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Calculus of Variations

Ricci curvature and monotonicity for harmonic functions

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Abstract In this paper we generalize the monotonicity formulas of "Colding (Acta Math 209:229–263, 2012)" for manifolds with nonnegative Ricci curvature. Monotone quantities play a key role in analysis and geometry; see, e.g., "Almgren (Preprint)", "Colding and Minicozzi II (PNAS, 2012)", "Garofalo and Lin (Indiana Univ Math 35:245–267, 1986)" for applications of monotonicity to uniqueness. Among the applications here is that level sets of Green's function on open manifolds with nonnegative Ricci curvature are asymptotically umbilic.

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1 Introduction

The paper [3] proved three new monotonicity formulas for harmonic functions on manifolds with nonnegative Ricci curvature. We will see that each of these formulas sits in a two-parameter family of monotonicity formulas, with one parameter corresponding to the equation and the second to the power of the gradient in the formula. Furthermore, various limiting cases of these formulas carry important geometric information. This paper deals with only one of the parameters, the power of the gradient; the other will be done in [5].

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1.1 Monotonicity formulas

Our monotonicity formulas will involve a one-parameter family A_β of scale-invariant quantities on an open manifold M^n. Namely, for a proper positive function $u : M \setminus \{x\} \to \mathbf{R}$, define

$$A_\beta(r) = r^{1-n} \int_{u=r} |\nabla u|^{1+\beta}. \tag{1.1}$$

To simplify notation, given a dimension $n > 2$ and an exponent $\beta \geq \frac{n-2}{n-1}$, then we define

$$\tilde{\beta} = \tilde{\beta}(n, \beta) \equiv 1 + (\beta - 1)(n - 1) \geq 0. \tag{1.2}$$

The next theorem shows that A_β is monotone when M has nonnegative Ricci curvature. The derivative is an integral depending on the trace-free second fundamental form II_0 of the level sets and the trace-free Hessian B of the function u^2 where u^{2-n} is a Green's function.[1]

Theorem 1.1 *Let M be nonparabolic[2] with nonnegative Ricci curvature and Green's function G. If $u = G^{\frac{1}{2-n}}$, then*

$$A'_\beta(r) = -\beta\, r^{n-3} \int_{r \leq u} u^{4-2n}\, |\nabla u|^\beta \left(|\text{II}_0|^2 + Ric(\mathbf{n}, \mathbf{n}) \right) - \frac{\beta\, r^{n-3}}{4(n-1)} \int_{r \leq u} u^{2-2n}\, |\nabla u|^{\beta-2} \left(\tilde{\beta}\, |B(\mathbf{n})|^2 + (n-2) \left| B(\mathbf{n})^T \right|^2 \right). \tag{1.3}$$

We used $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ to denote the unit normal to the level sets of u.

Theorem 1.1 illustrates the scale-invariance of $A_\beta(r)$: When u comes from the Euclidean Green's function, then the functionals are constant in r and, furthermore, the power r^{1-n} in the definition of A_β is the only one that makes it constant in r.

The other monotonicity formulas show that various combinations of different A_β's are monotone. These will be stated in Sect. 3.

1.2 Asymptotically umbilic

We say that the level sets of a proper function u are *asymptotically umbilic* if

$$r \int_r^{2r} \frac{1}{\text{Vol}(u=s)} \int_{u=s} |\text{II}_0|^2 \, ds \to 0 \text{ as } r \to \infty. \tag{1.4}$$

A consequence of our monotonicity formulas is that the level sets of u are asymptotically umbilic; cf. [2].

Monotone quantities play a key role in analysis and geometry; see e.g., [1,4,8] for applications of monotonicity to uniqueness

[1] Our Green's functions will be normalized so that on Euclidean space of dimension $n \geq 3$ the Green's function of the Laplacian is r^{2-n}.

[2] A complete manifold is nonparabolic if it admits a positive Green's function G. By a result of Varopoulos, [11], an open manifold with nonnegative Ricci curvature is nonparabolic if and only if $\int_1^\infty \frac{r}{\text{Vol}(B_r(x))}\, dr < \infty$. Combining the result of Varopoulos with work of Li and Yau [9], gives that $G = G(x, \cdot) \to 0$ at infinity.

Corollary 1.2 *If M has nonnegative Ricci curvature and is nonparabolic, then the level sets of $u = G^{\frac{1}{2-n}}$ are asymptotically umbilic.*

Monotone quantities play a key role in analysis and geometry; see, e.g., "Almgren (Preprint)", "Colding and Minicozzi II (PNAS, 2012)", "Garofalo and Lin (Indiana Univ Math 35:245–267, 1986)" for applications of monotonicity to uniqueness.

2 The trace-free Hessian

In this section, we will compute the Laplacian on combinations of u and $|\nabla u|$ that come up later in the monotonicity formulas. In order to squeeze as much as possible out of these formulas, we will express them in terms of the trace-free Hessian B and trace-free second fundamental form Π_0 which vanish in the model case.

We will several times later use the following elementary lemma:

Lemma 2.1 *Let $u : M \to \mathbf{R}$ be a smooth positive function. The following are equivalent:*

- $\Delta\, u^2 = 2n\, |\nabla u|^2$.
- $\Delta\, u = (n-1)\, \frac{|\nabla u|^2}{u}$.
- $\Delta\, u^{2-n} = 0$.

Proof For any positive function v and any real number α we have the following

$$\Delta v^{\alpha} = \alpha\, v^{\alpha-1} \left((\alpha-1)\, \frac{|\nabla v|^2}{v} + \Delta\, v \right). \tag{2.1}$$

The lemma easily follows. □

2.1 The trace free Hessian of u^2

Throughout this section, the function u satisfies

$$\Delta u^2 = 2n\, |\nabla u|^2. \tag{2.2}$$

The basic example is the function $|x|$ on Euclidean space $\mathbf{R}^n$. In that case, the full Hessian of $|x|$ satisfies a nice equation, not just the Laplacian of $|x|$. There are a number of ways of writing this, perhaps the simplest being that the trace-free Hessian of $|x|^2$ vanishes.

With this in mind, we define the tensor B to be the trace-free Hessian of u^2

$$B = \text{Hess}_{u^2} - 2\, |\nabla u|^2\, g\,, \tag{2.3}$$

where g is the Riemannian metric.

