface.

2.1 The Stratification of $\mathcal{S}$

When the initial hypersurface is mean convex, then all singularities are cylindrical; see [2, 14, 15, 17, 18, 24, 25]; cf. [3, 6].

The singular set $\mathcal{S}$ is stratified into subsets

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} = \mathcal{S}, \tag{2.1}$$

where $\mathcal{S}_k$ consists of all singularities where the tangent flow splits off a euclidean factor of dimension *at most k*. In particular, $\mathcal{S}_k \setminus \mathcal{S}_{k-1}$ is the set where the blowup

[2] The marriage ring is a thin mean convex torus of revolution in $\mathbb{R}^3$ where the MCF is smooth until it becomes extinct along a circle.

is $\mathbb{R}^k \times \mathbb{S}^{n-k}$. By [9], the Hessian has a special form at a critical point. Namely, if $p \in \mathcal{S}_k \setminus \mathcal{S}_{k-1}$, then

$$\mathrm{Hess}_u(p) = -\frac{1}{n-k}\,\Pi, \tag{2.2}$$

where Π is an orthogonal projection onto the orthogonal complement of the $\mathbb{R}^k$ factor. If $k \geq 1$, let Π_{axis} denote the orthogonal projection onto the k-plane tangent to the "axis."

It follows from upper semicontinuity of the density that the top strata $\mathcal{S} \setminus \mathcal{S}_{n-2}$ is compact. A priori, it is possible that a sequence of points in one of the lower strata might converge to a point in a higher strata. However, by (2.2), this is impossible when the arrival time is C^2:

LEMMA 2.1. *If u is C^2, then each strata $\mathcal{S}_k \setminus \mathcal{S}_{k-1}$ is compact.*

LEMMA 2.2. *If u is C^2 at $p \in \mathcal{S}_k \setminus \mathcal{S}_{k-1}$ with $k \geq 1$ and q_j is a sequence of regular points converging to p, then*

$$\Pi_{\mathrm{axis}}(\mathbf{n}(q_j)) \to 0. \tag{2.3}$$

PROOF. We will argue by contradiction, so suppose instead that there is a sequence $q_j \to p$ with $|\Pi_{\mathrm{axis}}(\mathbf{n}(q_j))| \geq \delta > 0$. Since $\mathbb{S}^n$ is compact, we can pass to a subsequence so that $\mathbf{n}(q_j) \to V \in \mathbb{S}^n$. In particular, we must have

$$|\Pi_{\mathrm{axis}}(V)| \geq \delta > 0. \tag{2.4}$$

Using the arrival time equation (1.2) at the smooth points q_j and then passing to limits since u is C^2, we get that

$$\begin{aligned} 0 &= \lim_{j\to\infty}\big(1 + \Delta u(q_j) - \mathrm{Hess}_u(q_j)(\mathbf{n}(q_j), \mathbf{n}(q_j))\big) \\ &= 1 + \Delta u(p) - \mathrm{Hess}_u(p)(V, V) \\ &= -\frac{1}{n-k} - \left[-\frac{1}{n-k}\,\langle \Pi(V), V\rangle\right] = -\frac{1}{n-k}\,|\Pi_{\mathrm{axis}}(V)|^2. \end{aligned} \tag{2.5}$$

This contradicts (2.4), giving the lemma. □

The next lemma, which does not assume that u is C^2, shows that a plane orthogonal to the axis of a singularity contains a point q where $\Pi(\nabla u(q)) = 0$.

LEMMA 2.3. *Suppose that $\nabla u(0) = 0$ and $\mathrm{Hess}_u(0)$ has kernel K. There exist $\epsilon > 0$ and C so that if $p \in B_\epsilon \cap K$, then there exists $q \in B_{C|p|} \cap (p + K^\perp)$ with $\Pi(\nabla u(q)) = 0$.*

PROOF. By the uniqueness of [8], the flow is cylindrical at time $t = u(0) - \sqrt{\delta}$ in a ball $B_{C'\delta}(p)$ for every $\delta \in (0, \epsilon)$ for some $\epsilon > 0$ sufficiently small. Here C' is a large constant.[3] Thus, since $p \in B_\epsilon \cap K$, the level set $\{u = u(0) - \sqrt{|p|}\}$ is

