Transport coefficients for higher dimensional quantum Hall effect

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An effective action for the bulk dynamics of the quantum Hall effect in arbitrary, even spatial dimensions was obtained some time ago in terms of a Chern-Simons term associated with the Dolbeault index theorem. Here we explore further properties of this action, showing how electronic band structures can be incorporated, obtaining Hall currents and conductivity (for arbitrary dimensions) in terms of integrals of Chern classes for the bands. We also derive the expression for Hall viscosity from the effective action. Explicit formulas for the Hall viscosity are given for $2 + 1$ and $4 + 1$ dimensions.

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I. INTRODUCTION

The quantum Hall effect has been intensively investigated for several decades from both theoretical and experimental points of view [1]. An interesting variant has been its generalization to higher dimensions [2–8]. Even though seemingly this is only of mathematical interest, it is intriguing that this may in fact be experimentally realizable using the idea of synthetic dimensions [9,10]. Shortly after the initial work on higher dimensional quantum Hall effect (QHE) [2], it was realized that complex manifolds present a class of spaces for which one can explicitly solve the Landau problem, construct states, analyze edge excitations, etc., and therefore uniformly extend the QHE to all even spatial dimensions [4–6]. Since holomorphicity is the key feature for states in the lowest Landau level, the Dolbeault index theorem provides a convenient mathematical technique for analyzing the phenomenon in arbitrary dimensions [11]. Some time ago we used this connection to construct the *topological* effective bulk action for a quantum Hall system *of integer filling fraction* in arbitrary even dimensions, including both gauge and gravitational fluctuations, in terms of a Chern-Simons action associated to the Dolbeault index density [12]. We were able to construct such effective actions for arbitrary integer filling fraction and Abelian and non-Abelian gauge fields, the latter being a novel possibility in higher dimensions. We may note here that effective actions in $2 + 1$ dimensions, including gravitational contributions, have been constructed by many authors; see, for example, Refs. [13–16].

In this paper we will investigate the general effective action obtained in Ref. [12], further focusing on the derivation of response functions and relevant transport coefficients in higher dimensions. For simplicity we will focus on the case of Abelian gauge fields but arbitrary even spatial dimensions. The response of the system to gauge and gravitational fluctuations is characterized by the electromagnetic current and the energy-momentum tensor, which can be straightforwardly derived from the effective action.

The transport coefficient related to the electromagnetic current is the Hall conductivity expressed in terms of the filling fraction. In the construction of the effective action from the Dolbeault index, the filling fraction was one of the input ingredients. However, in deriving the effective action we essentially considered an appropriate gauge-covariant Laplacian as the Hamiltonian for the Landau problem, which means that we formulated the action in terms of Landau levels for fermions in free space. More realistically, though, the fermions correspond to extended states in an energy band in the material. The Hall current and the filling fraction relevant to the quantum Hall state should thus be expressed in terms of the integrals of the Chern classes for the Berry curvature of the bands, as was done long ago for the two-dimensional case [17]. This is particularly important if we seek experimental realizations for the higher dimensional cases. This is the first problem we address in this paper. We show how the band structure can be incorporated in the effective action. The electromagnetic currents we obtain by this method agree with results obtained for four- and six-dimensional (4D, 6D) QHE using Hamiltonian perturbation theory for wave packets [18]. Furthermore, we obtain explicit expressions for the currents in arbitrary even dimensions, including contributions due to the spatial curvature.

Another transport coefficient of interest is the Hall viscosity. This is obtained from the response to perturbations of the metric, from the two-point function for the energy-momentum tensor [14,15]. We show how the Hall viscosity can be derived from the effective action, giving explicit formulas for $2 + 1$ and $4 + 1$ dimensions. While the Hall viscosity has been derived in $2 + 1$ dimensions by explicit calculation of the responses, our derivation places it within a uniform procedure easily applicable in any number of dimensions.

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II. REVIEW OF THE DERIVATION OF THE EFFECTIVE ACTION USING AN INDEX THEOREM

In this section we give a brief resume of the bulk effective action and how it is obtained from the index theorem [12]. Although we may need to consider general perturbations of the metric later, to begin with, the spatial manifold of interest for us is a complex Kähler manifold, such as $\mathbb{CP}^k$. The single-particle Hamiltonian is of the form $D_i \bar{D}_{\bar{i}}$, where D_i and $\bar{D}_{\bar{i}}$ are the holomorphic and antiholomorphic covariant derivatives, which include background gauge and gravitational fields. Thus the wave functions for the lowest Landau level obey the holomorphicity condition

$$\bar{D}_{\bar{i}} \Psi = 0. \tag{1}$$

The number of normalizable solutions to this equation is given by the index theorem for the twisted Dolbeault complex as

$$\text{Index}(\bar{D}) = \int_K \text{td}(T_c K) \wedge \text{ch}(V), \tag{2}$$

where $\text{td}(T_c K)$ is the Todd class on the complex tangent space of K, and $\text{ch}(V)$ is the Chern character of the relevant vector bundle [11].

An explanation of the various terms and terminology in (2) might be useful before we proceed. Generally the spin connections and curvatures take values in the Lie algebra of the holonomy group, which is $SO(2k)$ for a real manifold in $2k$ dimensions. For a complex manifold, coordinate transformations which preserve the complex structure are holomorphic transformations. This restricts the holonomy group to $U(k) \subset SO(2k)$. Correspondingly, the frame fields can be taken to be holomorphic and antiholomorphic 1-forms, which are combinations of the real ones given by the complex structure. The tangent space also has similar combinations which give $T_c K$. The Todd class is given in terms of the curvature 2-form for $T_c K$. It has the expansion [11]

$$\begin{aligned} \text{td} = 1 &+ \tfrac{1}{2} c_1 + \tfrac{1}{12}\left(c_1^2 + c_2\right) + \tfrac{1}{24} c_1 c_2 \\ &+ \tfrac{1}{720}\left(-c_4 + c_1 c_3 + 3 c_2^2 + 4 c_1^2 c_2 - c_1^4\right) + \cdots , \end{aligned} \tag{3}$$

where c_i are the Chern classes. For any vector bundle with curvature $\mathcal{F}$, the Chern classes are defined by[1]

$$\det\left(1 + \frac{i\mathcal{F}}{2\pi} t\right) = \sum_i c_i t^i. \tag{4}$$

Various terms in the expansion (3) can thus be expressed as powers of the curvature 2-form. The Todd class may also be represented, via the splitting principle, in terms of a generating function as

$$\text{td} = \prod_i \frac{x_i}{1 - e^{-x_i}}, \tag{5}$$

where x_i represents the "eigenvalues" of the curvature in a suitable canonical form (diagonal or the canonical antisymmetric form for real antisymmetric $i\mathcal{F}$). The expansion in (3) is obtained by using this generating function and rewriting it using traces of powers of curvatures.

In the case of the index as in (2), $\mathcal{F}$ is to be taken as the curvature 2-form R for $T_c K$. The first few Chern classes for the complex tangent space can then be explicitly written, using (4), as

$$\begin{aligned} c_1(T_c K) &= \text{Tr}\frac{iR}{2\pi} \\ c_2(T_c K) &= \frac{1}{2}\left[\left(\text{Tr}\frac{iR}{2\pi}\right)^2 - \text{Tr}\left(\frac{iR}{2\pi}\right)^2\right] \\ c_3(T_c K) &= \frac{1}{3!}\left[\left(\text{Tr}\frac{iR}{2\pi}\right)^3 - 3\,\text{Tr}\frac{iR}{2\pi}\,\text{Tr}\left(\frac{iR}{2\pi}\right)^2 \right. \\ &\quad \left. + 2\,\text{Tr}\left(\frac{iR}{2\pi}\right)^3\right] \\ c_4(T_c K) &= \frac{1}{4!}\left[\left(\text{Tr}\frac{iR}{2\pi}\right)^4 - 6\left(\text{Tr}\frac{iR}{2\pi}\right)^2 \text{Tr}\left(\frac{iR}{2\pi}\right)^2 \right. \\ &\quad + 8\,\text{Tr}\frac{iR}{2\pi}\,\text{Tr}\left(\frac{iR}{2\pi}\right)^3 + 3\,\text{Tr}\left(\frac{iR}{2\pi}\right)^2 \text{Tr}\left(\frac{iR}{2\pi}\right)^2 \\ &\quad \left. - 6\,\text{Tr}\left(\frac{iR}{2\pi}\right)^4\right]. \end{aligned} \tag{6}$$

The curvatures R take values in the Lie algebra of $U(k)$, which is the holonomy group for a complex manifold of real dimension $2k$. The traces in the above formula are thus over the $U(k)$ Lie algebra.

Turning to the other set of terms in the index formula (2), we may first note that the terminology of the Dolbeault complex being twisted, as often used in mathematics literature, is equivalent to saying that we have background gauge fields. In other words, the vector bundle V relevant for us is defined by the internal gauge symmetry structure of the fermion fields. The fermion wave functions are sections of this bundle. The Chern character is given by

$$\begin{aligned} \text{ch}(V) &= \text{Tr}(e^{i\mathcal{F}/2\pi}) \\ &= \dim V + \text{Tr}\frac{i\mathcal{F}}{2\pi} + \frac{1}{2!}\text{Tr}\frac{i\mathcal{F} \wedge i\mathcal{F}}{(2\pi)^2} + \cdots , \end{aligned} \tag{7}$$

where $\dim V$ is the dimension of the bundle V. Here $\mathcal{F}$ is the gauge field strength F. If spin is included, $\mathcal{F}$ will also include the curvature of the spin bundle.

By definition, the Dolbeault index gives the degeneracy of the lowest Landau level. If we consider a fully filled Landau level with filling fraction $\nu = 1$ and assign a unit charge to each fermion, the index will also give the total charge. The charge density may therefore be identified with the index density allowing us to construct an effective action. Thus in terms of an effective action S_{eff}, we can then write

$$\frac{\delta S_{\text{eff}}}{\delta A_0} = J_0 = \text{Index density}. \tag{8}$$

This shows that the leading term of the effective action may be taken as a Chern-Simons term $CS(A)$ whose variational derivative with respect to A_0 will give the index density.

[1]We start with connections and curvatures in an anti-Hermitian basis since they are natural, allowing us to write $F = dA + AA$, etc. This leads to some factors of i in various expressions at this stage. Later we will move to a Hermitian basis.