We will use that $\text{Hess}_{u^2} = 2\, u\, \text{Hess}_u + 2\, \nabla u \otimes \nabla u$, so that

$$2\, u\, \text{Hess}_u = \text{Hess}_{u^2} - 2\, \nabla u \otimes \nabla u = B + 2\, \left(|\nabla u|^2\, g - \nabla u \otimes \nabla u \right). \tag{2.4}$$

The next lemma computes the gradient of $|\nabla u|^2$ in terms of B.

Lemma 2.2 *We have $u\, \nabla |\nabla u|^2 = B(\nabla u)$, where $B(\nabla u)$ is given by $\langle B(\nabla u), v\rangle \equiv B(\nabla u, v)$.*

Proof Since $\nabla|\nabla u|^2 = 2\,\text{Hess}_u(\nabla u, \cdot)$, Eq. (2.4) gives

$$u\,\nabla|\nabla u|^2 = 2\,u\,\text{Hess}_u(\nabla u, \cdot) = B(\nabla u, \cdot) + 2\,\left(|\nabla u|^2\,\nabla u - |\nabla u|^2\,\nabla u\right) = B(\nabla u, \cdot). \tag{2.5}$$

□

Corollary 2.3 *We have* $2\,u\,\nabla|\nabla u| = B(\mathbf{n})$ *where* $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ *and* $4\,u^2\,|\nabla|\nabla u||^2 = |B(\mathbf{n})|^2$.

Proof Since $u\,\nabla|\nabla u|^2 = 2\,u\,|\nabla u|\,\nabla|\nabla u|$, this follows from Lemma 2.2. □

The next lemma computes the divergence of B.

Lemma 2.4 *The divergence of* B *is*

$$\delta B = Ric(\nabla u^2, \cdot) + (2n-2)\,\nabla|\nabla u|^2 = Ric(\nabla u^2, \cdot) + (2n-2)\,u^{-1}\,B(\nabla u). \tag{2.6}$$

Proof Fix a point $p \in M$ and let e_i be an orthonormal frame at p with $\nabla_{e_i} e_j(p) = 0$. Given a function w, the symmetry of the Hessian and the definition of the curvature give

$$w_{ijj} = w_{jij} = w_{jji} + \text{Ric}_{ij}\,w_j. \tag{2.7}$$

Using the definition of B, the fact that g is parallel, and (2.7) with $w = u^2$ gives

$$(\delta B)_i \equiv B_{ij,j} = (u^2)_{ijj} - 2\left(|\nabla u|^2\right)_i = \text{Ric}_{ij}\,(u^2)_j + \left(\Delta u^2\right)_i - 2\left(|\nabla u|^2\right)_i. \tag{2.8}$$

Thus, $\delta B = \text{Ric}(\nabla u^2, \cdot) + \nabla(\Delta u^2 - 2\,|\nabla u|^2)$. The lemma follows since $\Delta u^2 = 2n\,|\nabla u|^2$. □

Using this, we can compute the Laplacian of $|\nabla u|^2$.

Lemma 2.5 *We have*

$$\begin{aligned} u^2\,\Delta|\nabla u|^2 &= \frac{1}{2}\,|B|^2 + (2n-4)\,B(\nabla u, \nabla u) + \frac{1}{2}\,Ric(\nabla u^2, \nabla u^2) \\ &= \frac{1}{2}\,|B|^2 + (n-2)\,\langle \nabla|\nabla u|^2, \nabla u^2\rangle + \frac{1}{2}\,Ric(\nabla u^2, \nabla u^2). \end{aligned} \tag{2.9}$$

Proof Using the definition of the Laplacian, then Lemma 2.2, and then Lemma 2.4 gives

$$\begin{aligned} u^2\,\Delta|\nabla u|^2 &= u^2\,\text{div}\,\nabla|\nabla u|^2 = u^2\,\text{div}\,\left(u^{-1}\,B(\nabla u)\right) \\ &= u\,\langle \delta B, \nabla u\rangle + \langle B, u\,\text{Hess}_u\rangle - B(\nabla u, \nabla u) \\ &= u\,\text{Ric}(\nabla u^2, \nabla u) + (2n-2)\,B(\nabla u, \nabla u) \\ &\quad + \left\langle B, \left\{\frac{1}{2}\,B + \left(|\nabla u|^2\,g - \nabla u \otimes \nabla u\right)\right\}\right\rangle - B(\nabla u, \nabla u). \end{aligned} \tag{2.10}$$

Using that $\langle B, g\rangle = 0$ (since B is trace-free) and $\langle B, \nabla u \otimes \nabla u\rangle = B(\nabla u, \nabla u)$ gives

$$u^2\,\Delta|\nabla u|^2 = \frac{1}{2}\,\text{Ric}(\nabla u^2, \nabla u^2) + (2n-4)\,B(\nabla u, \nabla u) + \frac{1}{2}\,|B|^2.$$

This gives the first equality. The second uses that $u\,\nabla|\nabla u|^2 = B(\nabla u)$ by Lemma 2.2. □

More generally, we compute the Laplacian of powers of $|\nabla u|$.

Proposition 2.6 *If* $\alpha \neq 0$, *then*

$$\frac{2}{\alpha}\,|\nabla u|^{2-\alpha}\,\Delta|\nabla u|^{\alpha} = \frac{|B|^2 + (\alpha-2)\,|B(\mathbf{n})|^2 + Ric(\nabla u^2, \nabla u^2)}{2u^2} + \frac{n-2}{u^2}\,\langle \nabla|\nabla u|^2, \nabla u^2\rangle. \tag{2.11}$$

Proof We have

$$2\,\nabla\,|\nabla u|^{\alpha} = 2\,\alpha\,|\nabla u|^{\alpha-1}\,\nabla|\nabla u| = \alpha\,|\nabla u|^{\alpha-2}\,\nabla|\nabla u|^2. \tag{2.12}$$

