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An enriched cut finite element method for Stokes interface equations[☆]

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ABSTRACT

In this paper, we consider an enriched cut finite element method (ECFEM) with interface-unfitted meshes for solving Stokes interface equations consisting of two incompressible fluids with different viscosities. By approximating the velocity with the enriched $\mathbb{P}_1$ element and the pressure with the $\mathbb{P}_0$ element, and stabilizing the Galerkin variational formulation with suitable ghost penalty terms, we propose the new ECFEM and prove that it is well-posed and has the optimal *a priori* error estimate in the energy norm. All derived results are independent of the interface position. Moreover, compared with other conforming finite element methods with the optimal rate in convergence, the proposed scheme here not only has the minimum degrees of freedom, but also avoids using the derivative of the pressure in the penalty term. The presented numerical examples validate the theoretical predictions.

1. Introduction

In this paper, we consider the Stokes interface equations as follows:

$$-\nabla \cdot (\nu\epsilon(\mathbf{u}) - p\mathbf{I}) = \mathbf{f}, \quad \text{in } \Omega_1 \cup \Omega_2, \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega_1 \cup \Omega_2, \tag{1b}$$

$$[\mathbf{u}]_\Gamma = 0, \quad \text{on } \Gamma, \tag{1c}$$

$$[(\nu\epsilon(\mathbf{u}) - p\mathbf{I})\mathbf{n}_\Gamma]_\Gamma = \sigma\kappa\mathbf{n}_\Gamma, \text{ on } \Gamma, \tag{1d}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega. \tag{1e}$$

Here, Ω is an open bounded domain in $\mathbb{R}^2$ with a polyhedral boundary $\partial\Omega$, $\Omega_i \subset \Omega$ $(i = 1, 2)$ satisfying $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is a smooth interface separating the domain Ω, $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u_1(x_1, x_2), u_2(x_1, x_2))^\top$ is the velocity, $p = p(\mathbf{x}) = p(x_1, x_2)$ is the pressure, $\epsilon(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)/2$ is the strain rate tensor, $\nu = 2\nu_i$ on Ω_i $(i = 1, 2)$ is a piecewise constant, $\mathbf{I}$ is the identity tensor, $[\cdot]_\Gamma$ denotes the jump on the interface Γ which will be defined in the following, σ is the surface tension coefficient, κ is the curvature of the interface and $\mathbf{n}_\Gamma$ is the outward unit normal on the interface pointing from Ω_1 to Ω_2. The equations are usually used to model the two-phase incompressible flows when the viscosities are large.

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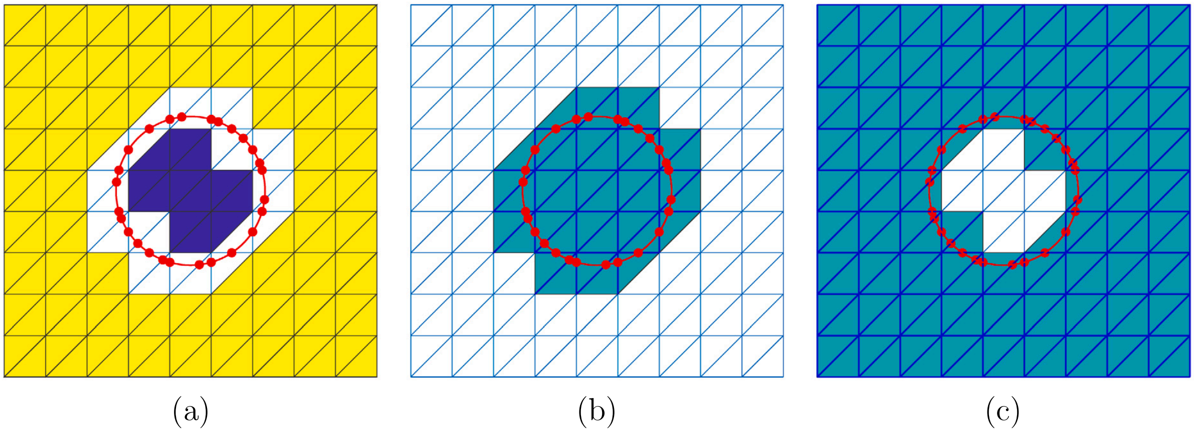


Fig. 1. Illustration of triangulations: (a). $\mathcal{T}_h^\Gamma$ (non-shaded triangles), ω_h^1 (red triangles), ω_h^2 (yellow triangles); (b) Ω_h^1 (colored triangles); (c) Ω_h^2 (colored triangles). Here Ω is a unit square, Γ is a circle, Ω_1 is the subdomain inside the circle, and $\Omega_2 = \Omega \backslash \Omega_1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Due to the discontinuities of the stress tensor across the interface in the equations, the standard finite element method cannot capture the solution of the problem (1) very well with an interface-unfitted mesh [25]. However, generating the interface-fitted meshes is usually a nontrivial task for the complicated interface problem (see interfaces in Section 5), especially for the problems with moving interfaces (see [11,26,31]). To deal with this difficulty, many attentions have been attracted in the past decades. One of the popular ways is to use the extended finite element method (XFEM) [3,12], in which the computational accuracy at the interface can be improved by adding extra basis functions for elements intersected by the interface based on interface-unfitted meshes. However, the linear system generated by this method may be generally sensitive to the interface position. Another widely investigated approach is the Nitsche's method. In this method, by enforcing the jump conditions via a variant of Nitsche's method and adapting the parameter in the numerical fluxes on each element, the discontinuity could be successfully captured [2,15]. But, as that in the XFEM, the corresponding linear system might be still ill-conditioned. Later, many combined methods with respect to the Nitsche's method and the XFEM were studied. By using overlapping fictitious domains and the ghost penalty stabilized technique, and applying the $\mathbb{P}_1$-iso-$\mathbb{P}_2/\mathbb{P}_1$ finite element pair for the velocity and pressure in the Eqs. (1) respectively, a cut finite element method (also called Nitsche-XFEM) was considered in [16]. The well-posedness and error estimates for the proposed method were also deduced in the paper. Then, other frequently used finite element methods for the Stokes interface equations were investigated under this frame, such as the $\mathbb{P}_2/\mathbb{P}_1$ elements in [22], $\mathbb{P}_1 b$(MINI)/$\mathbb{P}_1$ elements in [6,30] and $\mathbb{P}_{k+1}/\mathbb{P}_k$ ($k \geq 1$) in [24]. At the same, this idea were extended to the equal order finite element pair $\mathbb{P}_1/\mathbb{P}_1$ [27,29] which does not satisfy the inf–sup condition (Babuška-Brezzi condition [13]) in the classical sense, the hybrid higher-order method in [5], the nonconforming finite element in [18,28] and general cases in [14]. Moreover, the Navier–Stokes equations with interfaces was also simulated in [8,11].

In this paper, we consider the lowest order $\mathbb{P}_1/\mathbb{P}_0$ finite element method for the Stokes interface equations. Generally, this finite element pair does not satisfy the inf–sup condition. To make this approximation method to be well-posed, some stabilized terms are required in this scheme. Many works have been done in this topic, such as [17,19,20] by using the macroelement penalty terms in the variational formulation and [7,21,32] by using the enriched technique in the basis function. Here, we utilize the latter, i.e., the enriched $\mathbb{P}_1$ element for the velocity and the $\mathbb{P}_0$ element for the pressure. Different from the $\mathbb{P}_1 b$ element, the enriched $\mathbb{P}_1$ element consisting of the linear piecewise polynomial enriched by a discontinuous, piecewise linear and mean-zero vector function on each element, and only one degree of freedom per element is increased [7,21,32]. Therefore, compared with other conforming finite element pairs considered for the Eqs. (1) in the references, this one has the minimum degree of freedom among the schemes with the optimal rate in convergence. Here, the optimal rate in convergence means the error can be bounded by the best possible error. For example, if the degree kth polynomial has been employed in the simulation, if the solution is smooth, the expected optimal L^2-error in convergence is at the order $\mathcal{O}(h^{k+1})$ with h being the spatial mesh. For example, the Stokes stable MINI-element employs the continuous $\mathbb{P}_1$ element for pressure approximation. However, the numerical analysis for MINI element shows that $\|p - p_h\|_0 \approx \mathcal{O}(h)$, which is not the optimal rate in convergence. In contrast, our proposed enriched Galerkin scheme $\mathbb{P}_1/\mathbb{P}_0$ shows the velocity error measured in H^1-norm converges at the rate $\mathcal{O}(h)$ and the pressure error measured in L^2-norm converges at the rate $\mathcal{O}(h)$, which are the optimal rates. By adding some suitable penalty terms in the variational formulation to deal with the discontinuity at the interface, we propose a new enriched cut finite element method (ECFEM) for solving the Stokes interface equations, which is not only applicable to interface problems with an interface-unfitted mesh, but also avoids using the derivative of the pressure in the ghost penalty term. Furthermore, the well-posedness and the optimal rate of errors measured in the energy norm are deduced, too.

The paper is organized as follows. After introducing the ECFEM in Section 2, we deduce the well-posedness and the error estimate in Sections 3 and 4, respectively. Then, in Section 5, we present some numerical examples to validate the theoretical predictions. Finally, conclusions are made in Section 6.

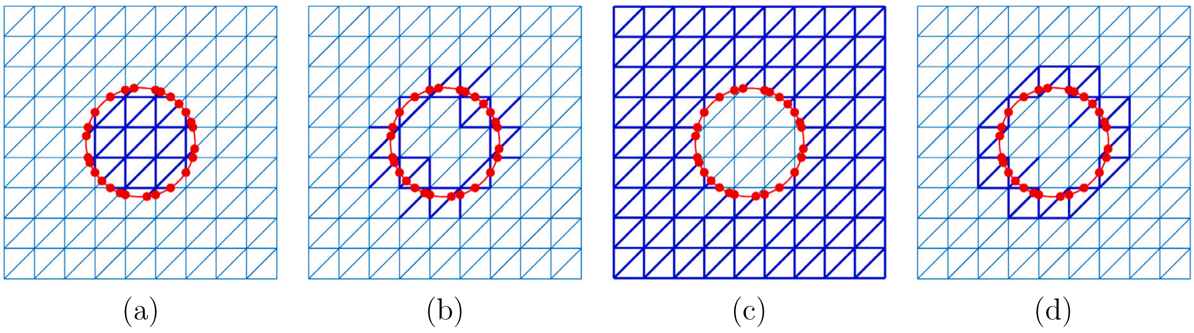


Fig. 2. Illustration of edges: (a) $\mathcal{F}_{h,1}$ (thick lines); (b) $\mathcal{G}_{h,1}$ (thick lines); (c) $\mathcal{F}_{h,2}$ (thick lines); (d) $\mathcal{G}_{h,2}$ (thick lines).

2. Enriched cut finite element method

In this section, after introducing some notations for the partition mesh, we will propose the ECFEM based on the lowest order $\mathbb{P}_1/\mathbb{P}_0$ enriched Galerkin finite elements.

2.1. Variational formulation

Defining

$$\mathbf{X} = \{\mathbf{u} \in [H^1(\Omega)]^2 : \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\},$$
$$Q = \{p \in L^2(\Omega) : (\nu^{-1}p, 1) = 0\},$$

the variational formulation of the Eqs. (1) is: find $(\mathbf{u}, p) \in \mathbf{X} \times Q$ such that for any $(\mathbf{v}, q) \in \mathbf{X} \times Q$,

$$(\nu\epsilon(\mathbf{u}), \epsilon(\mathbf{v}))_{\Omega_1\cup\Omega_2} - (\nabla\cdot\mathbf{v}, p)_{\Omega_1\cup\Omega_2} = (\mathbf{f}, \mathbf{v})_{\Omega_1\cup\Omega_2} + \langle\sigma\kappa, \mathbf{v}\cdot\mathbf{n}_\Gamma\rangle_\Gamma, \tag{2a}$$

$$(\nabla\cdot\mathbf{u}, q)_{\Omega_1\cup\Omega_2} = 0, \tag{2b}$$

where $(\cdot,\cdot)_{\Omega_1\cup\Omega_2}$ denotes the inner product in $\Omega_1 \cup \Omega_2$ and $\langle\cdot,\cdot\rangle_\Gamma$ denotes the dual product on Γ.

2.2. Mesh

Let $\mathcal{T}_h$ be a triangulation partition of Ω, which is generated independently of the location of the interface Γ. Denote

$$\mathcal{T}_h^i := \{T \in \mathcal{T}_h : |\partial T \cap \bar{\Omega}_i| > 0\}, \qquad \Omega_h^i := \cup_{T\in\mathcal{T}_h^i} T,$$
$$\omega_h^i := \cup_{T\in\mathcal{T}_h^i, T\subset\Omega_i} T, \; i = 1, 2,$$
$$\mathcal{T}_h^\Gamma := \{T \in \mathcal{T}_h : |\bar{T} \cap \Gamma| > 0\}.$$

It is easy to check that $\omega_h^i \subset \Omega^i \subset \Omega_h^i$, and $\mathcal{T}_h^\Gamma$ is the set of elements cutting the interface Γ (see Fig. 1).

On the other hand, we define the following sets of edges for $i = 1, 2$ (see Fig. 2):

$$\mathcal{F}_{h,i} := \{F : F \cap \Omega_i, F \text{ is an edge of the element } T \text{with } T \in \Omega_h^i\},$$
$$\mathcal{G}_{h,i} := \{F : F \subset \partial T, T \in \mathcal{T}_h^\Gamma, F \cap \bar{\Omega}_i \neq \emptyset\}.$$

Moreover, letting $h = \max_T h_T$ be the global parameter of the triangulation where h_T is the diameter of the simplex $T \in \mathcal{T}_h$, we also make the following assumptions for the computational mesh as that in [16]:

- **Assumption 1**: Assume that there exist two positive constants $\check{c}$ and $\hat{c}$ such that

$$\check{c}h \le h_T \le \hat{c}h, \qquad \forall T \in \mathcal{T}_h.$$

- **Assumption 2**: Assume that Γ either intersects the boundary ∂T ($T \in \mathcal{T}_h^\Gamma$) exactly twice with each edge at most once, or that $\Gamma \cap \bar{T}$ coincides with an edge of the element.
- **Assumption 3**: Assume that $\Gamma_T = \Gamma \cap T$ is a function of the length on $\Gamma_{T,h}$ (where $\Gamma_{T,h}$ is the straight line segment connecting the points of intersection between Γ and ∂T), i.e., there hold in local coordinates

$$\Gamma_{T,h} = \{(\xi,\eta) : 0 < \xi < |\Gamma_{T,h}|, \; \eta = 0\},$$

and

$$\Gamma_T = \{(\xi,\eta) : 0 < \xi < |\Gamma_{T,h}|, \; \eta = \delta(\xi)\}.$$

- **Assumption 4**: Assume that for each $T \in \mathcal{T}_h^\Gamma$, there are elements $T^i \subset \Omega_i, i = 1, 2$ such that $\bar{T} \cap \bar{T}^i \neq \emptyset$.
- **Assumption 5**: Assume that the mesh coincides with the outer boundary $\partial\Omega$.

