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Gluing Affine Vortices

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Abstract We provide an analytical construction of the gluing map for stable affine vortices over the upper half plane with the Lagrangian boundary condition. This result is a necessary ingredient in studies of the relation between gauged sigma model and nonlinear sigma model, such as the closed or open quantum Kirwan map.

Keywords Affine vortices, gauged linear sigma model, adiabatic limit, symplectic reduction

MR(2010) Subject Classification 53D20, 53D37

1 Introduction

Vortices are local minima of the Yang–Mills–Higgs functional. For example, given a Hermitian line bundle L over a compact Riemann surface Σ, for a unitary connection $A \in \mathcal{A}(L)$ and a smooth section $u \in \Gamma(L)$, the Yang–Mills–Higgs functional reads

$$\mathcal{YMH}(A, u) = \frac{1}{2}\int_\Sigma \left[|d_A u|^2 + \frac{1}{4\epsilon^2}(|u|^2 - 1)^2 + \epsilon^2 |F_A|^2 \right] \mathrm{dvol}_\Sigma, \ (\epsilon > 0)$$

and the vortex equation, which is the equation of motion for this functional, reads

$$\overline{\partial}_A u = 0, \quad *F_A - \frac{\boldsymbol{i}}{2\epsilon^2}(|u|^2 - 1) = 0. \tag{1.1}$$

Vortices appears in many areas of mathematics and physics, and have been generalized to the case of nonabelian gauge groups and nonlinear target spaces. To the author the most important appearance of vortices is in the two-dimensional *gauged linear sigma model* (GLSM), and the motivation of this work mainly comes from this perspective, as is explained below.

Vortices share many similar features with J-holomorphic curves. So it is natural to consider using vortices to define numerical invariants (similar to Gromov–Witten invariants) for a symplectic manifold X with a Hamiltonian action by a compact Lie group K. In this setting the equation is also called the "symplectic vortex equation", which was firstly considered by [3, 14, 15]. Moreover, such invariants (called gauged GW invariants or Hamiltonian GW invariants, see for example [2, 15]) is closely related to the ordinary GW invariants of the symplectic quotient $\bar{X} = X /\!\!/ K$. This relation is unveiled by looking at the "adiabatic limit" of the symplectic vortex equation (i.e., $\epsilon \to 0$ in (1.1)). The first mathematical discussion of adiabatic limit of the symplectic vortex equation and the relation between the gauged invariants of X and

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the GW invariants of $\bar{X}$ was given in [9], preceded by a similar result in gauge theory in [5]. In principle, using the symplectic vortex equation one can also define other symplectic invariants of a Hamiltonian K-manifold, for example, gauged Floer homology (see [6, 36]). Such invariants are related to their counterparts of $\bar{X}$ defined by J-holomorphic curves, via the $\epsilon \to 0$ adiabatic limit.

One cannot expect the invariants for X and $\bar{X}$ are trivially identified. This is because in the adiabatic limit process, vortices converge to J-holomorphic curves modulo "affine vortex" bubbles, which are solutions to the vortex equation over $\boldsymbol{C}$. Different from the bubbling phenomena in J-holomorphic curve theory or Donaldson theory, the bubbling of affine vortices is a codimension zero phenomenon. This is because the affine vortex equation has only translation invariance but not conformal invariance. Therefore, the invariants for X and $\bar{X}$ (i.e., the gauged Gromov–Witten invariants and ordinary Gromov–Witten invariants) are identified only after a "coordinate transformation" defined by counting affine vortices. Such coordinate transformation is often referred to as the *quantum Kirwan map*, which is a deformation of the classical Kirwan map $H_K^*(X) \to H^*(\bar{X})$. For Gromov–Witten theory, the principle of the quantum Kirwan map has been explained in [16, 32, 39]. In the case that X is a projective manifold and the K-action extends to an algebraic action by $K^{\mathbb{C}}$, the quantum Kirwan map is constructed in [32]. While in the symplectic setting, Ziltener has his (paused-for-long) project on this subject (see [37–39]).

We would like to mention the parallel developments in physics. Witten invented the framework of gauged linear sigma model (GLSM) in [30], which provides a way of completing nonlinear sigma model (NLSM) in the ultraviolet direction. Consider the A-twisted topological theories. In the infrared limit, GLSM converges to NLSM with instanton corrections, while the corrections come from the counting of *point-like instantons*. With this intuitive picture, this correspondence has been further developed in [13]. In the case when the superpotential of GLSM is zero, the moduli spaces of the topological theories are just vortices and holomorphic curves respectively, and the point-like instantons are exactly affine vortices.

One of the main motivations of the current work is from the project with Woodward, which aims at extending the above picture to the open string case. In the adiabatic limit of vortices over surfaces with boundary imposing Lagrangian boundary condition, there also appear affine vortices over the upper half plane $\boldsymbol{H}$. In the same spirit as in the closed case, counting affine vortices over $\boldsymbol{H}$ with Lagrangian boundary condition leads certain nontrivial relations between open-string invariants of Lagrangian submanifolds. Such an idea was firstly brought in by Woodward [31] and a precisely stated conjecture can be found in the introduction of [26]. The upshot of our project is to define an A_∞ morphism, which we call the *open quantum Kirwan map*, between two versions of Fukaya A_∞ algebras associated to a Lagrangian brane in a GIT quotient (see the paper [33]).

In the symplectic setting, in order to define the (open or closed) quantum Kirwan map by virtual integration over the moduli space and in order to prove its properties, one has to understand its compactification and the associated gluing. A Gromov type compactification of affine vortex moduli has been constructed in [39] for the closed case, and the open case is essentially covered by the main theorem of [26]. Compared to pseudoholomorphic curves, affine vortices have more involved types of degenerations. Energy concentration does cause bubbling of holomorphic spheres or disks, similar to pseudoholomorphic curves. Besides, the energy can also be separated to regions that are arbitrarily far away (w.r.t. the Euclidean metric), or escape from infinity.

In this paper we construct the gluing map for stable $\boldsymbol{H}$-vortices. We do not consider the most general configurations of stable affine vortices, which may contain arbitrary bubble trees of holomorphic disks or spheres in both X and $\bar{X}$. Instead, we only glue those configurations consisting of several affine vortex components connected by a single disk component. Such a configuration is illustrated in Figure 1, which we call a “simple” configuration (see Section 5). We also add marked points to stabilize the domains. We remark that this special case is the only new piece which has not been understood and this special case is enough for the application in [33]; moreover, the analysis can also be generalized to glue a general configuration, and can be used to construct any Kuranishi type local chart.

Our main theorem is the following, whose precise version is Theorem 5.7.

Theorem 1.1 *Let (X, ω, μ) be a Hamiltonian K-manifold equipped with a K-invariant, ω-compatible almost complex structure J. Assume that 0 is a regular value of μ and K acts freely on $\mu^{-1}(0)$. Let $L \subset X$ be a K-invariant embedded Lagrangian submanifold which is contained in $\mu^{-1}(0)$.*

Let $l, \underline{l} \geq 0$, $l + \underline{l} \geq 1$. Let $\overline{\mathcal{M}}_{l,\underline{l}}(\boldsymbol{H}; X, L)$ be (an open set of) the moduli space of gauge equivalence classes of perturbed stable affine vortices over $\boldsymbol{H}$ with l interior marked points and $\underline{l}$ boundary marked points, equipped with a natural topology. Let $\clubsuit$ be a simple combinatorial type, which labels a stratum $\mathcal{M}_{\clubsuit}(\boldsymbol{H}; X, L) \subset \overline{\mathcal{M}}_{l,\underline{l}}(\boldsymbol{H}; X, L)$. Here the perturbation is parametrized by an open subset of a finite dimensional real vector space $W_{\rm per}$ as well as the deformation space of the underlying marked curves.

Given $[\boldsymbol{w}_{\bullet}, \boldsymbol{v}_{\bullet}] \in \mathcal{M}_{\clubsuit}(\boldsymbol{H}; X, L)$, under a certain transversality assumption, there exist an open neighborhood $\mathcal{U}_{\clubsuit} \subset \mathcal{M}_{\clubsuit}(\boldsymbol{H}; X, L)$ of $[\boldsymbol{w}_{\bullet}, \boldsymbol{v}_{\bullet}]$, a real number $\epsilon_0 > 0$, and a continuous map

$$\mathrm{Glue} : \mathcal{U}_{\clubsuit} \times [0, \epsilon_0) \to \overline{\mathcal{M}}_{l,\underline{l}}(\boldsymbol{H}; X, L)$$

which is a homeomorphism onto an open subset.

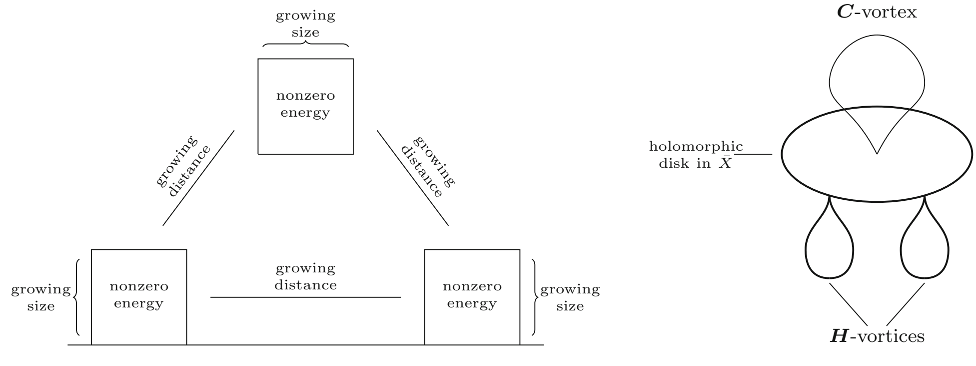

Figure 1 The degeneration of affine vortices over $\boldsymbol{H}$ towards a simple stable affine vortex. The picture on the left can also represent the degeneration of solutions to other types of equations satisfying translation invariance and energy quantization property

One of the technical difficulties in the analytical study of affine vortices is from their mixed geometric behaviors. In a region with large area, the image of u is close to the level set $\mu^{-1}(0)$, where the tangent bundle TX splits into $H_X \oplus G_X$. Here G_X is the direction of group action, and H_X is roughly the tangent bundle of the symplectic quotient. In the H_X direction u is close to a holomorphic curve while the behavior in the G_X direction is different. Therefore, especially for gluing, one has to be very careful in choosing the weighted Sobolev norms which match the behaviors in the two orthogonal directions. Another difficulty is due to the noncompactness of the domain; although there has been a great amount of details for adiabatic limit over compact Riemann surfaces in [9], we have to extend almost everything to the noncompact setting. One novelty of this paper is to use a special weighted Sobolev norms for the involved functional analysis (see the discussion at the beginning of Subsection 3.2). This choice of norm releases us from heavy symbol manipulations in doing concrete estimates.

1.1 Extensions and Applications

As we have explained, the immediate motivation for studying the gluing of affine vortices is from the project of the author with Woodward [33], which aims at defining the open quantum Kirwan map. Moreover, using the technique and analytical setting of this paper, one can construct the gluing map for affine vortices over $\boldsymbol{C}$, and the gluing map w.r.t. the adiabatic limit. This would be an important step towards the resolution of Salamon's quantum Kirwan map conjecture in the symplectic setting, initiated in [37, 39].

In symplectic geometry and gauge theory there are other types of objects which are translation invariant rather than conformal invariant. The figure 8 bubble, appeared in the strip shrinking limits of pseudoholomorphic quilts (see [1, 28, 29]), is such an example. There are also infinite dimensional examples, such as the anti-self-dual equation over $\boldsymbol{C} \times \Sigma$ (see [27]). We hope that the technique of this paper can be used in the gluing construction for other translation invariant equations.

In the joint project with G. Tian (see [20–23]), we are developing a mathematical theory of gauged linear sigma model. To prove the relation between GLSM correlation functions and GW invariants, we have to consider the gluing of point-like instantons. As we have indicated in this introduction, these instantons are generalizations of affine vortices with extra terms coming from a superpotential W; or in other words, solutions to the gauged Witten equation over $\boldsymbol{C}$, which is translation invariant. We will carry out this gluing construction in the future and certain strategies and technical results from this paper can definitely be useful in that case.

2 Moduli Space of Domain Curves

Recall that the pseudoholomorphic curve equation is invariant under conformal (biholomorphic) domain automorphisms. Therefore the study of the moduli space of stable marked complex curves (the Deligne–Mumford space) $\overline{\mathcal{M}}_{g,n}$ is crucial in Gromov–Witten theory. On the other hand, the affine vortex equation (over $\boldsymbol{A} = \boldsymbol{C}$, the complex plane, or $\boldsymbol{A} = \boldsymbol{H}$, the upper half plane) is only invariant under translations of the domain. This type of symmetry corresponds to a different moduli of marked curves, i.e., moduli spaces of configurations of n marked points in $\boldsymbol{A}$ modulo translations, and their compactifications.