Taking the divergence of this and using Lemma 2.5 gives

$$\begin{aligned}\frac{2}{\alpha}\,|\nabla u|^{3-\alpha}\,\Delta|\nabla u|^{\alpha} &= |\nabla u|^{3-\alpha}\,\mathrm{div}\,\left(|\nabla u|^{\alpha-2}\,\nabla|\nabla u|^2\right)\\ &= |\nabla u|\,\Delta|\nabla u|^2 + (\alpha-2)\,\langle\nabla|\nabla u|^2, \nabla|\nabla u|\rangle\\ &= \frac{|\nabla u|}{2u^2}\,\mathrm{Ric}(\nabla u^2, \nabla u^2) + \frac{|\nabla u|}{2u^2}\,|B|^2 + (n-2)\,\frac{|\nabla u|}{u^2}\,\langle\nabla|\nabla u|^2, \nabla u^2\rangle\\ &\quad + 2\,(\alpha-2)\,|\nabla u|\;|\nabla|\nabla u||^2\,. \end{aligned} \tag{2.13}$$

Since $4\,u^2\;|\nabla|\nabla u||^2 = |B(\mathbf{n})|^2$ by Corollary 2.3, simplifying this gives the proposition. □

When $\alpha = 1$, we get the following corollary:

Corollary 2.7 *We have*

$$2\,|\nabla u|\,\Delta|\nabla u| = \frac{1}{2u^2}\,\left(|B|^2 - |B(\mathbf{n})|^2 + Ric(\nabla u^2, \nabla u^2)\right) + \frac{n-2}{u^2}\,\langle\nabla|\nabla u|^2, \nabla u^2\rangle. \tag{2.14}$$

2.2 The trace-free second fundamental form

The second fundamental form II of the level sets of u is given by

$$\mathrm{II}(e_i, e_j) \equiv \langle\nabla_{e_i}\mathbf{n}, e_j\rangle\,, \tag{2.15}$$

where e_i is a tangent frame and $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ is the unit normal.

Lemma 2.8 *The trace-free second fundamental form II_0 is given by*

$$2\,u\,|\nabla u|\,II_0 = B_0 + \frac{B(\mathbf{n},\mathbf{n})}{n-1}\,g_0, \tag{2.16}$$

where B_0 is the restriction of B to tangent vectors and g_0 is the metric on the level set.

Proof Using that ∇u is normal, we can rewrite II as

$$2\,u\,|\nabla u|\,\mathrm{II}(e_i, e_j) = \langle\nabla_{e_i}\nabla u^2, e_j\rangle = \mathrm{Hess}_{u^2}(e_i, e_j). \tag{2.17}$$

The mean curvature H is the trace of II over the e_i's. We have

$$\begin{aligned} 2\,u\,|\nabla u|\,H &= \Delta u^2 - \mathrm{Hess}_{u^2}(\mathbf{n},\mathbf{n}) = 2n\,|\nabla u|^2 - \mathrm{Hess}_{u^2}(\mathbf{n},\mathbf{n})\\ &= 2(n-1)\,|\nabla u|^2 + \left(2\,|\nabla u|^2 - \mathrm{Hess}_{u^2}(\mathbf{n},\mathbf{n})\right)\\ &= 2(n-1)\,|\nabla u|^2 - B(\mathbf{n},\mathbf{n}). \end{aligned} \tag{2.18}$$

Thus, the trace-free second fundamental form II_0 is given by

$$\begin{aligned} 2\,u\,|\nabla u|\,\mathrm{II}_0 &= 2\,u\,|\nabla u|\,\left(\mathrm{II} - \frac{H}{n-1}\,g_0\right) = \mathrm{Hess}_{u^2} - 2\,|\nabla u|^2\,g_0 + \frac{B(\mathbf{n},\mathbf{n})}{n-1}\,g_0\\ &= B_0 + \frac{B(\mathbf{n},\mathbf{n})}{n-1}\,g_0, \end{aligned} \tag{2.19}$$

where Hess_{u^2} is restricted to tangent vectors. □

Lemma 2.9 *We have*

$$4\,u^2\,|\nabla u|^2\ |II_0|^2 = |B_0|^2 - \tfrac{(B(\mathbf{n},\mathbf{n}))^2}{n-1} = |B|^2 - \tfrac{n}{n-1}\ |B(\mathbf{n})|^2 - \tfrac{n-2}{n-1}\ \left|B(\mathbf{n})^T\right|^2 . \qquad (2.20)$$

Proof Since B is trace-free, we get that

$$\langle B_0, g_0\rangle = \mathrm{Tr}(B) - B(\mathbf{n},\mathbf{n}) = -B(\mathbf{n},\mathbf{n}), \qquad (2.21)$$

Using this in Lemma 2.8 gives

$$\begin{aligned} 4\,u^2\,|\nabla u|^2\ |\mathrm{II}_0|^2 &= |B_0|^2 + \frac{(B(\mathbf{n},\mathbf{n}))^2}{(n-1)^2}\,|g_0|^2 + 2\,\frac{B(\mathbf{n},\mathbf{n})}{n-1}\langle B_0, g_0\rangle \\ &= |B_0|^2 + \frac{(B(\mathbf{n},\mathbf{n}))^2}{n-1} - 2\,\frac{(B(\mathbf{n},\mathbf{n}))^2}{n-1}. \end{aligned} \qquad (2.22)$$

This gives the first equality. To get the second equality, use the symmetry of B to get

$$\begin{aligned} |B|^2 &= \sum_{i,j\leq(n-1)} (B(e_i,e_j))^2 + 2\sum_{i\leq(n-1)} (B(\mathbf{n},e_i))^2 + (B(\mathbf{n},\mathbf{n}))^2 \\ &= |B_0|^2 + 2\left|B(\mathbf{n})^T\right|^2 + (B(\mathbf{n},\mathbf{n}))^2 \end{aligned} \qquad (2.23)$$

and note that $|B(\mathbf{n})|^2 = \left|B(\mathbf{n})^T\right|^2 + (B(\mathbf{n},\mathbf{n}))^2$. □

2.3 Divergence formulas

We will compute the divergence of various quantities involving u and $|\nabla u|$. We will need the following differential inequality for $|\nabla u|$.

