

# A Riemannian Smoothing Steepest Descent Method for Non-Lipschitz Optimization on Embedded Submanifolds of $\mathbb{R}^n$

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**Abstract.** In this paper, we study the generalized subdifferentials and the Riemannian gradient subconsistency that are the basis for non-Lipschitz optimization on embedded submanifolds of  $\mathbb{R}^n$ . We then propose a Riemannian smoothing steepest descent method for non-Lipschitz optimization on complete embedded submanifolds of  $\mathbb{R}^n$ . We prove that any accumulation point of the sequence generated by the Riemannian smoothing steepest descent method is a stationary point associated with the smoothing function employed in the method, which is necessary for the local optimality of the original non-Lipschitz problem. We also prove that any accumulation point of the sequence generated by our method that satisfies the Riemannian gradient subconsistency is a limiting stationary point of the original non-Lipschitz problem. Numerical experiments are conducted to demonstrate the advantages of Riemannian  $\ell_p$  ( $0 < p < 1$ ) optimization over Riemannian  $\ell_1$  optimization for finding sparse solutions and the effectiveness of the proposed method.

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**Keywords:** **embedded submanifolds of  $\mathbb{R}^n$  • non-Lipschitz • generalized subdifferentials • smoothing steepest descent method • Riemannian gradient subconsistency**

## 1. Introduction

We consider the Riemannian optimization problem

$$\min f(x), \quad x \in \mathcal{M}, \quad (1)$$

where  $\mathcal{M}$  is a complete embedded submanifold of  $\mathbb{R}^n$  and  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a proper lower semicontinuous function and possibly non-Lipschitz. It is worth mentioning that the results developed in this paper also work for matrix-variable problems, that is,  $f: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ . Such problems arise in a variety of applications in signal processing, computer vision, and data mining (Adler et al. [3], Bačák et al. [7], Jiang et al. [36], Sun et al. [50]).

Many classical algorithms for unconstrained and smooth optimization have been extended from Euclidean space to Riemannian manifolds, such as the gradient descent algorithm, the conjugate gradient algorithm, the quasi-Newton algorithm, and the trust region method (Absil and Gallivan [1], Absil et al. [2], Boumal [12], Huang et al. [35]). Recently, Riemannian optimization with a nonsmooth but locally Lipschitz continuous objective function was considered in the literature. Here the smoothness and locally Lipschitz continuity are interpreted when the function in question is considered in the ambient Euclidean space. The Riemannian Clarke subdifferential of functions over manifolds has been defined, and its properties have been discussed in Hosseini and Pouryayevali [31]. Several algorithms have been proposed based on the notion of Riemannian Clarke subdifferential. For example, Hosseini and Uschmajew [32] proposed the Riemannian gradient sampling algorithm. This algorithm approximates the subdifferential using the convex hull of transported gradients from tangent spaces of randomly generated nearby points to the tangent space of the current space. The  $\epsilon$ -subgradient

algorithm (Grohs and Hosseini [30]) is a steepest descent method where the descent directions are obtained by a computable approximation of the  $\epsilon$ -subdifferential. The line search algorithms (Hosseini et al. [33]) include the nonsmooth Riemannian BFGS algorithm as a special case. For both the  $\epsilon$ -subgradient algorithm and the line search algorithms, either the algorithms terminate after a finite number of iterations with the  $\epsilon$ -subgradient-oriented descent direction being zero, or any accumulation point is a Riemannian Clarke stationary point. Other methods for nonsmooth optimization over Riemannian manifolds include the Riemannian subgradient method (Li et al. [42]), the Riemannian ADMM (Kovnatsky et al. [37], Lai and Osher [38], Li et al. [41]), the manifold proximal gradient method (Chen et al. [17, 18], Huang and Wei [34], Wang et al. [52]), the manifold proximal point method (Chen et al. [15]), the manifold proximal linear method (Wang et al. [53]), the manifold augmented Lagrangian method (Chen et al. [19], Zhou et al. [60], Zhu et al. [61]), and zeroth-order algorithms over Riemannian manifolds (Li et al. [40]).

The Riemannian generalized subdifferentials have been studied in Azagra et al. [4] and Ledyayev and Zhu [39] and are expected to be useful for analyzing non-Lipschitz optimization. To the best of our knowledge, however, there do not exist optimization algorithms for solving Riemannian non-Lipschitz optimization problems with rigorous convergence results. Consequently, the Riemannian generalized subdifferentials developed in Azagra et al. [4] and Ledyayev and Zhu [39] have not yet been used to show the convergence results for non-Lipschitz optimization. Non-Lipschitz optimization in Euclidean space finds many important applications, including, but not limited to, finding sparse solutions in signal processing and data mining (Chen et al. [24, 26], Liu et al. [43, 44], Shang et al. [48]) and neat edges in image restoration (Bian and Chen [9], Chen et al. [22], Zeng et al. [57]). Smoothing methods with a proper updating scheme for the smoothing parameter are efficient for solving large-scale nonsmooth optimization in Euclidean space (Chen [20], Chen and Zhou [21], Chen et al. [22, 23], Zhang and Chen [58, 59]). With a fixed smoothing parameter, one solves the smoothed problem to update the iterate. Certain strategy is then applied to decide whether and how the smoothing parameter needs to be changed. Under the so-called gradient consistency property, it can be shown that any accumulation point of the smoothing method is a limiting stationary point of the original nonsmooth optimization problem; see, for example, the definition in Zhang and Chen [59, p. 14]. The gradient consistency naturally holds for smoothing functions arising in various real applications with nonsmooth and locally Lipschitz objective functions (Burke and Hoheisel [13], Burke et al. [14], Chen [20], Xu et al. [55], Zhang and Chen [58]). Smoothing methods have been widely used to solve unconstrained non-Lipschitz optimization problems (Chen et al. [23]) and constrained non-Lipschitz optimization problems with convex feasible sets (Zhang and Chen [59]). However, minimizing a non-Lipschitz function on a nonconvex set has not been widely considered in the literature. In Chen et al. [26], an augmented Lagrangian method for non-Lipschitz nonconvex programming was proposed where the constraint set is nonconvex.

In Zhang and Chen [58], a smoothing projected gradient method for minimizing a nonsmooth but locally Lipschitz function on a convex feasible set in  $\mathbb{R}^n$  was proposed (Zhang and Chen [58, algorithm 3.1]). In Zhang and Chen [59], a smoothing active set method for linearly constrained non-Lipschitz nonconvex optimization in  $\mathbb{R}^n$  (Zhang and Chen [59, algorithm 3.1]) was proposed. As mentioned in remark 3.2 of Zhang and Chen [59], a unified framework of smoothing methods can be obtained by slightly modifying algorithm 3.1 of Zhang and Chen [59] with the same convergence result developed in Zhang and Chen [59], including the smoothing steepest descent method if the feasible set is  $\mathbb{R}^n$ . In this paper, the objective function is not necessarily locally Lipschitz. The Riemannian smoothing steepest descent (RSSD) method as well as the convergence analysis that will be developed in this paper extend those from Zhang and Chen [59]. The RSSD method can be considered as an extension of the smoothing steepest descent method on  $\mathbb{R}^n$  from Zhang and Chen [59] to embedded submanifolds of  $\mathbb{R}^n$ ; see Remark 4 for details.

## 1.1. Main Contributions

Our contributions of this paper are as follows:

- i. We characterize the Riemannian generalized subdifferentials for proper lower semicontinuous functions. We define the notion of limiting stationary point of (1) whose objective function is allowed to be not locally Lipschitz. When the objective function of (1) is locally Lipschitz, a limiting stationary point is a Clarke stationary point, but a Clarke stationary point is not necessarily a limiting stationary point of (1). Compared with the results in Ledyayev and Zhu [39], Proposition 2 in this paper has not been considered, and Example 2 of this paper has not been given there.
- ii. We define the Riemannian subdifferential of  $f$  associated with a smoothing function  $\tilde{f}$ . We define a stationary point  $x^*$  of (1) associated with  $\tilde{f}$  and show that  $x^*$  being a stationary point of (1) associated with  $\tilde{f}$  is a necessary optimality condition for  $x^*$  being a local minimizer of (1).

iii. To build the relationship between the above two notions of stationary points of (1), associated with or without  $\tilde{f}$ , we define the Riemannian gradient subconsistency of  $\tilde{f}$  at  $x$  on  $\mathcal{M}$ . Under the Riemannian gradient subconsistency of  $\tilde{f}$ , any stationary point of (1) associated with  $\tilde{f}$  is a limiting stationary point of (1). These concepts and results in ii and iii are extensions of the corresponding counterparts from Zhang and Chen [59] for optimization in  $\mathbb{R}^n$  to Riemannian optimization on  $\mathcal{M}$ . We show that the Riemannian gradient subconsistency holds if the gradient subconsistency of  $\tilde{f}$  holds at  $x$  on  $\mathbb{R}^n$ , provided that  $f$  is locally Lipschitz near  $x$  on  $\mathbb{R}^n$ . We also show that for a class of non-Lipschitz functions on  $\mathbb{R}^n$ , the Riemannian gradient subconsistency of their smoothing functions holds on  $\mathcal{M}$ . These two results have not been considered in the existing literature before.

iv. We design a Riemannian smoothing steepest descent method for solving (1). It is an extension of the smoothing steepest descent method in  $\mathbb{R}^n$  from Zhang and Chen [59] to embedded submanifolds of  $\mathbb{R}^n$ ; see Remark 4 for details. The proposed RSSD method is easy to implement and converges to a stationary point  $x^*$  of (1) associated with  $\tilde{f}$  where the objective function is nonsmooth, possibly not even locally Lipschitz. Under Riemannian gradient subconsistency of  $\tilde{f}$ ,  $x^*$  is also a limiting stationary point of (1).

v. When the objective function is locally Lipschitz, the convergence result of our RSSD method is stronger than that of the aforementioned existing methods for Riemannian nonsmooth optimization with locally Lipschitz objective functions. This is because these existing methods can guarantee only that any accumulation point of the sequence is a Clarke stationary point, but our result guarantees that any accumulation point of the sequence is a limiting stationary point.

The rest of this paper is organized as follows. In Section 2, we give a brief review on some basic concepts and properties related to Riemannian manifolds, the generalized subdifferentials and smoothing functions. We define the generalized subdifferentials for non-Lipschitz functions on embedded submanifolds of  $\mathbb{R}^n$  that are motivated by Azagra et al. [5] and Ledynev and Zhu [39]. In Section 3, we discuss the properties of the generalized subdifferentials for non-Lipschitz functions on embedded submanifolds of  $\mathbb{R}^n$ . We also define and discuss the Riemannian gradient subconsistency that is essential to the convergence analysis of our method. In Section 4, we propose our RSSD method and analyze its convergence behavior. In Section 5, we conduct numerical experiments on two important applications: finding the sparsest vectors in a subspace, and the sparsely used orthogonal complete dictionary learning (ODL). Finally, we draw some concluding remarks in Section 6.

## 2. Preliminaries

We define some notation first. Throughout this paper, without specification,  $\mathcal{M}$  denotes a complete embedded submanifold of  $\mathbb{R}^n$ . Let  $x \in \mathcal{M}$  and  $T_x \mathcal{M}$  be the tangent space of  $\mathcal{M}$  at  $x$ . The cotangent space at  $x$  via the Riemannian metric is denoted by  $T_x^* \mathcal{M}$ . We use  $T\mathcal{M}$  to denote the tangent bundle, that is, the disjoint union of the tangent spaces of  $\mathcal{M}$ :  $T\mathcal{M} := \{(x, v) | x \in \mathcal{M} \text{ and } v \in T_x \mathcal{M}\}$ . We consider the Riemannian metric on  $\mathcal{M}$  that is induced from the Euclidean inner product; that is, for any  $\xi, \eta \in T_x \mathcal{M}$ , we have  $\langle \xi, \eta \rangle_x = \xi^\top \eta$  if  $\xi$  and  $\eta$  are two column vectors of the same dimension, and  $\langle \xi, \eta \rangle_x = \text{Tr}(\xi^\top \eta)$  if  $\xi$  and  $\eta$  are two matrices of the same dimension, where  $\text{Tr}(Z)$  denotes the trace of matrix  $Z$ . We use  $\|x\|$  to denote the Euclidean norm when  $x$  is a vector, and the Frobenius norm when  $x$  is a matrix. We use  $B_{x, \delta} = \{y | \|y - x\| \leq \delta\}$  to represent a neighborhood of  $x$  with radius  $\delta > 0$ . For a subset  $D \subseteq \mathbb{R}^n$  with nonempty interior, a function  $h \in C^1(D)$  means that  $h$  is smooth, that is, continuously differentiable on  $D$ . For each  $x \in \mathcal{M}$ , the Riemannian metric induces an isomorphism between  $T_x \mathcal{M}$  and  $T_x^* \mathcal{M}$  through the mapping  $T_x \mathcal{M}, \exists v \mapsto v^* = \langle v, \cdot \rangle_x \in T_x^* \mathcal{M}$ . We define the norm on  $T_x^* \mathcal{M}$  by  $\|v^*\|_x^2 = \|v\|_x^2 = \langle v, v \rangle_x$ . The subscript  $x$  in  $\langle \cdot, \cdot \rangle_x$  and  $\|\cdot\|_x$  may be omitted when there is no ambiguity.

We now give the definition of the retraction operation.

**Definition 1** (Retraction; Absil et al. [2]). A retraction on a manifold  $\mathcal{M}$  is a smooth mapping  $R : T\mathcal{M} \rightarrow \mathcal{M}$  with the following two properties, where  $R_x$  denotes the restriction of  $R$  to the tangent space  $T_x \mathcal{M}$ :

- i.  $R_x(0_x) = x$ , where  $0_x$  denotes the zero element of  $T_x \mathcal{M}$ ;
- ii. it holds that

$$dR_x(0_x) = id_{T_x \mathcal{M}},$$

where  $dR_x$  is the differential of  $R_x$ , and  $id_{T_x \mathcal{M}}$  denotes the identity map on  $T_x \mathcal{M}$ .

By the inverse function theorem, we know that  $R_x$  is a local diffeomorphism (see, e.g., Hosseini et al. [33]).

### 2.1. Locally Lipschitz Functions on $\mathcal{M}$

We adopt the definition of locally Lipschitz functions on  $\mathcal{M}$  in Hosseini and Uschmajew [32]. Let  $r : [0, 1] \rightarrow \mathcal{M}$  be a  $C^1$  curve. The length of  $r$  is defined as  $l(r) = \int_0^1 \|r'(s)\| ds$ . Let  $x, y \in \mathcal{M}$ . Denote the collection of  $C^1$  curves

joining  $x$  and  $y$  by  $C(x, y)$ . Then the Riemannian distance between  $x$  and  $y$  is defined by  $\text{dist}(x, y) := \inf\{l(r) : r \in C(x, y)\}$ .

Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$  with the Riemannian distance, and let  $U$  be an open subset of  $\mathcal{M}$ . According to Hosseini and Uschmajew [32],  $f : \mathcal{M} \rightarrow \mathbb{R}$  is said to satisfy a Lipschitz condition of constant  $J$  on  $U$  if for any  $x, y \in U$  it holds that

$$|f(x) - f(y)| \leq J \text{dist}(x, y).$$

A function  $f$  is said to be Lipschitz near  $x \in \mathcal{M}$  if it satisfies the Lipschitz condition of some constant on an open neighborhood of  $x$ . A function  $f$  is said to be locally Lipschitz on  $\mathcal{M}$  if  $f$  is Lipschitz near  $x$  for every  $x \in \mathcal{M}$ .

## 2.2. Generalized Subdifferentials on $\mathbb{R}^n$

In the case that  $f$  is nonsmooth but locally Lipschitz continuous near  $x$ , the Clarke subdifferential  $\partial^0 f(x)$  of  $f$  at  $x \in \mathbb{R}^n$  is often used. Let

$$\Omega_f := \{x \in \mathbb{R}^n \mid f \text{ is differentiable at } x\}.$$

According to Clarke [27, theorem 2.5.1, p. 63], for nonsmooth but locally Lipschitz continuous function  $f$ , we have

$$\partial^0 f(x) = \text{co} \left\{ \lim_{v \rightarrow \infty} \nabla f(x_v) \mid x_v \rightarrow x, x_v \in \Omega_f \right\}, \quad (2)$$

where “co” denotes the convex hull.

We now review some important concepts and properties related to generalized subdifferentials of non-Lipschitz functions in Euclidean space  $\mathbb{R}^n$  that are often used in nonsmooth analysis (Bolte et al. [11], Rockafellar and Wets [47]).

**Definition 2** (Subdifferentials). We consider a lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . Then,

- i. the regular (or Fréchet) subdifferential of  $f$  at  $x \in \mathbb{R}^n$  is defined as

$$\hat{\partial}f(x) := \{v \mid f(y) \geq f(x) + \langle v, y - x \rangle + o(\|y - x\|)\};$$

- ii. the limiting subdifferential of  $f$  at  $x$  is defined as

$$\partial f(x) := \left\{ \lim_{v \rightarrow \infty} v_v \mid \exists (x_v, f(x_v)) \rightarrow (x, f(x)), v_v \in \hat{\partial}f(x_v) \right\}.$$

For a lower semicontinuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , according to Rockafellar and Wets [47, definition 8.3, p. 301], the horizontal subdifferential of  $f$  at  $x$  is defined as

$$\partial^\infty f(x) := \left\{ \lim_{v \rightarrow \infty} t_v v_v \mid \exists (x_v, f(x_v)) \rightarrow (x, f(x)), t_v \downarrow 0, v_v \in \hat{\partial}f(x_v) \right\}, \quad (3)$$

and according to Bolte et al. [11, definition 1], the Clarke subdifferential of a non-Lipschitz function  $f$  at  $x$  is defined as

$$\partial^0 f(x) := \bar{\text{co}} \{ \partial f(x) + \partial^\infty f(x) \}, \quad (4)$$

where “ $\bar{\text{co}}$ ” denotes the closure of convex hull.

It is known that

$$\hat{\partial}f(\bar{x}) \subseteq \partial f(\bar{x}) \subseteq \partial^0 f(\bar{x}). \quad (5)$$

We have the equivalent characterization for the regular subdifferential in the following lemma from Rockafellar and Wets [47, proposition 8.5, p. 302].

**Lemma 1.** A vector  $v \in \mathbb{R}^n$  belongs to  $\hat{\partial}f(\bar{x})$  if and only if in some neighborhood of  $\bar{x}$ , there is a function  $h \leq f$  with  $h(\bar{x}) = f(\bar{x})$  such that  $h$  is differentiable at  $\bar{x}$  with  $\nabla h(\bar{x}) = v$ . Moreover,  $h$  can be smooth with  $h(x) < f(x)$  for all  $x \neq \bar{x}$  near  $\bar{x}$ .

## 2.3. Generalized Subdifferentials on $\mathcal{M}$

Let  $h \in C^1(\mathcal{M})$ . According to Boumal [12, definition 3.34, p. 35], the differential of  $h$  at  $x$ ,  $dh(x) \in T_x^* \mathcal{M}$ , is a linear operator defined by

$$dh(x)[v] = \frac{d}{dt} h(c(t)) \Big|_{t=0}, \quad (6)$$

where  $c$  is a smooth curve on  $\mathcal{M}$  passing through  $x$  at  $t = 0$  with velocity  $v$ . By Boumal [12, definition 3.58, p. 42], the Riemannian gradient of  $h$  is the vector field  $\text{grad}h$  on  $\mathcal{M}$  uniquely defined by these identities:

$$\forall(x, v) \in T\mathcal{M}, \quad dh(x)[v] = \langle v, \text{grad}h(x) \rangle. \quad (7)$$

In the case that  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a nonsmooth but locally Lipschitz continuous function, the Riemannian Clarke subdifferential has been studied and used in analyzing the convergence of algorithms (Grohs and Hosseini [30], Hosseini and Pouryayevali [31], Hosseini and Uschmajew [32], Hosseini et al. [33], Yang et al. [56]). Let

$$\Omega_{f, \mathcal{R}} := \{x \in \mathcal{M} \mid f \text{ is differentiable at } x\}.$$

The Riemannian Clarke subdifferential, denoted by  $\partial_{\mathcal{R}}^o f(x)$ , is defined as (Hosseini and Uschmajew [32])

$$\partial_{\mathcal{R}}^o f(x) := \text{co} \left\{ \lim_{v \rightarrow \infty} \text{grad}f(x_v) \mid x_v \rightarrow x, x_v \in \Omega_{f, \mathcal{R}} \right\}. \quad (8)$$

Recall that  $\lim_{v \rightarrow \infty} \text{grad}f(x_v)$  in (8) can be explained as follows (see Hosseini and Uschmajew [32]). Let  $\{(x_v, \xi_v)\} \subseteq T\mathcal{M}$ , where  $\xi_v \in T_{x_v}\mathcal{M}$ . We say  $\xi_v$  converges to  $\xi$ , denoted by  $\lim_{v \rightarrow \infty} \xi_v = \xi$ , if  $x_v \rightarrow x$  and if for any smooth vector field  $\zeta$  on  $\mathcal{M}$ , it holds that  $\langle \xi_v, \zeta(x_v) \rangle_{x_v} \rightarrow \langle \xi, \zeta(x) \rangle_x$ . An equivalent definition of  $\partial_{\mathcal{R}}^o f(x)$  (Hosseini and Uschmajew [32]) relying on the definition of the Clarke subdifferential on linear spaces is

$$\partial_{\mathcal{R}}^o f(x) = \partial^o(f \circ R_x)(0_x) \quad (9)$$

for any retraction  $R$ .

The Riemannian regular (or Fréchet) subdifferential for lower semicontinuous functions on Riemannian manifolds was developed in Azagra et al. [5]. Later, the Riemannian regular, limiting, and horizontal subdifferentials were well studied in Ledyayev and Zhu [39]. The Riemannian generalized subdifferentials (Azagra et al. [5], Ledyayev and Zhu [39]) can be considered as natural extensions of the generalized subdifferentials of lower semicontinuous functions on  $\mathbb{R}^n$ . We will use the following definition for Riemannian subdifferentials throughout this paper.

**Definition 3** (Riemannian Subdifferentials). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous function. Then,

- i. the Riemannian regular (or Fréchet) subdifferential of  $f$  at  $x \in \mathcal{M}$  is defined as

$$\begin{aligned} \hat{\partial}_{\mathcal{R}} f(x) := & \{ \text{grad}h(x) \mid \exists \delta = \delta(h) > 0 \text{ such that } h \in C^1(B_{x, \delta}) \\ & \text{and } f - h \text{ attains a local minimum at } x \text{ on } \mathcal{M} \}; \end{aligned} \quad (10)$$

- ii. the Riemannian limiting subdifferential of  $f$  at  $x \in \mathcal{M}$  is defined as

$$\partial_{\mathcal{R}} f(x) := \left\{ \lim_{v \rightarrow \infty} v_v \mid \exists (x_v, f(x_v)) \rightarrow (x, f(x)), v_v \in \hat{\partial}_{\mathcal{R}} f(x_v) \right\}. \quad (11)$$

**Remark 1.** The Riemannian regular (or Fréchet) subdifferential of  $f$  at  $x \in \mathcal{M}$  in definition 3.1 of Ledyayev and Zhu [39] is defined as

$$\partial_F f(x) := \{dh(x) \mid h \in C^1(\mathcal{M}) \text{ and } f - h \text{ attains a local minimum at } x\}. \quad (12)$$

The Riemannian regular subdifferential  $\hat{\partial}_{\mathcal{R}} f(x)$  in this paper is essentially related only to the local property of  $h$ . By Whitney's [54] extension theorem, any smooth function on  $B_{x, \delta} \cap \mathcal{M}$  can be extended to the whole Euclidean space  $\mathbb{R}^n$ . Therefore,

$$\begin{aligned} \hat{\partial}_{\mathcal{R}} f(x) = & \{ \text{grad}h(x) \mid h \in C^1(\mathcal{M}) \text{ and } f - h \text{ attains local minimum at } x\} \\ = & \{ \text{grad}h(x) \mid dh(x) \in \partial_F f(x) \}. \end{aligned} \quad (13)$$

Hence,  $\partial_F f(x)$  in (12) and  $\hat{\partial}_{\mathcal{R}} f(x)$  in (10) are essentially the same, through the one-to-one correspondence between  $\text{grad}h(x)$  in tangent space and  $dh(x)$  in cotangent space.

In the next section, we will show that a vector in the Riemannian regular subdifferential  $\hat{\partial}_{\mathcal{R}} f(x)$  can be computed via the projection of an arbitrary vector of the regular subdifferential  $\hat{\partial}f(x)$  onto  $T_x\mathcal{M}$ , if  $\mathcal{M}$  is a Riemannian submanifold. We thus prefer to express the condition for  $h$  in (10), because such  $h \in C^1(B_{x, \delta})$  is also suitable for defining  $\hat{\partial}f(x)$ . When  $\mathcal{M} = \mathbb{R}^n$  and  $f : \mathcal{M} \rightarrow \mathbb{R}$  is a nonsmooth but locally Lipschitz continuous function, the Riemannian Clarke subdifferential coincides with the Clarke subdifferential in  $\mathbb{R}^n$ . When  $\mathcal{M} = \mathbb{R}^n$ , the Riemannian regular and limiting subdifferentials coincide with the usual regular and limiting subdifferentials in  $\mathbb{R}^n$ .

In this paper, we consider Riemannian optimization with non-Lipschitz objective function  $f$ . We will explain later that  $f$  may not be locally Lipschitz at some points on  $\mathcal{M}$ . For this purpose, we give the characterizations of

locally Lipschitz functions on  $\mathcal{M}$  that are easily checkable in Ledyaev and Zhu [39], which need the concept of convexity on  $\mathcal{M}$ . According to definition 2.5 of Azagra et al. [4], a subset  $U$  of  $\mathcal{M}$  is convex if for any given two points  $x, y \in U$ , there exists a unique geodesic in  $U$  joining  $x$  and  $y$  such that the length of the geodesic is  $\text{dist}(x, y)$ . According to theorem 2.6 in Azagra et al. [4], we know that for every  $x \in \mathcal{M}$ , there exists an open convex set  $U$  of  $\mathcal{M}$  such that  $x \in U$ . Then, according to theorem 5.3 in Ledyaev and Zhu [39] and the relations in (7) and (13), we have the following characterizations for a function on  $\mathcal{M}$  to be locally Lipschitz.

**Lemma 2.** *Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$  with the Riemannian distance. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous function. Then the following statements are equivalent:*

- i.  $f$  is locally Lipschitz near  $x$  on  $\mathcal{M}$ ;
- ii.  $\hat{\partial}_R f$  is bounded in a neighborhood of  $x$  on  $\mathcal{M}$ .

## 2.4. Smoothing Function

We use the following definition of a smoothing function on  $\mathbb{R}^n$  as in Zhang and Chen [59].

**Definition 4** (Smoothing Function). A function  $\tilde{f}(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is called a smoothing function of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  if  $\tilde{f}(\cdot, \mu)$  is continuously differentiable in  $\mathbb{R}^n$  for any  $\mu \in \mathbb{R}_{++}$ ,

$$\lim_{z \rightarrow x, \mu \downarrow 0} \tilde{f}(z, \mu) = f(x), \quad (14)$$

and there exist a constant  $\kappa > 0$  and a function  $\omega : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  such that

$$|\tilde{f}(x, \mu) - f(x)| \leq \kappa \omega(\mu) \quad \text{with} \quad \lim_{\mu \downarrow 0} \omega(\mu) = 0. \quad (15)$$

In order to emphasize that  $\mu$  is a smoothing parameter, we sometimes also write  $\tilde{f}(\cdot, \mu)$  as  $\tilde{f}_\mu(\cdot)$  in this paper.

**Example 1.** We use the absolute value function  $|t|, t \in \mathbb{R}$  as an example to illustrate the smoothing function. We can use the so-called uniform smoothing function

$$s_\mu(t) = \begin{cases} |t| & \text{if } |t| \geq \frac{\mu}{2}, \\ \frac{t^2}{\mu} + \frac{\mu}{4} & \text{if } |t| < \frac{\mu}{2}, \end{cases} \quad (16)$$

with  $\kappa = 1/4$  and  $\omega(\mu) = \mu$  in (15).

We refer to Chen [20] for more examples of smoothing functions. For the non-Lipschitz term  $|t|^p$  where  $0 < p < 1$ , its smoothing function can be defined as  $(s_\mu(t))^p$ , with  $\kappa = (1/4)^p$  and  $\omega(\mu) = \mu^p$  in (15).

## 3. Riemannian Generalized Subdifferentials and Riemannian Gradient Subconsistency

In this section, we first discuss properties of several generalized subdifferentials. We then define and discuss properties of Riemannian gradient subconsistency of proper lower semicontinuous functions and related stationary points of (1). These concepts and properties play important roles in the convergence analysis of our RSSD method in the next section. They also provide some basics for minimizing a non-Lipschitz function on an embedded submanifold of  $\mathbb{R}^n$ .

### 3.1. Riemannian Generalized Subdifferentials

**Proposition 1.** *Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$ ,  $x \in \mathcal{M}$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous function. Suppose  $R : T\mathcal{M} \rightarrow \mathcal{M}$  is a retraction defined in Definition 1. Then*

- i.  $\hat{\partial}_R f(x) = \hat{\partial}(f \circ R_x)(0_x)$  and  $\partial_R f(x) = \partial(f \circ R_x)(0_x)$ ;
- ii.  $v \in \hat{\partial}_R f(x)$  if and only if  $v \in T_x \mathcal{M}$  and the following holds:

$$f \circ R_x(\eta_x) \geq f \circ R_x(0_x) + \langle v, \eta_x \rangle + o(\|\eta_x\|), \quad \forall \eta_x \in T_x \mathcal{M}. \quad (17)$$

**Proof.** Statement i for  $\hat{\partial}_R f(x)$  holds, according to theorem 4.3 of Azagra et al. [5], corollary 4.2 of Ledyaev and Zhu [39], Definition 1 for retraction  $R : T\mathcal{M} \rightarrow \mathcal{M}$ , and Remark 1 that  $\hat{\partial}_R f(x)$  in Definition 3 of this paper and  $\partial_F(x)$  given in definition 3.1 of Ledyaev and Zhu [39] are essentially the same. The equivalent characterization of  $\hat{\partial}_R f(x)$  in statement ii can be easily obtained from statement i.  $\square$

Now we give the following proposition about Riemannian regular subdifferential that is useful for computation and theoretical analysis; see the employment of Proposition 2 in Equation (37) of Example 4, in Equation (39) of Remark 3, and in Equations (34) and (52) in the proofs of Theorem 2 and 3, respectively.