2.3. Numerical method

Before presenting the numerical method, we introduce some notations. First, we define the average and jump operators which will be frequently used in the following. For any $v_h \in H^s(\mathcal{T}_h)$ with $s > 1/2$, assuming that $v_h^\pm(\mathbf{x}) = \lim_{t\to 0^+} v_h(\mathbf{x} \mp t\mathbf{n})$ with $\mathbf{n}$ being a fixed unit normal to the edge F, the average and jump of v_h along F, denoted by $\{\{\cdot\}\}$ and $[\![\cdot]\!]$, are defined by, respectively

$$\begin{aligned}\{\{v_h\}\} =&\frac{1}{2}(v_h^+ + v_h^-), \qquad [\![v_h]\!] = v_h^+ - v_h^- \qquad \text{on } F \in \mathcal{F}_{h,i}\backslash\partial\Omega,\\ \{\{v_h\}\} =&v_h, \qquad [\![v_h]\!] = v_h \qquad \text{on } F \in \partial\Omega.\end{aligned}$$

And the average and jump operators across the interface Γ are defined as

$$\{q_h\}_\Gamma = \frac{1}{2}\left(q_{1h} + q_{2h}\right), \qquad [q_h]_\Gamma = q_{1h} - q_{2h},$$

where $q_{ih} = q_h|_{\Omega_i}(i = 1, 2)$. These definitions can be naturally extended to vector- or tensor-valued functions, which will be used latter without distinguishing.

Then, based on a triangulation partition of Ω shown in the above subsection, setting

$$\begin{aligned}\mathbf{C}_{h,i} =&\{\mathbf{u}_h \in [H_0^1(\Omega_h^i)]^2 \mid \mathbf{u}_h|_T \in [\mathbb{P}_1(T)]^2, \ \forall T \in \mathcal{T}_h^i\},\\ \mathbf{D}_{h,i} =&\{\mathbf{u}_h \in [L^2(\Omega_h^i)]^2 \mid \mathbf{u}_h|_T = \hat{c}\left(\mathbf{x} - \mathbf{x}_T\right), \ \forall T \in \mathcal{T}_h^i\},\end{aligned}$$

where $\mathbf{x}_T$ is the centroid of the element $T \in \mathcal{T}_h^i$ and $\hat{c}$ are unknown constants to be determined by the numerical scheme, the enriched Galerkin finite element space considered here for the velocity is defined by [7,21,32]

$$\mathbf{X}_{h,i} = \mathbf{C}_{h,i} \oplus \mathbf{D}_{h,i}.$$

Therefore, for any $\mathbf{v}_h \in \mathbf{X}_{h,i}$, one has the unique decomposition $\mathbf{v}_h = \mathbf{v}_h^C + \mathbf{v}_h^D$ with $\mathbf{v}_h^C \in \mathbf{C}_{h,i}$ and $\mathbf{v}_h^D \in \mathbf{D}_{h,i}$.

Letting

$$\begin{aligned}\mathbf{X}_h =&\{\mathbf{u}_h = (\mathbf{u}_{1h}, \mathbf{u}_{2h}) : \mathbf{u}_{ih} \in \mathbf{X}_{h,i}, i = 1, 2\},\\ Q_h =&\{p_h = (p_{1h}, p_{2h}) : p_{ih} \in Q_{h,i}, i = 1, 2\},\end{aligned}$$

where $Q_{h,i}$ is the space of piecewise constant polynomials defined on $\mathcal{T}_h^i$ with $(\nu^{-1}p_h, 1)_{\Omega_1\cup\Omega_2} = 0$, the new enriched cut finite element method investigated here is: find $(\mathbf{u}_h, p_h) \in \mathbf{X}_h \times Q_h$ such that for any $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times Q_h$,

$$A_h(\mathbf{u}_h, \mathbf{v}_h) - B_h(\mathbf{v}_h, p_h) + \epsilon_\mathbf{u} J_\mathbf{u}(\mathbf{u}_h, \mathbf{v}_h) =(\mathbf{f}, \mathbf{v}_h)_{\Omega_1\cup\Omega_2} + \langle\sigma\kappa, \{\mathbf{v}_h \cdot \mathbf{n}_\Gamma\}_\Gamma\rangle_\Gamma, \tag{3}$$

$$B_h(\mathbf{u}_h, q_h) + \epsilon_p J_p(p_h, q_h) =0, \tag{4}$$

where

$$\begin{aligned}A_h(\mathbf{u}_h, \mathbf{v}_h) =& (\nu\epsilon(\mathbf{u}_h), \epsilon(\mathbf{v}_h))_{\Omega_1\cup\Omega_2}\\ &- \sum_{i=1}^{2}\Big(\langle\{\{\nu\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}, [\![\mathbf{v}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} + \langle\{\{\nu\epsilon(\mathbf{v}_{ih})\mathbf{n}\}\}, [\![\mathbf{u}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} - \frac{\rho}{h_T}\langle\{\{\nu\}\}[\![\mathbf{u}_{ih}]\!], [\![\mathbf{v}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}}\Big)\\ &- \langle\{\nu\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma, [\mathbf{v}_h]_\Gamma\rangle_\Gamma - \langle\{\nu\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_\Gamma, [\mathbf{u}_h]_\Gamma\rangle_\Gamma + \frac{\lambda_\Gamma}{h_T}\langle\{\nu\}_\Gamma[\mathbf{u}_h]_\Gamma, [\mathbf{v}_h]_\Gamma\rangle_\Gamma,\\ B_h(\mathbf{u}_h, q_h) =& (\nabla\cdot\mathbf{u}_h, q_h)_{\Omega_1\cup\Omega_2} - \sum_{i=1}^{2}\langle\{\{q_{ih}\}\}, [\![\mathbf{u}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}} - \langle\{q_h\}_\Gamma, [\mathbf{u}_h\cdot\mathbf{n}_\Gamma]_\Gamma\rangle_\Gamma,\\ J_\mathbf{u}(\mathbf{u}_h, \mathbf{v}_h) =& \sum_{i=1}^{2} h_T\langle\nu_i[\![\nabla\mathbf{u}_{ih}\mathbf{n}]\!], [\![\nabla\mathbf{v}_{ih}\mathbf{n}]\!]\rangle_{\mathcal{G}_{h,i}},\\ J_p(p_h, q_h) =& \sum_{i=1}^{2} h_T\langle\nu_i^{-1}[\![p_{ih}]\!], [\![q_{ih}]\!]\rangle_{\mathcal{G}_{h,i}},\end{aligned}$$

and $\epsilon_\mathbf{u}$, ϵ_p, ρ and λ_Γ are positive parameters which will be determined in the following.

Remark 2.1. Here we compare the degrees of freedom for condensed enriched Galerkin finite element method (CEG) [21], MINI element (MINI) [1] and stabilized P_1/P_1 element (SP_1P_1) [4]. Both of the later two finite element methods employ the continuous $\mathbb{P}_1$ element with stabilization techniques: MINI element adds piecewise element bubble functions (a local polynomial vanishing on the element boundary: $[\text{span}\{\lambda_1\lambda_2\lambda_3\}]^2$ for 2D; $[\text{span}\{\lambda_1\lambda_2\lambda_3\lambda_4\}]^3$ for 3D); SP_1P_1 with Brezzi–Pitkaranta stabilization [4] on all the domain. The later two finite elements have been used in [6,30] for Stokes interface problems. In order to distinguish our proposed algorithm and these two inf–sup stable methods, we list the DoFs and convergence orders in Table 1 for the standard

Table 1
Comparison in DoFs with respect to the optimal order in convergence.

	DoFs	DEG($\mathbf{u}_h$)	DEG(p_h)	Optimal order	Theoretical order
CEG	(d#V) + #T	$\mathbb{P}_1$	$\mathbb{P}_0$	$\mathcal{O}(h)/\mathcal{O}(h)$	$\mathcal{O}(h)/\mathcal{O}(h)$
MINI	(d#V + d#T) + #V	2D: $\mathbb{P}_3$	$\mathbb{P}_1$	$\mathcal{O}(h^3)/\mathcal{O}(h^2)$	$\mathcal{O}(h)/\mathcal{O}(h)$
		3D: $\mathbb{P}_4$	$\mathbb{P}_1$	$\mathcal{O}(h^4)/\mathcal{O}(h^2)$	$\mathcal{O}(h)/\mathcal{O}(h)$
SP_1P_1	(d#V) + #V	$\mathbb{P}_1$	$\mathbb{P}_1$	$\mathcal{O}(h)/\mathcal{O}(h^2)$	$\mathcal{O}(h)/\mathcal{O}(h)$

Stokes equations without an interface. Denote the dimensionality by d, number of elements and vertices by #T and #V. The highest degrees of polynomial for approximating the velocity and pressure are denoted as DEG($\mathbf{u}_h$) and DEG(p_h). Besides, we also list the optimal order (H^1-error for the velocity/L^2-error for the pressure) in convergence, which is computed by the corresponding degree. For example, if the degree kth polynomial has been employed in the simulation, if the solution is smooth, the desired optimal L^2-error in convergence is at the order $\mathcal{O}(h^{k+1})$; the desired optimal H^1-error in convergence is at the order $\mathcal{O}(h^k)$. The theoretical order is the theoretical order obtained in the Refs. [4,21]. There are super-convergence results proved on the structured grids for MINI element (for example, the order for pressure is at $\mathcal{O}(h^{3/2})$ on structured grids [10]), but the super-convergence results are out of scope for this paper. We shall only consider the rigorous theoretical conclusions on the general grids for Stokes equations. Similar results can be obtained by considering the number of interface elements.

Remark 2.2. We can see that $\mathbf{u}_h \in \mathbf{X}_h, p_h \in Q_h$ are double valued on elements in $\mathcal{T}_h^\Gamma$. The discontinuity is possible at the interface Γ in this approximation.

Remark 2.3. In this method, it should be noted that all volume integrals in $A_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ are computed over physical domains Ω_i rather than Ω_h^i.

Remark 2.4. The ghost penalty term $J_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h)$ is added to ensure the coercivity of the bilinear term $A_h(\cdot,\cdot)$. The ghost penalty term $J_p(p_h, q_h)$ is added to ensure the inf–sup condition for the bilinear term $B_h(\cdot,\cdot)$. Different from the ghost penalty terms for the pressure in other conforming finite element methods (see [6,16,22,24,30]), which need to penalize the derivatives of the pressure on the edges, the new method here avoids such complicate computations and provides a simpler scheme.

Remark 2.5. In this work, we only consider the case when the curvature κ is considered to be a known quantity. In practice (e.g., for moving interfaces) it has to be computed from the level set function or other interface representations [23]. A suitable discretization of the curvature term may be needed to guarantee an optimal order in convergence.

Remark 2.6. In this work, we only consider a simple average strategy (i.e., half strategy) for simplicity. However, in the cases with anisotropic viscosity values, the weighted average may provide a better performance.

3. Well-posedness

In this section, we will analyze the well-posedness of the ECFEM proposed in the above. Before proceeding the deduction, we introduce some mesh dependent norms as follows:

$$
\begin{aligned}
|||\mathbf{u}|||^2 :=& \|\nu^{1/2}\epsilon(\mathbf{u})\|^2_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^{2}\left(h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_i)\mathbf{n}\}\}\|^2_{0,\mathcal{F}_{h,i}} + h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i]\!]\|^2_{0,\mathcal{F}_{h,i}}\right) \\
&+ h_T\|\{\nu^{1/2}\epsilon(\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma\|^2_{0,\Gamma} + h_T^{-1}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}]_\Gamma\|^2_{0,\Gamma}, \\
|||\mathbf{u}_h|||_h^2 :=& |||\mathbf{u}_h|||^2 + J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h), \\
|||(\mathbf{u},p)|||^2 :=& |||\mathbf{u}|||^2 + \|\nu^{-1/2}p\|^2_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^{2} h_T\|\{\{\nu^{-1/2}p_i\}\}\|^2_{0,\mathcal{F}_{h,i}} + h_T\|\{\nu^{-1/2}p\}_\Gamma\|^2_{0,\Gamma}, \\
|||(\mathbf{u}_h,p_h)|||_h^2 :=& |||\mathbf{u}_h|||_h^2 + \|\nu^{-1/2}p\|^2_{0,\Omega_1\cup\Omega_2} + J_p(p_h,q_h).
\end{aligned}
$$

3.1. Relation between norms

We first recall the trace inequality as follows.

Lemma 3.1 (*[15,16]*). *For any* $w \in H^1(T)$ (or $[H^1(T)]^2$) *with* $T \in \mathcal{T}_h$, *there hold*

$$\|w\|_{0,\partial T} \le \bar{c}\left(h_T^{-1/2}\|w\|_{0,T} + h_T^{1/2}\|\nabla w\|_{0,T}\right), \tag{5}$$

$$\|w\|_{0,\Gamma\cap\bar{T}} \le \bar{c}\left(h_T^{-1/2}\|w\|_{0,T} + h_T^{1/2}\|\nabla w\|_{0,T}\right), \tag{6}$$

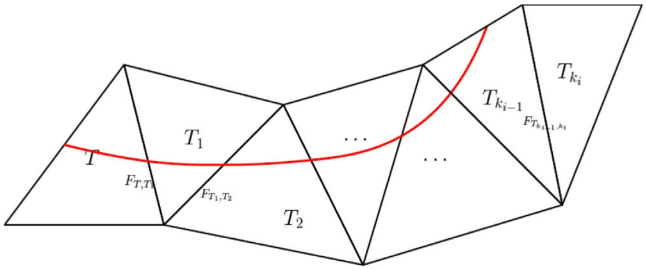


Fig. 3. Illustration of transmission edges passing from T to the adjacent element T_{k_i}.

hereafter, $\bar{c}$, c and c_i with i being an integer are general positive constants which are independent of the viscosity and the mesh, but may take different values at different occurrences.

Furthermore, for the discrete space defined above, there holds the following lemmas with respect to the inverse inequality and the partition domain.