Special cases of such moduli spaces have been given particular names. When $\boldsymbol{A} = \boldsymbol{H}$ and all markings are on the boundary, such a moduli space is Stasheff's multiplihedra J_n appeared in [18] where the identification was shown by Ma'u–Woodward [11]. When $\boldsymbol{A} = \boldsymbol{C}$, such a moduli space was called by Ma'u–Woodward a complexified associahedron. In this paper we need to treat a more general situation, namely, $\boldsymbol{A} = \boldsymbol{H}$ with not only boundary markings but also interior ones. However, we only give a modest treatment, which provides the necessities to study the gluing problem of affine vortices in an *ad hoc* way. The upshot is to give a convenient local universal family of a particular singular configuration.

2.1 Deformation of the Domain

Let $l, \underline{l} \geq 0$ such that $l + \underline{l} \geq 1$. Let $\mathcal{N}_{l,\underline{l}}$ be the moduli space of configurations of $l + \underline{l}$ distinct points in $\boldsymbol{H}$, such that l of them are in the interior and $\underline{l}$ of them are on the boundary. We use

$$\boldsymbol{y} := (y_1, \ldots, y_l; \underline{y}_1, \ldots, \underline{y}_{\underline{l}})$$

to denote such a configuration. Two configurations are regarded equivalent and representing the same point in $\mathcal{N}_{l,\underline{l}}$ if they are related by a translation $\mathsf{t} : \boldsymbol{H} \to \boldsymbol{H}$.

There are two other moduli spaces which we need to discuss. For $k \geq 1$, there is a moduli space $\mathcal{N}_k$ of configurations of k distinct points in $\boldsymbol{C}$, with equivalence induced from complex translations. A representative is just denoted as

$$\boldsymbol{x} = (x_1, \ldots, x_k).$$

For $m, \underline{m} \geq 0$ with $2m + \underline{m} \geq 2$, we also have the moduli space $\mathcal{M}_{m,\underline{m}}$ of marked disks with $\underline{m} + 1$ boundary markings and m interior markings. An element of $\mathcal{M}_{m,\underline{m}}$ is represented by a configuration

$$\boldsymbol{z} := (z_1, \ldots, z_m; \underline{z}_1, \ldots, \underline{z}_{\underline{m}})$$

where $z_i \in \operatorname{Int} \boldsymbol{H}$ and $\underline{z}_j \in \partial \boldsymbol{H}$. The $(\underline{m} + 1)$-st marked point is identified with the infinity of $\boldsymbol{H}$. Two configurations are equivalent, i.e., they represent the same point in $\mathcal{M}_{m,\underline{m}}$ if they are related by a Möbius transformation of $\boldsymbol{H}$ that fixes the infinity.

Notation 2.1 Whenever we have a collection of objects a_i (without underline) indexed by $i = 1, \ldots, m$ and a collection $\underline{a}_j$ (with underline) indexed by $j = 1, \ldots, \underline{m}$, in many situations, for convenience, we will denote

$$a_i = \underline{a}_{i-m}, \quad \forall i = n + 1, \ldots, m + \underline{m}.$$

We fix notations for tangent spaces of these moduli spaces. For $l, \underline{l} \geq 0$ with $\underline{l} + l \geq 1$, define

$$W_{l,\underline{l}} := \left\{ \boldsymbol{t} = (t_1, \ldots, t_l; \underline{t}_1, \ldots, \underline{t}_{\underline{l}}) \in \boldsymbol{C}^l \times \mathbb{R}^{\underline{l}} \,\middle|\, \operatorname{Re} \sum_{i=1}^{l} t_i + \sum_{j=1}^{\underline{l}} \underline{t}_j = 0 \right\}.$$

It can be identified with $\boldsymbol{C}^l \times \mathbb{R}^{\underline{l}}$ modulo real translations. For $k \geq 1$, define

$$W_k := \left\{ \boldsymbol{s} = (s_1, \ldots, s_k) \in \boldsymbol{C}^k \,\middle|\, \sum_{j=k}^{l} s_j = 0 \right\}.$$

It is identified with $\boldsymbol{C}^k$ modulo complex translations. Then $W_{l,\underline{l}}$ can be identified with a tangent space of $\mathcal{N}_{l,\underline{l}}$ and W_k is identified with a tangent space of $\mathcal{N}_k$.

The tangent spaces of $\mathcal{M}_{m,\underline{m}}$ cannot be uniformly described. However, fix a representative $\boldsymbol{z}$ of a point $[\boldsymbol{z}] \in \mathcal{M}_{m,\underline{m}}$, one can specify a real codimension 2 linear subspace $W_{m,\underline{m}}(\boldsymbol{z}) \subset \boldsymbol{C}^m \times \mathbb{R}^{\underline{m}}$, such that for all $\boldsymbol{q} \in W_{m,\underline{m}}(\boldsymbol{z})$ with $|\boldsymbol{q}|$ small, $\boldsymbol{z}+\boldsymbol{q}$ gives a parametrization of a neighborhood of $[\boldsymbol{z}]$ inside $\mathcal{M}_{m,\underline{m}}$.

We consider degenerations of configurations in $\mathcal{N}_{l,\underline{l}}$. The moduli space $\mathcal{N}_{l,\underline{l}}$ can be compactified by adding configurations that corresponding to cases when some points coming together or going away from each other (in different rates), and the compactified moduli space is denoted by $\overline{\mathcal{N}}_{l,\underline{l}}$. In this paper we do not need to consider compactifications of $\mathcal{N}_k$ or $\mathcal{M}_{m,\underline{m}}$.

We do not consider all possible degenerations of points of $\mathcal{N}_{l,\underline{l}}$, but will fix a "simple" stratum. Suppose we have a decomposition

$$(\clubsuit) \qquad l = \sum_{i=1}^{m} k_i + \sum_{j=1}^{\underline{m}} l_j, \quad \underline{l} = \sum_{j=1}^{\underline{m}} \underline{l}_j, \quad (k_i \geq 0,\ l_j \geq 0,\ \underline{l}_j \geq 0), \tag{2.1}$$

satisfying the stability condition

$$2m + \underline{m} \geq 2;\ k_i \geq 1,\ \forall i = 1, \ldots, m;\ l_j + \underline{l}_j \geq 1,\ \forall j = 1, \ldots, \underline{m}.$$

Then this decomposition, denoted by $\clubsuit$, gives a stratum $\mathcal{N}_\clubsuit \subset \overline{\mathcal{N}}_{l,\underline{l}}$. Every point of $\mathcal{N}_\clubsuit$ is represented by a collection of configurations

$$\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m; \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{\underline{m}}; \boldsymbol{z}).$$

Here $\boldsymbol{x}_i$ represents a point $\mathcal{N}_{k_i}$, $\boldsymbol{y}_j$ represents a point of $\mathcal{N}_{l_j,\underline{l}_j}$, and $\boldsymbol{z}$ represents a point of $\mathcal{M}_{m,\underline{m}}$.

We fix a point $[\boldsymbol{x}_\bullet] \in \mathcal{N}_\clubsuit$ and a representative $\boldsymbol{x}_\bullet = (\boldsymbol{x}_{\bullet,1}, \ldots, \boldsymbol{x}_{\bullet,m}; \boldsymbol{y}_{\bullet,1}, \ldots, \boldsymbol{y}_{\bullet,\underline{m}}; \boldsymbol{z}_\bullet)$. Then we can identify the tangent space at $[\boldsymbol{x}_\bullet]$, denoted for short by $W_{\rm def}$, with the product

$$\prod_{i=1}^{m} W_{k_i} \times \prod_{j=1}^{\underline{m}} W_{l_j,\underline{l}_j} \times W_{m,\underline{m}}(\boldsymbol{z}_\bullet).$$

A vector of $W_{\rm def}$ is denoted by $\boldsymbol{a}$ and for $|\boldsymbol{a}|$ small enough, denote by

$$\boldsymbol{x}_\bullet + \boldsymbol{a} \tag{2.2}$$

the deformed configuration defined in the obvious sense.

2.2 Local Model for Degeneration

The moduli $\overline{\mathcal{N}}_{l,\underline{l}}$ admits a universal curve

$$\begin{array}{c} \overline{\mathcal{U}}_{l,\underline{l}} \\ \downarrow \\ \overline{\mathcal{N}}_{l,\underline{l}} \end{array} . \tag{2.3}$$

We do not need the full description of $\overline{\mathcal{U}}_{l,\underline{l}}$, but only consider it near the point $[\boldsymbol{x}_\bullet]$. Introduce the gluing parameter $\epsilon \geq 0$. We omit the $\bullet$ from the notations temporarily.

For $\epsilon = 0$, set $\boldsymbol{x}_0(\boldsymbol{a}) = \boldsymbol{x} + \boldsymbol{a}$ as in (2.2). For $\epsilon > 0$, if $\boldsymbol{z} = (z_1, \ldots, z_m; \underline{z}_1, \ldots, \underline{z}_{\underline{m}})$, define

$$z_{i,\epsilon} := \frac{z_i}{\epsilon}, \quad i = 1, \ldots, m + \underline{m}. \tag{2.4}$$

Define a collection of $l+\underline{l}$ points $\boldsymbol{x}_\epsilon$ of $\boldsymbol{H}$ where the points in the interior are

$$(z_{i,\epsilon}+x_{i,\gamma},\ \underline{z}_{j,\epsilon}+y_{j,\nu})^{1\le j\le \underline{m},\ 1\le\nu\le l_j}_{1\le i\le m,\ 1\le\gamma\le k_i}; \tag{2.5}$$

the points on the boundary are

$$(\underline{z}_{j,\epsilon}+\underline{y}_{j,\nu})_{1\le j\le\underline{m},\ 1\le\nu\le\underline{l}_j}. \tag{2.6}$$

More generally, if $\boldsymbol{a}=(\boldsymbol{s}_1,\ldots,\boldsymbol{s}_m,\boldsymbol{t}_1,\ldots,\boldsymbol{t}_{\underline{m}},\boldsymbol{q})$ is a deformation of the singular curve, then $\boldsymbol{x}_\epsilon(\boldsymbol{a})$ is the collection of $l+\underline{l}$ points, where the points in the interior and on the boundary are

$$\left(\frac{z_i+q_i}{\epsilon}+x_{i,\gamma}+s_{i,\gamma},\ \frac{\underline{z}_j+\underline{q}_j}{\epsilon}+y_{j,\nu}+t_{j,\nu}\right)^{1\le j\le\underline{m},\ 1\le\nu\le l_j}_{1\le i\le m,\ 1\le\gamma\le k_i}, \tag{2.7}$$

$$\left(\frac{\underline{z}_j+\underline{q}_j}{\epsilon}+\underline{y}_{j,\nu}+\underline{t}_{j,\nu}\right)_{1\le j\le\underline{m},\ 1\le\nu\le\underline{l}_j}. \tag{2.8}$$

It is easy to see the following fact.

Lemma 2.2 *Given $[\boldsymbol{x}]\in\mathcal{N}_\clubsuit$ and a representative $\boldsymbol{x}$, for $r>0$ small enough, the map $[0,r)\times W^r_{\rm def}\to\overline{\mathcal{N}}_{l,\underline{l}}$ (where $W^r_{\rm def}\subset W_{\rm def}$ is the radius r ball) defined by*

$$(\epsilon,\boldsymbol{a})\mapsto[\boldsymbol{x}_\epsilon(\boldsymbol{a})]$$

is a homeomorphism onto an open neighborhood of $[\boldsymbol{x}]$.

Remark 2.3 Although the singular domain can have arbitrarily many components, but there is only one gluing parameter, and, turning on ϵ resolves all nodes at the same time.

Notice that on every marked curve $(\boldsymbol{H},\boldsymbol{x}_\epsilon(\boldsymbol{a}))$, there is the standard complex structure $j_\bullet$ and the standard volume form $ds\wedge dt$.

2.3 An Equivalent Local Model

Later when we do the pregluing construction, it is more convenient to fix the positions of the nodal points $z_{\bullet,i}$ on the disk component and regard the deformation of the marked disk as deformations of complex structures over a compact region disjoint from the nodal points and ∞. This is also the usual approach in gluing holomorphic curves in the symplectic setting when involved with deformations of complex structures.