In other words, we can "integrate up" the relation (8), *and appropriately covariantize* to obtain *the topological part of the action* $S_{\rm eff}$. (There can be subleading terms arising from dipole and higher multipole terms in J_0 which integrate to zero in the total charge. They can contribute terms involving derivatives of the fields in the effective action; they are also nontopological in nature. The Chern-Simons form associated to the index density is the leading term in the sense of a derivative expansion; see [12] for more details on this point.) This procedure, as described so far, does not determine the purely gravitational terms in $S_{\rm eff}$. However, even though we are interested in the bulk action, one could envisage the situation of a droplet with edge modes; these will generate a gravitational anomaly. The purely gravitational terms in $S_{\rm eff}$ will be determined by the gravitational anomaly *via the descent method used for anomalies* [19]. This was the procedure we used in [12] to obtain the effective bulk action for a higher dimensional quantum Hall effect.

In the case of a fully filled lowest Landau level, we derived a compact form for the topological effective action for a general complex manifold of even spatial dimension $2k$, given by

$$S_{\rm eff} = \int \left[{\rm td}(T_c K) \wedge \sum_p (CS)_{2p+1}(A) \right]_{2k+1} + 2\pi \int \Omega^{\rm grav}_{2k+1}. \tag{9}$$

Here $(CS)_{2p+1}(A)$ is the Chern-Simons term associated with just the gauge part and is defined by

$$\frac{1}{2\pi} d(CS)_{2p+1} = \frac{1}{(p+1)!} {\rm Tr}\left(\frac{iF}{2\pi}\right)^{p+1}. \tag{10}$$

One should expand the terms in the square brackets in (9) in powers of curvatures and F and pick out the term corresponding to the $(2k+1)$-form. This is indicated by the subscript $2k+1$ for the square brackets. The purely gravitational term $\Omega^{\rm grav}_{2k+1}$ in (9) is defined by

$$[{\rm td}(T_c K)]_{2k+2} = d\, \Omega^{\rm grav}_{2k+1}. \tag{11}$$

Notice that here we start with the $(2k+2)$-form and define the appropriate Chern-Simons form. The justification for this is via the well-known descent equations for anomalies, see [12,19].

These results can be further extended to include higher Landau levels for some special cases. For many manifolds, such as $\mathbb{CP}^k$, the degeneracy for spinless fermions in the sth Landau level is identical to the degeneracy for fermions of spin s in the lowest Landau level. One can then use the index theorem as before to construct the effective action. The procedure is exactly as outlined above, except that $\mathcal{F}$ in the Chern character now includes the curvature for the spin bundle as well. Explicitly, what this means is that

$$\mathcal{F} = F + \mathcal{R}_s = F + s\,R^0 \mathbb{1} + R^a T_a, \tag{12}$$

where R^0 and R^a denote the curvatures for the $U(1)$ and $SU(k)$ factors in $U(k) \subset SO(2k)$. T_a are the generators of $SU(k)$ in the appropriate representation of the appropriate spin. The general effective action has the form

$$S^{(s)}_{\rm eff} = \int \left[{\rm td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s + A) \right]_{2k+1} + 2\pi \int \Omega^{\rm grav}_{2k+1}, \tag{13}$$

where ω_s is the spin connection for $sR^0\mathbb{1} + R^a T_a$.

Specifically, the effective actions for $2+1$, $4+1$, and $6+1$ dimensions were worked out in detail. In $2+1$ dimensions, the result for the sth Landau level is

$$\begin{aligned} S^{(s)}_{2+1} &= \frac{i^2}{4\pi} \left\{ \int A\left[dA + 2\left(s+\frac{1}{2}\right) d\omega^0 \right] + \left[\left(s+\frac{1}{2}\right)^2 - \frac{1}{12} \right] \int \omega^0\, d\omega^0 \right\} \\ &= \frac{i^2}{4\pi} \int \left\{ \left[A + \left(s+\frac{1}{2}\right)\omega^0 \right] d\left[A + \left(s+\frac{1}{2}\right)\omega^0 \right] - \frac{1}{12}\omega^0\, d\omega^0 \right\}. \end{aligned} \tag{14}$$

In $4+1$ dimensions, we find

$$\begin{aligned} S^{(s)}_{4+1} = \frac{i^3(2j+1)}{(2\pi)^2} \int \bigg\{ &\frac{1}{3!}(A+(s+1)\omega^0)[d(A+(s+1)\omega^0)]^2 \\ &- \frac{1}{12}(A+(s+1)\omega^0)\left[(d\omega^0)^2 - \left[4j(j+1) - \frac{1}{2} \right] \frac{1}{2}(R^a \wedge R^a) \right] \bigg\}, \end{aligned} \tag{15}$$

where $j = s/2$. For a complex manifold such as $\mathbb{CP}^2$ in four dimensions, the holonomy group (which is where the gravitational curvatures and connections take values) is $U(2) \subset SO(4)$. In (15), ω^0 denotes the $U(1)$ spin connection, with $U(2) \sim SU(2)\times U(1)$, and R^a is the curvature for $SU(2)$.

In $6+1$ dimensions, the action for the lowest Landau level ($s=0$) was obtained as

$$S^{(0)}_{6+1} = \frac{1}{(2\pi)^3}\int \left\{ \frac{1}{4!}\left(A+\frac{3}{2}\omega^0\right)\left[d\left(A+\frac{3}{2}\omega^0\right)\right]^3 - \frac{1}{16}\left(A+\frac{3}{2}\omega^0\right)d\left(A+\frac{3}{2}\omega^0\right)\left[(d\omega^0)^2+\frac{1}{3}\mathrm{Tr}(\tilde{R}\wedge\tilde{R})\right] + \frac{1}{1920}\omega^0 d\omega^0[17(d\omega^0)^2+14\,\mathrm{Tr}(\tilde{R}\wedge\tilde{R})] + \frac{1}{720}\omega^0\mathrm{Tr}(\tilde{R}\wedge\tilde{R}\wedge\tilde{R})\right\} + \frac{1}{120}\int (CS)_7(\tilde{\omega}). \tag{16}$$

Here we are only displaying the result for the lowest Landau level for simplicity. Similar to the $4+1$-case, we have a $U(1)$ subgroup of the holonomy group with spin connection ω^0 and an $SU(3)$ subgroup with spin connection $\tilde{\omega}$. The curvature $\tilde{R}$ corresponds to $\tilde{\omega}$; it is in the fundamental representation of $SU(3)$, as given in Eqs. (A11) and (A12) in Appendix A. Also, $(CS)_7(\tilde{\omega})$ in (16) is the standard Chern-Simons 7-form for this spin connection.

III. HALL CURRENTS AND CHARACTERISTIC CLASSES FOR ELECTRONIC BANDS

In this section we describe how the effective actions derived in [12] (relevant for electrons in free space) should be modified to incorporate the geometrical properties of the electronic bands. We start by reiterating that the expressions for the effective actions (9), (13) are valid for both Abelian and non-Abelian gauge fluctuations. By construction, the variation of the effective action is of the form

$$\delta S_{\rm eff} = \int \delta A\,\mathcal{I}$$
$$\mathcal{I} = \sum_{l=0}^{k}\frac{1}{(k-l)!}\left(\frac{F}{2\pi}\right)^{k-l}\wedge[\mathrm{td}(T_cK)]_{2l}. \tag{17}$$

Here $\mathcal{I}$ is a differential $2k$-form. Formally, it is identical to the index density, but here we are interpreting it as a form in the $(2k+1)$-dimensional spacetime. Thus it defines a dual vector, which is the Hall current. In the following, for simplicity we only consider Abelian gauge fields and consider the electromagnetic Hall current, and focus on the lowest Landau level, $s=0$.

The expressions for the Hall current obtained as in (17) correspond to fermions in free space. To work out the required modification, taking account of the fact that the fermions are from an energy band appropriate to the material, we consider the motion of a particle, viewed as a wave packet with position x^i and momentum k^i. The dynamics of such a wave packet is described by the action [20]

$$S = \int k_i\dot{x}^i - E(k) + e\phi(x) - A_i\dot{x}^i + \mathcal{A}_i\dot{k}^i. \tag{18}$$

The wave packet is to be viewed as describing a single-particle extended state within an energy band, with energies given by $E(k)$. Here ϕ is the electrostatic potential, A_i is the magnetic vector potential, and $\mathcal{A}_i$ is the Berry connection defined by

$$\mathcal{A}_i(k) = \int [dx]\,\Psi_k^\dagger(x)\frac{\partial}{\partial k^i}\Psi_k(x). \tag{19}$$

We are considering a $2k$-dimensional spatial manifold, and $[dx]$ in (19) gives the appropriate volume element. The canonical symplectic structure associated with the action (18) is easily read off as

$$\omega_{\rm symp} = dk_i\,dx^i - \tfrac{1}{2}F_{ij}dx^i\,dx^j + \tfrac{1}{2}\Omega_{ij}dk^i\,dk^j, \tag{20}$$

where the gauge and Berry curvatures are given by

$$F_{ij} = \frac{\partial}{\partial x^i}A_j - \frac{\partial}{\partial x^j}A_i,$$
$$\Omega_{ij} = \frac{\partial}{\partial k^i}\mathcal{A}_j - \frac{\partial}{\partial k^j}\mathcal{A}_i. \tag{21}$$

The canonical structure (20) shows that the commutators $[x^i, x^j]$ and $[k_i, k_j]$ will be nonzero, since there are $dk^i\,dk^j$ and $dx^i\,dx^j$ terms in $\omega_{\rm symp}$, in addition to the standard term $dk_i\,dx^i$. Our aim now is to choose variables so as to eliminate mixing between the two sectors. Towards this we write

$$k_i = \lambda_{ij}p^j + \sigma_{ij}x^j. \tag{22}$$

This introduces p_i, which will take the place of k_i. In the following we consider F_{ij} and Ω_{ij} to be approximately constant; this is appropriate to the lowest order in what may be considered as a gradient expansion. With the substitution (22), $\omega_{\rm symp}$ becomes

$$\omega_{\rm symp} = [\lambda - \sigma^T\Omega\lambda]_{ij}\,dp^i\,dx^j + \left[-\sigma - \tfrac{1}{2}F + \tfrac{1}{2}\sigma^T\Omega\sigma\right]_{ij}dx^i\,dx^j + \tfrac{1}{2}[\lambda^T\Omega\lambda]_{ij}dp^i\,dp^j. \tag{23}$$

The term mixing x^i and p^i can be eliminated by imposing

$$\lambda - \sigma^T\Omega\lambda = 0. \tag{24}$$

The solution to this equation is given by $\sigma = -\Omega^{-1}$. We can then simplify ω as

$$\omega_{\rm symp} = \tfrac{1}{2}(\Omega^{-1}-F)_{ij}dx^i\,dx^j + \tfrac{1}{2}(\lambda^T\Omega\lambda)_{ij}dp^i\,dp^j. \tag{25}$$

We have separated the x- and p-dependent terms, but λ is not determined by our considerations so far. We will make the simplest choice that it is the identity matrix; this will suffice for our purpose. In this case we have

$$\omega_{\rm symp} = \tfrac{1}{2}(\Omega^{-1}-F)_{ij}dx^i\,dx^j + \tfrac{1}{2}\Omega_{ij}dp^i\,dp^j. \tag{26}$$

Recall that the lowest Landau-level dynamics for fermions in free space coupled to a magnetic field is governed by the canonical structure $\omega_{\rm symp} = -\frac{1}{2}F_{ij}dx^idx^j$. We see that the present case where we include the band structure is equivalent to using fermions in free space with a modified field, namely, $(F-\Omega^{-1})_{ij}$ in place of F. The phase volume corresponding to this $\omega_{\rm symp}$ is given by

$$d\mu = \frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k\frac{1}{k!}\left(\frac{F-\Omega^{-1}}{2\pi}\right)^k. \tag{27}$$

This shows that we also have an overall factor of $(\Omega)^k$ with integration over all momenta; this factor will take the place of the filling fraction.