Lemma 3.2 (*[16]*). *For any $w_h \in \mathbf{X}_h$ (or Q_h) and $T \in \mathcal{T}_h$, there hold*

$$\|\nabla w_h\|^2_{0,T} \le \bar{c} h_T^{-2} \|w_h\|^2_{0,T}, \tag{7}$$

$$\|w_h\|^2_{0,\partial T} \le \bar{c} h_T^{-1} \|w_h\|^2_{0,T}, \tag{8}$$

$$\|w_h\|^2_{0,\Gamma\cap\bar{T}} \le \bar{c} h_T^{-1} \|w_h\|^2_{0,T}. \tag{9}$$

Lemma 3.3. *For any $\mathbf{u}_h = (\mathbf{u}_{1h}, \mathbf{u}_{2h}) \in \mathbf{X}_h$, there hold*

$$\sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} \le c_1 \left(\sum_{i=1}^{2} \sum_{T\in\omega_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} + J_{\mathbf{u}}(\mathbf{u}_h, \mathbf{u}_h) \right), \tag{10}$$

$$\sum_{T\in\mathcal{T}_h^\Gamma} h_T \|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|^2_{0,\Gamma\cap\bar{T}} \le c_2 \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^\Gamma} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T}. \tag{11}$$

Proof. We prove the results following the process in, e.g., [16,28]. For any element $T \in \mathcal{T}_h^\Gamma$, let $\mathcal{E}_{T,T_{k_i}}$ be the set of transmission edges passing from T to the adjacent element $T_{k_i} \in \omega_h^i$ for $i = 1, 2$ and N_e denotes the number of transmission edges (see Fig. 3). The assumptions on the mesh guarantee that such elements T_{k_i} exist. Since $\nu_i^{1/2}\nabla\mathbf{u}_{ih}$ is a piecewise constant valued tensor, it is obvious

$$\epsilon(\mathbf{u}_{ih})|_T = \epsilon(\mathbf{u}_{ih})|_{T_{k_i}} + \sum_{F\in\mathcal{E}_{T,T_{k_i}}} \delta [\![\epsilon(\mathbf{u}_{ih})\mathbf{n}]\!]_F \mathbf{n},$$

where $\delta = \pm 1$ with the sign depending on the orientation of $\mathbf{n}$ so that the equality holds. Since the triangulation is quasi-uniform, the number N_e is bounded. Noting $\nabla\mathbf{u}_{ih}$ is a constant valued tensor in each T_{k_i}, and using Cauchy–Schwarz inequality and the geometric–arithmetic inequality, we can derive from the above equality that

$$\begin{aligned}\|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} \le& c\left(\frac{|T|}{|T_{k_i}|} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T_{k_i}} + N_e \sum_{F\in\mathcal{E}_{T,T_{k_i}}} \frac{|T|}{|F|} \|[\![\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}]\!]\|^2_{0,F} \right)\\ \le& c\left(\|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T_{k_i}} + \sum_{F\in\mathcal{E}_{T,T_{k_i}}} h_T \|[\![\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}]\!]\|^2_{0,F} \right).\end{aligned}$$

Summing over all elements in $\mathcal{T}_h^i$ $(i = 1, 2)$ gives

$$\begin{aligned}\sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} \le& c \sum_{i=1}^{2} \left(\sum_{T\in\omega_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} + \sum_{F\in\mathcal{G}_{h,i}} h_T \|\nu_i^{1/2}[\![\epsilon(\mathbf{u}_{ih})\mathbf{n}]\!]\|^2_{0,F} \right)\\ \le& c_1 \left(\sum_{i=1}^{2} \sum_{T\in\omega_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|^2_{0,T} + J_{\mathbf{u}}(\mathbf{u}_h, \mathbf{u}_h) \right).\end{aligned} \tag{12}$$

On the other hand, using the definition of the average operator on the interface Γ and (9), we get

$$h_T \|\{\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}_\Gamma\}_\Gamma\|^2_{0,\Gamma\cap\bar{T}} \le c h_T \left(\|\nu_1^{1/2}\epsilon(\mathbf{u}_{1h})\|^2_{0,\Gamma\cap\bar{T}} + \|\nu_2^{1/2}\epsilon(\mathbf{u}_{2h})\|^2_{0,\Gamma\cap\bar{T}} \right)$$

$$\leq c_2 \left(\|\nu_1^{1/2}\epsilon(\mathbf{u}_{1h})\|_{0,T}^2 + \|\nu_2^{1/2}\epsilon(\mathbf{u}_{2h})\|_{0,T}^2 \right).$$

Taking a summation over $\mathcal{T}_h^\Gamma$ in the above inequality, we complete the proof. □

Similar to the above lemma for the velocity, we can derive the following lemma for the pressure.

Lemma 3.4. *Let $p_h = (p_{1h}, p_{2h}) \in Q_h$, the following inequalities hold*

$$\sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \|\nu_i^{-1/2} p_{ih}\|_{0,T}^2 \leq c_3 \left(\sum_{i=1}^{2} \sum_{T\in\omega_h^i} \|\nu_i^{-1/2} p_{ih}\|_{0,T}^2 + J_p(p_h, p_h) \right), \tag{13}$$

$$\sum_{T\in\mathcal{T}_h^\Gamma} h_T \|\{\nu^{-1/2} p_h\}_\Gamma\|_{0,\Gamma\cap\bar{T}}^2 \leq c_4 \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^\Gamma} \|\nu_i^{-1/2} p_{ih}\|_{0,T}^2. \tag{14}$$

3.2. Property of the bilinear form $A_h(\cdot,\cdot)$

For the bilinear form $A_h(\cdot,\cdot)$ defined in Section 2, there hold the corresponding continuity and coercivity.

Lemma 3.5. *Let $\mathbf{u} \in \mathbf{X} + \mathbf{X}_h$, then there holds*

$$A_h(\mathbf{u}, \mathbf{v}_h) \leq c_5 |||\mathbf{u}||| \, |||\mathbf{v}_h|||, \qquad \forall \mathbf{v}_h \in \mathbf{X}_h. \tag{15}$$

Proof. By applying Cauchy–Schwarz inequality and noting that $\{\{\nu\epsilon(\mathbf{u})\mathbf{n}\}\} = \{\{\nu^{1/2}\epsilon(\mathbf{u})\mathbf{n}\}\}\{\{\nu\}\}^{1/2}$, we arrive at

$$\begin{aligned}
A_h(\mathbf{u}, \mathbf{v}_h) =& (\nu\epsilon(\mathbf{u}), \epsilon(\mathbf{v}_h))_{\Omega_1\cup\Omega_2} \\
&- \sum_{i=1}^{2} \left(\langle \{\{\nu\epsilon(\mathbf{u}_i)\mathbf{n}\}\}, [\![\mathbf{v}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} + \langle \{\{\nu\epsilon(\mathbf{v}_{ih})\mathbf{n}\}\}, [\![\mathbf{u}_i]\!]\rangle_{\mathcal{F}_{h,i}} - \frac{\rho}{h_T}\langle \{\{\nu\}\}[\![\mathbf{u}_i]\!], [\![\mathbf{v}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} \right) \\
&- \langle \{\nu\epsilon(\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma, [\mathbf{v}_h]_\Gamma\rangle_\Gamma - \langle \{\nu\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_\Gamma, [\mathbf{u}]_\Gamma\rangle_\Gamma + \frac{\lambda_\Gamma}{h_T}\langle \{\nu\}_\Gamma [\mathbf{u}]_\Gamma, [\mathbf{v}_h]_\Gamma\rangle_\Gamma, \\
\leq& \|\nu^{1/2}\epsilon(\mathbf{u})\|_{0,\Omega_1\cup\Omega_2} \|\nu^{1/2}\epsilon(\mathbf{v}_h)\|_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^{2} \|\{\{\nu^{1/2}\epsilon(\mathbf{u}_i)\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}} \|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}} \\
&+ \sum_{i=1}^{2} \|\{\{\nu^{1/2}\epsilon(\mathbf{v}_{ih})\}\}\|_{0,\mathcal{F}_{h,i}} \|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}} \\
&+ \sum_{i=1}^{2} \frac{\rho}{h_T} \|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}} \|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}} \\
&+ \|\{\nu^{1/2}\epsilon(\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma} \|\{\nu\}_\Gamma^{1/2}[\mathbf{v}_h]_\Gamma\|_{0,\Gamma} + \|\{\nu^{1/2}\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma} \|\{\nu\}_\Gamma^{1/2}[\mathbf{u}]_\Gamma\|_{0,\Gamma} \\
&+ \frac{\lambda_\Gamma}{h_T} \|\{\nu\}_\Gamma^{1/2}[\mathbf{u}]_\Gamma\|_{0,\Gamma} \|\{\nu\}_\Gamma^{1/2}[\mathbf{v}_h]_\Gamma\|_{0,\Gamma}
\end{aligned}$$

$$\begin{aligned}
\leq& \Bigg(\|\nu^{1/2}\epsilon(\mathbf{u})\|_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^{2} h_T^{1/2} \|\{\{\nu^{1/2}\epsilon(\mathbf{u}_i)\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}} + \sum_{i=1}^{2} h_T^{-1/2} \|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}} \\
&+ \sum_{i=1}^{2} \sqrt{\frac{\rho}{h_T}} \|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}} + h_T^{1/2} \|\{\nu^{1/2}\epsilon(\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma} \\
&+ h_T^{-1/2} \|\{\nu\}_\Gamma^{1/2}[\mathbf{u}]_\Gamma\|_{0,\Gamma} + \sqrt{\frac{\lambda_\Gamma}{h_T}} \|\{\nu\}_\Gamma^{1/2}[\mathbf{u}]_\Gamma\|_{0,\Gamma} \Bigg) \\
&\times \Bigg(\|\nu^{1/2}\epsilon(\mathbf{v}_h)\|_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^{2} h_T^{-1/2} \|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}} + \sum_{i=1}^{2} h_T^{1/2} \|\{\{\nu^{1/2}\epsilon(\mathbf{v}_{ih})\}\}\|_{0,\mathcal{F}_{h,i}} \\
&+ \sum_{i=1}^{2} \sqrt{\frac{\rho}{h_T}} \|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}} + h_T^{-1/2} \|\{\nu\}_\Gamma^{1/2}[\mathbf{v}_h]_\Gamma\|_{0,\Gamma} \\
&+ h_T^{1/2} \|\{\nu^{1/2}\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma} + \sqrt{\frac{\lambda_\Gamma}{h_T}} \|\{\nu\}_\Gamma^{1/2}[\mathbf{v}_h]_\Gamma\|_{0,\Gamma} \Bigg) \\
\leq& c_5 |||\mathbf{u}||| \, |||\mathbf{v}_h|||,
\end{aligned}$$

with $c_5 = c\max\{\sqrt{\rho}, \sqrt{\lambda_\Gamma}\}$. □

Lemma 3.6. *Assume that ρ and λ_Γ are sufficiently large positive constants, there holds*

$$c_6 |||\mathbf{u}_h|||_h^2 \le A_h(\mathbf{u}_h,\mathbf{u}_h) + \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h),\ \forall \mathbf{u}_h \in \mathbf{X}_h. \tag{16}$$

Proof. By noting the definition of $A_h(\cdot,\cdot)$, we have

$$\begin{aligned} A_h(\mathbf{u}_h,\mathbf{u}_h) =& \|\nu^{1/2}\epsilon(\mathbf{u}_h)\|_{0,\Omega_1\cup\Omega_2}^2 - \sum_{i=1}^2 \left(2\langle \{\{\nu\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}, [\![\mathbf{u}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} - \frac{\rho}{h_T}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2 \right) \\ &- 2\langle \{\nu\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma, [\mathbf{u}_h]_\Gamma\rangle_\Gamma + \frac{\lambda_\Gamma}{h_T}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|_{0,\Gamma}^2. \end{aligned} \tag{17}$$

By using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} -\sum_{i=1}^2 2\langle \{\{\nu\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}, [\![\mathbf{u}_{ih}]\!]\rangle_{\mathcal{F}_{h,i}} \le& \sum_{i=1}^2 2\left(h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 \right)^{1/2} \left(h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2 \right)^{1/2} \\ \le& \varepsilon_1 \sum_{i=1}^2 h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 + \sum_{i=1}^2 \frac{\varepsilon_1}{h_T}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2, \end{aligned}$$

and

$$\begin{aligned} -2\langle \{\nu\epsilon(\mathbf{u}_h)\mathbf{n}\}_\Gamma, [\mathbf{u}_h]_\Gamma\rangle_\Gamma \le& 2\left(h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 \right)^{1/2} \left(h_T^{-1}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|_{0,\Gamma}^2 \right)^{1/2} \\ \le& \varepsilon_2 h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 + \frac{\varepsilon_2^{-1}}{h_T}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|_{0,\Gamma}^2, \end{aligned}$$

with $\varepsilon_i (i = 1, 2)$ being positive constants. Substituting the above inequalities into (17), we get

$$\begin{aligned} A_h(\mathbf{u}_h,\mathbf{u}_h) \ge& \|\nu^{1/2}\epsilon(\mathbf{u}_h)\|_{0,\Omega_1\cup\Omega_2}^2 - \varepsilon_1 \sum_{i=1}^2 h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 + \sum_{i=1}^2 \frac{\rho-\varepsilon_1^{-1}}{h_T}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2 \\ &- \varepsilon_2 h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 + \frac{\lambda_\Gamma-\varepsilon_2^{-1}}{h_T}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|_{0,\Gamma}^2 \\ =& \|\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}\|_{0,\Omega_1\cup\Omega_2}^2 + \varepsilon_1 \sum_{i=1}^2 h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 \\ &+ \sum_{i=1}^2 \frac{\rho-\varepsilon_1^{-1}}{h_T}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2 + \varepsilon_2 h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 + \frac{\lambda_\Gamma-\varepsilon_2^{-1}}{h_T}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|_{0,\Gamma}^2 \\ &- 2\varepsilon_1 \sum_{i=1}^2 h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 - 2\varepsilon_2 h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2. \end{aligned} \tag{18}$$

Due to Lemmas 3.2 and 3.3, there holds that

$$\begin{aligned} 2\varepsilon_1 \sum_{i=1}^2 h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 \le& 2\varepsilon_1 \bar{c} \sum_{i=1}^2 \sum_{T\in\mathcal{T}_h^i} h_T\|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|_{0,T}^2 \\ \le& 2\varepsilon_1 \bar{c} c_1 \left(\sum_{i=1}^2 \sum_{T\in\omega_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|_{0,T}^2 + J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h) \right) \\ \le& 2\varepsilon_1 \bar{c} c_1 \left(\|\nu^{1/2}\epsilon(\mathbf{u}_h)\|_{0,\Omega_1\cup\Omega_2}^2 + J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h) \right), \end{aligned}$$

and

$$\begin{aligned} 2\varepsilon_2 h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 \le& 2\varepsilon_2 \sum_{T\in\mathcal{T}_h^\Gamma} h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma\cap\bar{T}}^2 \\ \le& 2\varepsilon_2 c_2 \sum_{T\in\mathcal{T}_h^\Gamma} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|_{0,T}^2 \\ \le& 2\varepsilon_2 c_2 \sum_{i=1}^2 \sum_{T\in\mathcal{T}_h^i} \|\{\{\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\}\}\|_{0,T}^2 \\ \le& 2\varepsilon_2 c_2 c_1 \left(\sum_{i=1}^2 \sum_{T\in\omega_h^i} \|\nu_i^{1/2}\epsilon(\mathbf{u}_{ih})\|_{0,T}^2 + J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h) \right) \\ \le& 2\varepsilon_2 c_2 c_1 \left(\|\nu^{1/2}\epsilon(\mathbf{u}_h)\|_{0,\Omega_1\cup\Omega_2}^2 + J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h) \right). \end{aligned}$$

Putting the above equalities into (18), we arrive at

$$\begin{aligned}A_h(\mathbf{u}_h,\mathbf{u}_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h)\geq&(1-2\varepsilon_1\bar{c}c_1-2\varepsilon_2c_2c_1)(\|\nu^{1/2}\epsilon(\mathbf{u}_h)\|^2_{0,\Omega_1\cup\Omega_2}+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h))\\&+\varepsilon_1\sum_{i=1}^{2}h_T\|\{\{\nu^{1/2}\epsilon(\mathbf{u}_{ih})\mathbf{n}\}\}\|^2_{0,\mathcal{F}_{h,i}}+\sum_{i=1}^{2}\frac{\rho-\varepsilon_1^{-1}}{h_T}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_{ih}]\!]\|^2_{0,\mathcal{F}_{h,i}}\\&+\varepsilon_2h_T\|\{\nu^{1/2}\epsilon(\mathbf{u}_h)\mathbf{n}_\Gamma\}_\Gamma\|^2_{0,\Gamma}+\frac{\lambda_\Gamma-\varepsilon_2^{-1}}{h_T}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u}_h]_\Gamma\|^2_{0,\Gamma}.\end{aligned}$$

Taking small enough $\varepsilon_i(i=1,2)$ such that $1-2\varepsilon_1\bar{c}c_1-2\varepsilon_2c_2c_1\geq c_0>0$ and large enough ρ and λ_Γ such that $\min\{\rho-\varepsilon_1^{-1},\lambda_\Gamma-\varepsilon_2^{-1}\}\geq c_0>0$, we get the result (16) with $c_6=\min\{1-2\varepsilon_1\bar{c}c_1-2\varepsilon_2c_2c_1,\varepsilon_1,\rho-\varepsilon_1^{-1},\lambda_\Gamma-\varepsilon_2^{-1},\varepsilon_2,\epsilon_{\mathbf{u}}\}$. □

3.3. Property of the bilinear form $B_h(\cdot,\cdot)$

Let the piecewise constant function be defined as

$$\bar{p}=\begin{cases}\nu_1|\Omega_1|^{-1}, & \text{in } \Omega_1,\\ -\nu_2|\Omega_2|^{-1}, & \text{in } \Omega_2,\end{cases}\tag{19}$$

and

$$M_0=\operatorname{span}\{\bar{p}\}\subset Q_h.$$

Then, there holds the decomposition (see [16,22,25])

$$p_h=p_h^0+(p_h^0)^\perp,\quad p_h^0\in M_0,\quad (p_h^0)^\perp\in M_0^\perp,$$

with

$$Q_h=M_0\oplus M_0^\perp \quad\text{and}\quad M_0^\perp:=\{(p_h^0)^\perp:\ ((p_h^0)^\perp,1)_{\Omega_i}=0\}.$$

Next, we shall prove the bilinear form $B_h(\cdot,\cdot)$ satisfies the inf–sup condition through three lemmas. We first consider $p_h^0\in M_0$ (Lemma 3.7), then $(p_h^0)^\perp\in M_0^\perp$ (Lemma 3.8), finally combine these results to obtain the desired result for $p_h\in Q_h$ (Lemma 3.9).