**Proposition 2.** *Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$ ,  $x \in \mathcal{M}$ , and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a lower semicontinuous function. Then*

$$\{\text{Proj}_{T_x \mathcal{M}} v \mid v \in \hat{\partial} f(x)\} \subseteq \hat{\partial}_{\mathcal{R}} f(x). \quad (18)$$

**Proof.** Using (10) in Definition 3 and that  $\mathcal{M}$  is a submanifold embedded in  $\mathbb{R}^n$  and  $h \in C^1(B_{x,\delta})$ , we have

$$\text{grad } h(x) = \text{Proj}_{T_x \mathcal{M}} \nabla h(x),$$

where  $\text{Proj}_{T_x \mathcal{M}} y$  denotes the projection of  $y \in \mathbb{R}^n$  onto  $T_x \mathcal{M}$ . Consequently,

$$\begin{aligned} \hat{\partial}_{\mathcal{R}} f(x) &= \{\text{Proj}_{T_x \mathcal{M}} \nabla h(x) \mid \exists \delta > 0 \text{ such that } h \in C^1(B_{x,\delta}) \text{ and} \\ &\quad f - h \text{ attains a local minimum at } x \text{ on } \mathcal{M}\}. \end{aligned} \quad (19)$$

Note that for any  $v \in \hat{\partial} f(x)$ , according to Lemma 1, there exists  $h \in C^1$ , such that  $f - h$  attains a local minimum at  $x$  on  $\mathbb{R}^n$ , which is sure to attain a local minimum at  $x$  on  $\mathcal{M} \subseteq \mathbb{R}^n$ . This, combining with (19), indicates that (18) holds.  $\square$

**Definition 5.** A point  $x \in \mathcal{M}$  is called a limiting stationary point of (1) if  $0 \in \partial_{\mathcal{R}} f(x)$ .

According to Proposition 1ii, we know that if  $\bar{x}$  is a local minimizer of  $f$  on  $\mathcal{M}$ , then  $0 \in \hat{\partial}_{\mathcal{R}} f(\bar{x})$ . By Definition 3, we have  $\hat{\partial}_{\mathcal{R}} f(\bar{x}) \subseteq \partial_{\mathcal{R}} f(\bar{x})$ . Hence,  $\bar{x}$  being a limiting stationary point of (1) is a necessary condition of  $f$  achieving a local minimum at  $\bar{x}$  on  $\mathcal{M}$ .

For a locally Lipschitz function  $f$ ,  $x \in \mathcal{M}$  is a Clarke stationary point of (1) if  $0 \in \partial_{\mathcal{R}}^o f(x)$ . The Clarke stationary point of (1) is widely used in the nonsmooth but locally Lipschitz Riemannian optimization literature (Grohs and Hosseini [30], Hosseini and Pouryayevali [31], Hosseini and Uschmajew [32], Hosseini et al. [33]). We show in the next proposition that a limiting stationary point is a Clarke stationary point.

**Proposition 3.** *Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$ , and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a locally Lipschitz function near  $x \in \mathcal{M}$ . Then  $\partial_{\mathcal{R}} f(x) \subseteq \partial_{\mathcal{R}}^o f(x)$ .*

**Proof.** The inclusion holds because

$$\partial_{\mathcal{R}}^o f(x) = \partial^o(f \circ R_x)(0_x) \supseteq \partial(f \circ R_x)(0_x) = \partial_{\mathcal{R}} f(x).$$

The first equality is due to (9), which transforms the Riemannian Clarke subdifferential of  $f$  at  $x$  to be the Clarke subdifferential of  $f \circ R_x$  at  $0_x$  on the Euclidean space  $T_x \mathcal{M}$ . The inclusion comes from (5). The last equality is obtained from Proposition 1i.  $\square$

We use the following example to show that for  $f$  being a locally Lipschitz function on  $\mathbb{R}^n$  and  $\mathcal{M}$  being an embedded submanifold of  $\mathbb{R}^n$ , a Clarke stationary point is not necessarily a limiting stationary point.

**Example 2.** Let us consider the Riemannian optimization problem

$$\min f(x_1, x_2) := \frac{1}{2} x_1^2 - x_1 - |x_2|, \quad x \in \mathcal{M}, \quad (20)$$

where  $\mathcal{M} = S^1 := \{x \in \mathbb{R}^2 \mid x^\top x = 1\}$  is the unit circle, and  $f$  is locally Lipschitz in  $\mathbb{R}^2$ . Let  $\bar{x} = (1, 0)^\top$  and  $\bar{x}_\epsilon = (\sqrt{1 - \epsilon^2}, \epsilon)^\top$ . It is clear that  $\bar{x}_\epsilon \rightarrow \bar{x}$  when  $\epsilon \rightarrow 0$ ,  $\|\bar{x}\| = \|\bar{x}_\epsilon\| = 1$ , and for any  $\epsilon \in (0, 1)$ ,

$$\begin{aligned} f(\bar{x}_\epsilon) &= \frac{1}{2}(1 - \epsilon^2) - \sqrt{1 - \epsilon^2} - \epsilon \\ &< \frac{1}{2} - (\sqrt{1 - \epsilon^2} + \epsilon) \\ &< \frac{1}{2} - 1 = -\frac{1}{2} = f(\bar{x}). \end{aligned}$$

Hence,  $\bar{x}$  is not a local minimizer of  $f$  on  $S^1$ .

For  $\mathcal{M} = S^1$ , we know from Absil et al. [2] that

$$\text{Proj}_{T_x \mathcal{M}} \xi = (I - xx^\top) \xi, \quad T_x \mathcal{M} = \{z \mid x^\top z = 0\}, \quad (21)$$

and for any  $x_\ell \in \Omega_{f, \mathcal{R}}$ ,

$$\text{grad}f(x_\ell) = \text{Proj}_{T_{x_\ell}\mathcal{M}} \nabla f(x_\ell) = (I - x_\ell x_\ell^\top) \nabla f(x_\ell).$$

By (8), we can calculate that

$$0 \in \partial_{\mathcal{R}}^0 f(\bar{x}) = \{(0, t)^\top \mid \forall t \in [-1, 1]\},$$

which indicates that  $\bar{x}$  is a Clarke stationary point of (20).

Using Proposition 1ii, we know that  $\hat{\partial}_{\mathcal{R}} f(\bar{x}) = \emptyset$ . Using Definition 3, and noting that there exists a neighborhood  $B_{\bar{x}, \delta}$  for some  $\delta > 0$ , such that  $f$  is continuously differentiable at any  $\bar{x}_\ell \neq \bar{x}$  in  $B_{\bar{x}, \delta} \cap S^1$ , we have

$$0 \notin \partial_{\mathcal{R}} f(\bar{x}) := \{(0, 1)^\top, (0, -1)^\top\}.$$

Hence,  $\bar{x}$  is not a limiting stationary point of (20).

The algorithm proposed in this paper is related to the smoothing function  $\tilde{f}$  that is employed. It is natural that the convergence result also relates to  $\tilde{f}$ . According to (3.7) of Zhang and Chen [59], given  $x \in \mathbb{R}^n$ , the subdifferential of  $f$  associated with  $\tilde{f}$  at  $x$  on  $\mathbb{R}^n$  is

$$G_{\tilde{f}}(x) = \{u \in \mathbb{R}^n \mid \nabla_x \tilde{f}(z_k, \mu_k) \rightarrow u \text{ for some } z_k \rightarrow x, \mu_k \downarrow 0\}. \quad (22)$$

We give the following definition for the Riemannian subdifferential of  $f$  associated with  $\tilde{f}$  at  $x$  on  $\mathcal{M}$ .

**Definition 6.** Given  $x \in \mathcal{M}$ , the Riemannian subdifferential of  $f$  associated with  $\tilde{f}$  at  $x$  on  $\mathcal{M}$  is

$$G_{\tilde{f}, \mathcal{R}}(x) = \{v \in T_x \mathcal{M} \mid \text{grad} \tilde{f}(z_k, \mu_k) \rightarrow v \text{ for some } z_k \in \mathcal{M}, z_k \rightarrow x, \mu_k \downarrow 0\}. \quad (23)$$

**Remark 2.** We require here that  $u \in G_{\tilde{f}}(x)$  and  $v \in G_{\tilde{f}, \mathcal{R}}(x)$  are vectors in the Euclidean space that  $f$  is defined on, and their entries are finite, that is, they are not  $\infty$  or  $-\infty$ . It is clear that if  $\mathcal{M}$  is the Euclidean space that  $f$  is defined on, then  $G_{\tilde{f}, \mathcal{R}}(x) = G_{\tilde{f}}(x)$ .

**Example 3.** For the smoothing function  $\tilde{f}_\mu(t) = (s_\mu(t))^p$  of  $f(t) = |t|^p$  with  $0 < p < 1$ , where  $s_\mu(t)$  is the uniform smoothing function of  $|t|$  defined in (16), we have

$$s'_\mu(t) = \begin{cases} \text{sign}(t) & \text{if } |t| \geq \frac{\mu}{2}, \\ \frac{2t}{\mu} & \text{if } |t| < \frac{\mu}{2}, \end{cases} \quad \text{and} \quad [(s_\mu(t))^p]' = p(s_\mu(t))^{p-1} s'_\mu(t).$$

Here,  $\text{sign}(t) = 1$  if  $t > 0$ ,  $\text{sign}(t) = -1$  if  $t < 0$ , and  $\text{sign}(t) = 0$  otherwise. For an arbitrary real number  $v \in \mathbb{R}$  and an arbitrarily chosen sequence  $\mu_k \downarrow 0$ , let  $t_k = a\mu_k^{2-p}$  with  $a = 4^{p-1}v/(2p)$ . It is easy to see that

$$\lim_{\mu_k \downarrow 0} [(s_{\mu_k}(t_k))^p]' = 2p4^{1-p}a = v.$$

Hence,  $G_{\tilde{f}}(0) = (-\infty, \infty)$ . For any point  $t \neq 0$ , we know that  $G_{\tilde{f}}(t) = p|t|^{p-1}\text{sign}(t)$ .

**Definition 7.** A point  $x \in \mathcal{M}$  is called a stationary point of (1) associated with  $\tilde{f}$  if  $0 \in G_{\tilde{f}, \mathcal{R}}(x)$ , that is,

$$\liminf_{z \rightarrow x, z \in \mathcal{M}, \mu \downarrow 0} \|\text{grad} \tilde{f}(z, \mu)\| = 0. \quad (24)$$

The following result is an extension of proposition 3.4 of Zhang and Chen [59] from  $\mathbb{R}^n$  to an embedded submanifold of  $\mathbb{R}^n$ . The key ingredient for the generalization to Riemannian manifold is to show that the sequence of the Riemannian gradients for the smoothing function has zero as one of its accumulation points.

**Theorem 1.** Let  $\mathcal{M}$  be an embedded submanifold of  $\mathbb{R}^n$ . For any smoothing function  $\tilde{f}$  of  $f$  as defined in Definition 4, if  $x^* \in \mathcal{M}$  is a local minimizer of  $f$  on  $\mathcal{M}$ , then  $x^*$  is a stationary point of (1) associated with  $\tilde{f}$ .

**Proof.** Because  $x^* \in \mathcal{M}$  is a local minimizer of  $f$  on  $\mathcal{M}$ , minima are preserved by composition with diffeomorphisms (see, e.g., the proof of (2)  $\Rightarrow$  (1) in proposition 2.2 of Azagra et al. [6]), we then know that  $0_{x^*}$  is a local minimizer of  $\hat{f} = f \circ R_{x^*}$  on the tangent space  $T_{x^*} \mathcal{M}$ . Hence, there exists a neighborhood  $B_{0_{x^*}, \delta}$  of  $0_{x^*}$  such that for any  $\eta \in T_{x^*} \mathcal{M} \cap B_{0_{x^*}, \delta}$ , it holds that  $\hat{f}(0_{x^*}) \leq \hat{f}(\eta)$ .

Let us denote  $\hat{f}_\mu = \tilde{f}_\mu \circ R_{x^*}$  for any fixed  $\mu > 0$ . We have

$$\begin{aligned}\hat{f}_\mu(0_{x^*}) &= \tilde{f}(x^*, \mu) \leq f(x^*) + \kappa\omega(\mu) \\ &= \hat{f}(0_{x^*}) + \kappa\omega(\mu) \\ &\leq \hat{f}(\eta) + \kappa\omega(\mu) \quad \text{for any } \eta \in B_{0_{x^*}, \delta} \\ &= f(x) + \kappa\omega(\mu) \quad \text{for } x = R_{x^*}(\eta) \\ &\leq \tilde{f}(x, \mu) + 2\kappa\omega(\mu) \\ &= \hat{f}_\mu(\eta) + 2\kappa\omega(\mu).\end{aligned}$$

Thus,

$$\hat{f}_\mu(0_{x^*}) \leq \hat{f}_\mu(\eta) + 2\kappa\omega(\mu), \quad \text{for any } \eta \in B_{0_{x^*}, \delta}. \quad (25)$$

For any  $\eta_z \in T_{x^*}\mathcal{M} \cap B_{0_{x^*}, \delta}$ , we define  $\eta_\mu = 0_{x^*} + \sqrt{\omega(\mu)}\eta_z \in T_{x^*}\mathcal{M} \cap B_{0_{x^*}, \delta}$  for all  $\mu$  sufficiently small, and  $\eta_\mu \rightarrow 0_{x^*}$  as  $\mu \downarrow 0$ . Because  $\hat{f}_\mu$  is continuously differentiable on  $T_{x^*}\mathcal{M}$ , by Taylor's expansion, we have

$$\hat{f}_\mu(0_{x^*}) = \hat{f}_\mu(\eta_\mu) + \langle \text{grad } \hat{f}_\mu(\eta_\mu), -\sqrt{\omega(\mu)}\eta_z \rangle_{x^*} + o\left(\sqrt{\omega(\mu)}\|\eta_z\|\right). \quad (26)$$

Substituting (26) into the left-hand side of (25), and replacing  $\eta$  by  $\eta_\mu$  with  $\mu$  that is sufficiently small, we get

$$\sqrt{\omega(\mu)}\langle \text{grad } \hat{f}_\mu(\eta_\mu), -\eta_z \rangle_{x^*} + o\left(\sqrt{\omega(\mu)}\|\eta_z\|\right) \leq 2\kappa\omega(\mu).$$