The effective action we obtained was the Chern-Simons action associated with the Dolbeault index density, considering the Landau problem as fermions in free space coupled to the magnetic field. In light of the modified structure (26) and (27), we see that we can transcribe the action to the case of interest (with fermions drawn from energy bands) by making two changes, namely, making the replacement $F \to F - \Omega^{-1}$ and including an integration over all momenta with the density $\frac{1}{k!}(\frac{\Omega}{2\pi})^k$. Thus if $S_{\rm eff}$ denotes the Chern-Simons action obtained from the Dolbeault index density as given in [12], the action for the case of fermions in electronic bands will be

$$S_{\rm eff/band} = \int_{BZ} \frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k S_{\rm eff}(F \to F - \Omega^{-1}). \tag{28}$$

The integration over the momenta is over the Brillouin zone of momentum states for the energy band. From this effective action one can easily extract a general expression for the Hall currents for a fully filled band with Abelian gauge fields. By taking the variation with respect to A_μ, we get

$$*J = \int_{BZ} \frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k [\mathcal{I}]_{F\to F-\Omega^{-1}},$$
$$J^\mu = \epsilon^{\mu\alpha_1\alpha_2\cdots\alpha_{2k}} \int_{BZ} \frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k [\mathcal{I}_{\alpha_1\alpha_2\cdots\alpha_{2k}}]_{F\to F-\Omega^{-1}}, \tag{29}$$

where $\mathcal{I}$ is given in (17).

This is the general expression, but it may be somewhat cryptic for straightforward application. Therefore we will first consider $S_{\rm eff}$ given in (14)–(16) for the case of $2+1$, $4+1$, and $6+1$ dimensions and work out the Hall currents. The general formula valid for all dimensions and with nonzero background curvatures will be worked out at the end of this section.

$(2+1)$ *dimensions.* The current is given by

$$J^i = \epsilon^{ij}\left(\frac{E_j}{2\pi} + \frac{1}{2}\frac{R_{j0}}{2\pi}\right)\nu_1, \tag{30}$$

where $E_i = F_{i0}$ and ν_1 is the integral of the first Chern class, given by

$$\nu_1 = \int_{BZ} \frac{\Omega}{2\pi}. \tag{31}$$

The curvature R_{j0} takes values in $U(1)$ as explained in the Introduction.

Expression (30) agrees with previous results in the case of flat two-dimensional (2D) spaces. An interesting feature of (30), which is also valid in all higher dimensions, is that *a Hall current can be generated from time variation of the metric even if there is no external electric field applied to the system.* While we make a note of this interesting fact, in deriving the Hall current in higher dimensions we will neglect this effect for simplicity and consider manifolds whose curvature is time independent.

$(4+1)$ *dimensions.* The relevant expression for $\mathcal{I}$ in $(4+1)$ dimensions is

$$\mathcal{I} = \frac{1}{2}\frac{F}{2\pi}\wedge\frac{F}{2\pi} + \frac{F}{2\pi}\wedge\frac{c_1}{2} + \frac{1}{12}(c_1^2 + c_2), \tag{32}$$

where c_1, c_2 are given in (6). Making the substitution $F \to F - \Omega^{-1}$ and integrating over the momentum space with density $\frac{1}{k!}(\frac{\Omega}{2\pi})^k$, as explained above in Eq. (29), produces the following expression for the electromagnetic Hall current:

$$J^i = \frac{1}{2}\frac{1}{(2\pi)^2}\epsilon^{ijkl}E_j\left(F_{kl} + \frac{{\rm Tr}\,R_{kl}}{2}\right)\nu_2 + \frac{1}{2\pi}E_j\nu_1^{ij}, \tag{33}$$

where ν_k and ν_1^{ij} are given by

$$\nu_k = \int_{BZ}\frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k, \tag{34}$$

$$\nu_1^{ij} = \int_{BZ}\frac{\Omega^{ij}}{(2\pi)^{2k-1}}d^{2k}p. \tag{35}$$

Here ν_k in (34) is the integral of the kth Chern class over the band of electronic states. In deriving (33) we used the relation

$$(\Omega^{-1})_{ij}\epsilon_{\alpha_1\alpha_2\cdots\alpha_{2k}}\Omega^{\alpha_1\alpha_2}\cdots\Omega^{\alpha_{2k-1}\alpha_{2k}}$$
$$= -2k\epsilon_{ij\alpha_1\cdots\alpha_{2k-2}}\Omega^{\alpha_1\alpha_2}\cdots\Omega^{\alpha_{2k-3}\alpha_{2k-2}}. \tag{36}$$

The expression (33) for the current agrees with the expression derived in [18] for flat manifolds, but now includes generalization to curved manifolds, in addition to the virtue of being derived purely from a topological point of view.

$(6+1)$ *dimensions.* The relevant index density in $(6+1)$ dimensions is

$$\mathcal{I} = \frac{1}{3!}\left(\frac{F}{2\pi}\right)^3 + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2\wedge\frac{c_1}{2}$$
$$+ \frac{F}{2\pi}\wedge\frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2). \tag{37}$$

Using Eq. (29), we obtain the following expression for the electromagnetic Hall current:

$$J^i = \epsilon^{ijklrs}\frac{1}{2^3(2\pi)^3}E_j\left[\left(F_{kl} + \frac{1}{2}{\rm Tr}R_{kl}\right)\left(F_{rs} + \frac{1}{2}{\rm Tr}R_{rs}\right)\right.$$
$$\left. - \frac{1}{12}{\rm Tr}(R_{kl}R_{rs})\right]\nu_3 + \frac{1}{2(2\pi)^2}E_j\left(F_{kl} + \frac{1}{2}{\rm Tr}R_{kl}\right)\nu_2^{ijkl}$$
$$+ \frac{1}{2\pi}E_j\nu_1^{ij}, \tag{38}$$

where ν_3 is the integral of the third Chern class, ν_1^{ij} is defined in (35), and ν_2^{ijkl} is given by

$$\nu_2^{ijkl} = \int\frac{\Omega^{ij}\Omega^{kl} + \Omega^{il}\Omega^{jk} + \Omega^{ik}\Omega^{lj}}{(2\pi)^4}d^6p. \tag{39}$$

In deriving (38) we have used a relation similar to (36). The general identity is given in (B8) in Appendix B. The specific cases we need here are

$$\nu_1^{ij} = \int_{BZ} \frac{1}{3!}\left(\frac{\Omega}{2\pi}\right)^3 \epsilon^{ij\alpha_1\alpha_2\alpha_3\alpha_4} \frac{1}{2!}\frac{(\Omega^{-1})^{\alpha_1\alpha_2}}{2(2\pi)}\frac{(\Omega^{-1})^{\alpha_3\alpha_4}}{2(2\pi)},$$
$$\nu_2^{ijkl} = -\int_{BZ} \frac{1}{3!}\left(\frac{\Omega}{2\pi}\right)^3 \epsilon^{ijklab}\frac{(\Omega^{-1})^{ab}}{2(2\pi)}. \quad (40)$$

Again, the expression (38) for the Hall current agrees with previous results derived for flat backgrounds [18] and further generalizes them to curved manifolds.

$(2k+1)$ *dimensions.* We now write down the general expression for the Hall current for an arbitrary $2k$-dimensional (complex) curved manifold:

$$J^i = \epsilon^{iji_1\cdots i_{2k-2}}\sum_{s=0}^{k-1}\frac{E_j(F^{k-s-1})_{i_1i_2\cdots i_{(2k-2s-2)}}}{(2\pi)^k 2^{k-1}(k-s-1)!}[(\mathrm{td})_s]_{i_{(2k-2s-1)}\cdots i_{(2k-2)}}\nu_k$$
$$+\sum_{s=0}^{k-1}\sum_{l=1}^{k-s-1}\frac{E_j(F^{k-s-l-1})_{i_1i_2\cdots i_{(2k-2s-2l-2)}}}{(2\pi)^{k-l}2^{k-l-1}(k-s-l-1)!}$$
$$\times\ [(\mathrm{td})_s]_{i_{(2k-2s-2l-1)}\cdots i_{(2k-2l-2)}}\nu_{k-l}^{iji_1\cdots i_{(2k-2l-2)}}, \quad (41)$$

where $[\mathrm{td}]_s$ is the $2s$-form in terms of curvatures in the expansion of the Todd class as given in (3). ν_k is the integral of the kth Chern class defined in (34), and the integrals ν_{k-l} over the subclasses are defined by

$$\nu_{k-l}^{iji_1\cdots i_{(2k-2l-2)}} = \int \frac{d^{2k}p}{(2\pi)^{k+l}}\sum_{\text{dist.perm.}}[\Omega^{ij}\Omega^{i_1i_2}\cdots\Omega^{i_{(2k-2l-3)}i_{(2k-2l-2)}}]. \quad (42)$$

The differential $2s$-form corresponding to the Todd class is written as

$$(\mathrm{td})_s = \frac{1}{2^s(2\pi)^s}[(\mathrm{td})_s]_{a_1\cdots a_{2s}}(dx^{a_1}\cdots dx^{a_{2s}}), \quad (43)$$

so that $[(\mathrm{td})_s]_{a_1\cdots a_{2s}}$ is given in terms of traces of powers of the curvature with indices as shown. Factors of 2 and 2π have been separately included in (41).

IV. HALL VISCOSITY IN HIGHER DIMENSIONS

The effective actions we have derived in [12] also allow for the direct evaluation of the Hall viscosity beyond two dimensions. Here we shall illustrate in detail the derivation of the Hall viscosity in the case of the two- and four-dimensional QHE, but similar arguments apply to all dimensions. (For the viscosity we consider QHE with electrons in free space.) Of course, in the case of the two-dimensional QHE, our results agree with previous results derived in Refs. [14–16].