Lemma 3.7. *Assume that the mesh size h is sufficient small, then, for $\forall p_h^0\in M_0$, there exists $\mathbf{v}_h\in\mathbf{X}_h$, such that the following inequalities hold,*

$$B_h(\mathbf{v}_h,p_h^0)\geq c_7\|\nu^{-1/2}p_h^0\|^2_{0,\Omega_1\cup\Omega_2},\tag{20}$$

$$\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{B_h(\mathbf{v}_h,p_h^0)}{|\!|\!|\mathbf{v}_h|\!|\!|}\geq c_8\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}.\tag{21}$$

Proof. It suffices to consider

$$p_h^0=\bar{p},\tag{22}$$

which is defined in (19). We will prove the result by 4 steps.

Step 1: We shall construct a function q_h, in which the standard inf–sup condition can be applied. Setting $\tilde{p}_h^0\in Q_{h,1}\times Q_{h,2}$ as

$$\tilde{p}_h^0:=\nu^{-1}\bar{p}=\begin{cases}|\Omega_1|^{-1}, & \text{in } \Omega_1,\\ -|\Omega_2|^{-1}, & \text{in } \Omega_2,\end{cases}\tag{23}$$

and $\mathbb{I}(\tilde{p}_h^0)$ as

$$\mathbb{I}(\tilde{p}_h^0):=\begin{cases}\tilde{p}_h^0, & T\in\mathcal{T}_h^i\backslash\mathcal{T}_h^\Gamma,\\ \frac{1}{|T|}\int_T\tilde{p}_h^0d\mathbf{x}, & T\in\mathcal{T}_h^\Gamma,\end{cases}\tag{24}$$

which takes the same value as $\tilde{p}_h^0$ in $\mathcal{T}_h^i\backslash\mathcal{T}_h^\Gamma$ and takes a different single value in the interface elements in $\mathcal{T}_h^\Gamma$, then, the desired function q_h can be defined by

$$q_h:=\mathbb{I}(\tilde{p}_h^0)-\alpha,$$

where α is the average of $\mathbb{I}(\tilde{p}_h^0)$ satisfying

$$\alpha=\frac{1}{|\Omega_1\cup\Omega_2|}\left(\mathbb{I}(\tilde{p}_h^0),1\right)_{\Omega_1\cup\Omega_2}.$$

It is easy to check that $(q_h, 1)_{\Omega_1\cup\Omega_2} = 0$, thus $q_h \in L_0^2(\Omega)$. By the standard inf–sup stability of the enriched Galerkin method [21], there exists $\mathbf{v}_h \in \mathbf{X}_h$ with $\mathbf{v}_h|_{\partial\Omega} = 0$ and $[\mathbf{v}_h]_\Gamma = 0$ such that

$$|||\mathbf{v}_h||| = \|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}, \qquad B_h(\mathbf{v}_h, q_h) \geq c\|q_h\|_{0,\Omega_1\cup\Omega_2}\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}. \tag{25}$$

Step 2: We shall rewrite the bilinear form $B_h(\mathbf{v}_h, p_h^0)$ into two terms. Noting that $p_h^0 = \bar{p} \in M_0$ and (23), with a simple calculation, we can derive that

$$\|\nu^{-1/2} p_h^0\|_{0,\Omega_1\cup\Omega_2}^2 = \tilde{c}(\nu, \Omega)\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}^2, \tag{26}$$

with

$$\tilde{c}(\nu, \Omega) = \frac{\nu_1|\Omega_1|^{-1} + \nu_2|\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} \geq \max(\nu_1, \nu_2) \min_{i=1,2} \frac{|\Omega_i|^{-1}}{|\Omega_1|^{-1} + |\Omega_1|^{-1}} := \bar{\bar{c}}. \tag{27}$$

where $\bar{\bar{c}}$ is a general positive constant independent of the viscosity and the mesh. Noting the definition of the bilinear form $B_h(\cdot,\cdot)$ and the fact $[\mathbf{v}_h]_\Gamma = 0$, and using (19), (22), (24) and the integration by parts, we have

$$\begin{aligned}
B_h(\mathbf{v}_h, p_h^0) &= \sum_{i=1}^{2}\left(\sum_{T\in\mathcal{T}_h^i}\int_{T\cap\Omega_i} p_{ih}^0 \nabla\cdot\mathbf{v}_{ih} d\mathbf{x} - \langle\{\{p_{ih}^0\}\}, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}\right)\\
&= \sum_{i=1}^{2}\left((\nabla\cdot\mathbf{v}_{ih}, \nu_i|\Omega_i|^{-1})_{\Omega_i} - \langle \nu_i|\Omega_i|^{-1}, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}\right)\\
&= \sum_{i=1}^{2}\nu_i\left((\nabla\cdot\mathbf{v}_{ih}, \tilde{p}_{ih}^0)_{\Omega_i} - \langle\tilde{p}_{ih}^0, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}\right)\\
&= \sum_{i=1}^{2}\nu_i\left(\sum_{T\in\mathcal{T}_h^i}\langle\tilde{p}_{ih}^0, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{F\subset\partial T, F\cap\Omega_i} + \langle\tilde{p}_{ih}^0, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\Gamma\cap T} - \langle\tilde{p}_{ih}^0, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{F\in\mathcal{F}_{h,i}}\right)\\
&= \langle[\nu\tilde{p}_h^0]_\Gamma, \{\mathbf{v}_h\cdot\mathbf{n}_\Gamma\}_\Gamma\rangle_\Gamma.
\end{aligned} \tag{28}$$

Similarly, there holds

$$B_h(\mathbf{v}_h, \tilde{p}_h^0) = \langle[\tilde{p}_h^0]_\Gamma, \{\mathbf{v}_h\cdot\mathbf{n}_\Gamma\}_\Gamma\rangle_\Gamma. \tag{29}$$

Due to (27), combining (28) with (29) yields

$$B_h(\mathbf{v}_h, p_h^0) = \tilde{c}(\nu, \Omega)B_h(\mathbf{v}_h, \tilde{p}_h^0) = \tilde{c}(\nu, \Omega)\left(B_h(\mathbf{v}_h, \tilde{p}_h^0 - q_h) + B_h(\mathbf{v}_h, q_h)\right). \tag{30}$$

Step 3: We shall estimate $B_h(\mathbf{v}_h, p_h^0)$ through estimating $B_h(\mathbf{v}_h, q_h)$ and $B_h(\mathbf{v}_h, \tilde{p}_h^0 - q_h)$, respectively.

(I). First, noting the construction of the operator $\mathbb{I}$ in **Step 1**, we can derive that

$$\|\mathbb{I}(\tilde{p}_h^0) - \tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} \leq ch^{1/2}.$$

Thus, the average α is bounded, by noting the fact $(\tilde{p}_h^0, 1)_{\Omega_1\cup\Omega_2} = 0$, as

$$\begin{aligned}
|\alpha| &= \frac{1}{|\Omega_1\cup\Omega_2|}\left|\int_{\Omega_1\cup\Omega_2}\mathbb{I}(\tilde{p}_h^0)d\mathbf{x}\right| = \frac{1}{|\Omega_1\cup\Omega_2|}\left|\int_{\Omega_1\cup\Omega_2}(\mathbb{I}(\tilde{p}_h^0) - \tilde{p}_h^0)d\mathbf{x}\right|\\
&\leq c\left(\int_{\Omega_1\cup\Omega_2}|\mathbb{I}(\tilde{p}_h^0) - \tilde{p}_h^0|^2 d\mathbf{x}\right)^{1/2} \leq ch^{1/2}.
\end{aligned}$$

Combining the above two equalities with the triangular inequality gives

$$\|q_h - \tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} \leq \|\mathbb{I}(\tilde{p}_h^0) - \tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} + \|\alpha\|_{0,\Omega_1\cup\Omega_2} \leq ch^{1/2}. \tag{31}$$

On the other hand, it is valid that

$$\begin{aligned}
B_h(\mathbf{v}_h, \tilde{p}_h^0 - q_h) &= \sum_{i=1}^{2}\left(\sum_{T\in\mathcal{T}_h^i}\int_{T\cap\Omega_i}(\tilde{p}_{ih}^0 - q_{ih})\nabla\cdot\mathbf{v}_{ih}d\mathbf{x} - \langle\{\{\tilde{p}_{ih}^0 - q_{ih}\}\}, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}\right)\\
&:= I_1 + I_2.
\end{aligned} \tag{32}$$

For I_1, by Schwarz inequality and (31), we have

$$\begin{aligned}
|I_1| &= \left|\sum_{i=1}^{2}\sum_{T\in\mathcal{T}_h^i}\int_{T\cap\Omega_i}(\tilde{p}_{ih}^0 - q_{ih})\nabla\cdot\mathbf{v}_{ih}d\mathbf{x}\right|\\
&\leq c\|\nabla\mathbf{v}_h\|_{0,\Omega_1\cup\Omega_2}\|\tilde{p}_h^0 - q_h\|_{0,\Omega_1\cup\Omega_2} \leq ch^{1/2}\|\nabla\mathbf{v}_h\|_{0,\Omega_1\cup\Omega_2} \leq ch^{1/2}|||\mathbf{v}_h|||.
\end{aligned} \tag{33}$$

For I_2, by Schwarz inequality, Lemma 3.2, and Poincare inequality, we get

$$\begin{aligned} |I_2| =& |\sum_{i=1}^{2}\langle\{\{\tilde{p}_{ih}^0 - q_{ih}\}\}, [\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}| \\ \leq& c\left(\sum_{i=1}^{2} h_T \|\{\{\nu^{-1/2}(\tilde{p}_{ih}^0 - q_{ih})\}\}\|_{0,\mathcal{F}_{h,i}}^2\right)^{1/2}\left(\sum_{i=1}^{2} h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}}^2\right)^{1/2} \\ \leq& c\left(\sum_{i=1}^{2}\sum_{T\in\mathcal{T}_h^i}\|\tilde{p}_{ih}^0 - q_{ih}\|_{0,T}^2\right)^{1/2} |\!|\!|\mathbf{v}_h|\!|\!|. \end{aligned} \tag{34}$$

Moreover, since q_h and $\tilde{p}_h^0$ are piecewise constants, it implies

$$\begin{aligned} \|\tilde{p}_{1h}^0 - q_{1h}\|_{0,\Omega_h^1} &= \sum_{T\in\mathcal{T}_h^1}\|\tilde{p}_{1h}^0 - q_{1h}\|_{0,T} = \sum_{T\in\omega_h^1}\|\tilde{p}_{1h}^0 - q_{1h}\|_{0,T} + \sum_{T\in\Omega_h^1\backslash\omega_h^1}\|\tilde{p}_{1h}^0 - q_{1h}\|_{0,T} \\ &\leq \left(1 + \frac{|\Omega_h^1\backslash\omega_h^1|}{|\omega_h^1|}\right)\|\tilde{p}_{1h}^0 - q_{1h}\|_{0,\omega_h^1} \leq c\|\tilde{p}_{1h}^0 - q_{1h}\|_{0,\omega_h^1}, \end{aligned}$$

and

$$\|\tilde{p}_{2h}^0 - q_{2h}\|_{0,\Omega_h^2} \leq c\|\tilde{p}_{2h}^0 - q_{2h}\|_{0,\omega_h^2},$$

which suggest that

$$\begin{aligned} \|\tilde{p}_{1h}^0 - q_{1h}\|_{0,\Omega_h^1} + \|\tilde{p}_{2h}^0 - q_h\|_{0,\Omega_h^2} \leq& c\left(\|\tilde{p}_{1h}^0 - q_{2h}\|_{0,\omega_h^1} + \|\tilde{p}_{2h}^0 - q_{2h}\|_{0,\omega_h^2}\right) \\ \leq& c\|\tilde{p}_h^0 - q_h\|_{0,\Omega_1\cup\Omega_2} \leq ch^{1/2}. \end{aligned} \tag{35}$$

Putting (33)–(35) into (32) and applying the first equality in (25), we arrive at

$$B_h(\mathbf{v}_h, \tilde{p}_h^0 - q_h) \geq -(|I_1| + |I_2|) \geq -ch^{1/2}|\!|\!|\mathbf{v}_h|\!|\!| = -ch^{1/2}\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}. \tag{36}$$

(II). Second, by using the standard inf–sup condition (25), the triangular inequality and (31), we have

$$\begin{aligned} B_h(\mathbf{v}_h, q_h) \geq& c\|q_h\|_{0,\Omega_1\cup\Omega_2}\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} \geq c\left(\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} - \|\tilde{p}_h^0 - q_h\|_{0,\Omega_1\cup\Omega_2}\right)\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} \\ \geq& c\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}\left(\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} - ch^{1/2}\right) \\ \geq& c\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}^2, \end{aligned} \tag{37}$$

by assuming that the mesh size h is sufficient small.

(III). Finally, substituting (36) and (37) into (30), and using the fact that $\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}$ is a constant (from (23)), we obtain, by assuming that the mesh size h is sufficient small, that

$$\begin{aligned} B_h(\mathbf{v}_h, p_h^0) =& \tilde{c}(\nu,\Omega)\left(B_h(\mathbf{v}_h, \tilde{p}_h^0 - q_h) + B_h(\mathbf{v}_h, q_h)\right) \\ \geq& \tilde{c}(\nu,\Omega)\left(-ch^{1/2}\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} + \|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}^2\right) \\ \geq& \tilde{c}(\nu,\Omega)\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}^2\left(c - \frac{ch^{1/2}}{\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2}}\right) \\ \geq& c_7\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}^2, \end{aligned}$$

which suggests (20).

Step 4: Utilizing the first equality in (25) and (26), we get

$$\begin{aligned} |\!|\!|\mathbf{v}_h|\!|\!| \leq& c\max(\nu_1,\nu_2)^{1/2}|\!|\!|\mathbf{v}_h|\!|\!| = c\max(\nu_1,\nu_2)^{1/2}\|\tilde{p}_h^0\|_{0,\Omega_1\cup\Omega_2} \\ \leq& c\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}. \end{aligned} \tag{38}$$

Combining (20) with (38) completes the proof. □

Let $\tilde{\mathbf{X}}_h$ be an analogue space just replacing Ω_h^i with ω_h^i in $\mathbf{X}_h$ defined in Section 2. Then there holds the following lemma.