Proposition 2.10 *If II_0 is the trace-free second fundamental form of the level set, then*

$$\Delta|\nabla u| = |\nabla u|\,|II_0|^2 + \frac{\mathrm{Ric}(\nabla u,\nabla u)}{|\nabla u|} + \frac{n-2}{u^2}\,\langle\nabla|\nabla u|,\nabla u^2\rangle + \frac{\left(|B(\mathbf{n})|^2 + (n-2)\ \left|B(\mathbf{n})^T\right|^2\right)}{4(n-1)\,|\nabla u|\,u^2}. \qquad (2.24)$$

Proof Corollary 2.7 gives that

$$|\nabla u|\ \Delta|\nabla u| = \tfrac{1}{4u^2}\ \left(|B|^2 - |B(\mathbf{n})|^2 + \mathrm{Ric}(\nabla u^2,\nabla u^2)\right) + \tfrac{n-2}{u^2}\ |\nabla u|\langle\nabla|\nabla u|,\nabla u^2\rangle. \qquad (2.25)$$

The next ingredient is Lemma 2.9 which gives that

$$4\,u^2\,|\nabla u|^2\ |\mathrm{II}_0|^2 = \left(|B|^2 - |B(\mathbf{n})|^2\right) - \tfrac{|B(\mathbf{n})|^2}{n-1} - \tfrac{n-2}{n-1}\ \left|B(\mathbf{n})^T\right|^2, \qquad (2.26)$$

so we see that

$$\frac{\left(|B|^2 - |B(\mathbf{n})|^2\right)}{4\,|\nabla u|\,u^2} = |\nabla u|\ |\mathrm{II}_0|^2 + \frac{\left(|B(\mathbf{n})|^2 + (n-2)\ \left|B(\mathbf{n})^T\right|^2\right)}{4(n-1)\,|\nabla u|\,u^2} \qquad (2.27)$$

□

Lemma 2.11 *Given* $p, \alpha \in \mathbf{R}$*, we have*

$$\operatorname{div}\left(u^{2p}\,|\nabla u|^{\alpha}\,\nabla u^2\right)=(2n+4p)\,u^{2p}\,|\nabla u|^{2+\alpha}+\alpha\,u^{2p}\,|\nabla u|^{\alpha-1}\,\langle\nabla|\nabla u|,\nabla u^2\rangle, \tag{2.28}$$

$$\begin{aligned}\operatorname{div}\left(u^{2p}\,|\nabla u|^{\alpha}\nabla|\nabla u|\right) &= u^{2p-2}\,(p+n-2)\,|\nabla u|^{\alpha}\,\langle\nabla u^2,\nabla|\nabla u|\rangle \\ &+u^{2p}\,|\nabla u|^{1+\alpha}\left(|\mathrm{II}_0|^2+\operatorname{Ric}(\mathbf{n},\mathbf{n})+\frac{(1+\alpha(n-1))\,|B(\mathbf{n})|^2+(n-2)\,\left|B(\mathbf{n})^T\right|^2}{4(n-1)\,|\nabla u|^2\,u^2}\right).\end{aligned} \tag{2.29}$$

Proof For the first claim, use $\Delta u^2=2n\,|\nabla u|^2$ to compute

$$\begin{aligned}\operatorname{div}\left(u^{2p}\,|\nabla u|^{\alpha}\,\nabla u^2\right) &= p\,u^{2p-2}\,|\nabla u|^{\alpha}\,\left|\nabla u^2\right|^2+u^{2p}\,|\nabla u|^{\alpha}\,\Delta u^2 \\ &\quad+\alpha\,u^{2p}\,|\nabla u|^{\alpha-1}\,\langle\nabla|\nabla u|,\nabla u^2\rangle \\ &=(2n+4p)\,u^{2p}\,|\nabla u|^{2+\alpha}+\alpha\,u^{2p}\,|\nabla u|^{\alpha-1}\,\langle\nabla|\nabla u|,\nabla u^2\rangle.\end{aligned} \tag{2.30}$$

To get the second claim, first use Proposition 2.10 to compute

$$\begin{aligned}\operatorname{div}\left(u^{2p}\,\nabla|\nabla u|\right) &= u^{2p}\,\Delta|\nabla u|+p\,u^{2p-2}\,\langle\nabla u^2,\nabla|\nabla u|\rangle \\ &=u^{2p}\left(|\nabla u|\,|\mathrm{II}_0|^2+|\nabla u|\operatorname{Ric}(\mathbf{n},\mathbf{n})+\frac{n-2}{u^2}\,\langle\nabla|\nabla u|,\nabla u^2\rangle\right. \\ &\quad\left.+\frac{\left(|B(\mathbf{n})|^2+(n-2)\,\left|B(\mathbf{n})^T\right|^2\right)}{4(n-1)\,|\nabla u|\,u^2}\right)+p\,u^{2p-2}\,\langle\nabla u^2,\nabla|\nabla u|\rangle \\ &=u^{2p}\,|\nabla u|\left(|\mathrm{II}_0|^2+\operatorname{Ric}(\mathbf{n},\mathbf{n})+\frac{\left(|B(\mathbf{n})|^2+(n-2)\,\left|B(\mathbf{n})^T\right|^2\right)}{4(n-1)\,|\nabla u|^2\,u^2}\right) \\ &\quad+u^{2p-2}\,(p+n-2)\,\langle\nabla u^2,\nabla|\nabla u|\rangle.\end{aligned} \tag{2.31}$$

The second claim follows from this since

$$\begin{aligned}\operatorname{div}\left(u^{2p}\,|\nabla u|^{\alpha}\nabla|\nabla u|\right) &= |\nabla u|^{\alpha}\operatorname{div}\left(u^{2p}\,\nabla|\nabla u|\right)+\alpha\,|\nabla u|^{\alpha-1}\,u^{2p}\,|\nabla|\nabla u||^2 \\ &=|\nabla u|^{\alpha}\operatorname{div}\left(u^{2p}\,\nabla|\nabla u|\right)+\frac{\alpha}{4}\,|\nabla u|^{\alpha-1}u^{2p-2}\,|B(\mathbf{n})|^2,\end{aligned} \tag{2.32}$$

where the last equality used that $4\,u^2\,|\nabla|\nabla u||^2=|B(\mathbf{n})|^2$ by Corollary 2.3. □

The previous divergence formulas allow us next to compute the Laplacian on various combinations of u and $|\nabla u|$. Recall that $\tilde{\beta}\geq 0$ was defined in (1.2).