[3] We can make C' as big as we want at the cost of decreasing ϵ.

an approximate cylinder about K in $B_{C'|p|}$. In particular, the intersection

$$\{u = u(0) - \sqrt{|p|}\} \cap (p + K^{\perp}) \tag{2.6}$$

is close to an $\mathbb{S}^{n-k+1}$, and, furthermore, u is strictly decreasing at each point of the intersection. Let $q \in (p + K^{\perp})$ be the point where u achieves its maximum inside the subset of $(p + K^{\perp})$ bounded by $\{u = u(0) - \sqrt{|p|}\}$. It follows that q is in the interior and, thus, $\nabla u(q)$ is orthogonal to $K^{\perp}$ as claimed. □

2.2 Local Lemma

In this subsection, we assume that u is C^2. The key to Theorem 1.1 is the following local proposition:

PROPOSITION 2.4. *Suppose that* $\nabla u(0) = 0$ *and* $\mathrm{Hess}_u(0)$ *has kernel* K. *Then there exists* $\epsilon > 0$ *so that* $B_\epsilon \cap \mathcal{S}$ *is the graph of a* C^1 *map*

$$f : \Omega \subset K \to K^{\perp}, \tag{2.7}$$

where Ω *is a connected open subset of* K *containing* 0. *Furthermore,* u *is constant on* $B_\epsilon \cap \mathcal{S}$.

PROOF. It follows from theorem 2.5 and corollary 4.5 in [11] that there is some $\delta > 0$ so that $B_\delta \cap \mathcal{S}$ is *contained in* the graph of a C^1 map[4]

$$f : \Omega \subset K \to K^{\perp}. \tag{2.8}$$

Moreover, $B_\delta \cap \mathcal{S}$ is automatically a (relatively) closed subset of this graph. To prove the first part of the proposition, we show that we can choose some $\epsilon \in (0, \delta]$ so $\mathcal{S}$ fills out the entire graph in B_ϵ. To do this, we must rule out the following possibility:

($\star$) There is a sequence $p_j \to 0$ of points $p_j \in K$ so that the plane P_j through p_j and parallel to $K^{\perp}$ misses $B_\delta \cap \mathcal{S}$.

We will show that ($\star$) leads to a contradiction. Namely, for each j, Lemma 2.3 gives a point $q_j \in B_{C|p_j|} \cap P_j$ with

$$\Pi(\nabla u(q_j)) = 0, \tag{2.9}$$

where Π is an orthogonal projection onto $K^{\perp}$. Since $\mathcal{S}$ does not intersect P_j, we know that $\nabla u(q_j) \neq 0$. Therefore, (2.9) gives that

$$\Pi(\mathbf{n}(q_j)) = 0. \tag{2.10}$$

However, this contradicts Lemma 2.2 since $q_j \to 0$. Thus, we get the desired $\epsilon > 0$. This gives the first part of the proposition.

Next, we must show that this graph is contained in a level set of u. This follows immediately from part (B) of theorem 1.2 in [11] since any two points in the graph can be connected by a C^1 curve in $\mathcal{S}$. □

[4] The main theorem of [11] states that the map f below is Lipschitz. However, the regularity of the distribution of k-planes implies that it is in fact C^1.

2.3 Local Extinction after Singularities

In the next lemma, $p \in \mathcal{S}_k \setminus \mathcal{S}_{k-1}$ is a singularity of the flow and $K_p^\perp$ is the $(n+1-k)$-dimensional plane through p orthogonal to the axis of the singularity.

LEMMA 2.5. *There exists $\epsilon > 0$, depending only on u and not on p, so that*

- *$B_\epsilon(p) \cap \{u > u(p)\}$ does not intersect $K_p^\perp$.*

PROOF. By the uniqueness of [8], the flow is cylindrical at time $t = u(p) - \sqrt{\delta}$ in a ball $B_{C\,\delta}(p)$ for every $\delta \in (0, \epsilon)$ for some $\epsilon > 0$ sufficiently small. Here $\epsilon > 0$ depends only on the cylindrical scale and, thus, is uniform in p by theorem 3.1 in [11] because each strata is compact by Lemma 2.1.