Let $\mathcal{V}_\bullet\subset\mathcal{M}_{m,\underline{m}}$ be an open neighborhood of the marked disk $[\boldsymbol{z}_\bullet]$. Let $\boldsymbol{q}\in\mathcal{V}_\bullet$ be the parameter. Then we have choose a family of representatives

$$(\boldsymbol{H},\boldsymbol{z}_{\boldsymbol{q}})=(\boldsymbol{H},z_{\boldsymbol{q},1},\ldots,z_{\boldsymbol{q},m},\underline{z}_{\boldsymbol{q},1},\ldots,\underline{z}_{\boldsymbol{q},\underline{m}}).$$

Choosing $\mathcal{V}_\bullet$ to be sufficiently small, there exist a smooth family of diffeomorphisms $\varphi_{\boldsymbol{q}}:\boldsymbol{H}\to\boldsymbol{H}$ such that $\varphi_\bullet={\rm Id}_{\boldsymbol{H}}$ and satisfying the following conditions.

(a) There exists a small $r>0$ such that for all $\boldsymbol{q}\in\mathcal{V}_\bullet$, the restriction

$$\varphi_{\boldsymbol{q}}:B_r(z_{\boldsymbol{q},i})\to\boldsymbol{H},\quad i=1,\ldots,m+\underline{m}$$

is the (unique) translation onto $B_r(z_{\bullet,i})$.

(b) There exists a large $R>0$ such that B_R contains all markings and the restriction of $\varphi_{\boldsymbol{q}}$ onto the complement $C_R:=\boldsymbol{H}\setminus B_R$ is always the identity.

We regard $\varphi_{\boldsymbol{q}}$ as a map between marked curves

$$\varphi_{\boldsymbol{q}}:(\boldsymbol{H},\ \boldsymbol{z}_{\boldsymbol{q}})\to(\boldsymbol{H},\ \boldsymbol{z}_\bullet).$$

Then $\varphi_{\boldsymbol{q}}$ pushes forward the standard Kähler structure $(j_{\bullet}, ds \wedge dt)$ another one, denoted by $(j_{\boldsymbol{q}}, \sigma_{\boldsymbol{q}} ds \wedge dt)$, which restricts to the standard ones in $B_r(z_{\bullet,i})$ and C_R.

Now we provide another universal family for the moduli $\overline{\mathcal{N}}_{l,\underline{l}}$ in this perspective. Given a gluing parameter $\epsilon > 0$, let $s_\epsilon : \boldsymbol{H} \to \boldsymbol{H}$ be the rescaling $z \mapsto \epsilon z$ and

$$j_{\boldsymbol{q}}^{\epsilon} = s_\epsilon^* j_{\boldsymbol{q}}, \quad \sigma_{\boldsymbol{q}}^{\epsilon} = s_\epsilon^* \sigma_{\boldsymbol{q}}^{\epsilon}.$$

Further, since these rescaled structures are the standard ones in a neighborhood (of radius proportional to ϵ^{-1}) of $\frac{z_{\bullet,i}}{\epsilon}$, using the same construction as before we can still have the position of the markings as in (2.5) and (2.6). Moreover, the diffeomorphism $\varphi_{\boldsymbol{q}}^{\epsilon} := s_\epsilon^* \varphi_{\boldsymbol{q}}$ maps the set of markings of the form (2.7) and (2.8) to the set of markings of the form (2.5) and (2.6), and pushes forward the standard Kähler structure to $(j_{\boldsymbol{q}}^{\epsilon}, \sigma_{\boldsymbol{q}}^{\epsilon} ds \wedge dt)$.

Notation 2.4 The diffeomorphism $\varphi_{\boldsymbol{q}}^{\epsilon}$ provides global holomorphic coordinates on $(\boldsymbol{H}, j_{\boldsymbol{q}}^{\epsilon})$ which differ from the standard one near $z_{\bullet,i}$ and near ∞ by translations. We can denote this global coordinate by $z_{\boldsymbol{q}}^{\epsilon}$. However, in most cases we still use z to denote this global coordinate on $\boldsymbol{H}$ although it depends on $\boldsymbol{q}$ and ϵ.

3 Recollections of Affine Vortices

In this section we review the basic knowledge about affine vortices, and explain how affine vortices can degenerate through two examples.

3.1 Preliminaries

We first recall the basic knowledge of vortices. Let K be a compact Lie group, with Lie algebra $\mathfrak{k}$ and complexification G. Let (X, ω) be a symplectic manifold. Assume there is a Hamiltonian K-action on (X, ω), with a moment map

$$\mu : X \to \mathfrak{k}^*.$$

Choose once and for all an Ad-invariant metric on $\mathfrak{k}$, so μ is viewed as $\mathfrak{k}$-valued. For $a \in \mathfrak{k}$, let $\mathcal{X}_a \in \Gamma(TX)$ be the associated infinitesimal action. Our convention is that $a \mapsto -\mathcal{X}_a$ is a Lie algebra homomorphism.

We make the following fundamental assumption.

Hypothesis 1 $0 \in \mathfrak{k}$ is a regular value of μ and K acts freely on $\mu^{-1}(0)$.[1)]

Let J be a K-invariant, ω-compatible almost complex structure. Then $g_X(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. We fix a K-invariant neighborhood U_X of $\mu^{-1}(0)$ such that K acts freely on U_X. Let $G_X \subset TX|_{U_X}$ be the distribution spanned by $\mathcal{X}_a$ and $J\mathcal{X}_a$ for all $a \in \mathfrak{k}$.

Let $L \subset X$ be an embedded K-invariant Lagrangian submanifold that is contained in $\mu^{-1}(0)$. Then $\bar{L} := L/K$ is an embedded Lagrangian submanifold of $\bar{X}$. We need a special type of Riemannian metric which does not necessarily coincide with g_X.

Lemma 3.1 *There exists a (J, L, μ)-admissible Riemannian metric h_X on X, i.e.,*

(a) *h_X is K-invariant.*

(b) *J is h_X-isometric.*

(c) *$J(TL)$ is orthogonal to TL w.r.t. h_X.*

1) Usually one imposes conditions on (X, ω, μ) to ensure C^0-compactness of moduli spaces. The C^0-compactness property is not necessary in constructing the gluing map.

(d) $T\mu^{-1}(0)$ *is orthogonal to* $J\mathcal{X}_a$ *w.r.t.* h_X*, for all* $a \in \mathfrak{k}$*.*

(e) L *is totally geodesic w.r.t. the Levi–Civita connection of* h_X*.*

(f) $\mu^{-1}(0)$ *is totally geodesic w.r.t. the Levi–Civita connection of* h_X*.*

Proof The idea of the proof essentially comes from the proof of [7, Lemma A.3], whose method was also used in proving [26, Lemma A.3]. Notice that if we obtain a metric h_X satisfying (b)–(f), then the metric obtained by averaging h_X over K satisfies (a)–(f). Hence we do not need to consider K-invariance in the construction.

By [26, Lemma A.3] there exists a metric $\tilde{h}_X$ on X satisfying (a)–(e) (notice that the compactness conditions of the symplectic quotient and L assumed in that paper are not necessary, since the construction of h_X is purely local). Consider an arbitrary metric h'_X whose values at $\mu^{-1}(0)$ coincide with the values of $\tilde{h}_X$ at $\mu^{-1}(0)$ and $h_X(v,w) := \frac{1}{2}(h'_X(v,w) + h'_X(Jv,Jw))$. Then any such h_X still satisfies (a)–(d) and L is totally geodesic inside $\mu^{-1}(0)$. The condition that $\mu^{-1}(0)$ being totally geodesic w.r.t. h_X is equivalent to a condition on the 1-jet of h_X at $\mu^{-1}(0)$, which is equivalent to another condition on the 1-jet of h'_X at $\mu^{-1}(0)$, via the relation between h_X and h'_X. This condition on the 1-jet of h'_X can be solved at least in local coordinates. Hence locally one can always find such metric h_X.

Moreover, the conditions (a)–(f) are intrinsic and they continue to hold under convex combinations and under multiplications by cut-off functions β whose derivative in the normal direction to $\mu^{-1}(0)$ is zero. Then patching the local constructions above using a partition of unity provides a metric satisfying the desired properties. □

Let us fix such a metric h_X. Let H_X be the h_X-orthogonal complement of G_X. Then $TX|_{U_X} = H_X \oplus G_X$ is a K-invariant splitting. Item (b) above implies that H_X is J-invariant; Item (d) above implies that $H_X|_{\mu^{-1}(0)} \subset T\mu^{-1}(0)$. It is also easy to see that $H_X|_{\mu^{-1}(0)}$ is the g_X-orthogonal complement of $G_X|_{\mu^{-1}(0)}$. Hence $H_X|_{\mu^{-1}(0)}$ is independent of the choice of h_X. It defines a connection on the K-bundle $\mu^{-1}(0)$.

Let $\boldsymbol{A}$ be either $\boldsymbol{C}$ or $\boldsymbol{H}$ with a given global holomorphic coordinate $z = s + \boldsymbol{i}t$ and volume form $ds \wedge dt$. We use Σ to denote an open subset of $\boldsymbol{C}$ or $\boldsymbol{H}$.

Definition 3.2

(a) *A smooth* **gauged map** *from* Σ *to* X *is a smooth map* $\boldsymbol{v} = (u, \phi, \psi) : \Sigma \to X \times \mathfrak{k} \times \mathfrak{k}$*. We often identify the triple* (u, ϕ, ψ) *with a pair* (u, a) *where* $a = \phi ds + \psi dt \in \Omega^1(\Sigma, \mathfrak{k})$*, and still abbreviate the pair by* $\boldsymbol{v}$*.*

(b) *Given a smooth gauged map* $\boldsymbol{v} = (u, \phi, \psi)$*, we denote*

$$d_a \boldsymbol{v} = ds \otimes \boldsymbol{v}_s + dt \otimes \boldsymbol{v}_t, \quad \text{where } \boldsymbol{v}_s = \partial_s u + \mathcal{X}_\phi(u),\ \boldsymbol{v}_t = \partial_t u + \mathcal{X}_\psi(u).$$

(c) *Let* $\sigma : \Sigma \to (0, +\infty)$ *be a smooth function. We denote by* $\tilde{\mathcal{M}}(\Sigma, \sigma; X, L)$ *the set of smooth solutions* $\boldsymbol{v} = (u, \phi, \psi)$ *to the equation*

$$\boldsymbol{v}_s + J\boldsymbol{v}_t = 0, \quad \partial_s \psi - \partial_t \phi + [\phi, \psi] + \sigma \mu(u) = 0, \quad u(\partial\Sigma) \subset L. \tag{3.1}$$

This is the symplectic vortex equation and solutions are called **vortices***. We always require the* **boundedness** *condition on vortices, namely, the image* $u(\Sigma)$ *has compact closure in* X *and the energy of* $\boldsymbol{v}$*, defined as follows, is finite*:

$$E(\boldsymbol{v}) := \frac{1}{2}[\|\boldsymbol{v}_s\|_{L^2}^2 + \|\boldsymbol{v}_t\|_{L^2}^2 + \|\partial_s \psi - \partial_t \phi + [\phi, \psi]\|_{L^2}^2 + \|\mu(u)\|_{L^2}^2].$$

Here the L^2-norm is defined with respect to the conformal metric on Σ corresponding to the volume form $\sigma ds \wedge dt$ (and any K-invariant metric on X, for example h_X or g_X). When $\partial\Sigma = \emptyset$, we abbreviate the set by $\tilde{\mathcal{M}}(\Sigma, \sigma; X)$; when σ is understood from the context, abbreviate as $\tilde{\mathcal{M}}(\Sigma; X, L)$ or $\tilde{\mathcal{M}}(\Sigma; X)$.

(d) *An* **affine vortex** *over $\boldsymbol{C}$ (usually called a $\boldsymbol{C}$-vortex) is an element of $\tilde{\mathcal{M}}(\boldsymbol{C}; X)$, i.e., smooth solutions $\boldsymbol{v} = (u, \phi, \psi) : \boldsymbol{C} \to X \times \mathfrak{k} \times \mathfrak{k}$ to the affine vortex equation*

$$\partial_s u + \mathcal{X}_\phi(u) + J(\partial_t u + \mathcal{X}_\psi(u)) = 0, \quad \partial_s \psi - \partial_t \phi + [\phi, \psi] + \mu(u) = 0. \tag{3.2}$$

An $\boldsymbol{H}$-vortex in (X, L) ($\boldsymbol{H}$-vortex for short) is an element of $\tilde{\mathcal{M}}(\boldsymbol{H}; X, L)$, i.e., smooth solutions $\boldsymbol{v} = (u, \phi, \psi)$ satisfying the same equation over $\boldsymbol{H}$ with the boundary condition $u(\partial\boldsymbol{H}) \subset L$. (Later we will consider a perturbed equation.)