Proposition 2.12 *Given* $q, \beta \in \mathbf{R}$*, we have*

$$\begin{aligned}\Delta\left(u^{2q}\,|\nabla u|^{\beta}\right) &= 2\,q\,(2q+n-2)\,u^{2q-2}\,|\nabla u|^{2+\beta}+\beta\,u^{2q}\,|\nabla u|^{\beta}\left(|\mathrm{II}_0|^2+\operatorname{Ric}(\mathbf{n},\mathbf{n})\right) \\ &\quad+\frac{\beta}{4(n-1)}\,u^{2q-2}\,|\nabla u|^{\beta-2}\left(\tilde{\beta}\,|B(\mathbf{n})|^2+(n-2)\,\left|B(\mathbf{n})^T\right|^2\right) \\ &\quad+\beta\,(2q+n-2)\,u^{2q-2}\,|\nabla u|^{\beta-1}\langle\nabla|\nabla u|,\nabla u^2\rangle.\end{aligned} \tag{2.33}$$

Proof The gradient is given by

$$\nabla\left(u^{2q}\,|\nabla u|^{\beta}\right)=q\,u^{2q-2}\,|\nabla u|^{\beta}\nabla u^2+\beta\,u^{2q}\,|\nabla u|^{\beta-1}\,\nabla|\nabla u|. \tag{2.34}$$

Taking the divergence of this and then applying the first claim in Lemma 2.11 with $p = q - 1$ and $\alpha = \beta$ and the second claim there with $q = p$ and $\alpha = \beta - 1$ gives

$$\begin{aligned}\Delta\left(u^{2q}\,|\nabla u|^{\beta}\right) &= q\,\mathrm{div}\left(u^{2q-2}\,|\nabla u|^{\beta}\nabla u^2\right) + \beta\,\mathrm{div}\left(u^{2q}\,|\nabla u|^{\beta-1}\,\nabla|\nabla u|\right)\\ &= q\,\left\{(2n+4q-4)\,u^{2q-2}\,|\nabla u|^{2+\beta} + \beta\,u^{2q-2}\,|\nabla u|^{\beta-1}\,\langle\nabla|\nabla u|,\nabla u^2\rangle\right\}\\ &\quad + \beta\,u^{2q}\,|\nabla u|^{\beta}\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right)\\ &\quad + \beta\,u^{2q-2}\,(q+n-2)\,|\nabla u|^{\beta-1}\langle\nabla|\nabla u|,\nabla u^2\rangle\\ &\quad + \frac{\beta\,u^{2q-2}\,|\nabla u|^{\beta-2}}{4(n-1)}\left(\tilde{\beta}\,|B(\mathbf{n})|^2 + (n-2)\left|B(\mathbf{n})^T\right|^2\right).\end{aligned} \tag{2.35}$$

□

We separately record the cases $2q = 2 - n$ and $q = 0$ next.

Corollary 2.13 *We have*

$$\begin{aligned}\Delta\left(u^{2-n}\,|\nabla u|^{\beta}\right) &= \beta\,u^{2-n}\,|\nabla u|^{\beta}\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right)\\ &\quad + \frac{\beta}{4(n-1)}\,u^{-n}\,|\nabla u|^{\beta-2}\left(\tilde{\beta}\,|B(\mathbf{n})|^2 + (n-2)\left|B(\mathbf{n})^T\right|^2\right),\end{aligned} \tag{2.36}$$

$$\begin{aligned}\Delta\,|\nabla u|^{\beta} &= \beta\,|\nabla u|^{\beta}\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right) + \beta\,(n-2)\,u^{-2}\,|\nabla u|^{\beta-1}\langle\nabla|\nabla u|,\nabla u^2\rangle\\ &\quad + \frac{\beta}{4(n-1)}\,u^{-2}\,|\nabla u|^{\beta-2}\left(\tilde{\beta}\,|B(\mathbf{n})|^2 + (n-2)\left|B(\mathbf{n})^T\right|^2\right).\end{aligned} \tag{2.37}$$

Proof Take $2q = 2 - n$ and $q = 0$ in Proposition 2.12. □

3 The monotonicity formulas

In this section, we use the formulas from the previous section to prove three one-parameter families of monotonicity formulas generalizing the three formulas from [3]. The parameter corresponds to the power of $|\nabla u|$ in the integrand and monotonicity holds as long as the parameter is at least a certain critical value. We will see in [5] that these formulas sit in two-parameter families where the functions then satisfy the nonlinear p-Laplace equation.

Throughout this section, $u > 0$ satisfies $\Delta u^2 = 2n\,|\nabla u|^2$ as in the previous section and, in addition, is proper and is normalized so that

$$\lim_{r\to 0}\frac{u}{r} = 1, \tag{3.1}$$

where r is the distance to a fixed point.

Recall that the scale-invariant quantity A_β is given by

$$A_\beta(r) = r^{1-n}\int_{u=r}|\nabla u|^{1+\beta}. \tag{3.2}$$

We also define a second family of scale-invariant quantities V_β by

$$V_\beta(r) = r^{2-n}\int_0^r\int_{u=s}\frac{|\nabla u|^{1+\beta}}{u^2}\,\mathrm{d}s = r^{2-n}\int_{u\le r}\frac{|\nabla u|^{2+\beta}}{u^2}, \tag{3.3}$$

where the second equality is the co-area formula. Differentiating V_β gives

$$r\, V_\beta'(r) = (2-n)\, V_\beta(r) + A_\beta(r). \tag{3.4}$$

The following simple lemma shows that both A_β and V_β are uniformly bounded:

Lemma 3.1 *If M is nonparabolic with nonnegative Ricci curvature, G is a Green's function, and $u = G^{\frac{1}{2-n}}$, then for all r*

$$A_\beta(r) \leq \mathrm{Vol}(\partial B_1(0)) \leq r^{1-n}\, \mathrm{Vol}(u=r)\,, \tag{3.5}$$

$$V_\beta(r) \leq \frac{\mathrm{Vol}(\partial B_1(0))}{n-2}. \tag{3.6}$$

In particular, both are uniformly bounded by their Euclidean values.

Proof We have that $|\nabla u| \leq 1$ by theorem 3.1 in [3]. Furthermore, Sect. 2 in [6] (cf. page 235 in [3]) gives that $r^{1-n} \int_{u=r} |\nabla u|$ is constant (independent of r); using the asymptotics near $u=0$ from lemma 2.1 in [3] to evaluate this constant gives

$$r^{1-n} \int\limits_{u=r} |\nabla u| = \mathrm{Vol}(\partial B_1(0)). \tag{3.7}$$

Both claims follow easily from these two facts. □

3.1 The first and second monotonicity formulas

We will now state and prove the first two monotonicity formulas. Recall that the constant $\tilde{\beta} \equiv 1 + (\beta-1)(n-1)$ is nonnegative since $n > 2$ and $\beta \geq \frac{n-2}{n-1}$.