Dividing both sides of the above inequality by  $\sqrt{\omega(\mu)}$  and taking the limit as  $\mu \downarrow 0$ , we get

$$\limsup_{\mu \downarrow 0} \langle \text{grad } \hat{f}_\mu(\eta_\mu), -\eta_z \rangle_{x^*} \leq 0,$$

which implies that

$$\liminf_{\eta \rightarrow 0_{x^*}, \eta \in T_{x^*}\mathcal{M}, \mu \downarrow 0} \langle \text{grad } \hat{f}_\mu(\eta), -\eta_z \rangle_{x^*} \leq 0. \quad (27)$$

Note that  $\eta_z \in T_{x^*}\mathcal{M} \cap B_{0_{x^*}, \delta}$  can be chosen arbitrarily. Let  $\mathcal{M}$  be a  $d$ -dimensional embedded submanifold of  $\mathbb{R}^n$ . We can choose  $E : \mathbb{R}^n \rightarrow T_{x^*}\mathcal{M}$  to be a linear bijection such that  $\{E(e_i)\}_{i=1}^d$  is an orthonormal basis of  $T_{x^*}\mathcal{M}$ , where  $e_i$  is the  $i$ th unit vector (see, e.g., Yang et al. [56, section 2]). Then

$$\text{grad } \hat{f}_\mu(\eta) = \sum_{i=1}^d \lambda_i^\mu E(e_i), \quad (28)$$

for some  $\lambda_i^\mu \in \mathbb{R}$ . Let us choose

$$\eta_z^{(i,1)} = \epsilon_i E(e_i), \quad \eta_z^{(i,2)} = -\epsilon_i E(e_i), \quad \text{for } i = 1, 2, \dots, d,$$

where  $\epsilon_i > 0$  is a sufficiently small constant such that  $\eta_z^{(i,1)}, \eta_z^{(i,2)} \in B_{0_{x^*}, \delta}$ . Substituting  $\text{grad } \hat{f}_\mu(\eta)$  in (27) by (28), and substituting  $\eta_z$  in (27) by  $\eta_z^{(i,1)}$  and  $\eta_z^{(i,2)}$ , respectively, we obtain

$$\liminf_{\mu \downarrow 0} -\epsilon_i \lambda_i^\mu \geq 0 \quad \text{and} \quad \liminf_{\mu \downarrow 0} \epsilon_i \lambda_i^\mu \geq 0.$$

The above two inequalities indicate

$$\lim_{\mu \downarrow 0} \lambda_i^\mu = 0.$$

Because  $i = 1, 2, \dots, d$  can be chosen arbitrarily, the above equality holds for each  $i$ . Hence, we get

$$\liminf_{\eta \rightarrow 0_{x^*}, \eta \in T_{x^*}\mathcal{M}, \mu \downarrow 0} \|\text{grad } \hat{f}_\mu(\eta)\| = \lim_{\mu \downarrow 0} \left\| \sum_{i=1}^d \lambda_i^\mu E(e_i) \right\| = 0. \quad (29)$$

According to Absil et al. [2, lemma 7.4.9, p. 153], we know that for any constant  $\tau > 1$ , there exist constants  $\bar{\delta} > 0$  and  $\bar{d} > 0$  such that for all  $\|\eta\| \leq \bar{d}$  and  $x = R_{x^*}(\eta) \in B_{x^*, \bar{\delta}} \cap \mathcal{M}$ ,

$$\|\text{grad } \tilde{f}(x, \mu)\| = \|\text{grad } \tilde{f}_\mu(R_{x^*}(\eta))\| \leq \tau \|\text{grad } \hat{f}_\mu(\eta)\|.$$

Here  $\bar{\delta}$  and  $\bar{d}$  relate only to  $\tau$  and the definitions of the retraction  $R$  and the Riemannian metric  $g$  of  $\mathcal{M}$ , which can be deduced from the proof of lemma 7.4.9 of Absil et al. [2]. Taking the limit  $\eta \rightarrow 0_{x^*}$ ,  $\eta \in T_{x^*}\mathcal{M}$ ,  $\mu \downarrow 0$  to both sides of the above inequality and using (29), we get

$$\liminf_{x \rightarrow x^*, x \in \mathcal{M}, \mu \downarrow 0} \|\text{grad } \tilde{f}(x, \mu)\| = 0,$$

and hence  $x^*$  is a stationary point of (1) associated with  $\tilde{f}$  as desired.  $\square$

We will show in Section 4 that any accumulation point of the proposed RSSD method is a stationary point of (1) associated with  $\tilde{f}$ . We will also show in Section 4 that any accumulation point of the proposed RSSD method is also a limiting stationary point of (1), provided  $\tilde{f}$  satisfies the Riemannian gradient subconsistency (to be defined in the next subsection) at the accumulation point.

### 3.2. Riemannian Gradient Subconsistency

Now we define the Riemannian gradient subconsistency of  $\tilde{f}$  at  $x \in \mathcal{M}$ , which makes a connection between the Riemannian subdifferential  $G_{\tilde{f}, \mathcal{R}}(x)$  associated with  $\tilde{f}$  and the Riemannian limiting subdifferential  $\partial_{\mathcal{R}}f(x)$ . The Riemannian gradient subconsistency is essential to show that any accumulation point of the RSSD method is a limiting stationary point of (1). Hence, when minimizing a nonsmooth but locally Lipschitz function on  $\mathcal{M}$ , the RSSD method has stronger convergence result than the existing methods that guarantee any accumulation point is a Clarke stationary point of (1), for example, the  $\epsilon$ -subgradient algorithm (Grohs and Hosseini [30]), line search algorithms (Hosseini et al. [33]), the Riemannian gradient sampling algorithm (Hosseini and Uschmajew [32]), and Riemannian proximal gradient methods (Huang and Wei [34]).

**Definition 8.** Given  $x \in \mathbb{R}^n$ , a smoothing function  $\tilde{f}$  of the function  $f$  is said to satisfy the gradient subconsistency at  $x$  on  $\mathbb{R}^n$  if

$$G_{\tilde{f}}(x) \subseteq \partial f(x). \quad (30)$$

Given  $x \in \mathcal{M}$ ,  $\tilde{f}$  is said to satisfy the Riemannian gradient subconsistency at  $x$  on  $\mathcal{M}$  if

$$G_{\tilde{f}, \mathcal{R}}(x) \subseteq \partial_{\mathcal{R}}f(x). \quad (31)$$

We say that  $\tilde{f}$  satisfies the gradient subconsistency on  $\mathbb{R}^n$  if (30) holds for any  $x \in \mathbb{R}^n$ , and that  $\tilde{f}$  satisfies the Riemannian gradient subconsistency on  $\mathcal{M}$  if (31) holds for any  $x \in \mathcal{M}$ .

Later we will show that if  $f$  is nonsmooth but locally Lipschitz near  $x$  on  $\mathbb{R}^n$ ,  $\tilde{f}$  is a smoothing function of  $f$ , and the gradient subconsistency of the smoothing function  $\tilde{f}$  at  $x$  on  $\mathbb{R}^n$  holds, then the Riemannian gradient subconsistency of  $\tilde{f}$  on  $\mathcal{M}$  holds. Furthermore, we also provide in (35) a Riemannian optimization problem that minimizes a non-Lipschitz function  $f$  on  $\mathcal{M}$ . We show that its smoothing function  $\tilde{f}$  defined in (42) satisfies the Riemannian gradient subconsistency on  $\mathcal{M}$ . It is worth mentioning that in numerical experiments of Section 5, both the problem (59) of finding the sparsest vectors in a subspace and the problem in (65) for sparsely used orthogonal complete dictionary learning using  $\ell_p$  ( $0 < p < 1$ ) regularization are examples of the model (35).

If the inclusion is substituted by the equality in (30), then we say  $\tilde{f}$  satisfies the gradient consistency at  $x$  on  $\mathbb{R}^n$ . If the inclusion is substituted by the equality in (31), then we say  $\tilde{f}$  satisfies the Riemannian gradient consistency at  $x$  on  $\mathcal{M}$ . Clearly, the gradient consistency indicates the gradient subconsistency. The gradient consistency of  $\tilde{f}$  on  $\mathbb{R}^n$  has been well studied in smoothing methods for nonsmooth optimization. For nonsmooth but locally Lipschitz function  $f$ , it has been shown that the gradient consistency property on  $\mathbb{R}^n$  holds for various smoothing functions in many real applications (Burke and Hoheisel [13], Burke et al. [14], Chen [20], Xu et al. [55], Zhang and Chen [58]).

The following theorem demonstrates that given an embedded submanifold  $\mathcal{M}$  of  $\mathbb{R}^n$ ,  $x \in \mathcal{M}$ , if the gradient subconsistency of  $\tilde{f}$  at  $x$  on  $\mathbb{R}^n$  holds, then the Riemannian gradient subconsistency of  $\tilde{f}$  holds at  $x$  on  $\mathcal{M}$ , provided that  $f$  is locally Lipschitz near  $x$  on  $\mathbb{R}^n$ .

**Theorem 2.** Given an embedded submanifold  $\mathcal{M}$  of  $\mathbb{R}^n$  and a vector  $x \in \mathcal{M}$ , let  $f$  be a locally Lipschitz function near  $x$  on  $\mathbb{R}^n$ , with  $\tilde{f}$  being a smoothing function of  $f$ . If the gradient subconsistency of  $\tilde{f}$  at  $x$  on  $\mathbb{R}^n$  holds, then the Riemannian gradient subconsistency of  $\tilde{f}$  at  $x$  on  $\mathcal{M}$  holds.

**Proof.** Let  $v \in G_{\tilde{f}, \mathcal{R}}(x)$ . Note that  $G_{\tilde{f}}(x) \subseteq \partial f(x)$  is bounded if  $f$  is a locally Lipschitz function near  $x$  on  $\mathbb{R}^n$ . Then there exist subsequences  $\{x_{\mu_k}\} \subset \mathcal{M}$ ,  $x_{\mu_k} \rightarrow x$ , and  $\{\mu_k\}$ ,  $\mu_k \downarrow 0$  as  $k \rightarrow \infty$ , and a vector  $u \in G_{\tilde{f}}(x)$  such that

$$u = \lim_{x_{\mu_k} \rightarrow x, x_{\mu_k} \in \mathcal{M}, \mu_k \downarrow 0} \nabla_{x_{\mu_k}} \tilde{f}(x_{\mu_k}, \mu_k), \quad (32)$$

and

$$\begin{aligned}
 v &= \lim_{x_{\mu_k} \rightarrow x, x_{\mu_k} \in \mathcal{M}, \mu_k \downarrow 0} \text{grad } \tilde{f}(x_{\mu_k}, \mu_k) \\
 &= \lim_{x_{\mu_k} \rightarrow x, x_{\mu_k} \in \mathcal{M}, \mu_k \downarrow 0} \text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} \nabla_x \tilde{f}(x_{\mu_k}, \mu_k), \\
 &= \text{Proj}_{T_x \mathcal{M}} u.
 \end{aligned} \tag{33}$$

The last equality holds because

$$\begin{aligned}
 &\|\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} \nabla_x \tilde{f}(x_{\mu_k}, \mu_k) - \text{Proj}_{T_x \mathcal{M}} u\| \\
 &\leq \|\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} \nabla_x \tilde{f}(x_{\mu_k}, \mu_k) - \text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} u\| + \|\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} u - \text{Proj}_{T_x \mathcal{M}} u\| \\
 &\leq \|\nabla_x \tilde{f}(x_{\mu_k}, \mu_k) - u\| + \|\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} u - \text{Proj}_{T_x \mathcal{M}} u\| \\
 &\rightarrow 0,
 \end{aligned}$$

as  $x_{\mu_k} \rightarrow x, x_{\mu_k} \in \mathcal{M}, \mu_k \downarrow 0$ . Here the second inequality comes from the fact that  $\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}}$  is nonexpansive. Moreover,  $\|\nabla_x \tilde{f}(x_{\mu_k}, \mu_k) - u\| \rightarrow 0$  by (32), and  $\|\text{Proj}_{T_{x_{\mu_k}} \mathcal{M}} u - \text{Proj}_{T_x \mathcal{M}} u\| \rightarrow 0$  because  $\text{Proj} : x \rightarrow \text{Proj}_{T_x \mathcal{M}}$  is continuously differentiable according to Boumal [12, exercise 3.66, p. 59].

Because the gradient subconsistency at  $x$  on  $\mathbb{R}^n$  holds, that is,  $G_{\tilde{f}}(x) \subseteq \partial f(x)$ , we know that  $u \in \partial f(x)$ . By the definition of limiting subdifferential of  $f$  on  $\mathbb{R}^n$ ,

$$\exists u_\ell \in \hat{\partial} f(x_\ell), (x_\ell, f(x_\ell)) \rightarrow (x, f(x)) \text{ such that } \lim_{\ell \rightarrow \infty} u_\ell = u.$$

By the characterization of Riemannian regular subdifferential in (18), we have

$$v_\ell = \text{Proj}_{T_{x_\ell} \mathcal{M}} u_\ell \in \hat{\partial} f(x_\ell), \tag{34}$$

and using the same arguments proving (33), we have

$$\lim_{\ell \rightarrow \infty} v_\ell = \lim_{\ell \rightarrow \infty} \text{Proj}_{T_{x_\ell} \mathcal{M}} u_\ell = \text{Proj}_{T_x \mathcal{M}} u = v.$$

This implies  $v \in \partial f(x)$ , and hence the smoothing function  $\tilde{f}$  of  $f$  satisfies the Riemannian gradient subconsistency at  $x$  on  $\mathcal{M}$ .  $\square$

Furthermore, we consider the following non-Lipschitz Riemannian minimization problem:

$$\min_{x \in \mathcal{M}} f(x) := \hat{f}(x) + \lambda \sum_{i=1}^m \varphi(|d_i^\top x|), \tag{35}$$

where  $\hat{f}$  is a continuously differentiable function,  $\mathcal{M}$  is an embedded submanifold of  $\mathbb{R}^n$ ,  $0 \neq d_i \in \mathbb{R}^n, i = 1, \dots, m$ , are nonzero vectors,  $\lambda > 0$  is a given constant, and  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a nonsmooth penalty function. The problem with  $\mathcal{M} = \mathbb{R}^n$  has been well investigated in Chen et al. [23], which includes many widely used nonsmooth penalty functions  $\varphi$  in variable selection, image restoration, and signal reconstruction.

If  $\varphi$  is nonsmooth but locally Lipschitz, it is easy to see that  $f$  is nonsmooth but locally Lipschitz. In this case, its Riemannian gradient subconsistency has been investigated in Theorem 2. Below we focus only on  $\varphi$  that is not locally Lipschitz. Motivated by assumption 1.1 in Chen et al. [23], we require  $\varphi$  to satisfy the following assumption.

**Assumption 1.** The function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous at zero with  $\varphi(0) = 0, \varphi'(0^+) = \infty$ , and  $\varphi$  is nonsmooth but locally Lipschitz in  $(0, \infty)$ .

For instance, the bridge penalty  $\varphi_1$  used in Chen et al. [22] and Chen et al. [23–25], the log penalty  $\varphi_2$  (Chen et al. [23]), and the penalty  $\varphi_3$  used in Alcantara et al. [4],

$$\varphi_1(t) = t^p, \varphi_2(t) = \log(\alpha t^p + 1), \varphi_3(t) = \min\{t^p, 1\}, \text{ for some } 0 < p < 1, \alpha > 0, \tag{36}$$

are not locally Lipschitz functions on  $\mathbb{R}_+ := \{t \in \mathbb{R} | t \geq 0\}$  that satisfy Assumption 1. If the objective function  $f$  is not locally Lipschitz on  $\mathbb{R}^n$ , it may also be not locally Lipschitz on  $\mathcal{M}$  as well (see the following example).