Remark 3.3 (a) Equation (3.1) is a special case of the symplectic vortex equation introduced by Mundet [14, 15] and Cieliebak–Gaio–Salamon [3], which can be written in a coordinate-free way over a Riemann surface Σ, where the variables are connections of a K-bundle over Σ and sections of the associated fibre bundle. Here we only consider affine vortices and skip the coordinate-free treatment.

(b) Vortices with Lagrangian boundary conditions are also studied in [34] for $K = S^1$, and in [6] for different types of Lagrangians (not K-invariant).

Equation (3.1) has a gauge symmetry. If $\boldsymbol{v} = (u, \phi, \psi)$ is an $\boldsymbol{A}$-vortex and $g : \boldsymbol{A} \to K$ is a gauge transformation, then

$$g \cdot \boldsymbol{v} := (g \cdot u, \mathrm{Ad}_g(\phi) - (\partial_s g) g^{-1}, \mathrm{Ad}_g(\psi) - (\partial_t g) g^{-1})$$

is also an $\boldsymbol{A}$-vortex and we say the two affine vortices are gauge equivalent [2)]. Let $\mathcal{M}(\Omega, \sigma; X, L)$ be the quotient of $\tilde{\mathcal{M}}(\Omega, \sigma; X, L)$ by gauge equivalence.

There is also a translation symmetry of (3.2). Suppose $\boldsymbol{v} = (u, \phi, \psi)$ is an $\boldsymbol{A}$-vortex. There is a group of translations isomorphic to $\boldsymbol{A}$, such that for any $\mathsf{t} \in \boldsymbol{A}$, $\mathsf{t} \cdot \boldsymbol{v} := (u \circ \mathsf{t}, \phi \circ \mathsf{t}, \psi \circ \mathsf{t})$ is also an $\boldsymbol{A}$-vortex. We regard $\mathsf{t} \cdot \boldsymbol{v}$ being equivalent to $\boldsymbol{v}$.

3.2 Linear Theory of Affine Vortices

In previous works on affine vortices, for example, [25, 26], to define weighted Sobolev norms on affine spaces, we choose parameters p and δ such that

$$p > 2, \quad 1 - \frac{2}{p} < \delta < 1. \tag{3.3}$$

In this paper, it will be useful and notationally convenient to choose more special values. We choose the following values of (p, δ) which satisfy (3.3):

$$2 < p < 4, \quad \delta = \delta_p := 2 - \frac{4}{p}. \tag{3.4}$$

3.2.1 Banach Manifolds

For $\boldsymbol{A}$ being either $\boldsymbol{C}$ or $\boldsymbol{H}$, choose a smooth weight function $\rho_{\boldsymbol{A}} : \boldsymbol{A} \to [1, +\infty)$ such that outside the unit disk, $\rho_{\boldsymbol{A}}(z) = |z|$. For an open subset $U \subset \boldsymbol{A}$ and a function $f : U \to \mathbb{R}$, define

$$\|f\|_{\tilde{L}^p(U)} = \left[\int_U |f(z)|^p [\rho_{\boldsymbol{A}}(z)]^{2p-4} ds dt \right]^{\frac{1}{p}}, \quad \|f\|_{\tilde{W}^{k,p}(U)} := \sum_{l=0}^{k} \|\nabla^l f\|_{\tilde{L}^p(U)}.$$

2) The notation indicates that the group of gauge transformations acts on the left

This is the weighted norm W^{k,p,δ_p} for δ_p given by (3.4).

Consider a smooth affine vortex $\boldsymbol{v} = (u, \phi, \psi)$. Following the analytic set-up of [39] and [25], we introduce the following norm on sections $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in W^{1,p}_{\rm loc}(U, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})$ by

$$\|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_m} := \|\xi\|_{L^\infty} + \|\nabla^a \boldsymbol{\xi}\|_{\tilde{L}^p} + \|d\mu \cdot \xi\|_{\tilde{L}^p} + \|d\mu \cdot J\xi\|_{\tilde{L}^p} + \|\eta\|_{\tilde{L}^p} + \|\zeta\|_{\tilde{L}^p}. \tag{3.5}$$

Here the symbol "m" stands for "mixed". Notice that this norm is gauge invariant. Let

$$W^{1,p}_{\rm loc}(U, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})_L \subset W^{1,p}_{\rm loc}(U, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})$$

be the subspace of sections (ξ, η, ζ) satisfying $\xi|_{\partial U} \subset TL$ and $\zeta|_{\partial U} = 0$. Let

$$\mathcal{B} := \mathcal{B}_{\boldsymbol{v}} \subset W^{1,p}_{\rm loc}(\boldsymbol{A}, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})_L$$

be the subspace of sections whose $\|\cdot\|_{\tilde{L}^{1,p}_m}$-norm are finite. It was proved in [39] and [25] that every $\mathcal{B}$ is a Banach space.

The norm (3.5) deserves a more geometric presentation in a suitable gauge. Denote

$$\Lambda_K = \{a \in \mathfrak{k} \mid \exp(2\pi a) = {\rm Id}_K\}.$$

When $\boldsymbol{A} = \boldsymbol{C}$, essentially by [9, Proposition 11.1], there exist $\lambda \in \Lambda_K$ and $x \in \mu^{-1}(0)$ such that, via a suitable smooth gauge transformation,

$$\lim_{r\to\infty} {\rm e}^{\lambda\theta} u(r{\rm e}^{{\rm i}\theta}) = x.$$

When $\boldsymbol{A} = \boldsymbol{H}$, by [26, Theorem 2.8], there exists $\underline{x} \in L$ such that via a suitable smooth gauge transformation,

$$\lim_{r\to\infty} u(z) = \underline{x}.$$

The difference between $\boldsymbol{C}$-vortices and $\boldsymbol{H}$-vortices in the asymptotic behavior is that at infinity, $\boldsymbol{H}$ has trivial topology but not $\boldsymbol{C}$. The element $\lambda \in \Lambda_K$ is called the holonomy of $\boldsymbol{v}$ at infinity. To unify the notations, we say the holonomy of an affine vortex over $\boldsymbol{H}$ at infinity is $\lambda = 0$.

We have a refined statement on the asymptotic behavior of the affine vortices.

Lemma 3.4 *Let $\boldsymbol{v} = (u, a)$ be a smooth affine vortex over $\boldsymbol{A}$. Assume $\delta \in (1 - \frac{2}{p}, 1)$. By applying a suitable gauge transformation of class $W^{2,p}_{\rm loc}$, the following condition holds.*

- *There exist $x \in X$, $\lambda \in \Lambda_K$ (which is zero if $\boldsymbol{A} = \boldsymbol{H}$) and $\check{\xi} \in W^{1,p}_{\rm loc}(C_R, T_xX)$ whose limit at ∞ is zero, such that if we define $(\check{u}, \check{a}) = {\rm e}^{\lambda\theta} \cdot (u, a)$ over C_R, the complement of the radius R open disk centered at the origin, then $\check{u}|_{C_R} = \exp_x \check{\xi}$ and*

$$\|\check{a}\|_{W^{1,p,\delta}(C_R)} + \|\nabla\check{\xi}\|_{L^{p,\delta}(C_R)} + \|d\mu(x)\cdot\check{\xi}\|_{L^{p,\delta}(C_R)} + \|d\mu(x)\cdot J\check{\xi}\|_{L^{p,\delta}(C_R)} < \infty.$$

Proof It is essentially proved in [25]. Indeed [25, Lemma 6.1] implies that after a gauge transformation, we have

$$\|\check{a}\|_{W^{1,p,\delta}(C_R)} + \|\nabla\check{\xi}\|_{W^{1,p,\delta}(C_R)} < \infty. \tag{3.6}$$

On the other hand, when $|\check{\xi}|$ is small, $d\mu(x)\cdot\check{\xi}$ is roughly comparable to $|\mu(u)|$ whose $L^{p,\delta}$-bound follows from the energy decay property of affine vortices. Lastly we can do a further gauge transformation to make $d\mu(x)\cdot J\check{\xi}|_{C_R} \equiv 0$ (similar gauge transformation is used in the proof of [25, Lemma 6.2]). The last gauge transformation is very small and does not alter the finiteness of (3.6). In particular we obtain the desired finiteness. □

The asymptotic behaviors of $\boldsymbol{v}$ implies that for R large, $u(C_R) \subset U_X$. Then for every $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \mathcal{B}_{\boldsymbol{v}}$, we can decompose ξ into the H_X component ξ^H and the G_X component ξ^G.

Lemma 3.5 *Suppose $\boldsymbol{v}$ is an affine vortex (of class $W^{1,p}_{\rm loc}$) satisfying the condition of Lemma 3.4 for some $\delta > \delta_p$ and $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \mathcal{B} = \mathcal{B}_{\boldsymbol{v}}$. Then*

(a) *There exists $\xi^H(\infty) \in H_{X,x}$ such that*

$$\lim_{z\to\infty} {\rm e}^{\lambda\theta}\xi^H(z) = \xi^H(\infty), \quad \lim_{z\to\infty} \xi^G(z) = 0.$$

(b) *The correspondence $\boldsymbol{\xi} \mapsto \xi^H(\infty)$ is bounded w.r.t. the norm $\|\cdot\|_{\tilde{L}^{1,p}_m}$.*

(c) *If there are $\boldsymbol{\xi}_\gamma \in \mathcal{B}$, $\gamma = 1, 2$ such that $\gamma_1^H(\infty) = \gamma_2^H(\infty)$, then*

$${\rm e}^{\lambda\theta}\xi_1^H - {\rm e}^{\lambda\theta}\xi_2^H \in L^{p,\delta_p-1}(C_R, u^*H_X).$$

In other words, after subtracting the limit at infinity, the H_X-component of ${\rm e}^{\lambda\theta}\xi$ is of class W^{1,p,δ_c} w.r.t. the cylindrical metric, where $\delta_c = \delta_p - 1 + \frac{2}{p} > 0$.

Proof The existence of limits of ${\rm e}^{\lambda\theta}\xi^H$ at ∞ follows from the Hardy-type inequality (see [39, Appendix A.4]), which says that if $f \in W^{1,p}_{\rm loc}(\boldsymbol{A})$ and $\nabla f \in L^{p,\delta}$, then f has a limit $f(\infty)$ at ∞ and $f - f(\infty) \in L^{p,\delta-1}$. Moreover, there is an equivalence of norms

$$\|f\|_{L^\infty} + \|\nabla f\|_{L^{p,\delta}} \approx |f(\infty)| + \|f - f(\infty)\|_{L^{p,\delta-1}} + \|\nabla f\|_{L^{p,\delta}}. \tag{3.7}$$

The limit of ξ^G at infinity vanishes by the choice of weight and Sobolev embedding. □

Definition 3.6 *Let $s_p > 0$ be the Sobolev constants making the following inequalities hold.*

(a) *Let $\boldsymbol{D}$ be the unit disk or half disk. Then for any $f \in W^{1,p}(\boldsymbol{D})$, we have*

$$\|f\|_{C^{0,1-\frac{2}{p}}} \leq s_p \|f\|_{W^{1,p}(\boldsymbol{D})}.$$

(b) *For any $f \in \tilde{L}^{1,p}_h(\boldsymbol{A})$, we have (see (3.7) for $\delta = \delta_p$)*

$$\|f - f(\infty)\|_{L^{p,\delta_p-1}} \leq s_p \|f\|_{\tilde{L}^{1,p}_h}.$$

Using the exponential map exp of h_X, one identifies $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \mathcal{B}$ whose norm is small with a nearby object

$$\boldsymbol{v}' := (u', \phi', \psi') := \exp_{\boldsymbol{v}} \boldsymbol{\xi} := (\exp_u \xi, \phi + \eta, \psi + \zeta). \tag{3.8}$$

Notice since L is totally geodesic w.r.t. h_X, $\boldsymbol{v}'$ still satisfies the boundary condition when $\boldsymbol{A} = \boldsymbol{H}$. Using (3.8) we identify a small ball centered at the origin of $\mathcal{B}$ with a Banach manifold of gauged maps near $\boldsymbol{v}$. By abuse of notation, we still use $\mathcal{B}$ to denote this Banach manifold. Then Lemma 3.5 implies that there is a *smooth* evaluation map

$${\rm ev}: \mathcal{B} \to \bar{X}, \text{ if } \boldsymbol{A} = \boldsymbol{C}; \quad {\rm ev}: \mathcal{B} \to \bar{L}, \text{ if } \boldsymbol{A} = \boldsymbol{H}. \tag{3.9}$$

3.2.2 Banach Vector Bundles

Define a vector bundle $\mathcal{E} \to \mathcal{B}$ whose fibre over $\boldsymbol{v}' = (u', \phi', \psi')$ is

$$\mathcal{E}|_{\boldsymbol{v}'} := \tilde{L}^p(\boldsymbol{A}, u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k}).$$

The vortex equation, plus a gauge fixing condition (which is called the Coulomb gauge) relative to $\boldsymbol{v}$ defines a section

$$\mathcal{F}: \mathcal{B} \to \mathcal{E}, \quad \mathcal{F}(\boldsymbol{v}') = \begin{bmatrix} \partial_s u' + \mathcal{X}_{\phi'}(u') + J(\partial_t u' + \mathcal{X}_{\psi'}(u')) \\ \partial_s \eta + [\phi, \eta] + \partial_t \zeta + [\psi, \zeta] + d\mu \cdot J\xi \\ \partial_s \psi' - \partial_t \phi' + [\phi', \psi'] + \mu(u') \end{bmatrix}. \tag{3.10}$$

For convenience, we placed the gauge fixing condition in the second entry. It is not hard to check that $\mathcal{F}(\boldsymbol{v}')$ indeed lies in the fibre of $\mathcal{E}$, i.e., having the required regularity at infinity.