The next theorem gives the first monotonicity formula.

Theorem 3.2 *We have*

$$\left(A_\beta - 2\,(n-2)\, V_\beta\right)'(r) = \frac{\beta}{r^{n-1}} \int\limits_0^r \int\limits_{u=s} |\nabla u|^{\beta-1} \left(|II_0|^2 + Ric(\mathbf{n},\mathbf{n})\right) \mathrm{d}s$$
$$+ \frac{\beta}{4(n-1)}\, r^{1-n} \int\limits_0^r \int\limits_{u=s} u^{-2}\, |\nabla u|^{\beta-3} \left(\tilde{\beta}\, |B(\mathbf{n})|^2 + (n-2) \left|B(\mathbf{n})^T\right|^2\right) \mathrm{d}s \tag{3.8}$$

Proof Differentiating A_β as in Sect. 2 of [6] (cf. [7] or page 235 in [3]) gives

$$A_\beta'(r) = r^{1-n} \int\limits_{u=r} \langle \nabla |\nabla u|^\beta, \mathbf{n} \rangle. \tag{3.9}$$

Using the asymptotics of the Green's function at the pole (cf. lemma 2.1 in [3]), $A_\beta(r)$ has a limit as $r \to 0$ and (using that $|\nabla u|$ is bounded away from 0 near $u=0$) is C^1 for $r > 0$. Hence, there is a sequence $r_i \to 0$ with[3]

$$r_i^{2-n} \left| \int\limits_{u=r_i} \langle \nabla |\nabla u|^\beta, \mathbf{n} \rangle \right| \equiv \left| r_i\, A_\beta'(r_i) \right| \to 0.$$

[3] If $f : [0,\epsilon) \to \mathbf{R}$ is continuous on $[0,\epsilon)$ and C^1 on $(0,\epsilon)$, then there exist $r_i \to 0$ with $r_i\, f'(r_i) \to 0$.

It follows that $\int_{u=r_i} \langle \nabla |\nabla u|^{\beta}, \mathbf{n}\rangle \to 0$ and, thus, applying Stokes' theorem and then using Corollary 2.13 gives

$$\begin{aligned} r^{n-1} A'_{\beta}(r) &= \int_{u\leq r} \Delta |\nabla u|^{\beta} \\ &= \int_{u\leq r} \left(\beta\, |\nabla u|^{\beta} \left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right) + \beta\,(n-2)\, u^{-2}\, |\nabla u|^{\beta-1} \langle \nabla |\nabla u|, \nabla u^2\rangle\right) \\ &\quad + \frac{\beta}{4(n-1)} \int_{u\leq r} u^{-2}\, |\nabla u|^{\beta-2} \left(\tilde{\beta}\, |B(\mathbf{n})|^2 + (n-2)\left|B(\mathbf{n})^T\right|^2\right). \end{aligned} \tag{3.10}$$

We apply the coarea formula on every term except the last term on the second line. For this, we use the first claim in Lemma 2.11 to get

$$\mathrm{div}\left(|\nabla u|^{\beta}\, \frac{\nabla u^2}{u^2}\right) = \frac{\beta |\nabla u|^{\beta-1}}{u^2} \langle \nabla |\nabla u|, \nabla u^2\rangle + (2n-4)\, \frac{|\nabla u|^{2+\beta}}{u^2}. \tag{3.11}$$

Since $n > 2$, the interior boundary integral goes to zero and the divergence theorem gives

$$\begin{aligned} \int_{u\leq r} \frac{\beta |\nabla u|^{\beta-1}}{u^2} \langle \nabla |\nabla u|, \nabla u^2\rangle &= \frac{2}{r} \int_{u=r} |\nabla u|^{1+\beta} - (2n-4) \int_{u\leq r} \frac{|\nabla u|^{2+\beta}}{u^2} \\ &= 2\, r^{n-1} \left(\frac{A_{\beta}(r) + (2-n)\, V_{\beta}(r)}{r}\right) = 2\, r^{n-1}\, V'_{\beta}(r), \end{aligned} \tag{3.12}$$

where the last equality used (3.4). Multiplying by $(n-2)\, r^{1-n}$ and putting this back into the formula for $A'_{\beta}(r)$ gives the theorem. □

The case $\beta = 2$ in Theorem 3.2 is the first monotonicity formula in [3]. We record the case $\beta = 1$ below separately as it seems to be of particular significance.

Corollary 3.3 *We have*

$$(A_1 - 2(n-2)\, V_1)'\,(r) \geq r^{1-n} \int_0^r \int_{u=s} \left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right) \mathrm{d}s. \tag{3.13}$$

Our second monotonicity formula follows.

Theorem 3.4 *If $r_1 < r_2$, then*

$$\begin{aligned} &(2-n)\, A_{\beta}(r_2) + r_2\, A'_{\beta}(r_2) - (2-n)\, A_{\beta}(r_1) - r_1\, A'_{\beta}(r_1) \\ &= \beta \int_{r_1\leq u\leq r_2} u^{2-n}\, |\nabla u|^{\beta} \left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right) \\ &\quad + \frac{\beta}{4(n-1)} \int_{r_1\leq u\leq r_2} u^{-n}\, |\nabla u|^{\beta-2} \left(\tilde{\beta}\, |B(\mathbf{n})|^2 + (n-2)\left|B(\mathbf{n})^T\right|^2\right). \end{aligned} \tag{3.14}$$

Moreover, we have

$$\left[r^{2-n}\,\left(A_\beta(r) - Vol(\partial B_1(0))\right)\right]' = \beta\, r^{1-n} \int_{u\le r} u^{2-n}\,|\nabla u|^\beta\,\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right)$$
$$+\frac{\beta}{4(n-1)} \int_{r_1\le u\le r_2} u^{-n}\,|\nabla u|^{\beta-2}$$
$$\times\left(\tilde\beta\,|B(\mathbf{n})|^2 + (n-2)\,\left|B(\mathbf{n})^T\right|^2\right), \tag{3.15}$$

where $B_1(0)$ is the Euclidean ball of radius one.