**Example 4.** Let us consider  $\mathcal{M} = S^1$ , which is the unit circle as in Example 2,  $\bar{x} = (\sqrt{2}/2, -\sqrt{2}/2)^\top$ , and

$$f(x) = |x_1 - x_2|^{\frac{1}{2}} + |x_1 + x_2|^{\frac{1}{2}}.$$

For each  $\gamma > 0$  let us define

$$h_\gamma(x) := |x_1 - x_2|^{\frac{1}{2}} + \gamma(x_1 + x_2), \quad \text{and} \quad \delta_\gamma := \frac{1}{2\gamma^2} > 0.$$

Then it is clear that  $f(\bar{x}) - h_\gamma(\bar{x}) = 0$ , and we claim that

$$f(x) - h_\gamma(x) = |x_1 + x_2|^{\frac{1}{2}} - \gamma(x_1 + x_2) \geq 0, \quad \text{for any } x \in B_{\bar{x}, \delta_\gamma}.$$

To see this, note that for any  $x \in B_{\bar{x}, \delta_\gamma}$ , if  $x_1 + x_2 \leq 0$ , then it is obvious that  $f(x) - h_\gamma(x) \geq 0$ . We then only need to consider  $x \in B_{\bar{x}, \delta_\gamma}$  and  $x_1 + x_2 > 0$ . In this case,  $f(x) - h_\gamma(x) \geq 0$  is equivalent to

$$(x_1 + x_2)^{\frac{1}{2}} \geq \gamma(x_1 + x_2),$$

that is,  $(x_1 + x_2)^{\frac{1}{2}} \leq \frac{1}{\gamma}$ . In view of  $x \in B_{\bar{x}, \delta_\gamma}$ , we know that

$$\max\{|x_1 - \bar{x}_1|, |x_2 - \bar{x}_2|\} \leq \sqrt{(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2} \leq \frac{1}{2\gamma^2},$$

which indicates

$$x_1 + x_2 = (x_1 - \bar{x}_1) + (x_2 - \bar{x}_2) \leq \frac{1}{\gamma^2},$$

and, consequently,  $(x_1 + x_2)^{\frac{1}{2}} \leq \frac{1}{\gamma}$ . Thus,  $f(x) - h_\gamma(x) \geq 0$  also holds in this case.

Therefore,  $f - h_\gamma$  attains a minimum at  $\bar{x}$  in a neighborhood  $B_{\bar{x}, \delta_\gamma}$  of  $\bar{x}$ . According to Lemma 1, we have, for any  $\gamma > 0$ ,

$$v_\gamma = \nabla h_\gamma(\bar{x}) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \hat{\partial} f(\bar{x}).$$

Thus, according to (21) and Proposition 2, we find for any  $\gamma > 0$ ,

$$\begin{aligned} u_\gamma &= \text{Proj}_{T_{\bar{x}}\mathcal{M}} v_\gamma = (I - \bar{x}\bar{x}^\top) \\ v_\gamma &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v_\gamma = \gamma \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \hat{\partial} f(\bar{x}). \end{aligned} \tag{37}$$

It is easy to see that  $\|u_\gamma\| \rightarrow \infty$  as  $\gamma \rightarrow \infty$ . In view of Lemma 2, we know that  $f$  is not locally Lipschitz on  $S^1$ .

**Remark 3.** We consider a general embedded submanifold  $\mathcal{M}$  of  $\mathbb{R}^n$ . Let  $\bar{x} \in \mathcal{M}$  and

$$I_{\bar{x}} = \{i \in \{1, \dots, m\} \mid d_i^\top \bar{x} \neq 0\} \text{ and } J_{\bar{x}} = \{i \in \{1, \dots, m\} \mid d_i^\top \bar{x} = 0\}. \tag{38}$$

Assume  $J_{\bar{x}} \neq \emptyset$ . Now we consider the model (35) with  $\varphi = \varphi_1$ . Using arguments similar to that in the above simple example, we choose an arbitrary  $i_0 \in J_{\bar{x}}$  and let

$$h_\gamma(x) = \hat{f}(x) + h_1(x) + h_{2,\gamma}(x),$$

where

$$h_1(x) = \lambda \sum_{i \in I_{\bar{x}}} |d_i^\top x|^p, \quad h_{2,\gamma}(x) = \lambda \gamma d_{i_0}^\top x \quad \text{for any } \gamma > 0.$$

It is easy to see that  $f - h_\gamma$  attains a local minimum in a neighborhood  $B_{\bar{x}, \delta_\gamma}$  for a positive constant  $\delta_\gamma$ , and  $f(\bar{x}) = h_\gamma(\bar{x})$ . Hence, by Lemma 1, we have  $\nabla h_\gamma(\bar{x}) \in \hat{\partial} f(\bar{x})$ , and, consequently, by Proposition 2, we find

$$u_\gamma = \text{Proj}_{T_{\bar{x}}\mathcal{M}} \nabla h_\gamma(\bar{x}) \in \hat{\partial} f(\bar{x}). \tag{39}$$

As long as there exists a point  $\bar{x} \in \mathcal{M}$  such that  $\text{Proj}_{T_{\bar{x}}\mathcal{M}} \nabla h_{2,\gamma}(\bar{x}) \neq 0$ , because  $\gamma > 0$  can be chosen arbitrarily large, we can conclude that  $f$  is not locally Lipschitz on  $\mathcal{M}$  according to Lemma 2.

For instance, if  $\mathcal{M}$  is the unit sphere in  $\mathbb{R}^n$ ,

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}, \quad (40)$$

then, as  $d_{i_0}^\top \bar{x} = 0$ , we have

$$\text{Proj}_{T_{\bar{x}}\mathcal{M}} \nabla h_{2,\gamma}(\bar{x}) = (I - \bar{x}\bar{x}^\top) \lambda \gamma d_{i_0} = \lambda \gamma d_{i_0} \neq 0. \quad (41)$$

Hence,  $f$  is not locally Lipschitz on  $\mathcal{M}$ .

Many applications can be formulated in the form of (35), such as finding the sparsest vectors in a subspace, and the sparsely used orthogonal complete dictionary learning that will be discussed later in Section 5.

Let  $\tilde{s}_\mu(t)$  be a smoothing function of  $|t|$ ,  $\tilde{\varphi}$  be a smoothing function of  $\varphi$  satisfying Definition 4, and the function

$$\tilde{f}(x, \mu) = \hat{f}(x) + \lambda \sum_{i=1}^m \tilde{\varphi}(\tilde{s}_\mu(d_i^\top x), \mu) \quad (42)$$

be a smoothing function of  $f$  defined in (35). For instance, for  $\varphi = \varphi_1$  and  $\varphi = \varphi_2$  in (36), we can choose

$$\tilde{\varphi}(t, \mu) = \varphi(t),$$

and for  $\varphi_3$ , we can use

$$\tilde{\varphi}_3(t, \mu) = \begin{cases} t^p - (t^p - 1)_+ & \text{if } |t^p - 1| \geq \frac{\mu}{2}, \\ t^p - \left( \frac{(t^p - 1)^2}{2\mu} + \frac{t^p - 1}{2} + \frac{\mu}{8} \right) & \text{if } |t^p - 1| < \frac{\mu}{2}. \end{cases}$$

**Theorem 3.** The smoothing function  $\tilde{f}$  that is constructed in (42) for the non-Lipschitz objective function  $f$  in (35) satisfies the Riemannian gradient subconsistency on  $\mathcal{M}$ .

**Proof.** For an arbitrary  $x \in \mathbb{R}^n$ , let the index sets  $I_x$  and  $J_x$  be defined as in (38) with  $\bar{x}$  being replaced by  $x$ . Let  $D_{J_x}$  be the matrix whose columns are  $d_i$ ,  $i \in J_x$ , that is,

$$D_{J_x} = (d_i)_{i \in J_x} \in \mathbb{R}^{n \times |J_x|}, \quad (43)$$

with  $|J_x|$  being the cardinality of the index set  $J_x$ .

If  $J_x = \emptyset$ , then  $f$  is locally Lipschitz near  $x$  on  $\mathbb{R}^n$ . It is clear that  $\tilde{f}$  satisfies the gradient subconsistency at  $x$  on  $\mathbb{R}^n$ . Thus,  $\tilde{f}$  satisfies the Riemannian gradient subconsistency at  $x$  on  $\mathcal{M}$  as shown in Theorem 2.

Otherwise,  $J_x \neq \emptyset$ . Define

$$\begin{aligned} f_1(z) &:= \lambda \sum_{i \in I_x} \varphi(|d_i^\top z|) \quad \text{and} \quad f_2(z) := \lambda \sum_{i \in J_x} \varphi(|d_i^\top z|), \\ \tilde{f}_1(z, \mu) &:= \lambda \sum_{i \in I_x} \tilde{\varphi}(\tilde{s}_\mu(d_i^\top z), \mu) \quad \text{and} \quad \tilde{f}_2(z, \mu) := \lambda \sum_{i \in J_x} \tilde{\varphi}(\tilde{s}_\mu(d_i^\top z), \mu). \end{aligned}$$

Clearly,

$$\lambda \sum_{i=1}^m \varphi(|d_i^\top z|) = f_1(z) + f_2(z) \quad \text{and} \quad \tilde{f}(z, \mu) = \hat{f}(z) + \tilde{f}_1(z, \mu) + \tilde{f}_2(z, \mu).$$

It is clear that

$$\nabla_x \tilde{f}(z_k, \mu_k) = \nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k) + \nabla_x \tilde{f}_2(z_k, \mu_k), \quad (44)$$

and

$$\lim_{k \rightarrow \infty} \nabla \hat{f}(z_k) = \nabla \hat{f}(x) \quad \text{and} \quad \lim_{k \rightarrow \infty} \nabla_x \tilde{f}_1(z_k, \mu_k) = \nabla f_1(x). \quad (45)$$

By direct computation,

$$\nabla_x \tilde{f}_2(z_k, \mu_k) = \sum_{i \in J_x} \tilde{\varphi}'(s_{\mu_k}(d_i^\top z_k), \mu_k) s'_{\mu_k}(d_i^\top z_k) d_i = D_{J_x} u_k, \quad (46)$$

where

$$u_k := u_k(z_k, \mu_k) = (\tilde{\varphi}'(s_{\mu_k}(d_i^\top z_k), \mu_k) s'_{\mu_k}(d_i^\top z_k))_{i \in J_x} \in \mathbb{R}^{|J_x|}.$$

Let  $v \in G_{\tilde{f}, \mathcal{R}}(x)$ . Then there exist infinite sequences  $\{z_k\} \subset \mathcal{M}$ ,  $z_k \rightarrow x$ , and  $\{\mu_k\}$ ,  $\mu_k \downarrow 0$  as  $k \rightarrow \infty$  such that

$$\begin{aligned} v &= \lim_{z_k \rightarrow x, z_k \in \mathcal{M}, \mu_k \downarrow 0} \text{grad } \tilde{f}(z_k, \mu_k) \\ &= \lim_{z_k \rightarrow x, z_k \in \mathcal{M}, \mu_k \downarrow 0} \text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}(z_k, \mu_k). \end{aligned} \quad (47)$$

For any  $g_k^1, g_k^2 \in \mathbb{R}^n$ , it is easy to see that

$$\begin{aligned} &\|\text{Proj}_{T_{z_k} \mathcal{M}} g_k^2\| - \|\text{Proj}_{T_{z_k} \mathcal{M}} (g_k^1 + g_k^2)\| \\ &\leq \|\text{Proj}_{T_{z_k} \mathcal{M}} (g_k^1 + g_k^2) - \text{Proj}_{T_{z_k} \mathcal{M}} g_k^2\| \leq \|g_k^1\|, \end{aligned}$$

which implies

$$\|\text{Proj}_{T_{z_k} \mathcal{M}} g_k^2\| \leq \|\text{Proj}_{T_{z_k} \mathcal{M}} (g_k^1 + g_k^2)\| + \|g_k^1\|. \quad (48)$$

By substituting  $g_k^1 = \nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k)$  and  $g_k^2 = \nabla_x \tilde{f}_2(z_k, \mu_k)$  into (48), we have

$$\|\text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}_2(z_k, \mu_k)\| \leq \|\text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}(z_k, \mu_k)\| + \|\nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k)\|.$$

The two terms on the right-hand side of the above inequality are bounded by noting (47) and (45). Thus,

$$\{\|\text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}_2(z_k, \mu_k)\|\} \text{ is bounded.} \quad (49)$$

We can write

$$D_{J_x} u_k = b_k^1 + b_k^2, \text{ where } b_k^1 \in T_{z_k} \mathcal{M}, b_k^2 \in (T_{z_k} \mathcal{M})^\perp; \quad (50)$$

$$\nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k) = a_k^1 + a_k^2, \text{ where } a_k^1 \in T_{z_k} \mathcal{M}, a_k^2 \in (T_{z_k} \mathcal{M})^\perp. \quad (51)$$

Here,  $(T_{z_k} \mathcal{M})^\perp$  is the orthogonal complement of  $T_{z_k} \mathcal{M}$ . By (49) and (46), we know that  $\{b_k^1\}$  is bounded.

Let  $r = \text{rank}(D_{J_x})$  be the rank of  $D_{J_x}$  and  $\text{Range}(D_{J_x})$  be the range of  $D_{J_x}$ . Let  $\{j_1, j_2, \dots, j_r\} \subseteq J_x$  such that  $\{d_{j_i}, i = 1, 2, \dots, r\}$  constitutes a basis for  $\text{Range}(D_{J_x})$ . We define  $\xi_i = d_{j_i}$ ,  $i = 1, 2, \dots, r$ . If  $r < n$ , we can find  $\xi_i \in \mathbb{R}^n$ ,  $i = r+1, \dots, n$ , such that  $\{\xi_1, \xi_2, \dots, \xi_n\}$  constitutes a basis for  $\mathbb{R}^n$ . Let us define the matrix  $\Xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^{n \times n}$  that is invertible. Then the linear system with unknown vector  $w$

$$\Xi w = b_k^1$$

is consistent and has a unique solution  $w_k = \Xi^{-1} b_k^1$ . It is clear that  $\{w_k\}$  is bounded.