Lemma 3.8. *Assume that the mesh size h is sufficiently small, for any $(p_h^0)^\perp = ((p_{1h}^0)^\perp, (p_{2h}^0)^\perp) \in M_0^\perp$, there exists $\tilde{\mathbf{v}}_h = (\tilde{\mathbf{v}}_{1h}, \tilde{\mathbf{v}}_{2h}) \in \tilde{\mathbf{X}}_h$ such that*

$$B_h(\tilde{\mathbf{v}}_h, (p_h^0)^\perp) \geq c_9\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2}^2 - J_p((p_h^0)^\perp, (p_h^0)^\perp), \tag{39}$$

$$|\!|\!|\tilde{\mathbf{v}}_h|\!|\!| \leq c_{10}\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_1\cup\Omega_2}. \tag{40}$$

Proof. On the one hand, defining $\alpha_i = \frac{1}{|\omega_h^i|}((p_{ih}^0)^\perp, 1)_{\omega_h^i}$ and $q_{ih} = (p_{ih}^0)^\perp - \alpha_i$ $(i = 1, 2)$, there holds that $(q_{ih}, 1)_{\omega_h^i} = 0$. From the inf–sup stability of the enriched Galerkin method on $\omega_h^i \subset \Omega_i$ [21], there exists $\tilde{\mathbf{v}}_{ih} \in \tilde{\mathbf{X}}_{h,i}$ with $\text{supp}(\tilde{\mathbf{v}}_{ih}) \subset \bar{\omega}_h^i$ satisfying

$$|\!|\!|\tilde{\mathbf{v}}_{ih}|\!|\!| = \|\nu_i^{-1/2} q_{ih}\|_{0,\omega_h^i} \quad \text{and} \quad B_h(\tilde{\mathbf{v}}_{ih}, q_{ih}) \geq c\|\nu_i^{-1/2} q_{ih}\|^2_{0,\omega_h^i}. \tag{41}$$

Following the proof in Lemma 3.8, we have

$$\begin{aligned}\|\nu_i^{-1/2} q_{ih}\|^2_{0,\Omega_h^i} \leq& c\left(\|\nu_i^{-1/2} q_{ih}\|^2_{0,\omega_h^i} + J_p(q_{ih}, q_{ih})\right) \\ \leq& c\left(B_h(\tilde{\mathbf{v}}_{ih}, q_{ih}) + J_p(q_{ih}, q_{ih})\right).\end{aligned} \tag{42}$$

On the other hand, $(p_h^0)^\perp \in M_0^\perp$, it is valid that

$$J_p(q_h, q_h) = J_p((p_h^0)^\perp, (p_h^0)^\perp). \tag{43}$$

Furthermore, since $\tilde{\mathbf{v}}_{ih} \in \tilde{\mathbf{X}}_{h,i}$ satisfies $\tilde{\mathbf{v}}_{ih} = 0$ on $\partial\omega_h^i$ and $(p_{ih}^0)^\perp - q_{ih} = \alpha_i$ is a constant, it is valid that

$$B_h(\tilde{\mathbf{v}}_{ih}, q_{ih}) = B_h(\tilde{\mathbf{v}}_{ih}, (p_{ih}^0)^\perp), \tag{44}$$

and

$$\begin{aligned}|\alpha_i| =& \frac{1}{|\omega_h^i|}|((p_{ih}^0)^\perp, 1)_{\omega_h^i}| = \frac{1}{|\omega_h^i|}|((p_{ih}^0)^\perp, 1)_{\Omega_i} - ((p_{ih}^0)^\perp, 1)_{\Omega_i\backslash\omega_h^i}| \\ =& \frac{1}{|\omega_h^i|}|((p_{ih}^0)^\perp, 1)_{\Omega_i\backslash\omega_h^i}| \leq \frac{|\Omega_i\backslash\omega_h^i|^{1/2}}{|\omega_h^i|}\|(p_{ih}^0)^\perp\|_{0,\Omega_i} \\ \leq& ch^{1/2}\|(p_{ih}^0)^\perp\|_{0,\Omega_i} \leq ch^{1/2}\|(p_{ih}^0)^\perp\|_{0,\Omega_h^i},\end{aligned} \tag{45}$$

where we used $((p_{ih}^0)^\perp, 1)_{\Omega_i} = 0$ and Schwarz inequality in the above proof, which follows that

$$\begin{aligned}\|\nu_i^{-1/2} q_{ih}\|_{0,\Omega_h^i} =& \|\nu_i^{-1/2}[(p_{ih}^0)^\perp - \alpha_i]\|_{0,\Omega_h^i} \geq \|\nu_i^{-1/2}(p_{ih}^0)^\perp\|_{0,\Omega_h^i} - c|\alpha_i| \\ \geq& \|\nu_i^{-1/2}(p_{ih}^0)^\perp\|_{0,\Omega_h^i}\left(1 - ch^{1/2}\right) \geq c\|\nu_i^{-1/2}(p_{ih}^0)^\perp\|_{0,\Omega_h^i},\end{aligned}$$

by assuming that the mesh size h is sufficiently small.

Therefore, putting the above inequality into (42) and using (43) -(44), we get

$$\begin{aligned}B_h(\tilde{\mathbf{v}}_{ih}, (p_{ih}^0)^\perp) + J_p((p_{ih}^0)^\perp, (p_{ih}^0)^\perp) =& B_h(\tilde{\mathbf{v}}_{ih}, q_{ih}) + J_p(q_{ih}, q_{ih}) \\ \geq& c\|\nu_i^{-1/2} q_{ih}\|^2_{0,\Omega_h^i} \geq c_9\|\nu_i^{-1/2}(p_{ih}^0)^\perp\|^2_{0,\Omega_h^i}.\end{aligned}$$

Moreover, it is easy to check that

$$\begin{aligned}|\!|\!|\tilde{\mathbf{v}}_{ih}|\!|\!| =& \|\nu_i^{-1/2} q_{ih}\|_{0,\omega_h^i} = \|\nu_i^{-1/2}\left((p_{ih}^0)^\perp - \alpha_i\right)\|_{0,\omega_h^i} \\ \leq& \|\nu_i^{-1/2}(p_{ih}^0)^\perp\|_{0,\Omega_i}(1 + ch^{1/2}) \leq c\|\nu_i^{-1/2}(p_{ih}^0)^\perp\|_{0,\Omega_i},\end{aligned}$$

Taking a summation with respect to i from 1 to 2 in the above two inequalities completes the proof. □

Lemma 3.9. *Assume that the mesh size h is sufficiently small, for any $p_h \in Q_h$, there exists $\mathbf{v}_h \in \mathbf{X}_h$ such that*

$$B_h(\mathbf{v}_h, p_h) \geq c_{11}\|\nu^{-1/2} p_h\|^2_{0,\Omega_1\cup\Omega_2} - c_{12} J_p(p_h, p_h), \tag{46}$$

$$|\!|\!|\mathbf{v}_h|\!|\!| \leq c_{13}\|\nu^{-1/2} p_h\|_{0,\Omega_1\cup\Omega_2}. \tag{47}$$

Proof. As we define above, for any $p_h \in Q_h$, we have the unique decomposition $p_h = p_h^0 + (p_h^0)^\perp$, where $p_h^0 \in M_0$ and $(p_h^0)^\perp \in M_0^\perp$. Estimates (20) and (39) suggest that there exist $\bar{\mathbf{v}}_h \in \mathbf{X}_h, \tilde{\mathbf{v}}_h \in \tilde{\mathbf{X}}_h$ satisfying

$$B_h(\bar{\mathbf{v}}_h, p_h^0) \geq c_7\|\nu^{-1/2} p_h^0\|^2_{0,\Omega_1\cup\Omega_2}, \tag{48a}$$

$$B_h(\tilde{\mathbf{v}}_h, (p_h^0)^\perp) \geq c_9\|\nu^{-1/2}(p_h^0)^\perp\|^2_{0,\Omega_h^1\cup\Omega_h^2} - J_p((p_h^0)^\perp, (p_h^0)^\perp). \tag{48b}$$

Noting that $\tilde{\mathbf{v}}_h \in \tilde{\mathbf{X}}_h$ and p_h^0 is a constant on ω_h^i, using the integration by parts, we have

$$B_h(\tilde{\mathbf{v}}_h, p_h^0) = \sum_{i=1}^{2}\left((\nabla\cdot\tilde{\mathbf{v}}_h, p_h^0)_{\omega_h^i} - \langle\{\{p_h^0\}\}, [\![\tilde{\mathbf{v}}_h\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}}\right) = 0.$$

For $\gamma > 0$, setting $\mathbf{v}_h := \bar{\mathbf{v}}_h + \gamma\tilde{\mathbf{v}}_h \in \mathbf{X}_h$ and using the above equality and (48), we have

$$B_h(\mathbf{v}_h, p_h) = B_h(\bar{\mathbf{v}}_h, p_h^0) + B_h(\bar{\mathbf{v}}_h, (p_h^0)^\perp) + \gamma B_h(\tilde{\mathbf{v}}_h, (p_h^0)^\perp)$$

$$\geq c_7\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}^2 + B_h(\bar{\mathbf{v}}_h,(p_h^0)^\perp) + \gamma\left(c_9\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2}^2 - J_p((p_h^0)^\perp,(p_h^0)^\perp)\right). \quad (49)$$

Considering the fact $[\bar{\mathbf{v}}_h]_\Gamma = 0$, and using the definition of $B_h(\cdot,\cdot)$, the trace inequality and (38), we can derive that

$$\begin{aligned} B_h(\bar{\mathbf{v}}_h,(p_h^0)^\perp) =&(\nabla\cdot\bar{\mathbf{v}}_h,(p_h^0)^\perp)_{\Omega_1\cup\Omega_2} - \sum_{i=1}^2\langle\{\{(p_{ih}^0)^\perp\}\},[\![\bar{\mathbf{v}}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}} \\ &\geq -c|\!|\!|\bar{\mathbf{v}}_h|\!|\!|\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2} - \sum_{i=1}^2 h_T^{1/2}\|\{\{(p_{ih}^0)^\perp\}\}\|_{0,\mathcal{F}_{h,i}}h_T^{-1/2}\|[\![\bar{\mathbf{v}}_{ih}]\!]\|_{0,\mathcal{F}_{h,i}} \\ &\geq -c|\!|\!|\bar{\mathbf{v}}_h|\!|\!|\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2} \geq -c_{10}\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2}. \end{aligned} \quad (50)$$

Thus,

$$\begin{aligned} B_h(\mathbf{v}_h,p_h) \geq& c_7\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}^2 - c_{10}\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2} \\ &+ \gamma\left(c_9\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2}^2 - J_p((p_h^0)^\perp,(p_h^0)^\perp)\right) \\ \geq& \left(c_7 - \frac{c_{10}\varepsilon_3}{2}\right)\|\nu^{-1/2}p_h^0\|_{0,\Omega_1\cup\Omega_2}^2 - \gamma J_p((p_h^0)^\perp,(p_h^0)^\perp) \\ &+ \left(\gamma c_9 - \frac{c_{10}}{2\varepsilon_3}\right)\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2}^2, \end{aligned}$$

with ε_3 being a positive constant. Taking ε_3 to be small enough such that $c_7 - c_{10}\varepsilon_3/2 \geq c_0 > 0$ and γ to be large enough such that $\gamma c_9 - \frac{c_{10}}{2\varepsilon_3} \geq c_0 > 0$, and noting (43) and $\|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_h^1\cup\Omega_h^2} \geq \|\nu^{-1/2}(p_h^0)^\perp\|_{0,\Omega_1\cup\Omega_2}$, we get (46).

Finally, (38) and (40) yield

$$|\!|\!|\mathbf{v}_h|\!|\!| \leq |\!|\!|\bar{\mathbf{v}}_h|\!|\!| + \gamma|\!|\!|\tilde{\mathbf{v}}_h|\!|\!| \leq c_{13}\|\nu^{-1/2}p_h\|_{0,\Omega_1\cup\Omega_2},$$

which completes the proof. □

3.4. Existence and uniqueness

Theorem 3.10. *Let $\mathbf{u}\in\mathbf{X}+\mathbf{X}_h, \mathbf{v}_h\in\mathbf{X}_h, p\in Q+Q_h$ and $q_h\in Q_h$, then there holds*

$$A_h(\mathbf{u},\mathbf{v}_h) - B_h(\mathbf{v}_h,p) + B_h(\mathbf{u},q_h) \leq c_{14}|\!|\!|(\mathbf{u},p)|\!|\!||\!|\!|(\mathbf{v}_h,q_h)|\!|\!|. \quad (51)$$

Proof. Noting the definition of $B_h(\cdot,\cdot)$ and $[u]_\Gamma = 0$, it is valid

$$\begin{aligned} &- B_h(\mathbf{v}_h,p) + B_h(\mathbf{u},q_h) \\ =& -(\nabla\cdot\mathbf{v}_h,p)_{\Omega_1\cup\Omega_2} + \sum_{i=1}^2\langle\{\{p_i\}\},[\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}} + \langle\{p\}_\Gamma,[\mathbf{v}_h\cdot\mathbf{n}_\Gamma]_\Gamma\rangle_\Gamma \\ &+ (\nabla\cdot\mathbf{u},q_h)_{\Omega_1\cup\Omega_2} - \sum_{i=1}^2\langle\{\{q_{ih}\}\},[\![\mathbf{u}_i^2\cdot\mathbf{n}]\!]\rangle_{\mathcal{F}_{h,i}} - \langle\{q_h\}_\Gamma,[\mathbf{u}\cdot\mathbf{n}_\Gamma]_\Gamma\rangle_\Gamma \\ \leq& c\|\nu^{1/2}\epsilon(\mathbf{v}_h)\|_{0,\Omega_1\cup\Omega_2}\|\nu^{-1/2}p\|_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^2\|\{\{\nu^{-1/2}p_i\}\}\|_{0,\mathcal{F}_{h,i}}\|\{\{\nu\}\}^{1/2}[\![\mathbf{v}_{ih}\cdot\mathbf{n}]\!]\|_{0,\mathcal{F}_{h,i}} \\ &+ \|\{\nu^{-1/2}p\}_\Gamma\|_{0,\Gamma}\|\{\nu\}_\Gamma^{1/2}[\mathbf{v}_h\cdot\mathbf{n}_\Gamma]_\Gamma\|_{0,\Gamma} \\ &+ c\|\nu^{1/2}\epsilon(\mathbf{u})\|_{0,\Omega_1\cup\Omega_2}\|\nu^{-1/2}q_h\|_{0,\Omega_1\cup\Omega_2} + \sum_{i=1}^2\|\{\{\nu^{-1/2}q_{ih}\}\}\|_{0,\mathcal{F}_{h,i}}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i\cdot\mathbf{n}]\!]\|_{0,\mathcal{F}_{h,i}}, \end{aligned}$$

which combining with Lemma 3.5 yields (51). □

Theorem 3.11. *Let $(\mathbf{u}_h,p_h)\in\mathbf{X}_h\times Q_h$, and assume that the mesh size h is sufficiently small and the parameters ρ, λ_Γ and ϵ_p are sufficiently large such that Lemma 3.6 holds, then there holds*

$$\sup_{(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h}\frac{A_h(\mathbf{u}_h,\mathbf{v}_h) - B_h(\mathbf{v}_h,p_h) + B_h(\mathbf{u}_h,q_h) + \epsilon_\mathbf{u}J_\mathbf{u}(\mathbf{u}_h,\mathbf{v}_h) + \epsilon_pJ_p(p_h,q_h)}{|\!|\!|(\mathbf{v}_h,q_h)|\!|\!|_h} \geq c_{15}|\!|\!|(\mathbf{u}_h,p_h)|\!|\!|_h. \quad (52)$$