We start by recalling that viscosity is defined in terms of the two-point function $\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\rangle$ for the energy-momentum tensor $T^{\mu\nu}$. We can identify the viscosity by considering the expansion of $T^{\mu\nu}$ (obtained from the effective action) in terms of powers of derivatives of the metric. The term involving the time derivative of the metric gives the viscosity. In mathematical terms,

$$T^{\mu\nu} = \eta^{\mu\nu}_{\rho\sigma}\dot{g}^{\rho\sigma} + \cdots, \quad (44)$$

and we identify $\eta^{\mu\nu}_{\rho\sigma}$ with the viscosity tensor. Since $T^{\mu\nu}$ is obtained from the variation of the action with respect to the (inverse) metric $g^{\mu\nu}$, we can also write this as

$$\langle T^{\mu\nu}(x)T_{\rho\sigma}(y)\rangle = 4g^{\mu\lambda}g^{\nu\tau}\frac{\delta^2 S_{\text{eff}}}{\delta g^{\rho\sigma}(y)\delta g^{\lambda\tau}(x)}$$
$$= \eta^{\mu\nu}_{\rho\sigma}\frac{\partial}{\partial x^0}\delta^{(2k+1)}(x,y)+\cdots. \quad (45)$$

This agrees with the usual definition in terms of the two-point correlation function for the energy-momentum tensor.

In using Eq. (44) for calculating the Hall viscosity, we should keep in mind that the effective action was obtained for complex manifolds. A general variation of the metric, which does not necessarily preserve the complex structure, is needed for the correlation function in Eqs. (44) or (45). Of course, one can, after identifying $\eta^{\mu\nu}_{\rho\sigma}$, set the background metric to its value appropriate for the complex manifold of interest. The spin connection involves the $U(k)$ subalgebra of the vector representation of the Lie algebra of the $SO(2k)$ holonomy group. Therefore, to carry out a general variation of the metric, we need the relation between the complex $U(k)$ spin connection (ω^0, ω^a) and the corresponding real $SO(2k)$ quantities $\omega^{\alpha\beta}$. This is worked out in detail in Appendix A for the two- and four-dimensional cases.

The $SO(2k)$ spin connection ω is related to the Christoffel symbols $\Gamma^\alpha_{\mu\,\beta}$ by[2]

$$\omega^{\alpha\beta}_\mu = e^\alpha_i\,\Gamma^i_{\mu j}(e^{-1})^{j\beta} - \partial_\mu e^\alpha_j\,(e^{-1})^{j\beta}. \quad (46)$$

In some formulas we will use the form notation $\omega^{\alpha\beta} = \omega^{\alpha\beta}_\mu\,dx^\mu$ and $\Gamma^i_j = \Gamma^i_{\mu\,j}\,dx^\mu$, where e^α_i is the frame field in general. Since we deal with a nonrelativistic system, we also have the specific values

$$g_{00} = 1,\quad g_{0i} = 0,\quad e^0_i = e^i_0 = 0. \quad (47)$$

The variation of the expression (46) gives

$$\delta\omega^{\alpha\beta} = e^\alpha_i\left(\delta\Gamma^i_j\right)(e^{-1})^{j\beta} - (d\Theta + \omega\,\Theta - \Theta\,\omega)^{\alpha\beta}$$
$$\delta\Gamma^i_{\mu\,j} = \tfrac{1}{2}g^{il}(-\nabla_l\delta g_{\mu j} + \nabla_\mu\delta g_{jl} + \nabla_j\delta g_{\mu l}) \quad (48)$$
$$\Theta = \delta e^\alpha_i\,(e^{-1})^{i\beta}.$$

A. Two-dimensional QHE

The topological bulk effective action in two dimensions for QHE for the sth Landau level is given by

$$S^{(s)}_{2+1} = \frac{1}{4\pi}\int\left[AdA + 2\bar{s}Ad\omega^0 + \left(\bar{s}^2 - \frac{1}{12}\right)\omega^0\,d\omega^0\right], \quad (49)$$

where we defined $\bar{s} = s + \frac{1}{2}$, and we switched to a Hermitian basis for the gauge and spin connections, which eliminates the factor of i^2 from (14). Further, as explained in Appendix A,

$$\omega^0 = \tfrac{1}{2}\epsilon^{\alpha\beta}\omega^{\alpha\beta},\quad \omega^0 d\omega^0 = -\tfrac{1}{2}(\omega^{\alpha\beta}d\omega^{\beta\alpha}). \quad (50)$$

[2]Our conventions, given in Appendix A, are that Greek letters from the beginning of the alphabet denote tangent frame indices, lowercase Roman letters indicate spatial components in the coordinate basis, and Greek letters from later in the alphabet denote coordinate basis again, but including space and time components.

Varying $\omega^{\alpha\beta}$, we get

$$\delta S_{3\mathrm{d}}^{(s)} = \frac{1}{4\pi}\int \bar{s}\, dA\epsilon^{\alpha\beta}\delta\omega^{\alpha\beta} - \left(\bar{s}^2 - \frac{1}{12}\right)(\delta\omega^{\alpha\beta}\, d\omega^{\beta\alpha}). \tag{51}$$

We will use (48) and evaluate the two terms in (51) separately. For the first one, we find

$$\int dA\epsilon^{\alpha\beta}\delta\omega^{\alpha\beta} = \int dA\epsilon^{\alpha\beta}(e\delta\Gamma e^{-1} - d\Theta - [\omega,\Theta])^{\alpha\beta} = \frac{1}{2}\int \epsilon^{\alpha\beta}e_i^\alpha e^{-1j\beta}\delta\Gamma^i_{\mu j}\epsilon^{\mu\nu\sigma}F_{\nu\sigma}d^3x. \tag{52}$$

In going from the first to the second equality in (52), we used the fact that $\epsilon^{\alpha\beta}[\omega,\Theta]^{\alpha\beta} = 0$ in two dimensions. We can evaluate the second term in (51) in a similar way:

$$\int(\delta\omega^{\alpha\beta}\, d\omega^{\beta\alpha}) = \frac{1}{2}\int e_i^\alpha e^{-1j\beta}\delta\Gamma^i_{\mu j}\epsilon^{\mu\nu\sigma}(R_{\nu\sigma})^{\beta\alpha}d^3x = \frac{1}{2}\int \delta\Gamma^i_{\mu j}\epsilon^{\mu\nu\sigma}(R_{\nu\sigma})_i^j d^3x. \tag{53}$$

The curvatures in (52) and (53) are defined by

$$F_{\nu\sigma} = \partial_\nu A_\sigma - \partial_\sigma A_\nu, \quad (R_{\nu\sigma})_i^j = (e^{-1})^{j\beta}\, e_i^\alpha\big(\partial_\nu\omega_\sigma^{\beta\alpha} - \partial_\sigma\omega_\nu^{\beta\alpha}\big). \tag{54}$$

Using $\delta\Gamma$ as given in (48), we find that the variation of the effective action (51) becomes

$$\begin{aligned}\delta S_{3\mathrm{d}}^{(s)} =& \frac{1}{16\pi}\int\big(-\nabla_l\delta g_{\mu j} + \nabla_\mu\delta g_{lj} + \nabla_j\delta g_{\mu l}\big)\left[e^{-1l\alpha}e^{-1j\beta}\epsilon^{\alpha\beta}\bar{s}F_{\nu\sigma} - \left(\bar{s}^2 - \frac{1}{12}\right)(R_{\nu\sigma})^{jl}\right]\epsilon^{\mu\nu\sigma}d^3x\\ =& -\frac{1}{8\pi}\int \delta g_{\mu l}\left[\bar{s}(J^0)^{lj}\nabla_jF_{\nu\sigma} - \left(\bar{s}^2 - \frac{1}{12}\right)\nabla_j(R_{\nu\sigma})^{jl}\right]\epsilon^{\mu\nu\sigma}d^3x.\end{aligned} \tag{55}$$

In obtaining the last line of (55), we have done an integration by parts and also defined the antisymmetric tensor:

$$(J^0)^{lj} = e^{-1l\alpha}e^{-1j\beta}\epsilon^{\alpha\beta}. \tag{56}$$

The energy-momentum tensor can be read off from the variation δS_{eff} using the usual formula

$$\delta S_{\mathrm{eff}} = -\frac{1}{2}\int \delta g_{ml}T^{ml}\sqrt{\det g}d^nx. \tag{57}$$

Comparing (55) and (57), we identify the energy-momentum tensor for the 2D QHE as

$$T^{ml} = -\frac{1}{4\pi\sqrt{\det g}}(g^{il}\epsilon^{mk} + g^{im}\epsilon^{lk})\left[\bar{s}(J^0)_i^j\nabla_jF_{0k} - \left(\bar{s}^2 - \frac{1}{12}\right)\nabla_j(R_{0k})_i^j\right]. \tag{58}$$

In order to calculate the Hall viscosity, we need to identify the terms in this expression that are proportional to the time derivative of the metric. For the covariant derivative of F we can use

$$\nabla_jF_{0k} = \partial_jF_{0k} - \Gamma^n_{j0}F_{nk} - \Gamma^n_{jk}F_{0n} = -\tfrac{1}{2}g^{nl}\dot{g}_{lj}F_{nk} + \cdots, \tag{59}$$

where the ellipsis indicates terms that do not contain $\dot{g}$. As for the curvature term, we find

$$(R_{0k})_i^j = \tfrac{1}{2}g^{jn}(\nabla_i\dot{g}_{nk} - \nabla_n\dot{g}_{ik}). \tag{60}$$

Therefore

$$\nabla_j(R_{0k})_i^j = \tfrac{1}{2}\nabla_j\nabla_i(g^{jn}\dot{g}_{nk}) - \tfrac{1}{2}g^{jn}\nabla_j\nabla_n\dot{g}_{ik} - \tfrac{1}{2}g^{nl}\dot{g}_{lj}(R_{nk})_i^j. \tag{61}$$

We can simplify this result further by commuting the covariant derivatives in the first term and writing

$$\nabla_j\nabla_i(g^{jn}\dot{g}_{nk}) = \nabla_i\nabla_j(g^{jn}\dot{g}_{nk}) + (R_{ji})_m^j(g^{mn}\dot{g}_{nk}) - (R_{ji})_k^m(g^{jn}\dot{g}_{nm}). \tag{62}$$

Using (62) in (61) we get

$$\nabla_j(R_{0k})_i^j = \tfrac{1}{2}\nabla_i\nabla_j(g^{jn}\dot{g}_{nk}) - \tfrac{1}{2}\nabla^2\dot{g}_{ik} - \tfrac{1}{2}g^{nl}\dot{g}_{lj}(R_{nk})_i^j + \tfrac{1}{2}g^{mn}\dot{g}_{nk}(R_{ji})_m^j - \tfrac{1}{2}g^{jn}\dot{g}_{nm}(R_{ji})_k^m. \tag{63}$$

An arbitrary perturbation of the metric is rather too general for our purpose, since some of it corresponds simply to a coordinate change or diffeomorphism. A suitable covariant gauge choice which restricts the variations appropriately is the de Donder gauge, which is given by [21]

$$\nabla_j(g^{jn}\dot{g}_{nk}) - \tfrac{1}{2}\nabla_k(g^{jr}\dot{g}_{jr}) = 0. \tag{64}$$

It is possible to choose such a gauge for the perturbations of the metric by using the freedom of coordinate transformations.