Let  $\bar{K} \subseteq K$  be an infinite sequence such that  $\lim_{k \rightarrow \infty, k \in \bar{K}} w_k = \bar{w}$ . By using (50) and (51), we get

$$\begin{aligned} \text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}(z_k, \mu_k) &= \text{Proj}_{T_{z_k} \mathcal{M}} (\nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k) + D_{J_x} u_k) \\ &= \text{Proj}_{T_{z_k} \mathcal{M}} (a_k^1 + a_k^2 + b_k^1 + b_k^2) \\ &= \text{Proj}_{T_{z_k} \mathcal{M}} (a_k^1 + b_k^1) = a_k^1 + b_k^1. \end{aligned}$$

Consequently,

$$\begin{aligned} v &= \lim_{z_k \rightarrow x, z_k \in \mathcal{M}, \mu_k \downarrow 0} \text{Proj}_{T_{z_k} \mathcal{M}} \nabla_x \tilde{f}(z_k, \mu_k) \\ &= \lim_{k \rightarrow \infty, k \in \bar{K}} (a_k^1 + b_k^1) \\ &= \lim_{z_k \rightarrow x, z_k \in \mathcal{M}, \mu_k \downarrow 0, k \in \bar{K}} \text{Proj}_{T_{z_k} \mathcal{M}} (\nabla \hat{f}(z_k) + \nabla_x \tilde{f}_1(z_k, \mu_k) + \Xi w_k) \\ &= \text{Proj}_{T_x \mathcal{M}} (\nabla \hat{f}(x) + \nabla f_1(x) + \Xi \bar{w}), \end{aligned}$$

where the last equality can be obtained using the arguments for (33).

We now define the function

$$\bar{h}(z) = \hat{f}(z) + f_1(z) + \sum_{i=1}^r \bar{w}_i d_j^T z + \sum_{i=r+1}^n \bar{w}_i \xi_i^T (z - x).$$

It is then easy to check that there exists a neighborhood  $B_{x,\delta}$  for some  $\delta > 0$  such that  $\bar{h}(z) \leq f(z)$  with  $\bar{h}(x) = f(x)$ , and  $\nabla \bar{h}(x) = \nabla \hat{f}(x) + \nabla f_1(x) + \Xi \bar{w}$ . Then, by Lemma 1,  $\nabla \bar{h}(x) \in \hat{\partial} f(x)$ . Hence,

$$v = \text{Proj}_{T_x \mathcal{M}} \left( \nabla \hat{f}(x) + \nabla f_1(x) + \Xi \bar{w} \right) \in \hat{\partial} f(x) \subseteq \partial f(x), \quad (52)$$

which indicates that  $\tilde{f}$  satisfies the Riemannian gradient subconsistency at  $x$  on  $\mathcal{M}$  in this case.

Because  $x \in \mathcal{M}$  is arbitrary, we have that  $\tilde{f}$  satisfies the Riemannian gradient subconsistency on  $\mathcal{M}$  as desired.  $\square$

At the end of this section, we make clear the relation between the set of limiting stationary points defined in Definition 5,

$$S_l = \{x^* \in \mathcal{M} \mid 0 \in \partial f(x^*)\},$$

and that of stationary points associated with  $\tilde{f}$  defined in Definition 7,

$$S_{\tilde{f}} = \{x^* \in \mathcal{M} \mid 0 \in G_{\tilde{f}, \mathcal{R}}(x^*)\}.$$

Any local minimizer of the Riemannian optimization problem (1) lies in both sets. If the Riemannian gradient subconsistency holds on  $\mathcal{M}$ , that is,  $G_{\tilde{f}, \mathcal{R}}(x^*) \subseteq \partial f(x^*)$  for any  $x^* \in \mathcal{M}$ , then  $S_{\tilde{f}} \subseteq S_l$ .

## 4. Riemannian Smoothing Steepest Descent Method

In this section, we present our RSSD method for solving the Riemannian optimization problem (1), which is detailed in Algorithm 1. The objective function in (1) is allowed to be non-Lipschitz on  $\mathcal{M}$ . We always assume there exists at least one global optimal solution of (1).

**Algorithm 1** (Riemannian Smoothing Steepest Descent Method for Solving (1))

```

1: Input:  $x_0 \in \mathcal{M}$ ,  $\delta_{opt} \geq 0$ ,  $\delta_0 > 0$ ,  $\mu_{opt} \geq 0$ ,  $\mu_0 > 0$ ,  $\sigma \in (0, 1)$ ,  $\beta \in (0, 1)$ ,  $\bar{\alpha} > 0$ ,  $\theta_\delta \in (0, 1)$ ,  $\theta_\mu \in (0, 1)$ .
2: for  $\ell = 0, 1, 2, \dots$  do
3:   Compute  $\eta_\ell = -\text{grad} \tilde{f}(x_\ell, \mu_\ell)$ .
4:   if  $\|\eta_\ell\| \leq \delta_{opt}$  and  $\mu_\ell \leq \mu_{opt}$  then
5:     return
6:   else if  $\|\eta_\ell\| \leq \delta_\ell$  then
7:      $\mu_{\ell+1} := \theta_\mu \mu_\ell$ ,  $\delta_{\ell+1} := \theta_\delta \delta_\ell$ ,
8:      $x_{\ell+1} := x_\ell$ .
9:   else
10:     $\mu_{\ell+1} = \mu_\ell$ ,  $\delta_{\ell+1} = \delta_\ell$ .
11:    Find  $t_\ell := \beta^{m_\ell} \bar{\alpha}$  where  $m_\ell$  is the smallest integer such that
12:       $\tilde{f}(R_{x_\ell}(\beta^{m_\ell} \bar{\alpha} \eta_\ell), \mu_\ell) \leq \tilde{f}(x_\ell, \mu_\ell) - \sigma \beta^{m_\ell} \bar{\alpha} \|\text{grad} \tilde{f}(x_\ell, \mu_\ell)\|^2$ . (53)
13:    Set  $x_{\ell+1} := R_{x_\ell}(t_\ell \eta_\ell)$ .
14:  end if
15: end for

```

A few remarks for Algorithm 1 are in order. First, the line search (53) is well defined, and  $t_\ell$  can be found in finite trials. To see this, note that for fixed  $\mu_\ell$ ,  $\tilde{f}(\cdot, \mu_\ell)$  is continuously differentiable. Clearly, we have

$$\lim_{t \downarrow 0} \frac{\tilde{f}_{\mu_\ell} \circ R_{x_\ell}(t \eta_\ell) - \tilde{f}_{\mu_\ell} \circ R_{x_\ell}(0_{x_\ell})}{t} = (\tilde{f}_{\mu_\ell} \circ R_{x_\ell})'(0_{x_\ell}, \eta_\ell) = \langle \text{grad} \tilde{f}(x_\ell, \mu_\ell), \eta_\ell \rangle.$$

Note that  $\eta_\ell = -\text{grad} \tilde{f}(x_\ell, \mu_\ell)$ . Thus, there exists  $\alpha > 0$  such that for all  $t \in (0, \alpha)$ ,

$$\tilde{f}_{\mu_\ell} \circ R_{x_\ell}(t \eta_\ell) \leq \tilde{f}_{\mu_\ell} \circ R_{x_\ell}(0_{x_\ell}) - t \sigma \|\text{grad} \tilde{f}(x_\ell, \mu_\ell)\|^2.$$

This guarantees that the line search step (53) is well defined.

The following result is an extension of theorem 3.5 together with remark 3.2 of Zhang and Chen [59]. The key ingredient for the extension is to show that the index set  $K$  related to the Riemannian gradient of the smoothing function defined in Theorem 4 is an infinite set.

**Theorem 4.** *Let  $K = \{\ell \mid \|\eta_\ell\| \leq \delta_\ell\}$  and  $\{x_\ell\}$  be an infinite sequence generated by Algorithm 1 with  $\delta_{opt} = \mu_{opt} = 0$ . Then the following statements hold:*

- i. *Any accumulation point  $x^*$  of  $\{x_\ell\}_{\ell \in K}$  is a stationary point of (1) associated with  $\tilde{f}$ .*
- ii. *In addition, if  $\tilde{f}$  satisfies the Riemannian gradient subconsistency at  $x^*$  on  $\mathcal{M}$ , then  $x^*$  is a limiting stationary point of (1).*

**Proof.** We first claim that if there exists an accumulation point  $x^* \in \mathcal{M}$ , then  $K$  is an infinite set, and

$$\lim_{\ell \rightarrow \infty, \ell \in K} \delta_\ell = 0 \quad \text{and} \quad \lim_{\ell \rightarrow \infty, \ell \in K} \mu_\ell = 0. \quad (54)$$

Suppose, on the contrary, that  $K$  is a finite set. This means there exists  $\bar{\ell}$  such that for all  $\ell \geq \bar{\ell}$ ,

$$\delta_\ell \equiv \delta_{\bar{\ell}}, \quad \mu_\ell \equiv \mu_{\bar{\ell}},$$

and

$$\eta_\ell = -\text{grad } \tilde{f}(x_\ell, \mu_{\bar{\ell}}), \quad \|\eta_\ell\| > \delta_{\bar{\ell}} > 0. \quad (55)$$

Therefore, for  $\ell \geq \bar{\ell}$ , we have  $x_{\ell+1} = R_{x_\ell}(t_\ell \eta_\ell)$ , where  $t_\ell$  is obtained by using the line search (53) with fixed  $\mu_{\bar{\ell}}$ . Then Algorithm 1 becomes a Riemannian steepest descent method for minimizing a smooth function  $\tilde{f}(\cdot, \mu_{\bar{\ell}})$  on  $\mathcal{M}$ . According to theorem 4.3.1 of Absil et al. [2], we have  $\text{grad } \tilde{f}(x^*, \mu_{\bar{\ell}}) = 0$ , which contradicts (55). Therefore,  $K$  is an infinite set. Note that for each  $\ell \in K$ , we have

$$\mu_{\ell+1} = \theta_\mu \mu_\ell \quad \text{and} \quad \delta_{\ell+1} = \theta_\delta \delta_\ell$$

with decaying factors  $\theta_\mu \in (0, 1)$  and  $\theta_\delta \in (0, 1)$ . This, together with  $K$  being an infinite set, yields (54) as desired.

By Algorithm 1, we have

$$\lim_{\ell \rightarrow \infty, \ell \in K} \|\text{grad } \tilde{f}(x_\ell, \mu_\ell)\| = \lim_{\ell \rightarrow \infty, \ell \in K} \|\eta_\ell\| \leq \lim_{\ell \rightarrow \infty, \ell \in K} \delta_\ell = 0.$$

Let  $\check{K}$  be a subsequence of  $K$  such that  $\lim_{\ell \rightarrow \infty, \ell \in \check{K}} x_\ell = x^*$ . The completeness of  $\mathcal{M}$  guarantees that  $x^* \in \mathcal{M}$ . Thus,

$$\liminf_{x \rightarrow x^*, x \in \mathcal{M}, \mu \downarrow 0} \|\text{grad } \tilde{f}(x, \mu)\| = 0, \quad \text{and} \quad 0 \in G_{\tilde{f}, \mathcal{R}}(x^*).$$

Hence,  $x^*$  is a stationary point of (1) associated with  $\tilde{f}$ ; that is, statement i holds.

In addition, if  $\tilde{f}$  satisfies the Riemannian gradient subconsistency at  $x^*$  on  $\mathcal{M}$ , then we know  $G_{\tilde{f}, \mathcal{R}}(x^*) \subseteq \partial_{\mathcal{R}} f(x^*)$ . Thus, we find  $0 \in \partial_{\mathcal{R}} f(x^*)$ . Hence,  $x^*$  is a limiting stationary point of (1). Consequently, statement ii holds.  $\square$

The sequence  $\{x_\ell\}$  generated by Algorithm 1 is guaranteed to have an accumulation point if the following assumption holds.

**Assumption 2.** *For any  $\bar{\mu} \in (0, \mu_0]$  and any given vector  $\bar{x} \in \mathcal{M}$ , the level set  $\mathcal{L}_{\bar{x}, \bar{\mu}} = \{x \in \mathcal{M} \mid \tilde{f}(x, \bar{\mu}) \leq \tilde{f}(\bar{x}, \bar{\mu})\}$  is compact.*

Assumption 2 holds if  $\mathcal{M}$  is compact. Assumption 2 also holds if  $f$  is coercive in  $\mathbb{R}^n$ , that is,  $|f(x)| \rightarrow \infty$  if  $\|x\| \rightarrow \infty$ , because for an arbitrary  $\bar{\mu} \in (0, \mu_0]$  and an arbitrary given vector  $\bar{x} \in \mathcal{M}$ , by using Definition 4 for smoothing function,  $x \in \mathcal{L}_{\bar{x}, \bar{\mu}}$  implies that

$$f(x) \leq f(\bar{x}) + 2\kappa\omega(\bar{\mu}),$$

which, together with the coercivity of  $f$ , yields that  $\mathcal{L}_{\bar{x}, \bar{\mu}}$  is compact.

Next, we explain how the RSSD method can be considered as an extension of the smoothing steepest descent method from  $\mathbb{R}^n$  to an embedded submanifold of  $\mathbb{R}^n$  in the following remark. Here, the smoothing steepest descent method comes from algorithm 3.1 and remark 3.2 of Zhang and Chen [59]. To be specific, we set  $\Omega = \mathbb{R}^n$  in algorithm 3.1 of Zhang and Chen [59], and we substitute “the active set method in Algorithm 2.1” in algorithm 3.1 of Zhang and Chen [59] by the well-known “steepest descent method with Armijo line search” (see, e.g., Bertsekas [8, subsection 1.2]). Then we get the smoothing steepest descent method on  $\mathbb{R}^n$ .

**Remark 4.** When  $\mathcal{M} = \mathbb{R}^n$ , let us set the parameters in our RSSD method to be

$$\theta_\mu = \theta_\delta = \zeta, \quad \delta_0 = \hat{\gamma} \mu_0,$$

where  $\zeta$  and  $\hat{\gamma}$  are the parameters in the smoothing steepest descent method on  $\mathbb{R}^n$ . On the other hand, for the smoothing steepest descent method on  $\mathbb{R}^n$ , let us set  $n_1 = 1$ , choose the steepest descent method with Armijo line search to be the same as (53) in our RSSD method, and

$$\mu_{k+1} = \zeta \mu_k.$$

Then, from the same initial point  $x_0$ , the sequence  $\{x_0, x_\ell\}_{\ell \in K}$  where  $K = \{\ell \mid \|\eta_\ell\| \leq \delta_\ell\}$  generated by our RSSD method coincides with the sequence generated by the smoothing steepest descent method.

In computation, we do not set  $\mu_{opt} = \delta_{opt} = 0$ , but instead set them to be small positive real numbers. For instance, we can set  $\mu_{opt} = \delta_{opt} = \epsilon$  for a given small positive real number  $\epsilon$ . Then we expect to get an  $\epsilon$ -approximate stationary point  $\hat{x}$  of (1) associated with  $\tilde{f}$  defined as follows, by implementing Algorithm 1.

**Definition 9.** We say that  $\hat{x} \in \mathcal{M}$  is an  $\epsilon$ -approximate stationary point of (1) associated with  $\tilde{f}$  if

$$\mu \leq \epsilon \quad \text{and} \quad \|\text{grad } \tilde{f}(\hat{x}, \mu)\| \leq \epsilon. \quad (56)$$

This definition of an  $\epsilon$ -approximate stationary point of (1) associated with  $\tilde{f}$  is motivated by Garmanjani and Vicente [29], where smoothing direct-search methods in nonsmooth optimization on  $\mathbb{R}^n$  have been developed to obtain an  $\epsilon$ -approximate solution.

Theorem 5 below is novel even when  $\mathcal{M} = \mathbb{R}^n$ . It has not been considered before for the smoothing steepest descent method in  $\mathbb{R}^n$ .