Proof. Taking $(\mathbf{v}_h,q_h) = (\mathbf{u}_h,0)$ and using Lemma 3.6, we have

$$A_h(\mathbf{u}_h,\mathbf{v}_h) - B_h(\mathbf{v}_h,p_h) + B_h(\mathbf{u}_h,q_h) + \epsilon_\mathbf{u}J_\mathbf{u}(\mathbf{u}_h,\mathbf{v}_h) + \epsilon_pJ_p(p_h,q_h)$$

$$=A_h(\mathbf{u}_h,\mathbf{u}_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h)$$
$$\geq c_6|||\mathbf{u}_h|||_h^2. \tag{53}$$

Setting $(\mathbf{v}_h,q_h)=(\mathbf{u}_h-\alpha\tilde{\mathbf{v}}_h,p_h)$ with $\alpha>0$ being a parameter and $\tilde{\mathbf{v}}_h$ satisfying the assumptions in Lemma 3.9, using (53), the inequality $J_{\mathbf{u}}(\tilde{\mathbf{v}}_h,\tilde{\mathbf{v}}_h)\leq|||\tilde{\mathbf{v}}_h|||^2$ and Lemmas 3.5 and 3.9, then we have

$$\begin{aligned}
&A_h(\mathbf{u}_h,\mathbf{v}_h)-B_h(\mathbf{v}_h,p_h)+B_h(\mathbf{u}_h,q_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{v}_h)+\epsilon_pJ_p(p_h,q_h)\\
=&A_h(\mathbf{u}_h,\mathbf{u}_h-\alpha\tilde{\mathbf{v}}_h)-B_h(\mathbf{u}_h-\alpha\tilde{\mathbf{v}}_h,p_h)+B_h(\mathbf{u}_h,p_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h-\alpha\tilde{\mathbf{v}}_h)+\epsilon_pJ_p(p_h,p_h)\\
=&A_h(\mathbf{u}_h,\mathbf{u}_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h)-\alpha(A_h(\mathbf{u}_h,\tilde{\mathbf{v}}_h)+\epsilon_{\mathbf{u}}J_{\mathbf{u}}(\mathbf{u}_h,\tilde{\mathbf{v}}_h))+\alpha B_h(\tilde{\mathbf{v}}_h,p_h)+\epsilon_pJ_p(p_h,p_h)\\
\geq&c_6|||\mathbf{u}_h|||_h^2-\alpha\max\{c_5,\epsilon_{\mathbf{u}}\}[|||\mathbf{u}_h|||+(J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{u}_h))^{1/2}][|||\tilde{\mathbf{v}}_h|||+(J_{\mathbf{u}}(\tilde{\mathbf{v}}_h,\tilde{\mathbf{v}}_h))^{1/2}]\\
&+\alpha(c_{11}\|\nu^{-1/2}p_h\|_{0,\Omega_1\cup\Omega_2}^2-c_{12}J_p(p_h,p_h))+\epsilon_pJ_p(p_h,p_h)\\
\geq&(c_6-\alpha\max\{c_5,\epsilon_{\mathbf{u}}\}/\varepsilon_4)|||\mathbf{u}_h|||_h^2+\alpha(c_{11}-2c_{13}^2\max\{c_5,\epsilon_{\mathbf{u}}\}\varepsilon_4)\|\nu^{-1/2}p_h\|_{0,\Omega_1\cup\Omega_2}^2\\
&+(\epsilon_p-\alpha c_{12})J_p(p_h,p_h),
\end{aligned} \tag{54}$$

with being a positive constant. Moreover, (47) and the triangle inequality follow by

$$|||(\mathbf{u}_h-\alpha\tilde{\mathbf{v}}_h,p_h)|||_h\leq c(1+\alpha)|||(\mathbf{u}_h,p_h)|||_h.$$

For a given $\epsilon_{\mathbf{u}}$, if ε_4 is small enough such that $\alpha(c_{11}-2c_{13}^2\max\{c_5,\epsilon_{\mathbf{u}}\}\varepsilon_4)\geq c_0>0$, α being small enough such that $c_6-\alpha\max\{c_5,\epsilon_{\mathbf{u}}\}/\varepsilon_4\geq c_0>0$ and ϵ_p being large enough such that $\epsilon_p-\alpha c_{12}\geq c_0>0$, combining the above inequality with (54) finishes the proof with $c_{15}=\min\{c_6-\alpha\max\{c_5,\epsilon_{\mathbf{u}}\}/\varepsilon_4,\alpha(c_{11}-2c_{13}^2\max\{c_5,\epsilon_{\mathbf{u}}\}\varepsilon_4),\epsilon_p-\alpha c_{12}\}/(c(1+\alpha))$. □

Remark 3.12. Theorems 3.10 and 3.11 indicate the existence and uniqueness of the numerical method (3)–(4).

4. Error analysis

In this section, we will derive the error estimate for the proposed ECFEM. In order to construct an interpolation operator, we recall the extension operators [9,16] $E^s_{\mathbf{u}_i}:[H^s(\Omega_i)]^2\to[H^s(\Omega)]^2$ and $E^s_{p_i}:H^s(\Omega_i)\to H^s(\Omega)$ $i=1,2,s\geq0$ such that

$$E^s_{\mathbf{u}_i}\mathbf{u}_i|_{\Omega_i}=\mathbf{u}_i,\quad\text{and}\quad\|E^s_{\mathbf{u}_i}\mathbf{u}_i\|_{s,\Omega}\leq c\|\mathbf{u}_i\|_{s,\Omega_i},\qquad\forall\mathbf{u}_i\in[H^s(\Omega_i)]^2, \tag{55}$$
$$E^s_{p_i}p_i|_{\Omega_i}=p_i,\quad\text{and}\quad\|E^s_{p_i}p_i\|_{s,\Omega}\leq c\|p_i\|_{s,\Omega_i},\qquad\forall p_i\in H^s(\Omega_i). \tag{56}$$

Here we shall introduce other finite element spaces. Let

$$\begin{aligned}
\mathbf{C}_h=&\{\mathbf{u}_h\in[H^1_0(\mathcal{T}_h)]^2\mid\ \mathbf{u}_h|_T\in[\mathbb{P}_1(T)]^2,\ \forall T\in\mathcal{T}_h\},\\
\mathbf{D}_h=&\{\mathbf{u}_h\in[L^2(\mathcal{T}_h)]^2\mid\mathbf{u}_h|_T=\mathbf{x}-\mathbf{x}_T,\ \forall T\in\mathcal{T}_h\},
\end{aligned}$$

where $\mathbf{x}_T$ is the centroid of the element $T\in\mathcal{T}_h$, the auxiliary finite element space is defined by

$$\hat{\mathbf{X}}_h=\mathbf{C}_h\oplus\mathbf{D}_h.$$

It can be seen that if $\hat{\mathbf{v}}_h\in\hat{\mathbf{X}}_h$, by rewriting $\mathbf{v}_h=(\hat{\mathbf{v}}_h|_{\mathcal{T}_h^1},\hat{\mathbf{v}}_h|_{\mathcal{T}_h^2})$, we have $\mathbf{v}_h\in\mathbf{X}_h$. Recall the interpolation operator $\Pi_h:[H^1(\Omega)]^2\to\hat{\mathbf{X}}_h$ defined in [21,33], and the local L^2-projection operator $\mathcal{P}_h:H^1(\Omega)\to Q_h$. There holds the following lemma.

Lemma 4.1 ([16,21]). *For the interpolation operator $\Pi_h:[H^1(\Omega)]^2\to\hat{\mathbf{X}}_h$, there hold*

$$(\nabla\cdot(\mathbf{u}-\Pi_h\mathbf{u}),1)_T=0,\qquad\forall T\in\mathcal{T}_h,\forall\mathbf{u}\in[H^1(\Omega)]^2, \tag{57}$$
$$|\mathbf{u}-\Pi_h\mathbf{u}|_{j,\Omega}\leq ch^{m-j}|\mathbf{u}|_{m,\Omega},\qquad0\leq j\leq m\leq2,\forall\mathbf{u}\in[H^2(\Omega)]^2, \tag{58}$$
$$\|\Pi_h\mathbf{u}\|_{j,\Omega}\leq c\|\mathbf{u}\|_{j,\Omega},\qquad0\leq j\leq2,\forall\mathbf{u}\in[H^1(\Omega)]^2, \tag{59}$$
$$\|p-\mathcal{P}_hp\|_{0,\Omega}\leq ch\|p\|_{1,\Omega},\qquad\forall p\in H^1(\Omega). \tag{60}$$

We define

$$\Pi^*_{ih}\mathbf{u}_i=\left(\Pi_hE^2_{\mathbf{u}_i}\mathbf{u}_i\right)|_{\Omega_h^i},\qquad\forall\mathbf{u}_i\in[H^2(\Omega_i)]^2, \tag{61}$$
$$\Pi^*_h\mathbf{u}=(\Pi^*_{1h}\mathbf{u}_1,\Pi^*_{2h}\mathbf{u}_2),\qquad\forall\mathbf{u}=(\mathbf{u}_1,\mathbf{u}_2)\ \text{ with }\ \mathbf{u}_i\in[H^2(\Omega_i)]^2, \tag{62}$$

and

$$\mathcal{P}^*_{ih}p_i=\left(\mathcal{P}_hE^1_{p_i}p_i\right)|_{\Omega_h^i},\qquad\forall p_i\in H^1(\Omega_i), \tag{63}$$
$$\mathcal{P}^*_hp=(\mathcal{P}^*_{1h}p_1,\mathcal{P}^*_{2h}p_2),\qquad\forall p=(p_1,p_2)\ \text{ with }\ p_i\in H^1(\Omega_i). \tag{64}$$

Lemma 4.2. *Let* $(\mathbf{u}, p) \in ([H_0^1(\Omega_1 \cup \Omega_2)]^2 \cap [H^2(\Omega_1 \cup \Omega_2)]^2) \times L_0^2(\Omega_1 \cup \Omega_2)$*, then there holds that*

$$|||(\mathbf{u} - \Pi_h^* \mathbf{u}, p - \mathcal{P}_h^* p)|||^2 \le c h^2 (\|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}^2 + \|\nu^{-1/2} p\|_{1,\Omega_1\cup\Omega_2}^2). \tag{65}$$

Proof. From the definition of the norm $|||\cdot|||$, we have

$$\begin{aligned}
&|||(\mathbf{u} - \Pi_h^* \mathbf{u}, p - \mathcal{P}_h^* p)|||^2 \\
=&\|\nu^{1/2}\epsilon(\mathbf{u} - \Pi_h^*\mathbf{u})\|_{0,\Omega_1\cup\Omega_2}^2 \\
&+ \sum_{i=1}^{2}\left(h_T \|\{\{\nu^{1/2}\epsilon(\mathbf{u}_i - \Pi_h^*\mathbf{u}_i)\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 + h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i - \Pi_h^*\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}}^2 \right) \\
&+ h_T \|\{\nu^{1/2}\epsilon(\mathbf{u} - \Pi_h^*\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 + h_T^{-1}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u} - \Pi_h^*\mathbf{u}]_\Gamma\|_{0,\Gamma}^2 + \|\nu^{-1/2}(p - \mathcal{P}_h^* p)\|_{0,\Omega_1\cup\Omega_2}^2 \\
&+ \sum_{i=1}^{2} h_T \|\{\{\nu^{-1/2}(p_i - \mathcal{P}_{ih}^* p_i)\}\}\|_{0,\mathcal{F}_{h,i}}^2 + h_T\|\{\nu^{-1/2}(p - \mathcal{P}_h^* p)\}_\Gamma\|_{0,\Gamma}^2.
\end{aligned} \tag{66}$$

Noting the definitions of $\{\{\cdot\}\}$, $[\![\cdot]\!]$ and Π_h^*, using Lemmas 3.1, 4.1 and (55), we obtain

$$\begin{aligned}
&h_T \|\{\{\nu_i^{1/2}\epsilon(\mathbf{u}_i - \Pi_h^*\mathbf{u}_i)\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 \\
=&\frac{1}{2} h_T \|\nu_i^{1/2}[\epsilon((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+)\mathbf{n} + \epsilon((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)\mathbf{n}]\|_{0,\mathcal{F}_{h,i}}^2 \\
\le&\frac{1}{2}\sum_{T\in\mathcal{T}_h^i} h_T \|\nu_i^{1/2}[\epsilon((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+)\mathbf{n} + \epsilon((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)\mathbf{n}]\|_{0,\partial T}^2 \\
\le&\frac{1}{2}\sum_{T\in\mathcal{T}_h^i} \Big(\|\nu_i^{1/2}[\epsilon((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) + \epsilon((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,T}^2 \\
&\qquad + h_T^2 \|\nu_i^{1/2}\nabla[\epsilon((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) + \epsilon((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,T}^2 \Big) \\
\le& c h^2 \|\nu_i^{1/2} E_{\mathbf{u}_i}^2 \mathbf{u}_i\|_{2,\Omega}^2 \\
\le& c h^2 \|\nu_i^{1/2}\mathbf{u}_i\|_{2,\Omega_i}^2,
\end{aligned}$$

and

$$\begin{aligned}
&h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i - \Pi_h^*\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}}^2 \\
=&h_T^{-1}\|\{\{\nu\}\}^{1/2}[((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) - ((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,\mathcal{F}_{h,i}}^2 \\
\le& c\sum_{T\in\mathcal{T}_h^i} h_T^{-1}\|\{\{\nu\}\}^{1/2}[((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) - ((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,\partial T}^2 \\
\le& c\sum_{T\in\mathcal{T}_h^i} \Big(h_T^{-2}\|\{\{\nu\}\}^{1/2}[((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) - ((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,T}^2 \\
&\qquad + \|\{\{\nu\}\}^{1/2}\nabla[((\mathbf{u}_i)^+ - \Pi_h^*(\mathbf{u}_i)^+) - ((\mathbf{u}_i)^- - \Pi_h^*(\mathbf{u}_i)^-)]\|_{0,T}^2 \Big) \\
\le& c h^2 \|\{\{\nu\}\}^{1/2} E_{\mathbf{u}_i}^2 \mathbf{u}_i\|_{2,\Omega}^2 \\
\le& c h^2 \|\nu_i^{1/2}\mathbf{u}_i\|_{2,\Omega_i}^2.
\end{aligned}$$

Similarly, it is valid that

$$h_T \|\{\{\nu^{-1/2}(p_i - \mathcal{P}_{ih}^* p_i)\}\}\|_{0,\mathcal{F}_{h,i}}^2 \le c h^2 \|\nu_i^{-1/2} p_i\|_{1,\Omega_i}^2.$$

The above three inequalities imply that

$$\begin{aligned}
&\sum_{i=1}^{2}\left(h_T \|\{\{\nu^{1/2}\epsilon(\mathbf{u}_i - \Pi_h^*\mathbf{u}_i)\mathbf{n}\}\}\|_{0,\mathcal{F}_{h,i}}^2 + h_T^{-1}\|\{\{\nu\}\}^{1/2}[\![\mathbf{u}_i - \Pi_h^*\mathbf{u}_i]\!]\|_{0,\mathcal{F}_{h,i}}^2 \right) \\
&\quad + \sum_{i=1}^{2} h_T \|\{\{\nu^{-1/2}(p_i - \mathcal{P}_{ih}^* p_i)\}\}\|_{0,\mathcal{F}_{h,i}}^2 \le c h^2 \|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}^2.
\end{aligned}$$