The intersection of the level set $u = u(p) - \sqrt{\delta}$ with $K_p^\perp$ is an $(n-k)$-sphere that separates $K_p^\perp$ (at least in the ball $B_\epsilon(p)$) into an inside containing p and an outside where the flow has recently gone through. Because the flow is monotone, it can never return to this outside region. By assumption, these inside regions shrink to p as $\delta \to 0$. □

The next corollary shows that if a critical time can be approached by future regular times, then each critical point at this time is a local maximum.

COROLLARY 2.6. *Suppose that u is C^2, $\nabla u(0) = 0$, and there exist $t_i > u(0)$ with $t_i \to u(0)$ and $\nabla u \neq 0$ on $\{u = t_i\}$. Then there exists $\delta > 0$ so that*

$$\sup_{B_\delta} u = u(0). \tag{2.11}$$

PROOF. Let $\epsilon > 0$ be from Lemma 2.5. We will argue by contradiction, so suppose instead that there is a sequence $p_j \to 0$ with $u(p_j) > u(0)$. By continuity of u, $u(p_j) \to u(0)$. Thus, after passing to subsequences for the p_j's and t_j's, we can assume that

$$u(p_1) > t_1 > u(p_2) > t_2 > \cdots \to u(0). \tag{2.12}$$

Suppose that i is large so that $|p_i| < \epsilon$. Since u is continuous and $u(p_i) > t_i > u(0)$, the line segment from 0 to p_i intersects $\{u = t_i\}$. Thus, we can choose q_i with

$$|\Pi_{\mathrm{axis}}(q_i)|^2 = \min\{|\Pi_{\mathrm{axis}}(q)|^2 \mid q \in B_\epsilon \text{ and } u(q) = t_i\} \le |p_i|^2. \tag{2.13}$$

This has two consequences:

$$|q_i|^2 \to 0, \tag{2.14}$$

$$\Pi(\mathbf{n}(q_i)) = 0. \tag{2.15}$$

To prove (2.14), use (2.13) to get that $|\Pi_{\mathrm{axis}}(q_i)|^2 \to 0$ and then use that the support of the flow for $u > u(0)$ must be close to K near 0 (by theorem 3.1 in [11]).

To see (2.15), let $h : u^{-1}(t_i) \to \mathbb{R}$ be given by $h(x) = |\Pi_{\text{axis}}(x)|^2$, so that

$$\frac{1}{2}\nabla_x h = \Pi_{\text{axis}}(x) - \langle \Pi_{\text{axis}}(x), \mathbf{n}(x)\rangle \mathbf{n}(x). \tag{2.16}$$

Since q_i is a minimum of h, we get that $\nabla_{q_i} h = 0$ and, therefore,

$$\Pi_{\text{axis}}(q_i) = \langle \Pi_{\text{axis}}(q_i), \mathbf{n}(q_i)\rangle \mathbf{n}(q_i). \tag{2.17}$$

It follows that $\Pi_{\text{axis}}(q_i) = \pm|\Pi_{\text{axis}}(q_i)|\mathbf{n}(q_i)$. This implies that

$$\Pi_{\text{axis}}(q_i) = 0 \quad \text{or} \quad \Pi(\mathbf{n}(q_i)) = 0. \tag{2.18}$$

Lemma 2.5 rules out the first possibility, so we get (2.15).

On the other hand, (2.14) allows us to apply Lemma 2.2 to get that

$$\Pi_{\text{axis}}(\mathbf{n}(q_i)) \to 0. \tag{2.19}$$

This contradicts (2.15), completing the proof. □

2.4 Proofs of the Main Results

We will prove one direction of Theorem 1.1 in the following proposition.