**Theorem 5.** Under Assumption 2, after finite iterations, Algorithm 1 with  $\mu_{opt} = \delta_{opt} = \epsilon$  will reach an iterate point that is an  $\epsilon$ -approximate stationary point of (1) with respect to  $\tilde{f}$ .

**Proof.** Let  $\theta = \max\{\theta_\mu, \theta_\delta\}$  and

$$n_K := \left\lceil \max \left\{ \log_\theta \frac{\epsilon}{\mu_0}, \log_\theta \frac{\epsilon}{\|\eta_0\|}, 1 \right\} \right\rceil.$$

Here,  $\lceil r \rceil$  refers to the smallest integer that is no less than the real number  $r$ .

We then have

$$\theta^{n_K} \mu_0 \leq \epsilon \quad \text{and} \quad \theta^{n_K} \|\eta_0\| \leq \epsilon.$$

Let us denote  $K_\epsilon = \{k_1, k_2, \dots, k_{n_K}\}$ , where  $k_i < k_j$  for  $1 \leq i < j \leq n_K$  such that  $K_\epsilon$  contains the first  $n_K$  elements that satisfy  $\|\eta_{k_i}\| \leq \delta_{k_i}$ . Thus,

$$\mu_{k_{n_K}} \leq \theta^{n_K} \mu_0 \leq \epsilon \quad \text{and} \quad \theta_{k_{n_K}} \leq \theta^{n_K} \|\eta_0\| \leq \epsilon,$$

and we get the iterate point  $x_{n_K}$  as an  $\epsilon$ -approximate stationary point of (1) associated with  $\tilde{f}$ .

From iterations  $x_{k_i}$  to  $x_{k_{i+1}}$  for  $1 \leq i \leq n_K - 1$ , the smoothing parameter keeps the same as  $\mu_{k_i}$ , and Algorithm 1 performs the iterates of the Riemannian steepest descent method for minimizing the smooth function  $\tilde{f}(x, \mu_{k_i})$  on  $\mathcal{M}$ . According to corollary 4.3.2 of Absil et al. [2],

$$\lim_{\ell \rightarrow \infty} \|\text{grad } \tilde{f}(x^\ell, \mu_{k_i})\| = 0.$$

This implies that after a finite number of steps (say,  $\hat{\ell}_i$ ), we will get  $\|\eta_{k_i + \hat{\ell}_i}\| = \|\text{grad } \tilde{f}(x^{k_i + \hat{\ell}_i}, \mu_{k_i})\| \leq \delta_{k_i}$ . Hence, after

$$k_{n_K} = \sum_{i=1}^{n_K-1} \hat{\ell}_i$$

steps, we will get an  $\epsilon$ -approximate stationary point of (1) associated with  $\tilde{f}$  as desired.  $\square$

**Remark 5.** Complexity for non-Lipschitz optimization in  $\mathbb{R}^n$  has been investigated in Bian and Chen [9, 10], Chen [20], and Chen et al. [25]. In Chen et al. [25], it was shown that solving a non-Lipschitz optimization problem is strongly NP-hard. In Bian and Chen [9, 10], a smoothing sequential quadratic regularization (SSQR) algorithm was proposed for solving non-Lipschitz optimization. The worst-case iteration complexity of the SSQR algorithm for finding an  $\epsilon$  affine-scaled stationary point is  $\mathcal{O}(\epsilon^{-2})$ . It is worth mentioning that the construction of a special strongly convex quadratic minimization problem and a special rule for updating the smoothing parameter at each iteration are essential to show the worst-case complexity in Bian and Chen [9, 10]. New techniques need to

be developed to obtain the iteration complexity of our RSSD method for Riemannian non-Lipschitz optimization, and we leave it for future work.

## 5. Numerical Experiments

In this section, we apply our RSSD method (Algorithm 1) to solve two problems: finding the sparsest vectors in a subspace (FSV) and the sparsely used orthogonal complete dictionary learning problem. A notebook with a 1.80 GHz central processing unit (CPU) and 16 GB of random access memory was used for the numerical experiments. We implemented Algorithm 1 in MATLAB (version R2018b).

### 5.1. Finding the Sparsest Vectors in a Subspace

The FSV problem seeks the sparsest vectors in an  $n$ -dimensional linear subspace  $W \subset \mathbb{R}^m$  ( $m > n$ ). This problem has been studied recently and it finds interesting applications and connection with sparse dictionary learning, sparse principal component analysis, and many other problems in signal processing and machine learning (Qu et al. [45, 46]). This problem is also known as dual principal component pursuit and finds applications in robust subspace recovery (Tsakiris and Vidal [51], Zhu et al. [63]). Let  $Q \in \mathbb{R}^{m \times n}$  denote a matrix whose columns form an orthonormal basis of  $W$ . The FSV problem can be formulated as

$$\min \|Qx\|_0, \quad \text{s.t. } x \in S^{n-1}, \quad (57)$$

where  $S^{n-1}$  is the unit sphere, and  $\|z\|_0$  counts the number of nonzero entries of  $z$ . Because of the combinatorial nature of the cardinality function  $\|\cdot\|_0$ , (57) is very difficult to solve in practice. In the literature, people have been focusing on its  $\ell_1$  norm relaxation given below (Qu et al. [45, 46], Tsakiris and Vidal [51], Zhu et al. [63]):

$$\min \|Qx\|_1, \quad \text{s.t. } x \in S^{n-1}, \quad (58)$$

where  $\|z\|_1 := \sum_i |z_i|$  is the  $\ell_1$  norm of vector  $z$ . Many algorithms have been proposed for solving (58), including the Riemannian gradient sampling algorithm (Hosseini and Uschmajew [32]), projected subgradient method (Zhu et al. [62]), Riemannian subgradient method (Li et al. [42]), manifold proximal point algorithm (Chen et al. [16]), and so on.

Moreover, for the compressive sensing problems that have the same objective functions as (57) and (58), people have found that using the  $\ell_p$  quasi-norm  $\|z\|_p^p := \sum_i |z_i|^p$  ( $0 < p < 1$ ) to replace  $\|z\|_1$  can help to promote the sparsity of  $z$  (Chartrand and Yin [15], Chen et al. [24, 25], Foucart and Lai [28], Liu et al. [43, 44]). Motivated by this, we propose the following  $\ell_p$  ( $0 < p < 1$ ) minimization model for the FSV problem:

$$\min f(x) := \|Qx\|_p^p, \quad \text{s.t. } x \in S^{n-1}. \quad (59)$$

We will illustrate that comparing with (58), (59) with proper choices of  $0 < p < 1$  is a better approximation to (57). To this end, we construct a simple example below for which the global minimizers of the Riemannian  $\ell_0$  model are known ahead of time.

Let  $V = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_5] \in \mathbb{R}^{35 \times 5}$  be a matrix, whose columns have five, six, seven, eight, and nine nonzero entries sequentially, and each nonzero entry of the column is the only nonzero entry in its row. Specifically, the nonzero entries of  $V$  are

$$\begin{aligned} V(1:5,1) &= (1^0 \ 2^0 \ 3^0 \ 4^0 \ 5^0)^T; \\ V(6:11,2) &= (1^1 \ 2^1 \ 3^1 \ 4^1 \ 5^1 \ 6^1)^T; \\ V(12:18,3) &= (1^2 \ 2^2 \ 3^2 \ 4^2 \ 5^2 \ 6^2 \ 7^2)^T; \\ V(19:26,4) &= (1^3 \ 2^3 \ 3^3 \ 4^3 \ 5^3 \ 6^3 \ 7^3 \ 8^3)^T; \\ V(27:35,5) &= (1^4 \ 2^4 \ 3^4 \ 4^4 \ 5^4 \ 6^4 \ 7^4 \ 8^4 \ 9^4)^T. \end{aligned}$$

Let the linear space  $W$  be the span of column vectors of  $V$ , and let  $Q = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_5] \in \mathbb{R}^{35 \times 5}$  be a matrix where  $\mathbf{q}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}$  for  $j = 1, \dots, 5$ . Clearly the columns of the matrix  $Q$  form an orthonormal basis of  $W$ .

Denote by  $\mathbf{e}_i \in \mathbb{R}^5$  the  $i$ th column of the identity matrix for  $i = 1, \dots, 5$ . It is then easy to see that for the Riemannian  $\ell_0$  model, the sparsest vector in the linear space  $W$  has five nonzero entries and  $\pm \mathbf{e}_1$  are the only two global minimizers, and  $\pm \mathbf{e}_i$ ,  $i = 2, 3, 4, 5$ , are local minimizers corresponding to vectors  $\pm Q\mathbf{e}_i \in \mathbb{R}^{35}$  in the linear space  $W$  with six, seven, eight, and nine nonzero entries, respectively. By direct computation, we list in Table 1 the objective values at  $\pm \mathbf{e}_i$  for  $i = 1, \dots, 5$  for the three models, respectively.

**Table 1.** Objective values of the three models at  $\pm \mathbf{e}_i$ ,  $i = 1, \dots, 5$ .

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\ Q(\pm \mathbf{e}_i)\ _0$	5	6	7	8	9
$\ Q(\pm \mathbf{e}_i)\ _1$	2.2361	2.2014	2.0473	1.9385	1.8631
$\ Q(\pm \mathbf{e}_i)\ _p^p$ , $p = 0.1$	4.6134	5.3531	5.8989	6.3284	6.6766
$\ Q(\pm \mathbf{e}_i)\ _p^p$ , $p = 0.01$	4.9599	5.9310	6.8764	7.8018	8.7088
$\ Q(\pm \mathbf{e}_i)\ _p^p$ , $p = 0.001$	4.9960	5.9931	6.9875	7.9798	8.9702

It is obvious that  $\pm \mathbf{e}_1$  are not global minimizers of the Riemannian  $\ell_1$  model, and  $\|Q(\pm \mathbf{e}_5)\|_1$  achieves the lowest values among the five objective values. For  $p = 0.1, 0.01, 0.001$ , we find that  $\pm \mathbf{e}_1$  achieve the lowest objective values among  $\pm \mathbf{e}_i$ ,  $i = 1, \dots, 5$ . In view of the facts that each entry of  $Q$  is nonnegative and each nonzero entry of the column of  $Q$  is the only nonzero entry in its row, we know that  $Q\mathbf{e}_i \geq 0$ ,  $i = 1, \dots, 5$ , and there is no index  $\bar{j}$  such that

$$(Q\mathbf{e}_{i_1})_{\bar{j}} > 0 \text{ and } (Q\mathbf{e}_{i_2})_{\bar{j}} > 0, \quad \forall i_1 \neq i_2, i_1, i_2 \in \{1, \dots, 5\}.$$

For any  $x \in S^4$ , we have  $x = \sum_{i=1}^5 x_i \mathbf{e}_i$ ,  $\sum_{i=1}^5 x_i^2 = 1$ , and  $|x_i| \geq x_i^2$  for  $i = 1, \dots, 5$ . It is easy to see that

$$|Qx| = \left| \sum_{i=1}^5 x_i Q\mathbf{e}_i \right| = \sum_{i=1}^5 |x_i| Q\mathbf{e}_i \geq \sum_{i=1}^5 x_i^2 Q\mathbf{e}_i,$$

and, consequently,

$$\begin{aligned} \|Qx\|_p^p &= \left\| |Qx| \right\|_p^p \geq \left\| \sum_{i=1}^5 x_i^2 Q\mathbf{e}_i \right\|_p^p \\ &\geq \sum_{i=1}^5 x_i^2 \|Q\mathbf{e}_i\|_p^p \geq \sum_{i=1}^5 x_i^2 \|Q\mathbf{e}_1\|_p^p = \|Q\mathbf{e}_1\|_p^p, \end{aligned}$$

where the second inequality is obtained from the concavity of  $\|\cdot\|_p^p$  for  $0 < p < 1$ . Therefore,  $\pm \mathbf{e}_1$  are the global minimizers of the Riemannian  $\ell_p$  model for  $p = 0.1, 0.01, 0.001$ , which coincide with the global minimizers of the original Riemannian  $\ell_0$  model. Moreover, the objective values at  $\pm \mathbf{e}_i$ ,  $i = 1, \dots, 5$  for the Riemannian  $\ell_p$  model keep increasing for  $i$  as that for the Riemannian  $\ell_0$  model.

There exists at least one nonzero row of  $Q$ , say,  $\hat{q}_{i_0}^\top$ , and there exists a vector  $\hat{x} \in S^{n-1}$  such that  $\hat{q}_{i_0}^\top \hat{x} = 0$ . Then, by the same arguments as in Remark 2, the objective function in (59) is not locally Lipschitz on  $S^{n-1}$ . Hence, algorithms proposed in Hosseini and Uschmajew [32], Li et al. [42], Chen et al. [16], and Zhu et al. [62] for solving (58) do not apply to (59). We propose to solve (59) using our RSSD method. We also use the proposed RSSD method to solve the Riemannian  $\ell_1$  norm minimization problem (58). Now we show the details below.

According to Absil et al. [2], the tangent space at  $x \in S^{n-1}$  is

$$T_x S^{n-1} := \{z \in \mathbb{R}^n \mid x^\top z = 0\},$$

and the projection of  $\xi \in \mathbb{R}^n$  onto the tangent space  $T_x S^{n-1}$  is

$$\text{Proj}_{T_x S^{n-1}} \xi = (I - xx^\top) \xi.$$

In our RSSD algorithm, we use  $R_x(\xi) = (x + \xi)/\|x + \xi\|$  as the retraction function. We use the following smoothing function for (59):

$$\tilde{f}(x, \mu) = \sum_{i=1}^m [s_\mu((Qx)_i)]^p, \quad (60)$$

where  $s_\mu(t)$  is the uniform smoothing function for  $|t|$  defined in (16).

The parameters of our RSSD method are set as

$$\mu_0 = 1, \delta_0 = 0.1, \theta_\mu = 0.5, \theta_\delta = 0.5. \quad (61)$$

We choose 50 initial points  $x_0$  from normally distributed random vectors, using MATLAB code

`randn('state',j); x0 = randn(n, 1); x0 = x0/norm(x0),`

for  $j = 1, \dots, 50$ .

**Table 2.** Frequencies of the computed solutions that fall in  $\pm \mathbf{e}_i$ ,  $i = 1, \dots, 5$ , from the 50 initial points, using the Riemannian  $\ell_1$  model and Riemannian  $\ell_p$  model, respectively.