Let ∇ be the Levi–Civita connection of the chosen K-invariant Riemannian metric h_X. Using ∇ one can trivialize $\mathcal{E}$ near $\boldsymbol{v}$. So we view $\mathcal{E}$ as a Banach space identified with its fibre over $\boldsymbol{v}$. A general vector of $\mathcal{E}$ is denoted by $\boldsymbol{\nu} = (\nu, \kappa, \varsigma)$. W.r.t. this trivialization one has the linearized operator of $\mathcal{F}$ at $\boldsymbol{v}$, denoted by $\mathcal{D}_{\boldsymbol{v}}: \mathcal{B} \to \mathcal{E}$, which reads

$$\mathcal{D}_{\boldsymbol{v}}(\xi, \eta, \zeta) = \begin{bmatrix} D(\xi) + \mathcal{X}_\eta + J\mathcal{X}_\zeta \\ \partial_s \eta + [\phi, \eta] + \partial_t \zeta + [\psi, \zeta] + d\mu(u) \cdot J\xi \\ \partial_s \zeta + [\phi, \zeta] - \partial_t \eta - [\psi, \eta] + d\mu(u) \cdot \xi \end{bmatrix}.$$

Here the first entry is the linearization of the Cauchy–Riemann operator and

$$D(\xi) = \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi + (\nabla_\xi J)(\partial_t u + \mathcal{X}_\psi) + J(\nabla_t \xi + \nabla_\xi \mathcal{X}_\psi).$$

It was proved in [39] (for $\boldsymbol{A} = \boldsymbol{C}$) and [25] (for $\boldsymbol{A} = \boldsymbol{H}$) that $\mathcal{D}$ is Fredholm.

We introduce a notation which will be used frequently. The connection form $a = \phi ds + \psi dt$ induces a deformed covariant derivative on $u^*TX \oplus \mathfrak{k} \oplus \mathfrak{k}$: for any section $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, define

$$\begin{aligned} \nabla_s^a \boldsymbol{\xi} &= (\nabla_s \xi + \nabla_\xi \mathcal{X}_\phi, \partial_s \eta + [\phi, \eta], \partial_s \zeta + [\phi, \zeta]), \\ \nabla_t^a \boldsymbol{\xi} &= (\nabla_t \xi + \nabla_\xi \mathcal{X}_\psi, \partial_t \eta + [\psi, \eta], \partial_t \zeta + [\psi, \zeta]). \end{aligned} \tag{3.11}$$

3.3 Local Model of the Moduli Space of Affine Vortices

The topology of the moduli space $\mathcal{M}(\boldsymbol{A}; X, L)$ is defined by uniform convergence with all derivatives over compact subsets of the domain, up to gauge transformation. This topology is called the **compact convergence topology**, abbreviated as c.c.t.. We say that a sequence $\boldsymbol{v}_n$ of affine vortices converge in c.c.t. to an affine vortex $\boldsymbol{v}_\infty$ if $\boldsymbol{v}_n$ converges to $\boldsymbol{v}_\infty$ uniformly with all derivatives over any compact subset of the domain $\boldsymbol{A}$. A sequence of points $[\boldsymbol{v}_n]$ converge to $[\boldsymbol{v}_\infty]$ if there is a sequence of smooth gauge transformations $g_n: \boldsymbol{A} \to K$ such that $g_n \cdot \boldsymbol{v}_n$ converge in c.c.t. to $\boldsymbol{v}_\infty$.

On the other hand, for the sake of Fredholm theory, on the linear level we are using the Banach space $\mathcal{B}_{\boldsymbol{v}}$ which requires certain regularity at ∞. To show that the linear Fredholm theory really describes the deformation of the moduli space $\mathcal{M}(\boldsymbol{A}; X, L)$, a necessary step is to prove the regularity results. Namely, if $[\boldsymbol{v}']$ is sufficiently close to $[\boldsymbol{v}]$ in the moduli space, then up to gauge transformations $\boldsymbol{v}'$ is inside the Banach manifold $\mathcal{B}_{\boldsymbol{v}}$. It implies that if the linearized operator $\mathcal{D}_{\boldsymbol{v}}$ is surjective, then its kernel provides a local manifold chart of $\mathcal{M}(\boldsymbol{A}; X, L)$ around the point $[\boldsymbol{v}]$. This is indeed a nontrivial problem, as the decaying property of affine vortices at ∞ is very complicated.

In [25], together with Venugopalan, the author constructed local models of affine vortices around a given affine vortex $\boldsymbol{v}$ under the assumption that the linearized operator $\mathcal{D}_{\boldsymbol{v}}: \mathcal{B}_{\boldsymbol{v}} \to \mathcal{E}_{\boldsymbol{v}}$

is surjective, based on Ziltener's proof of the Fredholmness of the linear operator[3]. Let us recall the precise statement of this result.

Theorem 3.7 ([25]) *Let $\boldsymbol{v} = (u, \phi, \psi)$ be an affine vortex over $\boldsymbol{A}$. Suppose $\mathcal{D}_{\boldsymbol{v}} : \mathcal{B}_{\boldsymbol{v}} \to \mathcal{E}_{\boldsymbol{v}}$ is surjective. Then there is a homeomorphism from a neighborhood of the origin of $\ker \mathcal{D}_{\boldsymbol{v}}$ onto a neighborhood of $[\boldsymbol{v}]$ in the moduli space $\mathcal{M}(\boldsymbol{A}; X, L)$. In particular, if $\boldsymbol{v}'$ is another affine vortex representing a point of $\mathcal{M}(\boldsymbol{A}; X, L)$ which is sufficiently close to $[\boldsymbol{v}]$, then there exists a gauge transformation of class $W^{2,p}_{\rm loc}$ which transforms $\boldsymbol{v}'$ to an affine vortex (which is still denoted by $\boldsymbol{v}'$) such that we can write $\boldsymbol{v}' = \exp_{\boldsymbol{v}} \boldsymbol{\xi}$ with $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \mathcal{B}_{\boldsymbol{v}}$ and $\|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_m}$ being sufficiently small. Moreover, we may require that $\boldsymbol{v}'$ is in Coulomb gauge with respect to $\boldsymbol{v}$, i.e.,*

$$\partial_s \eta + [\phi, \eta] + \partial_t \zeta + [\psi, \zeta] + d\mu(u) \cdot J\xi = 0.$$

The core of the proof of Theorem 3.7 is to deal with behaviors of affine vortices at infinity. Hence this result can certainly be generalized to the situation where the equation is perturbed by a compactly supported term. We will see and use such a generalization in later sections.

3.4 Examples of Degenerations

A sequence of affine vortices may degenerate in the limit to a stable affine vortex, which has multiple components. While the precise definition of sequential convergence in the perturbed case will be given in Definition 5.5, here we provide two examples on extremal cases of the degeneration for $\boldsymbol{A} = \boldsymbol{C}$, from which one can see the intuitive picture.

3.4.1 Example 1

Consider a sequence of monic polynomials of degree $d \geq 2$

$$f_i(z) = (z - z_i^1) \cdots (z - z_i^d).$$

By Taubes' theorem (see [10, 19]), there are unique solutions $h_i : \boldsymbol{C} \to \mathbb{R}$ to the equation

$$-\frac{\Delta h_i}{2\pi} + \frac{1}{2}(\mathrm{e}^{2h_i}|f_i(z)|^2 - 1) = 0$$

with an appropriate asymptotic constraint on h_i. Equivalently, $(\mathrm{e}^{h_i} f_i, -\partial_t h_i, \partial_s h_i)$ is an affine vortex with target $\mathbb{C}$ acted by $K = U(1)$, and his correspondence gives a homeomorphism

$$\mathcal{M}(\boldsymbol{C}; \mathbb{C}) \simeq \bigsqcup_{d \geq 0} \mathrm{Sym}^d \boldsymbol{C}.$$

Let us now moving the zeroes. Suppose there is a partition $\{1, \ldots, d\} = I_1 \sqcup \cdots \sqcup I_s$ by nonempty subsets, such that, as $i \to \infty$, $|z_i^\alpha - z_i^\beta|$ is bounded (resp. unbounded) when α, β belong to the same (resp. different) subsets of this partition. Then up to choosing a subsequence, one can show that the corresponding sequence of affine vortices will degenerate to a stable affine vortex having exactly s affine vortex components.

3.4.2 Example 2

Consider abelian vortices of higher rank. More precisely, consider a sequence $\vec{f_i}$, each of which is an N-tuple of polynomials

$$\vec{f_i} = (f_i^1, \ldots, f_i^N), \quad \max_\alpha \{\mathrm{deg} f_i^\alpha\} = d \geq 1.$$

3) See [39, Theorem 4]. Indeed Ziltener assumed that the symplectic quotient $\bar{X}$ has positive dimension, while in [25] we removed this assumption. See [25, Section 4].

Then by the main theorem of [35] (see also [24]), there exists a unique solution $h_i : \boldsymbol{C} \to \mathbb{R}$ to the equation

$$-\frac{\Delta h_i}{2\pi} + \frac{1}{2}\left(\mathrm{e}^{2h_i} \sum_{\alpha=1}^{N} |f_i^\alpha(z)|^2 - 1\right) = 0.$$

So such an N-tuple of polynomials gives an affine vortex with target $\mathbb{C}^N$ acted by $K = U(1)$.

For simplicity restrict to the special case that $N = 2$, $d = 1$ and

$$f_i^1(z) = z - n_i, \quad f_i^2(z) = z; \quad n_i \to \infty.$$

For the corresponding sequence of affine vortices in $\mathbb{C}^2$, one can argue that in the limit, there is no affine vortex component. Instead, after rescaling by the factor n_i, the sequence converges to the holomorphic sphere $z \mapsto [z-1, z]$ in $\mathbb{CP}^1$.

4 Holomorphic Disks Revisited

As we have seen from the last example, a sequence of affine vortices can converge to a holomorphic curve in $\bar{X}$. The latter is an object purely in nonlinear sigma model, where the gauge field completely depends on the matter field. This is actually a phenomenon similar to the case of Morse theory for Lagrange multipliers considered in [17]. For the purpose of gluing, one needs to better understand holomorphic curves in the quotient not only as maps into $\bar{X}$, but also as certain type of gauged maps into X.

4.1 Holomorphic Curves in the Quotient

Compare to the notion of gauged maps defined in Definition 3.2, a general gauged map over a Riemann surface Σ is a triple (P, A, u), where $P \to \Sigma$ is a K-bundle, $A \in \mathcal{A}(P)$ and $u \in \Gamma(P \times_K X)$. If P is trivialized and Σ has coordinate $z = s + \boldsymbol{i}t$, we can write $A = d + \phi ds + \psi dt$. In this case it coincides with our previous definition.

One can also use gauged maps to represent holomorphic curves in the quotient $\bar{X}$. For simplicity consider the case that Σ is a contractible subset of $\boldsymbol{H}$ or $\boldsymbol{C}$, equipped with a complex structure j_Σ and holomorphic coordinate $z = s + \boldsymbol{i}t$. Suppose $\bar{u} : \Sigma \to \bar{X}$ is a C^1-map satisfying the equation

$$\partial_s \bar{u} + \bar{J} \partial_t \bar{u} = 0, \quad \bar{u}(\partial\Sigma) \subset \bar{L}.$$

Here $\bar{J}$ is the induced almost complex structure on $\bar{X}$. Recall that on the K-bundle $\mu^{-1}(0) \to \bar{X}$, there is a connection given by the distribution $H_X|_{\mu^{-1}(0)}$ (which we call the *canonical connection*). Then one can pull back the K-bundle $\mu^{-1}(0) \to \bar{X}$ as well as the canonical connection, giving a bundle $P \to \Sigma$ and a connection $A \in \mathcal{A}(P)$. $\bar{u}$ also lifts to a section of $P \times_K X$. Since Σ is contractible, one can trivialize P so that A is written as $A = d + \phi ds + \psi dt$ and u is a genuine map $u : \Sigma \to \mu^{-1}(0)$. This gives us a gauged map $\boldsymbol{v} = (u, \phi, \psi)$. Moreover, if $\bar{u}$ is $J_{\bar{X}}$-holomorphic, then $\boldsymbol{v}$ satisfies the equation

$$\partial_s u + \mathcal{X}_\phi + J(\partial_t u + \mathcal{X}_\psi) = 0, \quad \mu(u) = 0, \quad u(\partial\Sigma) \subset L. \tag{4.1}$$

Different trivializations of the pull-back bundle $P \to \Sigma$ give gauge equivalent solutions to (4.1). On the other hand, each gauge equivalence class of solutions of (4.1) projects down to a $\bar{J}$-holomorphic map into $\bar{X}$.