PROPOSITION 2.7. *If u is C^2, then*

(1) *there is exactly one singular time T (where the flow becomes extinct), and*
(2) *the singular set $\mathcal{S}$ is a k-dimensional closed, connected, embedded C^1 submanifold of singularities where the blowup is a cylinder $\mathbb{S}^{n-k} \times \mathbb{R}^k$ at each point.*

Moreover, $\mathcal{S}$ is tangent to the $\mathbb{R}^k$ factor in (2).

PROOF. Fix a point $p \in \mathcal{S}$. Let k be the dimension of the kernel of $\text{Hess}_u(p)$, so p is cylindrical of type $\mathbb{S}^{n-k} \times \mathbb{R}^k$. Let $\mathcal{S}_p$ be the component of $\mathcal{S}$ containing p; note that each point in $\mathcal{S}_p$ must also be cylindrical of type $\mathbb{S}^{n-k} \times \mathbb{R}^k$ by Lemma 2.1. Given $q \in \mathcal{S}_p$, let $K_q^{\perp}$ be the k-dimensional kernel of $\text{Hess}_u(q)$.

Proposition 2.4 implies that each point q in $\mathcal{S}_p$ has an $\epsilon_q > 0$ so that

- $B_{\epsilon_q}(q) \cap \mathcal{S}$ is given as a C^1 graph over $K_q^{\perp}$.
- u is constant on this graph.

Since $\mathcal{S}_p$ is compact and connected, it follows that $\mathcal{S}_p$ is a closed, connected, embedded C^1 k-dimensional submanifold and $u \equiv u(p)$ on $\mathcal{S}_p$.

Since $\mathcal{S}$ is compact, we conclude that $\mathcal{S}$ is given as a finite collection of disjoint embedded C^1 closed submanifolds

$$\mathcal{S} = \bigcup_{j=1}^{N} \mathcal{S}_{p_j} \quad \text{with } u(\mathcal{S}_{p_j}) \equiv u(p_j). \tag{2.20}$$

Let T be the first singular time. In the remainder of the proof, we will show that

(A) T is also the extinction time and, thus, the only singular time.
(B) $\mathcal{S}$ has only one component.

Let $\mathcal{S}_T = \mathcal{S} \cap \{u = T\}$ be the union of the $\mathcal{S}_{p_j}$'s where $u(p_j) = T$. Note that $\mathcal{S}_T$ is compact and there exists $\kappa > 0$ so that

$$\mathcal{S} \cap \{T < u < T + \kappa\} = \varnothing \tag{2.21}$$

since there are only finitely many singular times. Thus, Corollary 2.6 gives $\delta > 0$ so that

$$\sup_{T_\delta(\mathcal{S}_T)} u = T, \tag{2.22}$$

where $T_\delta(\mathcal{S}_T)$ is the δ-tubular neighborhood of $\mathcal{S}_T$.

We can now prove (A) by contradiction. Namely, if (A) does not hold, then (2.22) and the monotonicity of the flow imply that $\{u = t\}$ intersects both inside and outside of $T_{\delta/2}(\mathcal{S}_T)$ for $t < T$. Since the initial hypersurface is connected and the flow is smooth before $u = T$, we know that $\{u = t\}$ is connected for each $t < T$. Thus, we get a sequence of points $z_j \in \partial T_{\delta/2}(\mathcal{S}_T)$ with $u(z_j) < T$ and $u(z_j) \to T$. By compactness, a subseqence of the z_j's converges to $z \in \partial T_{\delta/2}(\mathcal{S}_T)$. Continuity of u implies that $u(z) = T$ and, thus, (2.22) implies that z is a local maximum for u and $\nabla u(z) = 0$. This contradicts that $z \in \partial T_{\delta/2}(\mathcal{S}_T)$ is not a critical point, giving (A).

Now that we know that every point in $\{u = T\}$ is a critical point, the same argument that we used for (A) implies that $\mathcal{S} = \mathcal{S}_T$ is connected. This gives (B), completing the proof. □

3 The Arrival Time Is C^2 Away from the Axis

Throughout this section, u will be the arrival time for a mean convex flow in $\mathbb{R}^{n+1}$ starting from a smooth, closed mean convex hypersurface. By [9], u is twice differentiable everywhere with bounded Hess_u and is smooth away from the singular set where $\nabla u = 0$.