For the term with respect to the interface, we can deduce

$$\begin{aligned}
h_T^{-1}\|\{\nu\}_\Gamma^{1/2}[\mathbf{u} - \Pi_h^*\mathbf{u}]_\Gamma\|_{0,\Gamma}^2 =& h_T^{-1}\|\nu_1^{1/2}(\mathbf{u}_1 - \Pi_{1h}^*\mathbf{u}_1) - \nu_2^{1/2}(\mathbf{u}_2 - \Pi_{2h}^*\mathbf{u}_2)\|_{0,\Gamma}^2 \\
\le& c\sum_{i=1}^{2} h^{-1}\|\nu_i^{1/2}(\mathbf{u}_i - \Pi_{ih}^*\mathbf{u}_i)\|_{0,\Gamma}^2
\end{aligned}$$

$$
\begin{aligned}
&\le c \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} h_T^{-1} \|\nu_i^{1/2}(\mathbf{u}_i - \Pi_{ih}^* \mathbf{u}_i)\|_{0,\Gamma\cap\bar{T}}^2 \\
&\le c \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \left(h_T^{-2} \|\nu_i^{1/2}(\mathbf{u}_i - \Pi_{ih}^* \mathbf{u}_i)\|_{0,T}^2 + \|\nu_i^{1/2}\nabla(\mathbf{u}_i - \Pi_{ih}^* \mathbf{u}_i)\|_{0,T}^2 \right) \\
&\le c h^2 \sum_{i=1}^{2} \|\nu_i^{1/2} E_{\mathbf{u}_i}^2 \mathbf{u}_i\|_{2,\Omega}^2 \\
&\le c h^2 \|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}^2.
\end{aligned}
$$

The similar process as above follows that

$$
\begin{aligned}
h_T \|\{\nu^{1/2}\epsilon(\mathbf{u} - \Pi_h^*\mathbf{u})\mathbf{n}_\Gamma\}_\Gamma\|_{0,\Gamma}^2 &\le c h^2 \|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}^2, \\
h_T \|\{\{\nu^{-1/2}(p - \mathcal{P}_h^* p)\}\}_\Gamma\|_{0,\Gamma}^2 &\le c h^2 \|\nu^{1/2} p\|_{1,\Omega_1\cup\Omega_2}^2.
\end{aligned}
$$

Putting above inequalities into (66), and using (58) and (60), we get the result. □

Lemma 4.3. *Let $(\mathbf{u},p) \in ([H_0^1(\Omega_1\cup\Omega_2)]^2 \cap [H^2(\Omega_1\cup\Omega_2)]^2) \times L_0^2(\Omega_1\cup\Omega_2)$ and $(\mathbf{u}_h,p_h) \in \mathbf{X}_h \times Q_h$ be solutions of the problem (2) and (3)–(4), respectively, then there hold*

$$
A_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) - B_h(\mathbf{v}_h, p-p_h) = \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{X}_h, \tag{67}
$$

$$
B_h(\mathbf{u}-\mathbf{u}_h, q_h) = \epsilon_p J_p(p_h,q_h), \qquad \forall q_h \in Q_h. \tag{68}
$$

Proof. The proof is very similar to Lemma 3.1 in [16], which is omitted here. □

Theorem 4.4. *Let $(\mathbf{u},p) \in ([H_0^1(\Omega_1\cup\Omega_2)]^2 \cap [H^2(\Omega_1\cup\Omega_2)]^2) \times L_0^2(\Omega_1\cup\Omega_2)$ and $(\mathbf{u}_h,p_h) \in \mathbf{X}_h \times Q_h$ be solutions of the problems (2) and (3)–(4), respectively. Under the assumptions of Theorem 3.11, there holds*

$$
|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le c_{16} h (\|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2} + \|\nu^{-1/2}p\|_{1,\Omega_1\cup\Omega_2}). \tag{69}
$$

Proof. Let

$$
\mathbf{e}_h := \mathbf{u}-\mathbf{u}_h = (\mathbf{u} - \Pi_h^*\mathbf{u}) + (\Pi_h^*\mathbf{u} - \mathbf{u}_h) := -\xi_h + \theta_h.
$$

Using the triangle equality and the definitions of the norms and the operators Π_h^* and $\mathcal{P}_h^*$, we have

$$
|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le |||(\xi_h, p-\mathcal{P}_h^*p)||| + |||(\theta_h, \mathcal{P}_h^*p - p_h)|||_h. \tag{70}
$$

For the second term on the right hand side in the above inequality, Theorem 3.11 and Lemma 4.3 imply that

$$
\begin{aligned}
& c_{15} |||(\theta_h, \mathcal{P}_h^*p - p_h)|||_h \\
\le & \sup_{(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h} \frac{A_h(\theta_h,\mathbf{v}_h) - B_h(\mathbf{v}_h,\mathcal{P}_h^*p - p_h) + B_h(\theta_h,q_h) + \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\theta_h,\mathbf{v}_h) + \epsilon_p J_p(\mathcal{P}_h^*p - p_h, q_h)}{|||(\mathbf{v}_h,q_h)|||_h} \\
\le & \sup_{(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times Q_h} \frac{A_h(\xi_h,\mathbf{v}_h) - B_h(\mathbf{v}_h,\mathcal{P}_h^*p - p) + B_h(\xi_h,q_h) + \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\mathbf{v}_h) + \epsilon_p J_p(\mathcal{P}_h^*p, q_h)}{|||(\mathbf{v}_h,q_h)|||_h}.
\end{aligned} \tag{71}
$$

Using Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\mathbf{v}_h) &\le J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\Pi_h^*\mathbf{u})^{1/2} J_{\mathbf{u}}(\mathbf{v}_h,\mathbf{v}_h)^{1/2} \le J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\Pi_h^*\mathbf{u})^{1/2} |||(\mathbf{v}_h,q_h)|||_h, \\
J_p(\mathcal{P}_h^*p,q_h) &\le J_p(\mathcal{P}_h^*p,\mathcal{P}_h^*p)^{1/2} J_p(q_h,q_h)^{1/2} \le J_p(\mathcal{P}_h^*p,\mathcal{P}_h^*p)^{1/2} |||(\mathbf{v}_h,q_h)|||_h.
\end{aligned}
$$

Putting above estimates into (71) and using Lemma 3.5, we arrive at

$$
c_{15} |||(\theta_h, \mathcal{P}_h^*p - p_h)|||_h \le (c_5+2)\left(|||(\xi_h, p-\mathcal{P}_h^*p)||| + \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\Pi_h^*\mathbf{u})^{1/2} + \epsilon_p J_p(\mathcal{P}_h^*p,\mathcal{P}_h^*p)^{1/2} \right). \tag{72}
$$

On the other hand, for $\mathbf{u}_i \in [H_0^1(\Omega_i)]^2 \cap [H^2(\Omega_i)]^2$ and $E_{\mathbf{u}}^2 = (E_{\mathbf{u}_1}^2\mathbf{u}_1, E_{\mathbf{u}_2}^2\mathbf{u}_2)$, using Lemma 3.2, (58) and (55), we have

$$
\begin{aligned}
J_{\mathbf{u}}(\Pi_h^*\mathbf{u},\Pi_h^*\mathbf{u}) &= J_{\mathbf{u}}(E_{\mathbf{u}}^2\mathbf{u} - \Pi_h^*\mathbf{u}, E_{\mathbf{u}}^2 - \Pi_h^*\mathbf{u}) \\
&\le c \sum_{i=1}^{2} \sum_{F\in\mathcal{G}_{h,i}} h_T \|\nu_i^{1/2}\{\{\nabla(E_{\mathbf{u}_i}^2\mathbf{u}_i - \Pi_{ih}^*\mathbf{u}_i)\mathbf{n}\}\}\|_{0,F}^2 \\
&\le c \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \|\nu_i^{1/2}\{\{\nabla(E_{\mathbf{u}_i}^2\mathbf{u}_i - \Pi_{ih}^*\mathbf{u}_i)\mathbf{n}\}\}\|_{0,T}^2
\end{aligned}
$$

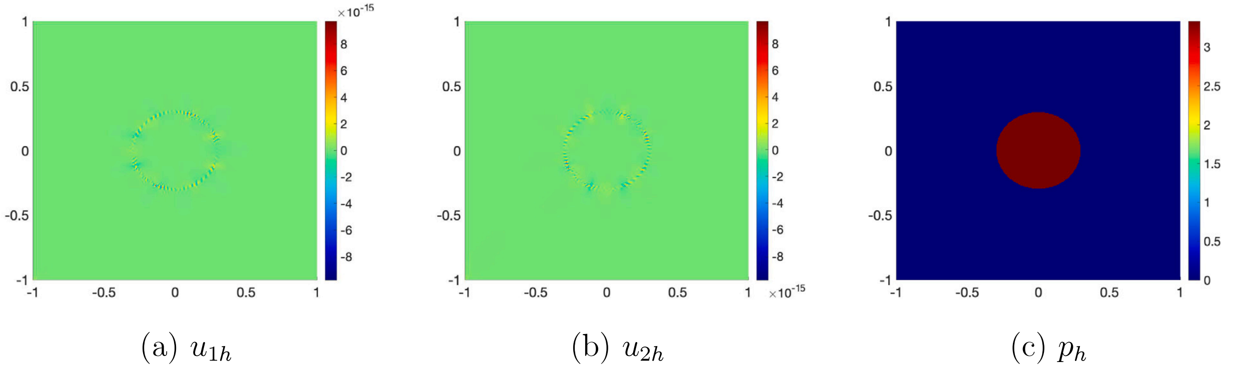


Fig. 4. Plot of the numerical velocity and pressure for Circular-type interface problem (*Case I*).

$$
\begin{aligned}
&\le ch^2 \sum_{i=1}^{2} \sum_{T\in\mathcal{T}_h^i} \|\nu_i^{1/2} E_{\mathbf{u}_i}^2 \mathbf{u}_i\|_{2,T}^2 \\
&\le ch^2 \sum_{i=1}^{2} \|\nu_i^{1/2} E_{\mathbf{u}_i}^2 \mathbf{u}_i\|_{2,\Omega}^2 \\
&\le ch^2 \sum_{i=1}^{2} \|\nu_i^{1/2} \mathbf{u}_i\|_{2,\Omega_i}^2 \le ch^2 \|\nu^{1/2}\mathbf{u}\|_{2,\Omega_1\cup\Omega_2}^2. \qquad (73)
\end{aligned}
$$

Similarly, there holds

$$J_p(\mathcal{P}_h^* p, \mathcal{P}_h^* p) = ch^2 \|\nu^{-1/2} p\|_{1,\Omega_1\cup\Omega_2}^2. \qquad (74)$$

Substituting (73) and (74) into (72) and using Lemma 4.2, we finish the proof with $c_{16} = c(c_5 + \epsilon_{\mathbf{u}} + \epsilon_p + 2)/c_{15}$. □

5. Numerical examples

In this section, we present some numerical experiments by using the proposed method (3)–(4) to verify the theoretical prediction. It should be noted that the interface is in general not available exactly. Here, the interface is represented by linear segments on $\mathcal{T}_h$ which results in an $\mathcal{O}(h^2)$ approximation of the interface Γ, and numerical integrations are conducted on the approximated sub-domains. Since we consider the enriched $\mathbb{P}_1/\mathbb{P}_0$ elements here, according to the error estimate stated in the above section, we know that this geometric approximating limitation for the interface will not effect the convergence order. The sub-domain inside Γ is denoted as Ω_1 and the sub-domain outside of Γ is denoted as Ω_2. And the errors, which we report in the numerical examples below, are all computed on the domains Ω_1 and Ω_2 that are separated by the discrete interface Γ_h. Moreover, in all tests, the stabilized parameters are taken as $\rho = 50$, $\epsilon_{\mathbf{u}} = 1E-3$, $\epsilon_p = 1$ and $\lambda_\Gamma = 50$, and the non-aligned mesh is used. In all the numerical tests, for simplicity, we discretize continuous curve interface by piecewise line segments and the associated errors for the interface elements are computed in the sub-triangles and sub-quadrilateral in the interface elements.

5.1. Circular-type interface

In this subsection, we consider a problem on the domain $\Omega = (-1,1)\times(-1,1)$ with a circular-type interface Γ. The viscosity is set to $\nu = 1$.

Case I. Let the radius $R = 0.3$, Ω_1 and Ω_2 be the sub-domains inside and outside the circular interface, and the exact solution be $\mathbf{u} = \mathbf{0}, p|_{\Omega_2} = 0, p|_{\Omega_1} = 1/R$. This corresponds to a circular fluid drop in the equilibrium with the surrounding fluid. It can be checked that the problem has an interface condition $[\mathbf{u}]_\Gamma = 0$ and a nonhomogeneous jump condition $[\nu\nabla\mathbf{u}\cdot\mathbf{n}_\Gamma - p\mathbf{n}_\Gamma]_\Gamma\cdot\mathbf{n} = [-p\mathbf{n}_\Gamma]_\Gamma\cdot\mathbf{n}_\Gamma = -1/R$. With the mesh size $h = 1/81$ and the stabilized parameters described above, we plot the simulation results in Fig. 4. We can find that the maximum values of u_{1h} and u_{2h} are $9.7715E-15$ and $9.8029E-15$ (Fig. 4(a)–(b)), which are matching $\mathbf{u} = \mathbf{0}$ within the machine accuracy. Besides, the discontinuity of the pressure inside and outside the circular drop can be well captured in this test.

Case II. Then, let the radius $R = 0.4$ and the exact solution be

$$
\mathbf{u} = \begin{cases} \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ \cos(\pi x_1)\cos(\pi x_2) \end{pmatrix} & \text{in } \Omega_1, \\ \begin{pmatrix} 4(x_1-1)^2(x_1+1)^2 x_2(x_2-1)(x_2+1) \\ -4x_1(x_1-1)(x_1+1)(x_2-1)^2(x_2+1)^2 \end{pmatrix} & \text{in } \Omega_2, \end{cases} \qquad p = 60x_1^2x_2 - 20x_2^3.
$$

Table 2
Errors and convergence orders for Circular-type interface problem (*Case II*).