Proof To keep notation simple, within this proof define $f(r)$ and $g(r)$ by

$$f(r) \equiv r^{2-n}\,A_\beta(r) = r^{1-n}\int_{u=r} u^{2-n}\,|\nabla u|^{1+\beta}\,, \tag{3.16}$$

$$g(r) \equiv r^{n-1}\,f'(r) = r^{n-1}\,\left(r^{2-n}\,A_\beta(r)\right)' = (2-n)\,A_\beta(r) + r\,A'_\beta(r). \tag{3.17}$$

Using the second expression for $f(r)$, we compute its derivative (cf. [6])

$$f'(r) = r^{1-n}\int_{u=r}\left\langle\nabla\left(u^{2-n}\,|\nabla u|^\beta\right),\mathbf{n}\right\rangle. \tag{3.18}$$

Thus, for $r_1 < r_2$ the divergence theorem and Corollary 2.13 give

$$g(r_2) - g(r_1) = r_2^{n-1}f'(r_2) - r_1^{n-1}f'(r_1) = \int_{r_1\le u\le r_2}\Delta\left(u^{2-n}\,|\nabla u|^\beta\right)$$
$$= \beta\int_{r_1\le u\le r_2} u^{2-n}\,|\nabla u|^\beta\,\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right)$$
$$+\frac{\beta}{4(n-1)}\int_{r_1\le u\le r_2} u^{-n}\,|\nabla u|^{\beta-2}\left(\tilde\beta\,|B(\mathbf{n})|^2 + (n-2)\,\left|B(\mathbf{n})^T\right|^2\right), \tag{3.19}$$

giving the first claim.

It follows from lemma 2.1 of [3] that there is a sequence $r_i \to 0$ so that

$$g(r_i) \to (2-n)\,\mathrm{Vol}(\partial B_1(0)). \tag{3.20}$$

Namely, the first two parts of lemma 2.1 of [3] (that give the asymptotics of both u and $|\nabla u|$ as u goes to zero) give that $A_\beta(r)$ goes to $\mathrm{Vol}(\partial B_1(0))$ as $r \to 0$ and, thus, there must exist a sequence $r_i \to 0$ with $r_i\,A'_\beta(r_i) \to 0$. This gives (3.20).

Putting (3.20) back into (3.19) and taking the limit as $r_i \to 0$ gives for $r > 0$ that

$$g(r) - (2-n)\,\mathrm{Vol}(\partial B_1(0)) = \beta\int_{u\le r} u^{2-n}\,|\nabla u|^\beta\,\left(|\mathrm{II}_0|^2 + \mathrm{Ric}(\mathbf{n},\mathbf{n})\right)$$
$$+\frac{\beta}{4(n-1)}\int_{u\le r} u^{-n}\,|\nabla u|^{\beta-2}$$
$$\times\left(\tilde\beta\,|B(\mathbf{n})|^2 + (n-2)\,\left|B(\mathbf{n})^T\right|^2\right). \tag{3.21}$$

The second claim follows from this since

$$r^{n-1}\left(r^{2-n}\left(A_\beta(r)-\mathrm{Vol}(\partial B_1(0))\right)\right)'=g(r)-(2-n)\,\mathrm{Vol}(\partial B_1(0)). \tag{3.22}$$

□

3.2 The third monotonicity formula

The following formula is the third monotonicity formula for A_β. We will see that this leads to the monotonicity of A_β itself.

Theorem 3.5 *If $r_1 < r_2$, then*

$$\begin{aligned} r_2^{3-n}\,A'_\beta(r_2)-r_1^{3-n}\,A'_\beta(r_1) &= \beta\int_{r_1\le u\le r_2} u^{4-2n}\,|\nabla u|^\beta\,\left(|II_0|^2+Ric(\mathbf{n},\mathbf{n})\right)\\ &\quad+\frac{\beta}{4(n-1)}\int_{r_1\le u\le r_2} u^{2-2n}\,|\nabla u|^{\beta-2}\\ &\quad\times\left(\tilde\beta\,|B(\mathbf{n})|^2+(n-2)\left|B(\mathbf{n})^T\right|^2\right). \end{aligned} \tag{3.23}$$

Proof Using the formula (3.9) for A'_β, we get

$$r^{3-n}\,A'_\beta(r)=r^{4-2n}\int_{u=r}\langle\nabla|\nabla u|^\beta,\mathbf{n}\rangle=\beta\int_{u=r}\left\langle u^{4-2n}\,|\nabla u|^{\beta-1}\,\nabla|\nabla u|,\mathbf{n}\right\rangle, \tag{3.24}$$

so that the divergence theorem gives

$$r_2^{3-n}\,A'_\beta(r_2)-r_1^{3-n}\,A'_\beta(r_1)=\beta\int_{r_1\le u\le r_2}\mathrm{div}\,\left(u^{4-2n}\,|\nabla u|^{\beta-1}\nabla|\nabla u|\right). \tag{3.25}$$

The theorem follows from this since the second claim in Lemma 2.11 with $\alpha=\beta-1$ and $p=2-n$ gives

$$\begin{aligned} \mathrm{div}\,\left(u^{4-2n}\,|\nabla u|^{\beta-1}\nabla|\nabla u|\right) &= u^{4-2n}\,|\nabla u|^\beta\,\left(|\mathrm{II}_0|^2+\mathrm{Ric}(\mathbf{n},\mathbf{n})\right)\\ &\quad+\frac{1}{4(n-1)}\,u^{2-2n}\,|\nabla u|^{\beta-2}\\ &\quad\times\left(\tilde\beta\,|B(\mathbf{n})|^2+(n-2)\left|B(\mathbf{n})^T\right|^2\right). \end{aligned} \tag{3.26}$$

□

3.3 Monotonicity of A_β

Finally, we turn to the monotonicity of A_β. For this, we will focus on infinity, instead of the point singularity of the Green's function.

Proof of Theorem 1.1 We show first that $A'_\beta(r)\le 0$. It is convenient to define $f(r)$ by

$$f(r)\equiv r^{3-n}\,A'_\beta(r). \tag{3.27}$$

Theorem 3.5 gives that $f(s)\ge f(r)$ whenever $s>r$ and, thus, that

$$A'_\beta(s)=s^{n-3}\,f(s)\ge s^{n-3}\,f(r). \tag{3.28}$$

Integrating this from r to R would give

$$A_\beta(R) - A_\beta(r) \geq f(r) \int_r^R s^{n-3}\, \mathrm{d}s. \tag{3.29}$$

However, if $f(r) > 0$, then the right-hand side is not integrable as $R \to \infty$, but this contradicts that $A_\beta(R)$ is uniformly bounded by Lemma 3.1. This contradiction shows that we must always have $f(r) \leq 0$, completing the first step of the proof.