We would like to remark that if we view both u and (ϕ, ψ) as independent variables of (4.1), then it is not an elliptic equation. The correct perspective is to regard ϕ and ψ as components of the pullback connection, so only u is the independent variable of (4.1). Another perspective is that ϕ and ψ are the unique functions satisfying

$$d\mu(u) \cdot J(\partial_s u + \mathcal{X}_\phi) = d\mu(u) \cdot J(\partial_t u + \mathcal{X}_\psi) = 0.$$

Then the G_X-component of (4.1) is automatically satisfied and (4.1) is equivalent to

$$P_H(\partial_s u + \mathcal{X}_\phi + J(\partial_t u + \mathcal{X}_\psi)) = 0, \quad \mu(u) = 0, \quad u(\partial\Sigma) \subset L. \tag{4.2}$$

where $P_H : H_X \oplus G_X \to H_X$ is the projection onto the first factor.

From now on we take the gauged map viewpoint of holomorphic curves in $\bar{X}$. In particular, if $\bar{u}$ is a holomorphic disk in $\bar{X}$ with boundary in $\bar{L}$, then we view it as a map $\bar{u} : (\boldsymbol{H}, \partial\boldsymbol{H}) \to (\bar{X}, \bar{L})$ with removable singularity at infinity, so it can be lifted to a gauge equivalence class of solutions to (4.1) with $\Sigma = \boldsymbol{H}$.

4.2 Weighted Sobolev Spaces

Since holomorphic disks we are interested in are limits of affine vortices near ∞, we prefer to work with the Euclidean coordinate of $\boldsymbol{H}$. We introduce the following weighted Sobolev norms.

Definition 4.1 *Choose a function $\rho_\infty : \boldsymbol{H} \to [1, +\infty)$ such that ρ_∞ equals $|z|$ outside a compact set. For $U \subset \Sigma$ and $f, g : U \to \mathbb{R}$ define*

$$\|f\|_{\tilde{L}_h^{1,p}(U)} = \|f\|_{L^\infty(U)} + \left[\int_U |\nabla f(z)|^p [\rho_\infty(z)]^{2p-4} ds dt\right]^{\frac{1}{p}}.$$

Here the subscript h stands for "horizontal", indicating the section take value in H_X.

By the Hardy-type inequality (see (3.7)), there is a natural equivalence of Banach spaces

$$\tilde{L}_h^{1,p} \simeq W^{1,p}(\boldsymbol{D}).$$

We can define a similar space $\tilde{L}_h^{1,p}(u_\infty^* H_X)$ of sections of $u_\infty^* H_X$. The norm is defined as

$$\|\xi_\infty^H\|_{\tilde{L}_h^{1,p}} = \|\xi_\infty^H\|_{L^\infty} + \left[\int_U |P_H \nabla^{a_\infty} \xi_\infty^H|^p [\rho_\infty(z)]^{2p-4} ds dt\right]^{\frac{1}{p}}. \tag{4.3}$$

Here ∇^{a_∞} is the deformed covariant derivative defined in (3.11). So $P_H \circ \nabla^{a_\infty} \circ P_H$ is a covariant derivative on $u_\infty^* H_X$. There is a canonical identification

$$\pi_* : u_\infty^* H_X \to \bar{u}_\infty^* T\bar{X}.$$

Notice that h_X induces a Riemannian metric on $\bar{X}$, which induces a connection $\bar{\nabla}$ on $u_\infty^* T\bar{X}$.

Lemma 4.2 *One has $\pi_* \circ P^H \nabla^{a_\infty} = \bar{\nabla} \circ \pi_*$. Hence π_* induces a natural isometry between $\tilde{L}_h^{1,p}(u_\infty^* H_X)$ and $\tilde{L}_h^{1,p}(\bar{u}_\infty^* T\bar{X})$ where the Sobolev norm of the latter is defined using $\bar{\nabla}$.*

Therefore, we have an equivalence of Banach spaces

$$\tilde{L}_h^{1,p}(\boldsymbol{H}, u_\infty^* H_X) \simeq W^{1,p}(\boldsymbol{D}, \bar{u}_\infty^* T\bar{X}). \tag{4.4}$$

Consider a small ball of $\tilde{L}_h^{1,p}(u_\infty^* H_X)$ centered at the origin, denoted by $\mathcal{B}_\infty^H$, and identify it via the exponential map of the metric h_X provided by Lemma 3.1 with a space of maps from Σ to $\mu^{-1}(0)$. The property of h_X implies that any map obtained in this way has image lying in $\mu^{-1}(0)$ and boundary values lying in L. Any $u'_\infty \in \mathcal{B}^H_\infty$ pulls back the canonical connection on $\mu^{-1}(0)$ to a gauge field $\phi'_\infty ds + \psi'_\infty dt$ of class L^p_{loc}. A corollary of (4.4) is the following.

Corollary 4.3 *There are smooth evaluation maps*

$$\mathrm{ev}^\infty_\infty : \mathcal{B}^H_\infty \to \bar{L}, \quad \mathrm{ev}^i_\infty : \mathcal{B}^H_\infty \to \bar{X}, \ 1 \le i \le m, \quad \underline{\mathrm{ev}}^j_\infty : \mathcal{B}^H_\infty \to \bar{L}, \ 1 \le j \le \underline{m}. \tag{4.5}$$

Define a Banach space bundle $\mathcal{E}^H_\infty \to \mathcal{B}^H_\infty$ whose fibre over u'_∞ is the space of sections of $(u'_\infty)^* H_X$ of class $\tilde{L}^p_h$. Using the parallel transport of H_X w.r.t. the connection $P_H \circ \nabla \circ P_H$, one can trivialize this bundle $\mathcal{E}^H_\infty$ so each fibre is identified with the fibre at the central element u_∞. Define a section $\mathcal{F}^H_\infty : \mathcal{B}^H_\infty \to \mathcal{E}^H_\infty$ as

$$\mathcal{F}^H_\infty(u'_\infty) := \mathcal{F}^H_\infty(\xi^H_\infty) = P_H(\partial_s u'_\infty + \mathcal{X}_{\phi'_\infty} + J(\partial_t u'_\infty + \mathcal{X}_{\psi'_\infty})). \tag{4.6}$$

Then Lemma 4.2 implies that $\mathcal{F}^H_\infty$ corresponds to the standard Cauchy–Riemann equation.

4.3 An Augmentation

Now we would like to include deformation of u_∞ in all directions in the space X. Recall that one has the splitting

$$u^*_\infty TX = u^*_\infty H_X \oplus u^*_\infty G_X.$$

We also want to include deformations of the gauge field. Denote

$$\boldsymbol{G}_X = G_X \oplus \mathfrak{k} \oplus \mathfrak{k}.$$

We would like to define another weighted Sobolev norms for sections of $u^*_\infty \boldsymbol{G}_X$.

Definition 4.4 *Let $U \subset \Sigma$ be an open subset.*

(a) *For functions $f : U \to \mathbb{R}$, define*

$$\|f\|_{\tilde{L}^{k,p}_g(U)} = \sum_{i=0}^k \|\nabla^i f\|_{\tilde{L}^p} = \sum_{i=0}^k \left[\int_U |\nabla^i f|^p [\rho_\infty(z)]^{2p-4} ds dt\right]^{\frac{1}{p}}.$$

(b) *Let $\tilde{L}^{k,p}_g(U, u^*_\infty G_X)$ (resp. $\tilde{L}^{k,p}_g(U, u^*_\infty \boldsymbol{G}_X)$) be the space of section of $u^*_\infty G_X$ (resp. $u^*_\infty \boldsymbol{G}_X$) of class $\tilde{L}^{k,p}_g$ w.r.t. the connection $P_G \circ \nabla^{a_\infty} \circ P_G$. We add a subscript "$_L$" to the notations to indicate the boundary condition, i.e., the ζ-component vanishes on the boundary.*

(c) *For any $\boldsymbol{\xi}_\infty = (\xi^H_\infty, \xi^G_\infty, \eta_\infty, \zeta_\infty) \in W^{1,p}_{\mathrm{loc}}(U, u^*_\infty H_X \oplus \mathfrak{k} \oplus \mathfrak{k})$, define*

$$\|\boldsymbol{\xi}_\infty\|_{\tilde{L}^{1,p}_m} = \|\xi^H_\infty\|_{\tilde{L}^{1,p}_h} + \|\boldsymbol{\xi}^G_\infty\|_{\tilde{L}^{1,p}_g}. \tag{4.7}$$

The norm (4.7) is defined using the diagonal part of the connection ∇^{a_∞} w.r.t. the splitting $H_X \oplus G_X$. The following lemma says that it is equivalent to use ∇^{a_∞}.

Lemma 4.5 *The norm on the direct sum*

$$\mathcal{B}_\infty = \tilde{L}^{1,p}_h(\Sigma, u^*_\infty H_X)_L \oplus \tilde{L}^{1,p}_g(\Sigma, u^*_\infty \boldsymbol{G}_X)_L \tag{4.8}$$

is equivalent to the norm defined by

$$\|\xi^H_\infty\|_{L^\infty} + \|\boldsymbol{\xi}^G_\infty\|_{\tilde{L}^p_g} + \|\nabla^{a_\infty} \xi^H_\infty\|_{\tilde{L}^p} + \|\nabla^{a_\infty} \boldsymbol{\xi}^G_\infty\|_{\tilde{L}^p}. \tag{4.9}$$

Namely, the norms defined by using ∇ and by using the diagonal part of ∇ are equivalent.

Proof The direct sum norm on $\mathcal{B}_\infty$ is no greater than the norm (4.9). It remains to show that the norm (4.9) (actually only the last two parts) can be controlled by the direct sum norm.

The difference between these two norms comes from the fact that the connection ∇^{a_∞} does not necessarily preserve the splitting $H_X \oplus G_X$. w.r.t. this splitting, we can write

$$\nabla = \begin{bmatrix} \nabla^H & E' \\ E'' & \nabla^G \end{bmatrix}$$

where E', E'' are smooth and K-invariant. Then we have

$$\nabla^{a_\infty}_{s/t} \xi^H_\infty - P_H \nabla^{a_\infty}_{s/t} \xi^H_\infty = E''_{\boldsymbol{v}_{\infty,s/t}}(\xi^H_\infty), \quad \nabla^{a_\infty}_{s/t} \xi^G_\infty - P_G \nabla^{a_\infty}_{s/t} \xi^G_\infty = E'_{\boldsymbol{v}_{\infty,s/t}}(\xi^G_\infty),$$

where the notation indicates that E' and E'' are linear in $\boldsymbol{v}_{\infty,s}$ and $\boldsymbol{v}_{\infty,t}$. By straightforward calculation, we have

$$\|\boldsymbol{v}_{\infty,s/t}\|_{\tilde{L}^p_g}, \quad \|\boldsymbol{v}_{\infty,s/t}\|_{\tilde{L}^p_h} < +\infty. \tag{4.10}$$

Moreover, we have Sobolev embedding type estimate

$$\|\xi^H_\infty\|_{L^\infty} \lesssim \|\xi^H_\infty\|_{\tilde{L}^{1,p}_h} \leq \|\xi^H_\infty\|_{\tilde{L}^{1,p}_m}, \quad \|\xi^G_\infty\|_{L^\infty} \lesssim \|\xi^G_\infty\|_{\tilde{L}^{1,p}_g} \lesssim \|\xi^G_\infty\|_{\tilde{L}^{1,p}_m}. \tag{4.11}$$

The former follows from the definition of $\tilde{L}^{1,p}_h$ and the latter follows from the fact that $P_G \nabla^{a_\infty}$ is a connection on $u^* \boldsymbol{G}_X$ that preserves the metric (see [12, Remark 3.5.1]). Hence we have

$$\|\nabla^{a_\infty} \xi^H_\infty\|_{\tilde{L}^p} = \|P_H \nabla^{a_\infty} \xi^H_\infty\|_{\tilde{L}^p} + \|E'' \xi^H_\infty\|_{\tilde{L}^p} \lesssim \|\xi^H_\infty\|_{\tilde{L}^{1,p}_h} + \|\xi^H_\infty\|_{L^\infty} \|d_{a_\infty} \boldsymbol{v}_\infty\|_{\tilde{L}^p} \lesssim \|\xi^H_\infty\|_{\tilde{L}^{1,p}_h},$$

$$\|\nabla^{a_\infty} \xi^G_\infty\|_{\tilde{L}^p} = \|P_G \nabla^{a_\infty} \xi^G_\infty\|_{\tilde{L}^p} + \|E' \xi^G_\infty\|_{\tilde{L}^p_h} \lesssim \|\xi^G_\infty\|_{\tilde{L}^{1,p}} + \|\xi^G_\infty\|_{L^\infty} \|d_{a_\infty} \boldsymbol{v}_\infty\|_{\tilde{L}^p} \lesssim \|\xi^G_\infty\|_{\tilde{L}^{1,p}_g}.$$

4.4 The Linear Operators

Now we consider the deformation theory of $\boldsymbol{v}_\infty$. As discussed in Subsection 4.1, we should regard u_∞ as the independent variable and a_∞ is the pull-back of the connection on $\mu^{-1}(0) \to \bar{X}$ by u_∞. Then modulo gauge transformation, the space infinitesimal deformations of u_∞ can be identified with $\tilde{L}^{1,p}_h(\Sigma, u^*_\infty H_X)_L$.