Using (59), (63), and (64) in the expression (58) for the energy-momentum tensor, we find that the term linear in $\dot{g}$ is of the form

$$T^{ml} = \frac{1}{8\pi\sqrt{\det g}}(g^{il}\epsilon^{mk} + g^{im}\epsilon^{lk})\left[\left[\bar{s}(J^0)_i^j g^{rs} F_{sk}\right]\dot{g}_{rj} + \left(\bar{s}^2 - \frac{1}{12}\right)\left(\frac{1}{2}\nabla_i\nabla_k(g^{rn}\dot{g}_{rn}) - \nabla^2\dot{g}_{ik} - g^{nl}\dot{g}_{lj}\,(R_{nk})_i^j + g^{mn}\dot{g}_{nk}(R_{ji})_m^j - g^{jn}\dot{g}_{nm}(R_{ji})_k^m\right)\right]. \tag{65}$$

In two dimensions there are further simplifications, since the Riemann tensor has the form

$$R_{ijkl} = \frac{R}{2}(g_{ik}g_{jl} - g_{il}\,g_{jk}), \tag{66}$$

where R is the Ricci scalar curvature. Furthermore,

$$(J^0)^{lj} = e^{-1l\alpha}e^{-1j\beta}\epsilon^{\alpha\beta} = \frac{\epsilon^{lj}}{\sqrt{\det g}}. \tag{67}$$

By use of these expressions, the result (65) for the energy-momentum tensor can be simplified as

$$T^{ml} = \frac{1}{8\pi\sqrt{\det g}}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\left\{\left[\bar{s}B + \left(\bar{s}^2 - \frac{1}{12}\right)\left(\frac{R}{2} - \nabla^2\right)\right]\dot{g}_{ki} + \frac{1}{2}\left(\bar{s}^2 - \frac{1}{12}\right)\nabla_i\nabla_k(g^{rn}\dot{g}_{rn})\right\}. \tag{68}$$

In this expression, we used the magnetic field B given by

$$F_{ij} = \epsilon_{ij}B\sqrt{\det g}. \tag{69}$$

Comparing (68) with the expression (44) of the energy-momentum tensor in terms of the Hall viscosity, we see that we can write

$$\sqrt{\det g}T^{ml} = \tfrac{1}{2}\,\eta_H\,(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\dot{g}_{ki} + \tfrac{1}{2}\,\eta_H^{(2)}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\nabla_i\nabla_k(g^{rn}\dot{g}_{rn}), \tag{70}$$

where the coefficients can be read off as

$$\eta_H = \frac{1}{4\pi}\left[\bar{s}B + \left(\bar{s}^2 - \frac{1}{12}\right)\left(\frac{R}{2} + \vec{k}^2\right)\right]$$
$$\eta_H^{(2)} = \frac{1}{8\pi}\left(\bar{s}^2 - \frac{1}{12}\right). \tag{71}$$

The magnetic field and curvature-dependent terms of η_H, and the structure of T^{ml} as in (70), are in agreement with [15].[3] Notice that the coefficient of $\dot{g}_{rs}$ in (68) and (70) is an operator, so an expansion in terms of the eigenmodes of the covariant Laplacian will be needed to identify numerical values. For purposes of comparison, we have indicated the eigenvalue of $-\nabla^2$ as $\vec{k}^2$, which would be appropriate in the flat space limit.

B. Four-dimensional QHE

Turning to $4+1$ dimensions, we write the action (15) as

$$S_{4+1}^{(s)} = \frac{(s+1)}{(2\pi)^2}\int\left\{\frac{1}{3!}(A + (s+1)\omega^0)[d(A + (s+1)\omega^0)]^2 - \frac{1}{12}(A + (s+1)\omega^0)\left[(d\omega^0)^2 + \frac{1}{4}R^a \wedge R^a - s\left(\frac{s}{2} + 1\right)R^a \wedge R^a\right]\right\}, \tag{72}$$

where we have removed the overall factor of i^3 by going over to the Hermitian forms of the connections. We have also used $j = s/2$. The relation between the $U(2)$ spin connections and curvatures used in (72) and the corresponding $SO(4)$ quantities is derived in (A27) and is given by

$$\omega^0 = \tfrac{1}{4}\epsilon^{\alpha\beta}\omega^{\alpha\beta}, \quad R^0 = \tfrac{1}{4}\epsilon^{\alpha\beta}d\omega^{\alpha\beta}$$
$$R^aR^a = -4R^0R^0 - R^{\alpha\beta}R^{\beta\alpha}. \tag{73}$$

Using these relations, the effective action (72) can be expressed as

$$S_{4+1}^{(s)} = \frac{(s+1)}{(2\pi)^2}\int\left[\frac{1}{3!}\mathcal{A}d\mathcal{A}d\mathcal{A} - \frac{1}{12}\int\mathcal{A}\left[4s\left(\frac{s}{2}+1\right)d\omega^0 d\omega^0 + \left(s\left(\frac{s}{2}+1\right) - \frac{1}{4}\right)R^{\alpha\beta}\wedge R^{\beta\alpha}\right]\right], \tag{74}$$

where $\mathcal{A} = A + (s+1)\omega^0$.

[3] We thank A. G. Abanov for discussions resolving a slight discrepancy in the coefficient of $\vec{k}^2$ in η_H and of $\eta_H^{(2)}$ between (71) and [15]. Equation (71) is the correct expression.

The variation of the effective action (74) naturally splits into two types of terms of the form

$$\delta S = \delta S^{(1)} + \delta S^{(2)}, \tag{75}$$

$$\delta S^{(1)} = \frac{(s+1)}{(2\pi)^2}\int \delta\omega^0\, K, \tag{76}$$

$$\delta S^{(2)} = -\frac{(s+1)}{12(2\pi)^2}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]\int \mathcal{A}\,\delta(R^{\alpha\beta}R^{\beta\alpha}), \tag{77}$$

$$K = \frac{s+1}{2}dAdA + \frac{2}{3}\left[(s+1)^2+\frac{1}{2}\right]dAd\omega^0 + \frac{s+1}{2}d\omega^0 d\omega^0 - \frac{s+1}{12}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]R^{\alpha\beta}R^{\beta\alpha}. \tag{78}$$

The variation and simplification of these terms will proceed along lines similar to the $(2+1)$-dimensional case. Using (48) and an integration by parts for the $d\Theta$ term, we find

$$\int \delta\omega^0 K = \frac{1}{4}\int [(J^0)^{lj}\nabla_j\delta g_{\mu l} - (\delta e e^{-1})^{\alpha\beta}([\omega_\mu,\epsilon])^{\beta\alpha}]K^\mu\sqrt{\det g}, \tag{79}$$

where

$$K^\mu\sqrt{\det g} = \epsilon^{\mu\nu\sigma\tau\rho}\left\{\frac{s+1}{2}\partial_\nu A_\sigma\,\partial_\tau A_\rho + \frac{2}{3}\left[(s+1)^2+\frac{1}{2}\right]\partial_\nu A_\sigma\,\partial_\tau\omega^0_\rho + \frac{s+1}{2}\partial_\nu\omega^0_\sigma\,\partial_\tau\omega^0_\rho - \frac{s+1}{48}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]R^{\alpha\beta}_{\nu\sigma}R^{\beta\alpha}_{\tau\rho}\right\}, \tag{80}$$

and $(J^0)^{lj}$ is given in (56). The contribution of the second term in (79) to the symmetrized version of the energy-momentum tensor is zero. (The variation of the frame field as in $(\delta e e^{-1})^{\alpha\beta}$ can be related to the variation of the metric (which is the symmetric combination) and an antisymmetric part. It is the symmetric part which is relevant for the energy-momentum tensor.[4]) With an integration by parts, (76) simplifies as

$$\delta S^{(1)} = -\frac{s+1}{8(2\pi)^2}\int \delta g_{ml}\nabla_j[(J^0)^{lj}K^m + (J^0)^{mj}K^l]\sqrt{\det g}. \tag{81}$$

Comparing this with (57), we find that the contribution to the energy-momentum tensor from (81) is

$$(T^{ml})^{(1)} = \frac{(s+1)}{4(2\pi)^2}\nabla_j[(J^0)^{lj}K^m + (J^0)^{mj}K^l]. \tag{82}$$

For the evaluation of the second type of terms, namely, Eq. (77), we notice that

$$\delta\mathrm{tr}(R\wedge R) = d\,\mathrm{tr}\,\delta\big(\omega d\omega + \tfrac{2}{3}\omega^3\big) = 2d\,\mathrm{tr}(\delta\omega R). \tag{83}$$

[The trace here, indicated by tr, is for ω, R in the vector representation of $SO(2k)$.] Using Eqs. (83) and (48) in Eq. (77), we then get

$$\delta S^{(2)} = -\frac{(s+1)}{6(2\pi)^2}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]\int d\mathcal{A}\mathrm{tr}[\delta\Gamma e^{-1}Re + \delta e e^{-1}(dR+[\omega,R])]. \tag{84}$$

The last term in (84) vanishes by the Bianchi identity $dR+[\omega,R]=0$. After writing $\delta\Gamma$ in terms of variations of the metric and carrying out a partial integration we find

$$\delta S^{(2)} = \frac{(s+1)}{12(2\pi)^2}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]\int \delta g_{\mu l}\nabla_j[(R_{\nu\sigma})^{jl}\partial_\tau\mathcal{A}_\rho]\epsilon^{\mu\nu\sigma\tau\rho}d^5x. \tag{85}$$

Comparing this with (57), we find that the contribution to the energy-momentum tensor from (85) is

$$(T^{ml})^{(2)} = -\frac{(s+1)}{12(2\pi)^2}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]\left[\nabla_j[(R_{\nu\sigma})^{jl}\partial_\tau\mathcal{A}_\rho]\frac{\epsilon^{m\nu\sigma\tau\rho}}{\sqrt{\det g}} + (m\leftrightarrow l)\right]. \tag{86}$$

In order to identify the Hall viscosity, we have to extract terms linear in $\dot g$ in (82) and (86). To simplify the calculation we will neglect terms of the form $\partial\dot g$, which will produce momentum-dependent terms for the Hall viscosity.