#N	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$	Order	$\|\mathbf{u}-\mathbf{u}_h\|$	Order	$\|p-p_h\|$	Order
11	6.17E−01	–	1.36E−01	–	4.37E+00	–
21	3.05E−01	1.09E+00	3.45E−02	2.12E+00	2.20E+00	1.06E+00
41	1.63E−01	9.39E−01	9.48E−03	1.93E+00	1.11E+00	1.02E+00
81	8.51E−02	9.54E−01	2.14E−03	2.19E+00	5.59E−01	1.01E+00
161	4.44E−02	9.46E−01	5.27E−04	2.04E+00	2.81E−01	9.99E−01
321	2.25E−02	9.87E−01	1.30E−04	2.03E+00	1.42E−01	9.93E−01

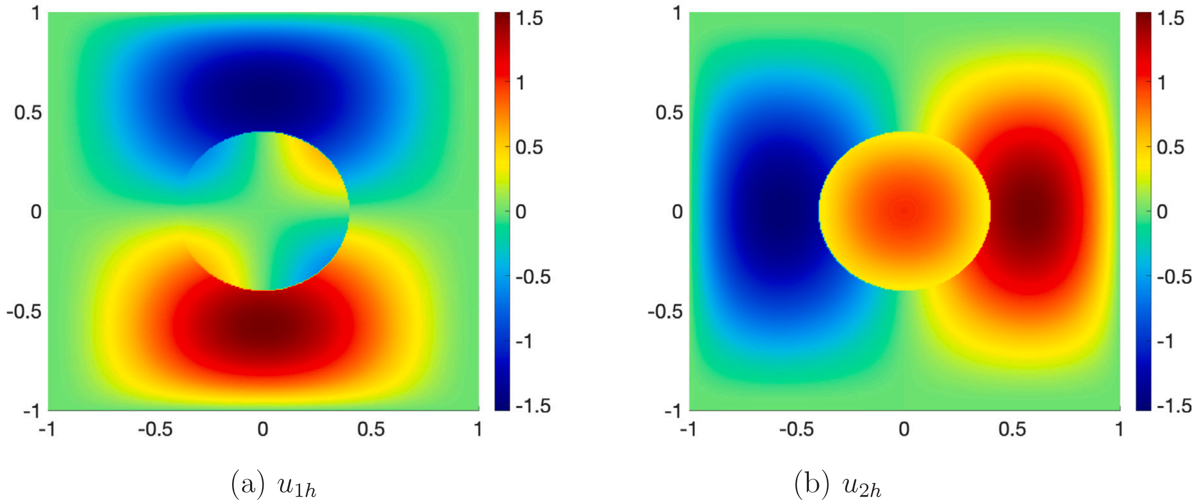


Fig. 5. Plot of the numerical velocity $\mathbf{u}_h = (u_{1h}, u_{2h})^\top$ for Circular-type interface (*Case II*).

It is easy to check that $[\mathbf{u}]_\Gamma \neq 0$ and $[(\nu\epsilon(\mathbf{u}))\,\mathbf{n}_\Gamma]_\Gamma \neq 0$, which means that this is a problem with nonhomogeneous interface conditions. Let $\Xi = [\mathbf{u}]_\Gamma$ be the discontinuity of $\mathbf{u}$ crossing Γ, then the right-hand term in (3) should be changed to

$$\begin{aligned}&A_h(\mathbf{u}_h,\mathbf{v}_h) - B_h(\mathbf{v}_h,p_h) + \epsilon_{\mathbf{u}} J_{\mathbf{u}}(\mathbf{u}_h,\mathbf{v}_h)\\ =&(\mathbf{f},\mathbf{v}_h)_{\Omega_1\cup\Omega_2} + \langle\sigma\kappa,\mathbf{v}_h\cdot\mathbf{n}_\Gamma\rangle_\Gamma - \langle\{\nu\epsilon(\mathbf{v}_h)\mathbf{n}_\Gamma\}_\Gamma,\Xi\rangle_\Gamma + \frac{\lambda_\Gamma}{h_T}\langle\{\nu\}_\Gamma\Xi,[\mathbf{v}_h]_\Gamma\rangle_\Gamma.\end{aligned}$$

The errors and the convergence results, which are reported in Table 2, indicate that the optimal rates in convergence are obtained for the velocity in H^1-seminorm and L^2-norm and the pressure in L^2-norm. These further confirm the theoretical conclusions. Furthermore, the numerical velocity is plotted in Fig. 5 with the mesh size $h = 1/161$, in which the discontinuity of the velocity is simulated very well, too.

5.2. Ellipse-type interface

In this test, we consider a problem with a ellipse-type interface as $\frac{x_1^2}{0.7^2} + \frac{x_2^2}{0.2^2} = 1$. Setting the exact solution for $\mathbf{u}$ to be the same as that in Case II in Section 5.1 and the pressure to be

$$p = \begin{cases} 10 & \text{in } \Omega_1,\\ 60x_1^2x_2 - 20x_2^3 & \text{in } \Omega_2,\end{cases}$$

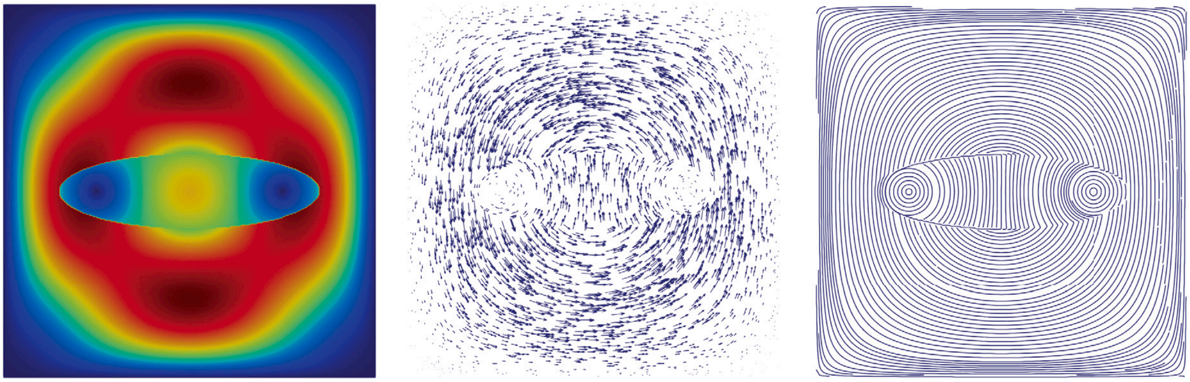
and the viscosity $\nu_1 = 0.1$ and $\nu_2 = 1$, we first collect the errors and convergence results in Table 3. Again, the optimal rates in convergence can be observed for all tested cases. Then, we plot the magnitude of the magnitude, the velocity field, and the streamline in Fig. 6, which are all in well agreement with the exact solutions.

5.3. Bean-type interface

In this example, we consider a problem on the domain $\Omega = (-1,1)\times(-1.5,1.5)$ with the interface Γ being $-r-\cos(2\theta) = 0$ (r, θ are variables in the polar coordinate). With the same exact solutions as that in the Case II in Section 5.2 and $\nu = 1$, we show the errors and convergence results in Table 4 and the numerical velocity in Fig. 7. The desired approximation has been achieved in this test, too.

Table 3
Errors and convergence orders for Ellipse-type interface problem.

#N	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$	Order	$\|\mathbf{u}-\mathbf{u}_h\|$	Order	$\|p-p_h\|$	Order
11	5.97E−01	–	3.15E−01	–	9.62E+00	–
21	3.20E−01	9.62E−01	7.19E−02	2.28E+00	6.56E+00	5.91E−01
41	1.62E−01	1.02E+00	1.71E−02	2.15E+00	2.37E+00	1.52E+00
81	8.72E−02	9.11E−01	3.99E−03	2.14E+00	1.17E+00	1.04E+00
161	4.50E−02	9.61E−01	1.01E−03	2.00E+00	5.47E−01	1.11E+00
321	2.25E−02	1.00E+00	2.46E−04	2.04E+00	2.74E−01	1.00E+00

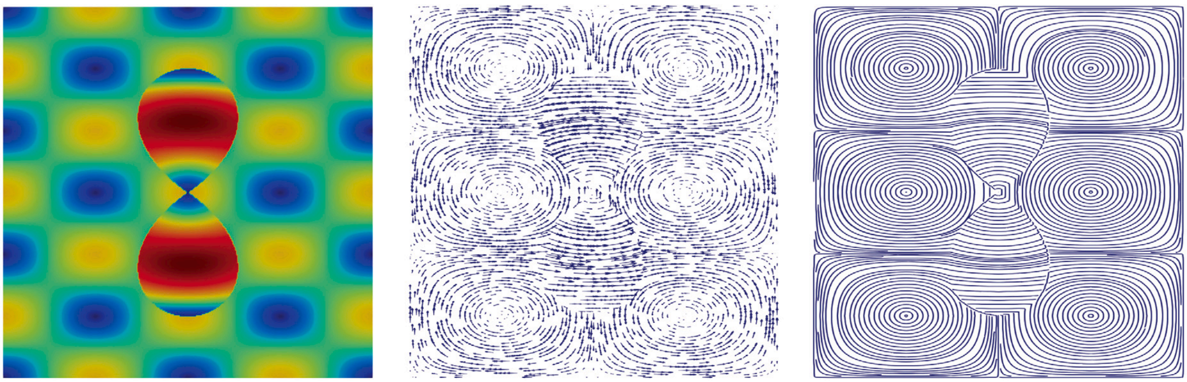


(a)Magnitude (b)Velocity field (c)Streamline

Fig. 6. Plot of the magnitude, field and streamline of the numerical velocity for Ellipse-type interface problem.

Table 4
Errors and convergence orders for Bean-type interface problem.

#N	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|$	Order	$\|\mathbf{u}-\mathbf{u}_h\|$	Order	$\|p-p_h\|$	Order
11	4.36E+00	–	7.49E−01	–	1.31E+01	–
21	2.21E+00	1.05E+00	1.48E−01	2.50E+00	5.53E+00	1.33E+00
41	9.21E−01	1.31E+00	3.85E−02	2.02E+00	5.15E+00	1.06E−01
81	3.98E−01	1.23E+00	9.68E−03	2.03E+00	2.57E+00	1.02E+00
161	1.95E−01	1.04E+00	2.52E−03	1.96E+00	1.27E+00	1.02E+00
321	9.09E−02	1.10E+00	6.34E−04	2.00E+00	6.40E−01	9.99E−01



(a)Magnitude (b)Velocity field (c)Streamline

Fig. 7. Plot of the magnitude, field and streamline of the numerical velocity for Bean-type interface problem.

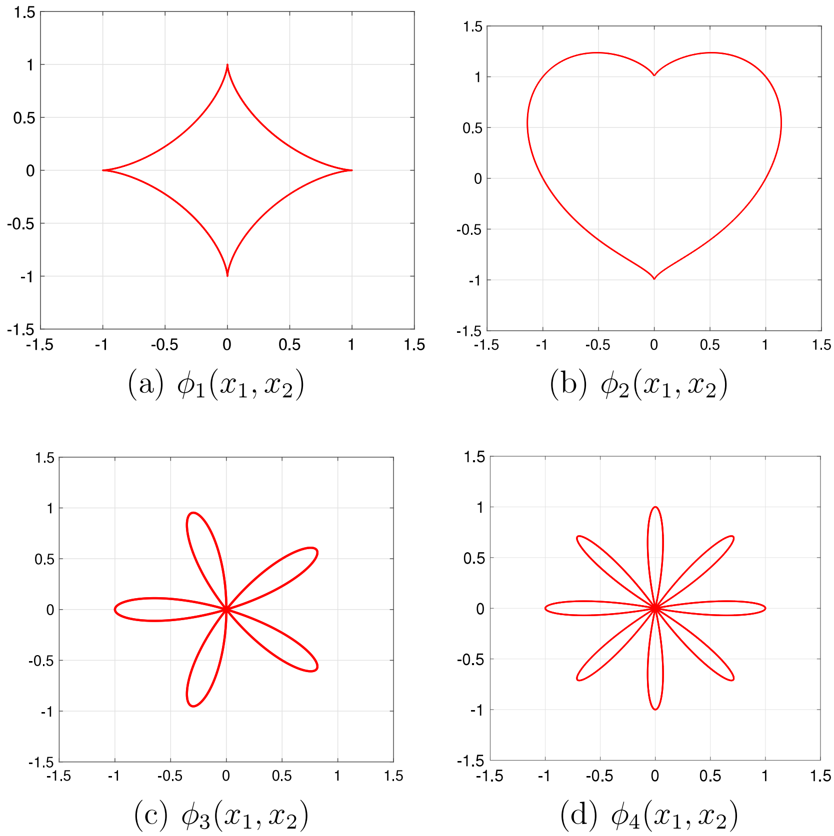


Fig. 8. Different interfaces.

5.4. Some other interfaces

Moreover, setting the domain to be $\Omega = (-1.5, 1.5) \times (-1.5, 1.5)$ and choose the exact solutions to be chosen as

$$\mathbf{u} = \begin{cases} \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ \cos(\pi x_1)\cos(\pi x_2) \end{pmatrix} + 10 & \text{in } \Omega_1, \\ \begin{pmatrix} 4(x_1-1)^2(x_1+1)^2 x_2(x_2-1)(x_2+1) \\ -4x_1(x_1-1)(x_1+1)(x_2-1)^2(x_2+1)^2 \end{pmatrix} & \text{in } \Omega_2, \end{cases} \qquad p = 60x_1^2 x_2 - 20x_2^3.$$

We investigate problems with four other interfaces, respectively. Defining the interface as (see Fig. 8)

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - |x_1^{2/3}| - |x_2^{2/3}|, \\ \phi_2(x_1, x_2) &= (x_2^2 + x_1^2 - 1)^3 - x_1^2 x_2^3, \\ \phi_3(r, \theta) &= r + \cos(5\theta), \\ \phi_4(r, \theta) &= r - \cos(8\theta), \end{aligned}$$

we present the numerical streamlines in Fig. 9. From this figure, we can observe the well agreement of the numerical solutions and exact solutions, which validate the correctness and reliability of the proposed method again.

6. Conclusions

In this work, we proposed an enriched finite element method for solving Stokes interface problems. By adding some penalty terms in the enriched $\mathbb{P}_1/\mathbb{P}_0$ finite element pair, the proposed method is able to solve the interface problem on the non-aligned grids with the optimal rate in convergence. Rigorous analysis has been established to show the well-posedness and the error estimate. Furthermore, several numerical experiments have been tested to validate the theoretical conclusions. As the future work, we may consider the related fast solver and develop the efficient pre-conditioner to further reduce the computational cost for the practical three dimensional applications.

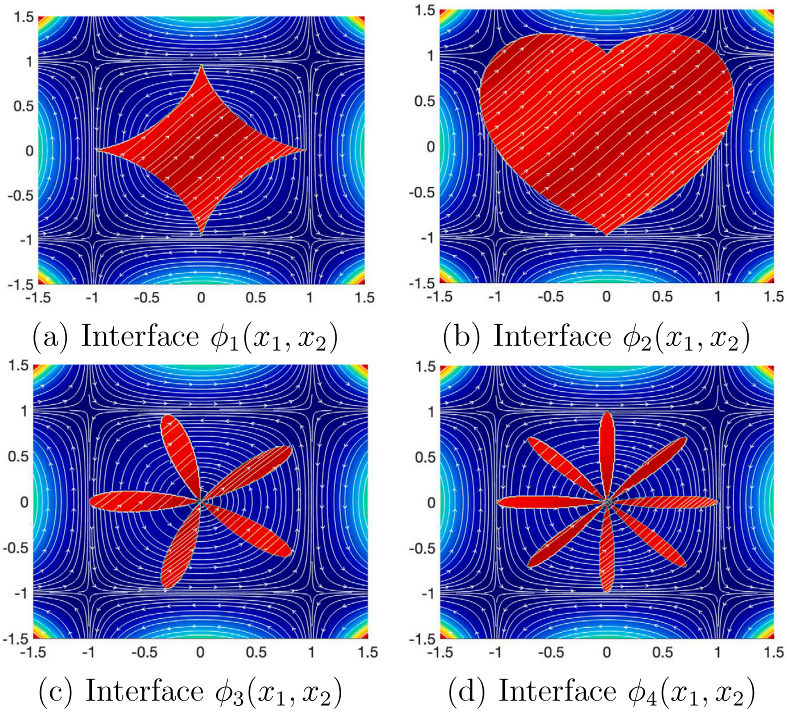


Fig. 9. Plot of streamlines of the numerical velocity with different interfaces.

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