We use the connection $\nabla^H := P^H \circ \nabla \circ P^H$ to trivialize $\mathcal{E}^H_\infty$ near u_∞. Let the linearization of $\mathcal{F}^H_\infty$ at u_∞ be

$$d\mathcal{F}^H_\infty(u_\infty) : \tilde{L}^{1,p}_h(\Sigma, u^*_\infty H_X)_L \to \tilde{L}^p(\Sigma, u^*_\infty H_X) = \mathcal{E}^H_\infty|_{u_\infty}. \tag{4.12}$$

On the other hand, u_∞ is also a solution to

$$\overline{\partial}_{a_\infty} u_\infty := \partial_s u_\infty + \mathcal{X}_{\phi_\infty}(u_\infty) + J(\partial_t u_\infty + \mathcal{X}_{\psi_\infty}(u_\infty)).$$

We use the Levi–Civita connection ∇ of h_X to formally define a linearization, which reads

$$D_\infty(\xi) = \nabla_s \xi + \nabla_\xi \mathcal{X}_{\phi_\infty} + (\nabla_\xi J)(\partial_t u_\infty + \mathcal{X}_{\psi_\infty}) + J(\nabla_t \xi + \nabla_\xi \mathcal{X}_{\psi_\infty}), \quad \forall \xi_\infty \in \Gamma(u^*_\infty TX). \tag{4.13}$$

Using the decomposition $H_X \oplus G_X$, one can write D_∞ in the block matrix form

$$D_\infty = \begin{bmatrix} D^H_\infty & E^1_\infty \\ E^2_\infty & D^G_\infty \end{bmatrix}. \tag{4.14}$$

Lemma 4.6 *$D^H_\infty = d\mathcal{F}^H_\infty$ where $\mathcal{F}^H_\infty$ is defined by (4.6).*

Proof Take $\xi_\infty^H \in u_\infty^* H_X$ and define $u_\infty^\varepsilon = \exp_{u_\infty} \varepsilon \xi_\infty^H$. Let the pullback connection form be $\phi_\infty^\varepsilon ds + \psi_\infty^\varepsilon dt$. Then we have

$$\partial_s u_\infty^\varepsilon + \mathcal{X}_{\phi_\infty^\varepsilon}, \partial_t u_\infty^\varepsilon + \mathcal{X}_{\psi_\infty^\varepsilon} \in H_X. \tag{4.15}$$

Let P_ε be the parallel transport w.r.t. ∇ and P_ε^H be the parallel transport w.r.t. ∇^H. When restrict to the distribution H_X, one can see that $P_\varepsilon^H = P_H \circ P_\varepsilon$. Then by (4.15), one has

$$\begin{aligned} d\mathcal{F}_\infty^H(\xi_\infty^H) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} P_\varepsilon^H(\partial_s u_\infty^\varepsilon + \mathcal{X}_{\phi_\infty^\varepsilon} + J(\partial_t u_\infty^\varepsilon + \mathcal{X}_{\psi_\infty^\varepsilon})) \\ &= P_H \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} P_\varepsilon(\partial_s u_\infty^\varepsilon + \mathcal{X}_{\phi_\infty^\varepsilon} + J(\partial_t u_\infty^\varepsilon + \mathcal{X}_{\psi_\infty^\varepsilon})) \\ &= P_H D_\infty(\xi_\infty^H) + P_H(\mathcal{X}_{\frac{d\phi_\infty^\varepsilon}{d\varepsilon}} + J\mathcal{X}_{\frac{d\psi_\infty^\varepsilon}{d\varepsilon}}) = P_H D_\infty(\xi_\infty^H). \end{aligned}$$

Later we will need the following fact.

Lemma 4.7 D_∞^G, E_∞^1, E_∞^2 *define bounded operators*

$$\begin{aligned} &D_\infty^G : \tilde{L}_g^{1,p}(u_\infty^* G_X)_L \to \tilde{L}^p(u_\infty^* G_X); \\ &E_\infty^1 : \tilde{L}_g^{1,p}(u_\infty^* G_X)_L \to \tilde{L}^p(u_\infty^* H_X), \quad E_\infty^2 : \tilde{L}_h^{1,p}(u_\infty^* H_X)_L \to \tilde{L}^p(u_\infty^* G_X). \end{aligned}$$

Proof Consider

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} P_\varepsilon(\partial_s u_\infty^\varepsilon + \mathcal{X}_{\phi_\infty}(u_\infty^\varepsilon) + J(\partial_t u_\infty^\varepsilon + \mathcal{X}_{\psi_\infty}(u_\infty^\varepsilon))) = \nabla_s^{a_\infty}\xi_\infty + J\nabla_t^{a_\infty}\xi_\infty + (\nabla_{\xi_\infty} J)\boldsymbol{v}_{\infty,t}.$$

If ξ_∞ is a section of $u_\infty^* H_X$, then we have

$$\|D_\infty^G(\xi_\infty)\|_{\tilde{L}^p} \le \|\xi_\infty\|_{\tilde{L}_g^{1,p}} + \|P_G(\nabla_{\xi_\infty} J)\boldsymbol{v}_{\infty,t}\|_{\tilde{L}^p} \lesssim \|\xi_\infty\|_{\tilde{L}_g^{1,p}} + \|\xi_\infty\|_{L^\infty}\|\boldsymbol{v}_{\infty,t}\|_{\tilde{L}^p} \lesssim \|\xi_\infty\|_{\tilde{L}_g^{1,p}}.$$

Therefore D_∞^G is a bounded operator. On the other hand, it is easy to see that E_1^∞ and E_2^∞ are linear in $ds \otimes \boldsymbol{v}_{\infty,s} + dt \otimes \boldsymbol{v}_{\infty,t}$. Therefore, we have the following estimates. First,

$$\begin{aligned} \|E_2^\infty(\xi_\infty^H)\|_{\tilde{L}^p} &\le \|\xi_\infty^H\|_{L^\infty}\left[\int_\Sigma |d_{a_\infty} u_\infty(z)|^p[\rho_\infty(z)]^{2p-4} dsdt\right]^{\frac{1}{p}}, \\ \|E_1^\infty(\xi_\infty^G)\|_{\tilde{L}^p} &\le \|\xi_\infty^G\|_{L^\infty}\left[\int_U |d_{a_\infty} u_\infty(z)|^p[\rho_\infty(z)]^{2p-4} dsdt\right]^{\frac{1}{p}}, \end{aligned}$$

where the integrals are finite due to the asymptotic property of $\boldsymbol{v}_\infty$. □

4.5 Deformations of Disks

Now we fix a holomorpic disk $\bar{u}_\infty : \boldsymbol{D} \to \bar{X}$ and consider a nearby one which can be written as

$$\bar{u}'_\infty = \overline{\exp}_{\bar{u}_\infty} \bar{\xi}_\infty.$$

If we lift $\bar{u}_\infty$ to a map $u_\infty : \boldsymbol{D} \to \mu^{-1}(0)$, then a convenient lift of $\bar{u}'_\infty$ is

$$u'_\infty = \exp_{u_\infty} \xi_\infty^H$$

where $\xi_\infty^H \in \Gamma(\boldsymbol{D}, u_\infty^* H_X)$ is the horizontal lift of $\bar{\xi}_\infty$, over the disk we have elliptic estimate and hence there are estimates

$$\|\xi_\infty^H\|_{W^{k,p}(\boldsymbol{D})} \le c_k \|\xi_\infty^H\|_{W^{1,p}(\boldsymbol{D})}.$$

Now consider the associated gauge fields $a_\infty, a'_\infty \in \Omega^1(\boldsymbol{D}, \mathfrak{k})$. It is easy to see that

$$\|a'_\infty - a_\infty\|_{W^{1,p}(\boldsymbol{D})} \lesssim \|\nabla^H \xi_\infty^H\|_{W^{1,p}(\boldsymbol{D})}.$$

We would like to write this estimate in terms of norms $\|\cdot\|_{\tilde{L}_g^{1,p}(\boldsymbol{H})}$ and $\|\cdot\|_{\tilde{L}_h^{1,p}(\boldsymbol{H})}$.

Since the Euclidean metric on compact regions of $\boldsymbol{H}$ is comparable to the standard metric on the corresponding region of $\boldsymbol{D}$, the only possible divergence may happen near ∞. Let $C_R \subset \boldsymbol{H}$ be the complement of a large ball B_R centered at the origin. Then we have

$$\begin{aligned}\|a'_\infty - a_\infty\|_{\tilde{L}^p} &= \left[\int_{C_R} |a'_\infty - a_\infty|^p [\rho_\infty(z)]^{2p-4} dsdt\right]^{\frac{1}{p}} \\ &\approx \left[\int_{C_R} |a'_\infty - a_\infty|_{\boldsymbol{D}}^p |z|^{-2p} [\rho_\infty(z)]^{2p-4} dsdt\right]^{\frac{1}{p}} \approx \|a'_\infty - a_\infty\|_{L^p(\boldsymbol{D})}.\end{aligned}$$

Here $|\cdot|_{\boldsymbol{D}}$ means the norm on tensors with respect to the metric on $\boldsymbol{D}$. On the other hand, over C_R, since the gauge field a_∞ is bounded with respect to the Euclidean metric, one has

$$|\nabla^{a_\infty}(a'_\infty - a_\infty)| \approx |a'_\infty - a_\infty| + |\nabla(a'_\infty - a_\infty)| \approx |z|^{-2}|a'_\infty - a_\infty| + |z|^{-4}|\nabla(a'_\infty - a_\infty)|_{\boldsymbol{D}}.$$

It follows that $\|\nabla^{a_\infty}(a'_\infty - a_\infty)\|_{\tilde{L}^p}$ is also bounded by $\|a'_\infty - a_\infty\|_{L^{1,p}(\boldsymbol{D})}$. We summarize this estimate as follows.

Lemma 4.8 *Let $u_\infty : \boldsymbol{H} \to \mu^{-1}(0)$ project down to a holomorphic disk in $\bar{X}$ and $u'_\infty = \exp_{u_\infty} \xi_\infty^H$ be another holomorphic disk where $\xi_\infty^H \in \Gamma(\boldsymbol{H}, u_\infty^* H_X)$. Let a_∞ and a'_∞ be the gauge fields on $\boldsymbol{H}$ by pulling back the canonical connection via u_∞ and u'_∞ respectively. Then there is a constant $c > 0$ such that if the two holomorphic disks are sufficiently close, one has*

$$\|a'_\infty - a_\infty\|_{\tilde{L}_g^{1,p}} + \|\xi_\infty^H\|_{\tilde{L}_h^{1,p}} \leq c\|\bar{\xi}_\infty^H\|_{W^{1,p}(\boldsymbol{D})}.$$

5 Statement of the Gluing Theorem

In this section we state the main theorem of this paper, under a precise version of the transversality assumption.