Focusing first on $(T^{ml})^{(1)}$ and using the fact that

$$\nabla_\mu(J^0)^{lj} = e^{-1l\alpha}[\epsilon,\omega_\mu]^{\alpha\beta}e^{-1j\beta}, \tag{87}$$

[4]The antisymmetric part can be related to spin densities and can be relevant for some other transport coefficient related to the correlation function for the energy-momentum tensor and the spin density. This is not our focus at this stage.

we find

$$\nabla_j((J^0)^{lj}K^m) = e^{-1l\alpha}e^{-1j\beta}\big([\epsilon,\omega_j]^{\alpha\beta}K^m + \epsilon^{\alpha\beta}\partial_j K^m + \epsilon^{\alpha\beta}\Gamma^m_{j\rho}K^\rho\big). \tag{88}$$

The first term in (88) vanishes if ω preserves the $U(2)$ structure as expressed in (A21). For a constant magnetic field the second term in (88) will contribute only momentum-dependent terms, of the form $\partial\dot{g}$, to the expression for the Hall viscosity. The third term leads to

$$(T^{ml})^{(1)} = \frac{s+1}{8(2\pi)^2}[g^{mn}(J^0)^{lj} + g^{ln}(J^0)^{mj}]\dot{g}_{nj}K^0. \tag{89}$$

Turning to $(T^{ml})^{(2)}$, we notice that

$$\tilde{K}^{jlm} = [(R_{\nu\sigma})^{jl}\partial_\tau\mathcal{A}_\rho]\frac{\epsilon^{m\nu\sigma\tau\rho}}{\sqrt{\det g}} \tag{90}$$

transforms as a rank-3 contravariant tensor, so its covariant derivative is easy to write down. If any of the indices ν,σ,τ,ρ is taken to be the time-component, the corresponding contribution will involve covariant derivatives of $\dot{g}$. Since we are not including them here, the only contribution is from $\Gamma^m_{j0}\tilde{K}^{jl0}$ in the expression for the covariant derivative. So the term linear in $\dot{g}$ in (86) (without covariant derivatives on it) is of the form

$$(T^{ml})^{(2)} = \frac{(s+1)}{24(2\pi)^2}\left[s\left(\frac{s}{2}+1\right) - \frac{1}{4}\right][g^{mn}(R_{rs})^{lj} + g^{ln}(R_{rs})^{mj}]\dot{g}_{nj}\partial_p\mathcal{A}_q\frac{\epsilon^{rspq}}{\sqrt{\det g}}. \tag{91}$$

These two expressions, namely, Eqs. (89) and (91), give us the momentum-independent terms of the Hall viscosity. To simplify further, we will consider it in two particular limits: (1) the flat limit where the $\mathbb{CP}^2$ radius becomes very large and the curvature vanishes, and (2) on the manifold $\mathbb{CP}^2$ with curvatures set to the values appropriate to this background.

In the flat limit, $\mathbb{CP}^2$ space decomposes into $\mathbb{C}\times\mathbb{C}$, corresponding to the planes (1,2) and (3,4). The flat limit may be the most pertinent case for the current experimental setups [9,10]. Each plane carries a constant perpendicular magnetic field $F_{12} = F_{34} = B = n/2r^2$. Also, we can write $(J^0)^{ij}\to\epsilon^{ij}$. Since the curvature terms vanish in this limit, the contribution from $(T^{ml})^{(2)}$ is zero. The contribution from $(T^{ml})^{(1)}$ is of the form

$$T^{ml} = \frac{(s+1)^2}{8(2\pi)^2}(g^{mi}\epsilon^{lk} + g^{li}\epsilon^{mk})\dot{g}_{ki}B^2. \tag{92}$$

Comparing with (70), we find that the Hall viscosity in this limit is

$$\eta_H = \frac{1}{4}\left(\frac{(s+1)B}{2\pi}\right)^2. \tag{93}$$

We have not evaluated the momentum-dependent terms, so there is no result for such terms in the Hall viscosity.

Turning to the Hall viscosity for the $\mathbb{CP}^2$ background, it is useful to write the expression for K^0 and $(T^{ml})^{(2)}$ in terms of differential forms:

$$d^4x\sqrt{\det g}\,K^0 = \frac{s+1}{2}dAdA + \frac{2}{3}\left[(s+1)^2+\frac{1}{2}\right]dAd\omega^0 + \frac{s+1}{2}d\omega^0\,d\omega^0 - \frac{s+1}{12}\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]R^{\alpha\beta}R^{\beta\alpha}. \tag{94}$$

From the relations given in Appendix A, $dA = F = n\,\Omega_K$, $d\omega^0 = R^0 = (3/2)\Omega_K$, and

$$R^{\alpha\beta}R^{\beta\alpha} = -4R^0R^0 - R^aR^a = -6\,\Omega_K^2, \tag{95}$$

where Ω_K is the Kähler 2-form for $\mathbb{CP}^2$. Using these relations, K^0 becomes

$$K^0 = \left\{(s+1)(n/2)^2 + \left[(s+1)^2+\frac{1}{2}\right](n/2) + \frac{s+1}{8}[(s+1)^2+3]\right\}. \tag{96}$$

Similarly, we write $(T^{ml})^{(2)}$ as

$$d^4x\sqrt{\det g}\,(T^{ml})^{(2)} = \frac{s+1}{12(2\pi)^2}\left\{\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right](g^{mn}R^{lj} + g^{ln}R^{mj})\dot{g}_{nj}d\mathcal{A}\right\}. \tag{97}$$

A useful relation is to note that $R^a\wedge\Omega_K = 0$, as shown in Appendix A. Also, $d\mathcal{A} = dA + (s+1)d\omega^0 = [n+\frac{3}{2}(s+1)]\Omega_K$. The expression for $(T^{ml})^{(2)}$ then simplifies as

$$(T^{ml})^{(2)} = \frac{s+1}{8(2\pi)^2}\left\{\left[s\left(\frac{s}{2}+1\right)-\frac{1}{4}\right]\left[\frac{n}{2}+\frac{3}{4}(s+1)\right]\right\}[g^{mi}(J^0)^{lk} + g^{li}(J^0)^{mk}]\dot{g}_{ki}. \tag{98}$$

Using K^0 from (96) in (89) and adding $(T^{ml})^{(2)}$ from the equation given above, we get

$$T^{ml} = \frac{(s+1)}{8(2\pi)^2}[g^{mi}(J^0)^{lk} + g^{li}(J^0)^{mk}]\dot{g}_{ki}\left\{(s+1)\left(\frac{n}{2}\right)^2 + \left[\frac{3}{2}(s+1)^2-\frac{1}{4}\right]\left(\frac{n}{2}\right) + \frac{(s+1)}{2}\left[(s+1)^2-\frac{3}{8}\right]\right\}. \tag{99}$$

We notice that the tensorial structure of the energy-momentum tensor in (99) is appropriately modified, with the metric and the covariant version of J^0, for the $\mathbb{CP}^2$ background compared to (92) and that the curvature terms do contribute to the overall factor. However, the contribution from the curvature terms is relatively negligible in the large-B limit, as expected.

V. DISCUSSION

In this paper we have considered some of the transport properties of quantum Hall systems in arbitrary, even spatial dimensions. The effective action obtained in [12] provides a uniform approach and a convenient starting point for this, as transport coefficients can be obtained by varying this action with respect to the external fields and the metric. Specifically, we focus on the Hall conductivity and the Hall viscosity, which are the transport properties most relevant from an experimental point of view. Towards this, we first generalized the effective action from [12] to take account of the fact that electrons belong to an energy band in a solid, rather than being in free space. We derived an expression for the electromagnetic Hall current, valid for any even spatial dimension, displaying various terms proportional to integrals of the Chern classes of the Berry curvature of the electronic bands. Additionally, our expressions include the contributions due to the spatial curvature. We expect that these expressions, with or without the spatial curvature, will be directly relevant for proposed experimental realizations in higher dimensions [9,10].

We have also given explicit expressions for the Hall viscosity in two and four spatial dimensions, including terms which depend on the curvature. While the result for two dimensions agrees with previous work on the calculation of responses, it should be emphasized that our approach places it within a uniform method of derivation. The results for four dimensions are obviously new.

An important point worth noting is that, in general, there are several additional transport coefficients or response functions possible. Already in two spatial dimensions, we see from (30) that the second term is of the form

$$J^i = \frac{\nu_1}{4\pi}\epsilon^{ij} R_{j0} = \frac{\nu_1}{8\pi}\epsilon^{ij}\epsilon_{kl}\frac{\nabla^k(g^{ln}\dot{g}_{nj})}{\sqrt{\det g}}. \quad (100)$$

This shows that there is a new transport coefficient ζ^i_{mn} we can define by

$$\langle J^i(x)\, T_{mn}(y)\rangle = \zeta^i_{mn}\frac{1}{\sqrt{\det g}}\partial_0\delta^{(3)}(x-y)$$
$$\zeta^i_{mn} = -\frac{\nu_1}{8\pi}\epsilon^{ij}(g_{mj}\epsilon_{kn} + g_{nj}\epsilon_{km})g^{kl}\nabla_l. \quad (101)$$

(As written, ζ^i_{mn} is an operator and must be interpreted in terms of eigenfunctions of the gradient operator or in terms of Fourier components.) This transport coefficient exists for the higher dimensional cases as well, although we have not calculated explicit formulas for it.

Higher dimensions also allow for the possibility of non-Abelian background gauge fields. The responses to varying the non-Abelian gauge field background will constitute another set of transport coefficients. Finally, we have already noted, in the footnote after Eq. (80), that one can also have transport coefficients with correlation functions involving the spin density. In principle, all such additional transport coefficients can be calculated using the effective action from [12], but we leave this to future work.

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APPENDIX A: BASIC FEATURES AND GEOMETRY OF $\mathbb{CP}^k$ SPACES

Let t_A denote the generators of $SU(k+1)$ as matrices in the fundamental representation, normalized so that $\mathrm{Tr}(t_At_B) = \frac{1}{2}\delta_{AB}$. These generators are classified into three groups. The ones corresponding to the $SU(k)$ part of $U(k) \subset SU(k+1)$ will be denoted by t_a, $a = 1, 2, \ldots, k^2-1$, while the generator for the $U(1)$ direction of the subgroup $U(k)$ will be denoted by t_{k^2+2k}. The $2k$ remaining generators of $SU(k+1)$ which are not in $U(k)$ are the coset generators, denoted by t_α, $\alpha = k^2, \ldots, k^2+2k-1$. (To distinguish the various components, we use Greek letters from the beginning of the alphabet here; the corresponding E's defined below will be the components in the tangent frame, not the coordinate frame. Lowercase Roman letters from the middle of the alphabet onwards will denote components in the coordinate frame, for spatial directions only. When spacetime coordinate frames are involved, we use Greek letters from later in the alphabet for the coordinate frame.)