	$\pm \mathbf{e}_1$	$\pm \mathbf{e}_2$	$\pm \mathbf{e}_3$	$\pm \mathbf{e}_4$	$\pm \mathbf{e}_5$
$\ell_1$ model	0	4	15	10	21
$\ell_p$ model, $p = 0.1$	1	4	16	9	20
$\ell_p$ model, $p = 0.01$	6	5	14	8	17
$\ell_p$ model, $p = 0.001$	10	9	10	5	16

We terminate each instance when the CPU time reaches 50 seconds and find that all the 50 computed solutions fall in  $\{\pm \mathbf{e}_i, i = 1, \dots, 5\}$  corresponding to  $p = 0.1, 0.01, 0.001$ . The CPU time is measured with the MATLAB command “cputime.” Here we say  $\hat{x}$  fall in  $\{\pm \mathbf{e}_i\}$  for  $i = 1, \dots, 5$  if

$$\text{gap}(\hat{x}, \pm \mathbf{e}_i) = \min\{\|\hat{x} - \mathbf{e}_i\|, \|\hat{x} + \mathbf{e}_i\|\} \leq 10^{-8}. \quad (62)$$

We record in Table 2 the frequencies of the computed solutions that fall in  $\{\pm \mathbf{e}_i, i = 1, \dots, 5\}$ , respectively. We can conclude that the Riemannian  $\ell_p$  model with  $p = 0.001$  succeeds in finding the true global minimizers of the original Riemannian  $\ell_0$  model 10 times from the 50 initial points. In contrast, the Riemannian  $\ell_1$  minimization model does not find the true global minimizers from the 50 initial points.

This example demonstrates that there indeed exists a problem for which the Riemannian  $\ell_1$  model fails to find the sparsest vector in a subspace, whereas the Riemannian  $\ell_p$  model with suitable  $0 < p < 1$  can find the sparsest vector in a subspace. Hence, it is useful to develop an algorithm for solving Riemannian non-Lipschitz optimization problems with rigorous convergence results. This is the main motivation of this paper.

## 5.2. Sparsely Used Orthogonal Complete Dictionary Learning

Given a set of data  $Y = [\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_m] \in \mathbb{R}^{n \times m}$ , the sparsely used orthogonal complete dictionary learning seeks a dictionary that can sparsely represent  $Y$ . More specifically, ODL seeks an orthogonal matrix  $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \in \mathbb{R}^{n \times n}$  and a sparse matrix  $S \in \mathbb{R}^{n \times m}$  such that  $Y \approx XS$ . The matrix  $X$  is called an orthogonal dictionary. We refer to Sun et al. [50] for more details of this model. This problem can be modeled as the Riemannian  $\ell_0$  minimization problem (Spielman et al. [49]):

$$\min \frac{1}{m} \sum_{i=1}^m \|\mathbf{y}_i^\top X\|_0, \quad \text{s.t. } X \in \text{St}(n, n), \quad (63)$$

where  $\text{St}(n, n) = \{X \in \mathbb{R}^{n \times n} | X^\top X = I_n\}$  is the orthogonal group, which is a special case of the Stiefel manifold. To overcome the computational difficulty of the Riemannian  $\ell_0$  minimization model, the  $\ell_0$  term is usually replaced by the  $\ell_1$  norm in the literature, which leads to the following Riemannian  $\ell_1$  minimization problem for ODL (Spielman et al. [49], Sun et al. [50]):

$$\min \frac{1}{m} \sum_{i=1}^m \|\mathbf{y}_i^\top X\|_1, \quad \text{s.t. } X \in \text{St}(n, n). \quad (64)$$

Here we again consider the Riemannian  $\ell_p$  ( $0 < p < 1$ ) quasi-norm minimization model

$$\min \frac{1}{m} \sum_{i=1}^m \|\mathbf{y}_i^\top X\|_p^p, \quad \text{s.t. } X \in \text{St}(n, n), \quad (65)$$

and apply the RSSD method to solve it. We now specify the details. The tangent space of the Stiefel manifold  $\text{St}(n, n)$  is

$$T_X \text{St} := \{\xi \in \mathbb{R}^{n \times n} : \xi^\top X + X^\top \xi = 0\}.$$

The projection of  $Z \in \mathbb{R}^{n \times n}$  onto the tangent space  $T_X \text{St}(n, n)$  is

$$\text{Proj}_{T_X \text{St}} Z = Z - \frac{1}{2} X(X^\top Z + Z^\top X). \quad (66)$$

We use the QR factorization as the retraction on the Stiefel manifold, which is given by  $R_X(\xi) = \text{qf}(X + \xi)$ . Here,  $\text{qf}(A)$  denotes the  $Q$  factor of the QR decomposition of  $A$ .

Li et al. [42] proposed a Riemannian subgradient method and its variants—the Riemannian incremental subgradient method and Riemannian stochastic subgradient method—for solving the Riemannian  $\ell_1$  minimization

**Table 3.** Average sparsity levels of computed solutions from 50 instances.

Initial points	$\ell_1$ minimization model			$\ell_p$ model, $p = 0.001$ RSSD
	R-Full	R-Inc	R-Sto	
$x_0^{\text{Gauss}}, \tau = 10^{-4}$	0.3727	0.3857	0.3456	0.5000
$x_0^{\text{Gauss}}, \tau = 10^{-5}$	0.3697	0.3852	0.3450	0.4895
$x_0^{\text{Uniform}}, \tau = 10^{-4}$	0.3727	0.3784	0.3234	0.5000
$x_0^{\text{Uniform}}, \tau = 10^{-5}$	0.3675	0.3773	0.3222	0.4915

problem (64). In this section, we use our RSSD to solve the Riemannian  $\ell_p$  minimization model (65) with  $p = 0.001$  and compare its performance with the algorithms proposed in Li et al. [42] for solving (64). We thus generate the synthetic data for ODL in a similar manner as in Li et al. [42], which is detailed below. We first generate the underlying orthogonal dictionary  $X^* \in \text{St}(n, n)$  with  $n = 30$ , whose entries are drawn according to standard Gaussian distribution. The number of samples  $m = \lfloor 10 \cdot n^{1.5} \rfloor = 1643$ . The sparse matrix  $S^* \in \mathbb{R}^{n \times m}$  is generated such that the entries follow the Bernoulli–Gaussian distribution with parameter 0.5. Finally, we set  $Y = X^*S^*$ . We generate 50 instances using this procedure. For each instance, we generate two different initial points: one is a standard Gaussian random vector denoted by  $x_0^{\text{Gauss}}$ , and the other one is a uniform random vector denoted by  $x_0^{\text{uniform}}$ . For the ease of presentation, we refer to the three algorithms in Li et al. [42]—the Riemannian subgradient method, Riemannian incremental subgradient method, and Riemannian stochastic subgradient method—as R-Full, R-Inc, and R-Sto, respectively. We use the same parameters in (61) for RSSD. The codes for R-Full, R-Inc, and R-Sto were downloaded from the author’s web page.<sup>1</sup>

For each instance, we terminate each method if the CPU time reaches 50 seconds. We truncate the entries of  $Y^\top \hat{X}$  as

$$(Y^\top \hat{X})_{ij} = 0, \quad \text{if } |(Y^\top \hat{X})_{ij}| < \tau,$$

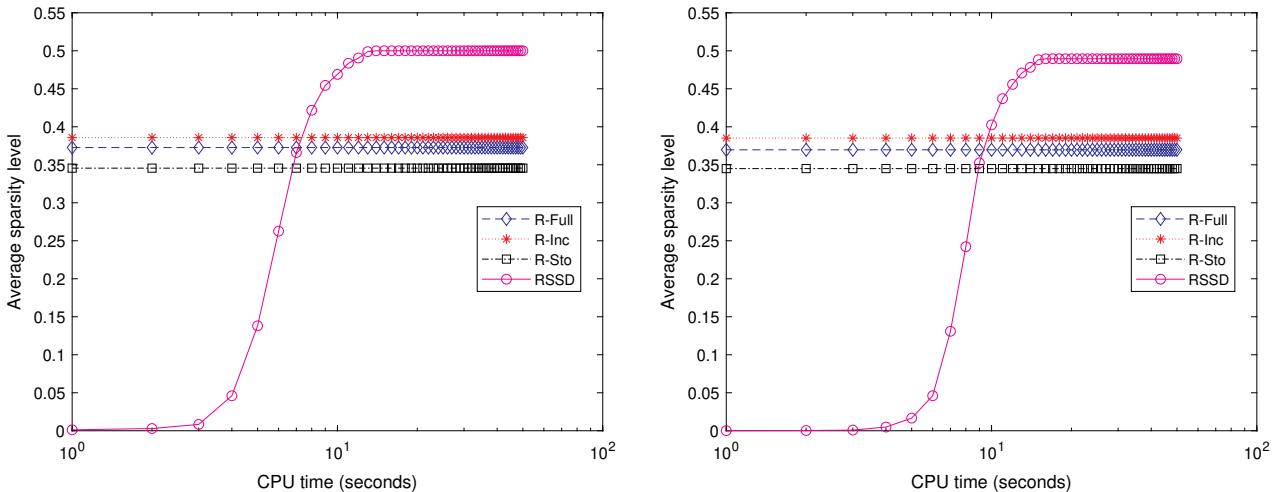
where  $\tau > 0$  is a pre-given tolerance, and  $\hat{X}$  is the computed solution. We report the average of the sparsity level of  $Y^\top \hat{X}$  over 50 instances in Table 3, where the sparsity level is computed by

$$\text{sparsity level} = \frac{\text{number of zero entries of } Y^\top \hat{X}}{mn}.$$

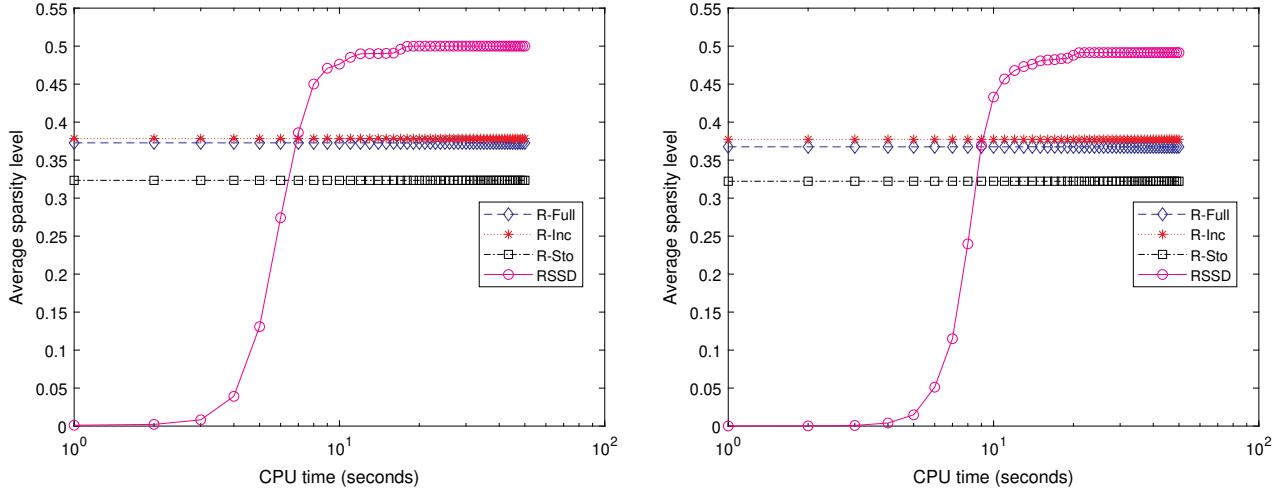
Note that the desired sparsity level of  $Y^\top \hat{X}$  is 0.5 because of the way that  $S^*$  was generated. We see from Table 3 that the  $\ell_p$  minimization model with  $p = 0.001$  solved by our RSSD method provides the best results in terms of the sparsity level.

For each instance, we compute the sparsity level at the latest iterate point obtained by each method when the CPU time reaches  $t = 1, 2, \dots, 50$  seconds, respectively. We then compute the corresponding average sparsity level of the 50 instances and plot the trajectory of the sparsity level at  $t = 1, 2, \dots, 50$  seconds in Figures 1 and 2. We use a log scale on the  $x$ -axes in Figures 1 and 2. From these figures, it is clear that the  $\ell_p$  minimization model (65)

**Figure 1.** (Color online) Average sparsity level vs. CPU time of 50 instances using Gaussian initial points. In the left panel,  $\tau = 10^{-4}$ . In the right panel,  $\tau = 10^{-5}$ . A log scale is used on the  $x$ -axes.



**Figure 2.** (Color online) Average sparsity level vs. CPU time of 50 instances using uniform initial points. In the left panel,  $\tau = 10^{-4}$ . In the right panel,  $\tau = 10^{-5}$ . A log scale is used on the  $x$ -axes.



with  $p = 0.001$  solved by the RSSD method provides the best results in terms of sparsity level. More specifically, the RSSD method can improve the sparsity to the desired level 0.5 after about 15 seconds, whereas the other three algorithms stopped making progress after about one second, and none of the three algorithms achieved a sparsity level higher than 0.4. This example also demonstrates the necessity of developing the RSSD method for solving Riemannian non-Lipschitz optimization problems.

We admit that from our numerical experience, the computation cost of our RSSD method for each iteration is higher than those of the Riemannian subgradient-type methods, including the R-Full, R-Inc, and R-Sto methods in Li et al. [42]. The reason is that the Armijo line search is used in the RSSD method, whereas no line search strategy is adopted in the Riemannian subgradient-type methods in Li et al. [42]. It is also possible to develop more efficient smoothing algorithms than the RSSD method for solving Riemannian non-Lipschitz optimization problems, by making use of the theoretical analysis of Riemannian generalized subdifferentials and Riemannian gradient subconsistency property developed in this paper.

Recall that we use the CPU time budget (50 seconds) to terminate each method in numerical experiments. For our RSSD method, we record  $\mu_\ell$  and  $\|\eta_\ell\| = \|\text{grad } \tilde{f}(x_\ell, \mu_\ell)\|$  when the sparsity level becomes stable, and try the values around them. We find that  $\mu_{opt} \in [2 \times 10^{-3}, 3 \times 10^{-3}]$  and  $\delta_{opt} \in [4 \times 10^{-3}, 10^{-2}]$  are suitable as stopping criterion of the RSSD method for the sparsely used ODL problem, and in this case, the sparsity level keeps almost the same but the number of iterations is around one-third of that given by the CPU budget (50 seconds). If smaller values of  $\mu_{opt}$  and  $\delta_{opt}$  are also used as a stopping criteria, together with the CPU budget (50 seconds), the RSSD method will often reach the CPU time budget, but the sparsity level keeps almost the same.

## 6. Concluding Remarks

In this paper, we study the Riemannian generalized subdifferentials and Riemannian gradient subconsistency relating to non-Lipschitz optimization on embedded submanifolds of  $\mathbb{R}^n$ . We then develop RSSD, a novel Riemannian smoothing steepest descent method, for minimizing a non-Lipschitz function over embedded submanifolds of  $\mathbb{R}^n$ . We prove that any accumulation point generated by our RSSD method is a stationary point of (1) associated with the smoothing function employed in the method, which is necessary for local optimality of (1). Moreover, we also prove that any accumulation point is a limiting stationary point of (1) if the Riemannian gradient subconsistency property holds at the accumulation point. We show that smoothing functions satisfy the Riemannian gradient subconsistency under mild conditions. Numerical results on finding the sparsest vectors in a subspace and the sparsely used orthogonal complete dictionary learning demonstrate the necessity of studying non-Lipschitz optimization on embedded submanifolds of  $\mathbb{R}^n$  and the effectiveness of our RSSD method for solving non-Lipschitz optimization on embedded submanifolds of  $\mathbb{R}^n$ .

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## Endnote

<sup>1</sup> See <https://github.com/lixiao0982/Riemannian-subgradient-methods>.

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