The second step is to show that there is a sequence $r_j \to \infty$ with

$$\left| f(r_j) \right| \to 0. \tag{3.30}$$

However, this follows immediately from A' having a sign and $|A|$ being bounded.

The theorem follows from (3.30), Theorem 3.5 and the monotone convergence theorem. □

Proof of Corollary 1.2 By Theorem 1.1, $A_1(r)$ is non-increasing and

$$\begin{aligned} A_1(R/2) - A_1(R) &\geq -\int_{R/2}^R A_1'(r)\, \mathrm{d}r \geq \int_{R/2}^R r^{n-3} \int_{r \leq u} u^{4-2n}\, |\nabla u|\, |\mathrm{II}_0|^2\, \mathrm{d}r \\ &\geq \int_{R/2}^R \left(\frac{R}{2}\right)^{n-3} \int_{R \leq u} u^{4-2n}\, |\nabla u|\, |\mathrm{II}_0|^2\, \mathrm{d}r \\ &= \left(\frac{R}{2}\right)^{n-2} \int_{R \leq u} u^{4-2n}\, |\nabla u|\, |\mathrm{II}_0|^2 \\ &\geq (2R)^{2-n} \int_{R \leq u \leq 2R} |\nabla u|\, |\mathrm{II}_0|^2 = (2R)^{2-n} \int_R^{2R} \int_{u=s} |\mathrm{II}_0|^2\, \mathrm{d}s, \end{aligned} \tag{3.31}$$

where the last equality is the coarea formula.

Since A_1 is nonnegative and non-increasing, it has a limit and, thus,

$$\lim_{R \to \infty} \left\{ R^{2-n} \int_R^{2R} \int_{u=s} |\mathrm{II}_0|^2\, \mathrm{d}s \right\} = 0. \tag{3.32}$$

The corollary follows from this since $\mathrm{Vol}(u = s) \geq s^{n-1}\, \mathrm{Vol}(\partial B_1(0))$ by Lemma 3.1 □

4 Examples

In this section we will give some simple examples that illustrate some of the results of the previous sections.

Example 4.1 Suppose that M^2 is a surface and $u : M \to \mathbf{R}$ is a smooth function. If $s \in \mathbf{R}$ is a regular value of u, then the level set is umbilic since there is only one principal curvature. On the other hand if M is flat Euclidean space $\mathbf{R}^n$ with $n \geq 3$ and $u : \mathbf{R}^n \to \mathbf{R}$ is a proper smooth function all of whose level sets are umbilic and s is a regular value of u, then $u^{-1}(s)$ is a round sphere; see, for instance, [10] Vol. IV, p. 11.

Obviously, there are many different functions on $\mathbf{R}^n$ all of whose level sets are round spheres. For instance, in addition to distance functions to fixed points, then the function

$$u(x) = \sqrt{|x|^2 + x_1^2} - x_1 \tag{4.1}$$

has level sets that are round spheres

$$u^{-1}(s) = \{x \in \mathbf{R}^n \mid |x - (s, 0, \dots, 0)|^2 = 2\, s^2\}. \tag{4.2}$$

This also shows that, even though all the level sets of u are round spheres, the metric on $\mathbf{R}^n$ may not be written as $\mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2$, where θ are coordinates on the level sets of u.

Example 4.2 Let M be the flat 4-dimensional manifold $\mathbf{R}^3 \times \mathbf{S}^1$ with coordinates $x = (x_1, x_2, x_3, \theta)$ and let $u_1, u_2 : M \to \mathbf{R}$ be nonnegative proper functions on M given by

$$u_1(x) = \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{4}}, \tag{4.3}$$

$$u_2(x) = \mathrm{dist}_M(0, x). \tag{4.4}$$

Then

$$\liminf_{r\to\infty} \frac{r^2}{\mathrm{Vol}(u_1 = \sqrt{r})} \int_{u_1=\sqrt{r}} |\mathrm{II}_0|^2 \;\; > \;\; 0, \tag{4.5}$$

$$\limsup_{r\to\infty} \frac{r^2}{\mathrm{Vol}(u_2 = r)} \int_{u_2=r} |\mathrm{II}_0|^2 = 0. \tag{4.6}$$

In particular, the level sets are asymptotically umbilic for u_2 but not for u_1^2. Note that u_2 and u_1^2 are proportional to the distance function r to 0 when r is large and thus, in particular,

$$\lim_{|x|\to\infty} \frac{u_1^2(x)}{u_2(x)} = 1. \tag{4.7}$$

Note that both u_1^{-2} and u_2^{-2} (where smooth) are proper positive harmonic functions. We saw earlier that, by Corollary 1.2, for any proper positive harmonic function u^{2-n} on a manifold with nonnegative Ricci curvature and Euclidean volume growth

$$\liminf_{r\to\infty} \frac{r^2}{\mathrm{Vol}(u = r)} \int_{u=r} |\mathrm{II}_0|^2 = 0. \tag{4.8}$$

Example 4.3 Let M be a smooth n-dimensional manifold with a warped product metric of the form

$$\mathrm{d}r^2 + f^2(r, \theta)\, g, \tag{4.9}$$

where g is a metric on a smooth $(n-1)$-dimensional manifold N. The second fundamental form of the level sets of $u = r$ is given by

$$\mathrm{II} = \partial_r \log f\, g. \tag{4.10}$$

In particular, the level sets are umbilic. However, if we also require that $\Delta r^2 = 2n\, |\nabla r|^2 = 2n$, then $f = C\, r$ for some constant C and thus M with the metric is part of a metric cone. To see this, note that

$$\Delta r^2 = 2 + 2\, r\, \Delta r = 2 + 2(n-1)\, r\, \partial_r \log f. \tag{4.11}$$

Thus if $\Delta r^2 = 2n\, |\nabla r|^2 = 2n$, then $1 = r\, \partial_r \log f$. Or, in other words, $f = C\, r$.

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