We can now use a $(k+1)\times(k+1)$ matrix g in the fundamental representation of $SU(k+1)$ to coordinatize $\mathbb{CP}^k$, with the identification $g \sim gh$, where $h \in U(k)$. We can expand $g^{-1}dg$, which is an element of the Lie algebra, as

$$g^{-1}dg = (-iE^{k^2+2k}t_{k^2+2k} - iE^a t_a - iE^\alpha t_\alpha)$$
$$= (-iE^{k^2+2k}t_{k^2+2k} - iE^a t_a - iE^{+I}t_{+I} - iE^{-I}t_{-I}), \quad (A1)$$

where

$$E^{+I} = E^{k^2+2I-2} - iE^{k^2+2I-1},$$
$$E^{-I} = E^{k^2+2I-2} + iE^{k^2+2I-1}, \quad I = 1, \ldots, k. \quad (A2)$$

E^α are 1-forms corresponding to the frame fields in terms of which the Cartan-Killing metric on $\mathbb{CP}^k$ is given by

$$ds^2 = g_{ij}dx^i dx^j = E^\alpha_i E^\alpha_j dx^i dx^j. \quad (A3)$$

The Kähler 1-form on $\mathbb{CP}^k$ is given by

$$\alpha = i\sqrt{\frac{2k}{k+1}}\,\mathrm{Tr}(t_{k^2+2k}g^{-1}dg) = \sqrt{\frac{k}{2(k+1)}}\,E^{k^2+2k}. \quad (A4)$$

The corresponding Kähler 2-form $\Omega_K = d\alpha$ is given by

$$\Omega_K = -i\sqrt{\frac{2k}{k+1}}\mathrm{tr}(t_{k^2+2k}g^{-1}dg\wedge g^{-1}dg) = -\frac{1}{4}\sqrt{\frac{2k}{k+1}}f_{(k^2+2k)\alpha\beta}E^\alpha\wedge E^\beta = -\frac{1}{4}\epsilon_{\alpha\beta}E^\alpha\wedge E^\beta. \tag{A5}$$

f_{ABC} are the $SU(k+1)$ structure constants, defined by $[t_A, t_B] = if_{ABC}t_C$. In deriving the last line of (A5), we used the fact that $f_{(k^2+2k)\alpha\beta} = \sqrt{\frac{k+1}{2k}}\epsilon_{\alpha\beta}$, where

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = 1 \quad \text{for} \quad \alpha = k^2+2I-2, \quad \beta = k^2+2I-1, \quad I = 1,2,\dots,k$$
$$= 0 \quad \text{for all other choices.} \tag{A6}$$

The volume of $\mathbb{CP}^k$ is normalized so that

$$\int_{\mathbb{CP}^k}\left(\frac{\Omega_K}{2\pi}\right)^k = 1. \tag{A7}$$

The Maurer-Cartan identity $d(g^{-1}dg) = -g^{-1}dgg^{-1}dg$, along with (A1), leads to

$$dE^{k^2+2k} = -\frac{1}{2}f^{(k^2+2k)\alpha\beta}E^\alpha\wedge E^\beta = 2\sqrt{\frac{k+1}{2k}}\Omega_K$$
$$dE^a + \frac{1}{2}f^{abc}E^b\wedge E^c = -\frac{1}{2}f^{a\alpha\beta}E^\alpha\wedge E^\beta \tag{A8}$$
$$dE^\alpha = -f^{\alpha A\beta}E^A\wedge E^\beta.$$

The $U(k)$ spin connection ω^{IJ} is defined in terms of the holomorphic frame fields E^{+I} by

$$dE^{+I} + \omega^{IJ}E^{+J} = 0, \quad I = 1,\dots,k. \tag{A9}$$

ω^{IJ} takes values in the Lie algebra of $U(k)$, so one can write

$$\omega^{IJ} = -i(\omega^0\mathbf{1} + \omega^a t_a). \tag{A10}$$

The curvature 2-form is given by

$$R = d\omega + \omega\wedge\omega = -i(R^0\mathbf{1} + R^a t_a), \tag{A11}$$

$$R^0 = d\omega^0$$
$$R^a = d\omega^a + \tfrac{1}{2}f^{abc}\omega^b\omega^c. \tag{A12}$$

Even though equations up to (A8) used specific properties of $\mathbb{CP}^k$, Eqs. (A9)–(A12) hold for any manifold with a complex structure so that the holonomy group is $U(k)$. In deriving the effective actions, including gauge and gravitational fluctuations, for higher dimensional QHE we have used the topological property of the Dolbeault index to move away from the specific gauge and curvature background values. So the equations which hold, in general, are (A9)–(A12).

If we now specialize to the case of $\mathbb{CP}^k$, using Maurer-Cartan identities (A8) we can identify the ω^{IJ} as

$$\omega = \bar{\omega} = -i\left(\sqrt{\frac{k+1}{2k}}E^{k^2+2k}\mathbf{1} + E^a t_a\right) \equiv -i\left(\omega^0\mathbf{1} + \omega^a t_a\right). \tag{A13}$$

The curvature components for $\mathbb{CP}^k$ are then given by

$$\bar{R}^0 = \frac{k+1}{k}\Omega_K$$
$$\bar{R}^a = -\frac{1}{2}f^{a\alpha\beta}E^\alpha\wedge E^\beta, \tag{A14}$$

where we have indicated the background values with an overbar, as in $\bar{R}$. Notice that in the tangent frame, the curvatures are given in terms of the $U(k)$ structure constants.

We can now use the freedom of h transformations to parametrize g in terms of complex coordinates $z^i, \bar{z}^i$. We choose a parametrization such that

$$g_{i,k+1} = \frac{z_i}{\sqrt{1+z\cdot\bar{z}}}, \quad i = 1,\dots,k, \quad g_{k+1,k+1} = \frac{1}{\sqrt{1+z\cdot\bar{z}}}. \tag{A15}$$

Using this parametrization, one can write the Kähler 2-form Ω_K in (A5) in terms of the local complex coordinates in the following, more familiar form:

$$\Omega_K = i\left[\frac{dz\cdot d\bar{z}}{1+z\cdot\bar{z}} - \frac{\bar{z}\cdot dzz\cdot d\bar{z}}{(1+z\cdot\bar{z})^2}\right]. \tag{A16}$$

We further choose the relation between the complex coordinates and the real ones to be the usual one:

$$z^i = x^{2i-1} + ix^{2i}, \quad \bar{z}^i = x^{2i-1} - ix^{2i}, \quad i = 1,\dots,k. \tag{A17}$$

The parametrizations (A17) and (A15) determine the appropriate choice for the Cartesian frame fields in the following way. For simplicity we will work with $\mathbb{CP}^1 = S^2$, but the argument works in general. Using Eq. (A15), one finds that the complex frame field $E^+ = E^1 - iE^2 = idz/(1+z\bar{z})$. In the flat limit, where the radius of the sphere becomes large, $E^+ = E^1 - iE^2 = i(e^1 + ie^2) \sim i(dx + idy)$, where $(e^1, e^2) = (-E^2, -E^1)$. It is the e's that provide the conventional Cartesian frame fields given the choices (A17) and (A15). More generally, for $\mathbb{CP}^k$,

$$(e^{2I-1}, e^{2I}) = (-E^{k^2+2I-1}, -E^{k^2+2I-2}), \quad I = 1,\dots,k. \tag{A18}$$

In terms of the Cartesian frame fields e, the Kähler 2-form Ω_K and the metric g_{ij} can be written as

$$\Omega_K = \tfrac{1}{4}\epsilon^{\alpha\beta}e^\alpha\wedge e^\beta, \quad g_{ij} = e_i^\alpha e_j^\alpha, \tag{A19}$$

where $\alpha = 1,\dots,2k$ and $\epsilon^{12} = \epsilon^{34} = \cdots = \epsilon^{2I-1,2I} = 1$ for $I = 1,\dots,k$.

The spin connection and the corresponding curvature defined in Eqs. (A9)–(A12) involve the $U(k)$ subalgebra of the vector representation of the Lie algebra of the $SO(2k)$ holonomy group. In the effective actions for quantum Hall effect in higher dimensions we have considered fluctuations of the holomorphic $U(k)$ spin connection away from their background values. Further, in obtaining the energy-momentum tensor from the effective actions, we must consider arbitrary variations of the metric. For this we will need to consider the connection and curvature in $SO(2k)$. Thus it is important to know how the $U(k)$ spin connection and curvature are embedded in $SO(2k)$. We will now derive this relation for the four-dimensional quantum Hall effect on $\mathbb{CP}^2$, although similar expressions hold for all $\mathbb{CP}^k$.

The real components of the $SO(4)$ spin connection can be identified via

$$de^\alpha + \omega^{\alpha\beta} e^\beta = 0. \tag{A20}$$

Using (A2), (A9), and (A18), we find

$$\begin{aligned} \omega^{\alpha\beta} &= \epsilon^{\alpha\beta}\omega^0 + (J^a)^{\alpha\beta}\omega^a \\ R^{\alpha\beta} &= \epsilon^{\alpha\beta}R^0 + (J^a)^{\alpha\beta}R^a, \end{aligned} \tag{A21}$$

where ω^0, R^0 and ω^a, R^a are the $U(1)$ and $SU(2)$ components of the complex spin connection and curvature as defined in Eqs. (A10) and (A11). The $(4{\times}4)$ matrices J^a are related to the $SU(3)$ structure constants $f^{a\alpha\beta}$ via

$$(J^a)^{\alpha\beta} = (f^1, -f^2, f^3)^{3+\alpha,3+\beta}. \tag{A22}$$

In particular,

$$\begin{aligned} J^1 &= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \\ J^2 &= \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ J^3 &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{A23}$$

Similar expressions hold for all $\mathbb{CP}^k$ manifolds.