PROOF OF THEOREM 1.3. It follows from [11] that the region in (1.3) intersects the singular set only at 0 for $\delta > 0$ small enough. Thus, by [9], we need only show that any sequence $q_j \to 0$ in (1.3) must have $\mathrm{Hess}_u(q_j) \to \mathrm{Hess}_u(0)$. Furthermore, by lemma 2.11 of [9], $u(x) \leq u(0)$ in the region (1.3) with equality only for $x = 0$.

If $e_1, \dots, e_n$ is an orthonormal frame for the level sets of u, then

$$\mathrm{Hess}_u(e_i, e_j) = \frac{A(e_i, e_j)}{H}, \tag{3.1}$$

$$\mathrm{Hess}_u(\mathbf{n}, \mathbf{n}) = \nabla_{\mathbf{n}} |\nabla u| = -\frac{\partial_t H}{H^3} = -\frac{(\Delta + |A|^2) H}{H^3}, \tag{3.2}$$

$$\mathrm{Hess}_u(e_i, \mathbf{n}) = \nabla_{e_i} |\nabla u| = -\frac{H_i}{H^2}. \tag{3.3}$$

In the region (1.3), the uniqueness of [8] gives that the rescaled level set flow converges to cylinders with axis K. If we let ρ denote the distance to K, then

$$\frac{\nabla u}{|\nabla u|} \to \partial_\rho \quad \text{and} \quad \frac{1}{H\rho} = \frac{|\nabla u|}{\rho} \to 1, \tag{3.4}$$

$$-\frac{A}{H} \to \frac{1}{n-k}\,\Pi \text{ restricted to the tangent space,} \tag{3.5}$$

$$\frac{\left|\nabla \frac{A}{H}\right|}{H}, \quad \frac{|\nabla H|}{H^2}, \quad \text{and} \quad \frac{|\Delta H|}{H^3} \to 0. \tag{3.6}$$

The first three claims are immediate from the uniqueness of the blowup. The last three claims follow from the smooth convergence of the rescaled level sets to the cylinder (where each of these quantities is 0); the powers of H are the appropriate scaling factors.

Combining these facts shows that Hess_u is continuous in this conical region. □

PROOF OF THEOREM 1.1. One direction is given by Proposition 2.7. We will suppose therefore that (1) and (2) hold and show that u must be C^2. By [9], u is twice differentiable everywhere and smooth away from the singular set $\mathcal{S}$. Thus, we must show that Hess_u is continuous at each point of $\mathcal{S}$.

Using the form of the Hessian, it follows that if $p, \tilde{p} \in \mathcal{S}$, then

$$|\mathrm{Hess}_u(p) - \mathrm{Hess}_u(\tilde{p})| \le C \operatorname{dist}(T_p\mathcal{S}, T_{\tilde{p}}\mathcal{S}). \tag{3.7}$$

Fix a point $p \in \mathcal{S}$ and let $q_j \to p$ be any sequence. We must show that $\mathrm{Hess}_u(q_j) \to \mathrm{Hess}_u(p)$. For each j, let p_j be a closest point in $\mathcal{S}$ to q_j. It follows that

- $|p_j - q_j| \le |p - q_j| \to 0$ and
- $\langle (p_j - q_j), T_{p_j}\mathcal{S}) \rangle = 0$.

The second property allows us to apply Theorem 1.3 to get that

$$|\mathrm{Hess}_u(p_j) - \mathrm{Hess}_u(q_j)| \to 0. \tag{3.8}$$

Finally, since $\mathcal{S}$ is C^1 and $p_j \to p$, (3.7) implies that

$$|\mathrm{Hess}_u(p_j) - \mathrm{Hess}_u(p)| \to 0. \tag{3.9}$$

□

Acknowledgments. The authors were partially supported by National Science Foundation grants DMS 1404540 and DMS 1206827. This material is based upon work supported by National Science Foundation Grant DMS 1440140, while T.H.C. was in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, CA, during the spring of 2016.

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Received June 2016.