The matrices J^a form a basis for the Lie algebra of $SU(2)$, obeying the commutation rules:

$$[J^a, J^b] = \epsilon^{abc}J^c. \tag{A24}$$

Further, they satisfy the following relations:

$$\begin{aligned} \epsilon^{\alpha\beta}(J^a)^{\beta\alpha} &= 0, \quad \mathrm{Tr}(J^aJ^b) = -\delta^{ab} \\ (J^a)^{\alpha\beta}(J^a)^{\gamma\delta} &= \tfrac{1}{4}(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}) - \tfrac{1}{4}\epsilon^{\alpha\beta\gamma\delta}. \end{aligned} \tag{A25}$$

Using (A25), we can write the relation between the complex $U(k)$ components of the spin connection and curvature in terms of the real $SO(2k)$ components of the corresponding quantities. In particular, we have the following relations:

$\mathbb{CP}^1$ *case*:

$$\omega^0 = \tfrac{1}{2}\epsilon^{\alpha\beta}\omega^{\alpha\beta} = \omega^{12} \tag{A26}$$

$\mathbb{CP}^2$ *case*:

$$\begin{aligned} R^0 &= \tfrac{1}{4}\epsilon^{\alpha\beta}R^{\alpha\beta} \\ R^aR^a &= -4R^0R^0 - R^{\alpha\beta}R^{\beta\alpha} \end{aligned} \tag{A27}$$

In formulating QHE on $\mathbb{CP}^k$, one has to choose the background values for the gauge fields as well. We take the $U(1)$ and $SU(k)$ background gauge fields as proportional to $E_i^{k^2+2k}$ and E_i^a. Specifically,

$$\begin{aligned} \bar{A}^{k^2+2k} &= -in\sqrt{\frac{2k}{k+1}}\mathrm{Tr}(t_{k^2+2k}g^{-1}dg) = \frac{n}{2}\sqrt{\frac{2k}{k+1}}E^{k^2+2k} \\ \bar{A}^a &= E^a = 2i\mathrm{Tr}(t^ag^{-1}dg). \end{aligned} \tag{A28}$$

The corresponding $U(1)$ and $SU(k)$ background field strengths are

$$\bar{F} = n\,\Omega_{\mathrm{K}}, \quad \bar{F}^a = \bar{R}^a. \tag{A29}$$

We see from (A29) that the background field strengths are proportional to the background curvature components, which are constant in the appropriate frame basis, proportional to the $U(k)$ structure constants (A14). It is in this sense that the field strengths in (A29) correspond to uniform magnetic fields appropriate in defining QHE.

In terms of the frame fields e^α, the curvatures for $\mathbb{CP}^2$ are given by

$$\begin{aligned} R^0 &= \tfrac{3}{2}\Omega_{\mathrm{K}} = \tfrac{3}{4}(e^1e^2 + e^3e^4) \\ R^1 &= \tfrac{1}{2}(e^1e^4 - e^2e^3), \quad R^2 = -\tfrac{1}{2}(e^2e^4 + e^1e^3) \\ R^3 &= \tfrac{1}{2}(e^1e^2 - e^3e^4). \end{aligned} \tag{A30}$$

In particular, we have the relation $R^a \wedge \Omega_{\mathrm{K}} = 0$.

A few other relations, which might be of interest for $\mathbb{CP}^k$, for arbitrary k are the following:

$$\int_{\mathbb{CP}^k} \mathrm{td}\,(T_cK)|_{2k} = 1, \tag{A31}$$

where $\mathrm{td}\,(T_cK)$ is the Todd class in the complex tangent space, and in (A31) the $2k$-form is selected as the integrand. Explicitly, the Todd class has the expansion given in (3) as

$$\begin{aligned} \mathrm{td} = 1 &+ \tfrac{1}{2}c_1 + \tfrac{1}{12}\left(c_1^2 + c_2\right) + \tfrac{1}{24}c_1c_2 \\ &+ \tfrac{1}{720}(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4) + \cdots, \end{aligned} \tag{A32}$$

where c_i are the Chern classes. The first few Chern classes can be easily evaluated using (A14) as

$$\begin{aligned} c_1 &= \mathrm{Tr}\frac{iR}{2\pi} = (k+1)\frac{\Omega_{\mathrm{K}}}{2\pi} \\ c_2 &= \frac{1}{2}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2 - \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right] = \tfrac{1}{2}k(k+1)\left(\frac{\Omega_{\mathrm{K}}}{2\pi}\right)^2. \end{aligned} \tag{A33}$$

In deriving the expression for c_2, we used the fact that

$$\begin{aligned} R^a \wedge R^a &= \frac{1}{4}f^{a\alpha\beta}f^{a\gamma\delta}E^\alpha E^\beta E^\gamma E^\delta = -2\frac{k+1}{k}\Omega_{\mathrm{K}}^2 \\ \mathrm{Tr}[iR \wedge iR] &= k(R^0)^2 + \frac{1}{2}(R^a)^2 = (k+1)\Omega_{\mathrm{K}}^2. \end{aligned} \tag{A34}$$

These can be easily shown using completeness relations for the matrices t^A in the fundamental representation. More generally, the Chern classes for $\mathbb{CP}^k$ can be written as

$$c_i = \frac{k!}{i!(k-i)!}\left(\frac{\Omega_{\mathrm{K}}}{2\pi}\right)^i. \tag{A35}$$

Using Eqs. (A7) and (A33), we can easily check the validity of (A31) for $\mathbb{CP}^1$, $\mathbb{CP}^2$, and $\mathbb{CP}^3$, the needed integrals being

$$\int_{\mathbb{CP}^1} c_1 = 2\int \frac{\Omega_{\rm K}}{2\pi} = 2$$
$$\int_{\mathbb{CP}^2} c_1^2 + c_2 = (3^2+3)\int\left(\frac{\Omega_{\rm K}}{2\pi}\right)^2 = 12 \qquad \text{(A36)}$$
$$\int_{\mathbb{CP}^3} c_1 c_2 = 4\times 6\int\left(\frac{\Omega_{\rm K}}{2\pi}\right)^3 = 24.$$

APPENDIX B: AN IDENTITY ON DETERMINANT OF Ω

In this Appendix we give a derivation of the identity (40) for $k=3$ and the more general case used in text. [In what follows, Ω is the Berry curvature given in (21).] Consider Grassmann variables Q_a and η_a, $a = 1, 2, \ldots, 2k$, and start with the identity

$$\int [dQ] e^{Q\Omega Q} e^{\eta\cdot Q} = \mathcal{K}\, e^{\eta\Omega^{-1}\eta/4}$$
$$\mathcal{K} = \left[\frac{1}{k!}\epsilon_{a_1a_2\cdots a_{2k}}\Omega^{a_1a_2}\Omega^{a_3a_4}\cdots\Omega^{a_{2k-1}a_{2k}}\right]. \qquad \text{(B1)}$$

We equate the term with $2l$ powers of η on both sides. We also carry out the integration on the Q's on the left-hand side by expanding $e^{Q\Omega Q}$. This gives us the relation

$$\left[\frac{(-1)^l}{(2l)!(k-l)!}\right]\eta_{a_1}\cdots\eta_{a_{2l}}\,\epsilon_{a_1\cdots a_{2l}a_{2l+1}\cdots a_{2k}} \times (\Omega^{a_{2l+1}a_{2l+2}}\cdots\Omega^{a_{2k-1}a_{2k}})$$
$$= \mathcal{K}\frac{1}{4^l l!}\eta_{a_1}\cdots\eta_{a_{2l}}\,[(\Omega^{-1})^{a_1a_2}\cdots(\Omega^{-1})^{a_{2l-1}a_{2l}}]. \qquad \text{(B2)}$$

We can remove the η's by writing $[(\Omega^{-1})^{a_1a_2}\cdots(\Omega^{-1})^{a_{2l-1}a_{2l}}]$ in the fully antisymmetrized form, so that

$$\left[\frac{(-1)^l}{(2l)!(k-l)!}\right]\epsilon_{a_1\cdots a_{2l}a_{2l+1}\cdots a_{2k}}(\Omega^{a_{2l+1}a_{2l+2}}\cdots\Omega^{a_{2k-1}a_{2k}})$$
$$= \mathcal{K}\frac{1}{4^l l!}[(\Omega^{-1})^{a_1a_2}\cdots(\Omega^{-1})^{a_{2l-1}a_{2l}}]_{\rm antisym}. \qquad \text{(B3)}$$

This is the basic identity. In calculating the Hall currents, we get this expression multiplied by a factor of $\epsilon^{ijk\cdots a_1\cdots a_{2l}}$. This allows us to write the identity

$$\left[\frac{(-1)^l}{(2l)!(k-l)!}\right](2l)!\,\delta^{ij\cdots}_{a_{2l+1}\cdots a_{2k}}(\Omega\,\Omega\cdots)^{a_{2l+1}\cdots a_{2k}}$$
$$= \mathcal{K}\frac{1}{4^l l!}\epsilon^{ij\cdots a_1\cdots a_{2l}}[(\Omega^{-1})^{a_1a_2}\cdots(\Omega^{-1})^{a_{2l-1}a_{2l}}]. \qquad \text{(B4)}$$

Because of the antisymmetry of $\delta^{ij\cdots}_{a_{2l+1}\cdots a_{2k}}$, we get all permutations of all indices in $(\Omega\,\Omega\cdots)^{a_{2l+1}\cdots a_{2k}}$ on the left-hand side of this equation. Since permutations of the Ω's themselves [$(k-l)!$ of these] and the permutation of the two indices on each Ω (2^{k-l} of these) do not change the expression, we can write

$$\delta^{ij\cdots}_{a_{2l+1}\cdots a_{2k}}(\Omega\,\Omega\cdots)^{a_{2l+1}\cdots a_{2k}} = (k-l)!\,2^{k-l}\sum_{\rm dist.perm.}(\Omega\,\Omega\cdots)^{ij\cdots}. \qquad \text{(B5)}$$

The number of terms in the sum in this equation is given by

$$\text{Number of distinct permutations} = \frac{(2k-2l)!}{2^{k-l}\,(k-l)!}. \qquad \text{(B6)}$$

Since

$$\frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k = \frac{1}{2^k(2\pi)^k}\mathcal{K}\,d^{2k}p, \qquad \text{(B7)}$$

we can use (B5) to bring (B4) to the form

$$\int\frac{1}{k!}\left(\frac{\Omega}{2\pi}\right)^k\epsilon^{ij\cdots a_1\cdots a_{2l}}\left\{\frac{(-1)^l}{l!}\left[\left(\frac{(\Omega^{-1})^{a_1a_2}}{2(2\pi)}\right)\cdots\left(\frac{(\Omega^{-1})^{a_{2l-1}a_{2l}}}{2(2\pi)}\right)\right]\right\} = \nu^{ij\cdots}_{k-l}, \qquad \text{(B8)}$$

where we define

$$\nu^{ij\cdots}_{k-l} = \int\frac{1}{(2\pi)^{k+l}}\sum_{\rm dist.perm.}(\Omega\,\Omega\cdots)^{ij\cdots}. \qquad \text{(B9)}$$

Notice that there are $2k-2l$ indices in this expression, so we can make this explicit by writing it out as

$$\nu^{i_1i_2\cdots i_{(2k-2l)}}_{k-l} = \int\frac{1}{(2\pi)^{k+l}}\sum_{\rm dist.perm.}(\Omega^{i_1i_2}\,\Omega^{i_3i_4}\cdots\Omega^{i_{(2k-2l-1)}i_{(2k-2l)}}). \qquad \text{(B10)}$$

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