5.1 Perturbation

Let us fix a representative of a point $[\boldsymbol{x}_\bullet] \in \overline{\mathcal{N}}_{l,\underline{l}}$ of type ♣ denoted by

$$\boldsymbol{x}_\bullet = (\boldsymbol{x}_{\bullet,1}, \ldots, \boldsymbol{x}_{\bullet,m}, \boldsymbol{y}_{\bullet,1}, \ldots, \boldsymbol{y}_{\bullet,\underline{m}}, \boldsymbol{z}_\bullet).$$

Recall that we have a universal family for the moduli $\overline{\mathcal{N}}_{l,\underline{l}}$. Consider a small neighborhood $\overline{\mathcal{Q}}_\bullet^\epsilon$ which by Lemma 2.2 is identified with $[0,\epsilon) \times W_{\rm def}^\epsilon$, and the origin is identified with $[\boldsymbol{x}_\bullet]$. The restriction of the universal curve to $\overline{\mathcal{Q}}_\bullet^\epsilon$ is denoted by $\overline{\mathcal{U}}_\bullet^\epsilon$. There is a closed subset $\overline{\mathcal{U}}_\bullet^{\epsilon,{\rm sing}} \subset \overline{\mathcal{U}}_\bullet^\epsilon$ corresponding to the marked points and nodal points of the fibres.

Definition 5.1 *A perturbation datum near $[\boldsymbol{x}_\bullet]$ consists of a finite dimensional vector space $W_{\rm per}$ and for some small ϵ_0, a (not necessarily linear) smooth map*

$$\iota : W_{\rm per} \to \Gamma_c((\overline{\mathcal{U}}_\bullet^{\epsilon_0} \setminus \overline{\mathcal{U}}_\bullet^{\epsilon_0,{\rm sing}}) \times X, TX). \tag{5.1}$$

Here Γ_c means smooth sections whose fibrewise supports are compact. Moreover we require that ι satisfies the following conditions.

(a) ι *is* K*-equivariant, namely, for* $\boldsymbol{e} \in W_{\rm per}$*,* $z \in \overline{\mathcal{U}}_{\bullet}^{\epsilon_0} \setminus \overline{\mathcal{U}}_{\bullet}^{\epsilon_0, {\rm sing}}$*,* $g \in K$ *and* $x \in X$*,*

$$\iota(\boldsymbol{e}, z, gx) = g_* \iota(\boldsymbol{e}, z, x) \in T_{gx} X.$$

(b) ι *is supported in an open neighborhood of* $\mu^{-1}(0)$ *where we can decompose* TX *as the direct sum of* H_X *and* G_X*. Moreover, when restricted to disk components (i.e., the root component of the configurations which are supposed to be mapped into* $\bar{X}$*), the values of* ι *are contained in* H_X*.*

Notation Given a perturbation data ι, suppose u is a smooth or continuous map u from the fibre of $\overline{\mathcal{U}}_{\bullet}^{\epsilon} \to \overline{\mathcal{Q}}_{\bullet}^{\epsilon}$ over $\boldsymbol{x}_{\bullet, \epsilon}(\boldsymbol{a})$, and $\boldsymbol{e} \in W_{\rm per}$. Then they induce a section of $u^* TX$ (the inhomogeneous term) denoted by

$$\iota(\boldsymbol{e}, \boldsymbol{x}_{\bullet, \epsilon}(\boldsymbol{a}), u).$$

In many situations we combine $\boldsymbol{e}$ and $\boldsymbol{a}$ and denote $\boldsymbol{w} = (\boldsymbol{e}, \boldsymbol{a})$. So the induced inhomogeneous term is also denoted by $\iota_\epsilon(\boldsymbol{w}, u)$. Moreover, when $\epsilon = 0$, u has multiple components. For each component u_i or u_∞, denote the restriction to that component by $\iota(\boldsymbol{e}, \boldsymbol{x}_\bullet + \boldsymbol{a}, u_i)$ or $\iota(\boldsymbol{e}, \boldsymbol{x}_\bullet + \boldsymbol{a}, u_\infty)$.

Remark 5.2 In the setting of [33], where we will use the stabilizing divisor technique of Cieliebak–Mohnke [4], we consider almost complex structures J_z that depend on the point z on the universal curve over $\overline{\mathcal{N}}_{l, \underline{l}}$. Then the difference of J_z from the fixed almost complex structure is a perturbation data ι satisfying the above definition.

On the other hand, in the Kuranishi setting, we can choose perturbation for each domain component independently. In that case $W_{\rm per}$ is the direct sum of different summands.

5.2 Stable Affine Vortices

We only consider a "partial" compactification of the moduli space of perturbed $\boldsymbol{H}$-vortices over a smooth domain.

Definition 5.3 *A stable* $\boldsymbol{H}$*-vortex of type* ♣ *consists of the following objects*

(a) *A representative* $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m; \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{\underline{m}}; \boldsymbol{z})$ *representing a point* $[\boldsymbol{x}] \in \overline{\mathcal{Q}}_{\bullet}^{\epsilon_0} \cap \mathcal{N}_{\clubsuit}$*, which can be written as* $\boldsymbol{x} = \boldsymbol{x}_\bullet + \boldsymbol{a}$ *for certain small deformation parameter* $\boldsymbol{a} \in W_{\rm def}$*.*

(b) *An element* $\boldsymbol{e} \in W_{\rm per}$*.*

(c) *A collection of gauged maps* $\boldsymbol{v} = (\boldsymbol{v}_\infty, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m, \underline{\boldsymbol{v}}_1, \ldots, \underline{\boldsymbol{v}}_{\underline{m}})$ *where* $\boldsymbol{v}_\infty$ *and* $\underline{\boldsymbol{v}}_1, \ldots, \underline{\boldsymbol{v}}_{\underline{m}}$ *are defined on* $\boldsymbol{H}$ *with boundaries lie in* L *and* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ *are defined on* $\boldsymbol{C}$*.*

They satisfy the following conditions. (Here we use the alternate local model introduced in Subsection 2.3*, so we regard* $\boldsymbol{z}$ *as the same as* $\boldsymbol{z}_\bullet$ *but on* $\boldsymbol{H}$ *equipped with a different complex structure* $j_{\boldsymbol{q}}$ *and different holomorphic coordinate* $z_{\boldsymbol{q}}$*.)*

(a) **(Perturbed Equation)** $\boldsymbol{v}_\infty = (u_\infty, \phi_\infty, \psi_\infty)$ *satisfies*

$$\partial_s u_\infty + \mathcal{X}_{\phi_\infty} + J(\partial_t u_\infty + \mathcal{X}_{\psi_\infty}) + \iota(\boldsymbol{e}, \boldsymbol{x}, u_\infty) = 0, \quad \mu(u_\infty) = 0, \quad u_\infty(\partial \boldsymbol{H}) \subset L. \tag{5.2}$$

For $i = 1, \ldots, m + \underline{m}$*,* $\boldsymbol{v}_i = (u_i, \phi_i, \psi_i)$ *satisfies*

$$\partial_s u_i + \mathcal{X}_{\phi_i} + J(\partial_t u_i + \mathcal{X}_{\psi_i}) + \iota(\boldsymbol{e}, \boldsymbol{x}, u_i) = 0, \ \partial_s \psi_i - \partial_t \phi_i + [\phi_i, \psi_i] + \mu(u_i) = 0, \ u_i(\partial \boldsymbol{H}) \subset L. \tag{5.3}$$

(b) **(Matching condition)** $\boldsymbol{v}_i(\infty) = \bar{u}_\infty(z_i) \in \bar{X}$ *for* $i = 1, \ldots, m$ *and* $\underline{\boldsymbol{v}}_j(\infty) = \bar{u}_\infty(\underline{z}_j) \in \bar{L}$ *for* $j = 1, \ldots, \underline{m}$*. Here* $\bar{u}_\infty$ *is the map to* $\bar{X}$ *induced from* $\boldsymbol{v}_\infty$*.*

Remark 5.4 There are two reasons why we do not consider the case of having unstable components. If there are unstable components one just adds extra marked points to stabilize, which does not make an essential difference. On the other hand, in the case of using stabilizing divisors and domain dependent almost complex structures (which is the approach of [33]), there cannot be unstable components in a stable object.

Let $\tilde{\mathcal{M}}^{\epsilon_0}_{\clubsuit}(\boldsymbol{H}; X, L)$ be the set of stable $\boldsymbol{H}$-vortices of type $\clubsuit$ as defined above. The corresponding objects on smooth domains are defined as follows. Let $\tilde{\mathcal{M}}^{\epsilon_0}_{l,\underline{l}}(\boldsymbol{H}; X, L)$ consist of triples $(\boldsymbol{e}, \boldsymbol{x}_{\bullet,\epsilon}(\boldsymbol{a}), \boldsymbol{v})$ where $\boldsymbol{e} \in W_{\rm per}$, $\epsilon \in (0, \epsilon_0]$, $\boldsymbol{a} \in W^{\epsilon_0}_{\rm def}$ and $\boldsymbol{v} = (u, \phi, \psi)$ is a gauged map from $\boldsymbol{H}$ to X, which satisfies the perturbed equation:

$$\partial_s u + \mathcal{X}_\phi + J(\partial_t u + \mathcal{X}_\psi) + \iota_\epsilon(\boldsymbol{e}, \boldsymbol{a}, u) = 0, \quad \partial_s \psi - \partial_t \phi + [\phi, \psi] + \mu(u) = 0, \quad u(\partial \boldsymbol{H}) \subset L.$$

Definition 5.5 *Let $(\boldsymbol{e}_n, \boldsymbol{x}_{\bullet,\epsilon_n}(\boldsymbol{a}_n), \boldsymbol{v}_n)$, $n = 1, 2, \ldots$ be a sequence of elements in $\tilde{\mathcal{M}}^{\epsilon_0}_{l,\underline{l}}(\boldsymbol{H}; X, L)$ and $(\boldsymbol{e}, \boldsymbol{x}_\bullet + \boldsymbol{a}, \boldsymbol{v})$ be an element of $\tilde{\mathcal{M}}^{\epsilon_0}_{\clubsuit}(\boldsymbol{H}; X, L)$ where $\boldsymbol{v}$ has components $\boldsymbol{v}_\infty$, $\boldsymbol{v}_1$, …, $\boldsymbol{v}_m$, $\underline{\boldsymbol{v}}_1$, …, $\underline{\boldsymbol{v}}_{\underline{m}}$, as described in Definition 5.3. We say that $(\boldsymbol{e}_n, \boldsymbol{x}_{\bullet,\epsilon_n}(\boldsymbol{a}_n), \boldsymbol{v}_n)$ converges to $(\boldsymbol{e}, \boldsymbol{x}_\bullet(\boldsymbol{a}), \boldsymbol{v})$ if after removing finitely many terms in this sequence, the following conditions are satisfied.*

(a) *ϵ_n converges to 0, $\boldsymbol{a}_n$ converges to $\boldsymbol{a}$ and $\boldsymbol{e}_n$ converges to $\boldsymbol{e}$.*

(b) *For $i = 1, \ldots, m + \underline{m}$, define the translation*

$$\mathsf{t}_{i,n}(z) = z - \frac{z_{\bullet,i}}{\epsilon_n}.$$

Then the sequence of gauged maps $\boldsymbol{v}_n \circ \mathsf{t}_{i,n}$ converges in c.c.t. on $\boldsymbol{C}$ or $\boldsymbol{H}$ to $\boldsymbol{v}_i$.

(c) *Define $\varphi_n(z) = \epsilon_n z$. Then $u_n \circ \varphi_n^{-1}$ converges uniformly on compact subsets of $\boldsymbol{H} \setminus \{z_1, \ldots, z_m, \underline{z}_1, \ldots, \underline{z}_{\underline{m}}\}$ to u_∞ (we do not require the convergence of gauge fields).*

(d) *There is no energy loss, i.e.,*

$$\lim_{n\to\infty} E(\boldsymbol{v}_n) = \sum_{i=1}^{m} E(\boldsymbol{v}_i) + \sum_{j=1}^{\underline{m}} E(\underline{\boldsymbol{v}}_j) + E(\boldsymbol{v}_\infty).$$

By the K-equivariance of the perturbation ι, one can define an equivalence relation among perturbed stable affine vortices using gauge transformations. We omit the detailed definition. For any $\epsilon \in (0, \epsilon_0]$, let $\mathcal{M}^{\epsilon}_{l,\underline{l}}(\boldsymbol{H}; X, L)$ (resp. $\mathcal{M}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L)$) be the set of gauge equivalence classes in $\tilde{\mathcal{M}}^{\epsilon}_{l,\underline{l}}(\boldsymbol{H}; X, L)$ (resp. $\tilde{\mathcal{M}}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L)$). Define

$$\overline{\mathcal{M}}^{\epsilon}_{l,\underline{l}}(\boldsymbol{H}; X, L) := \mathcal{M}^{\epsilon}_{l,\underline{l}}(\boldsymbol{H}; X, L) \sqcup \mathcal{M}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L).$$

where the former component is called the "top stratum" and the latter called the "lower stratum". Definition 5.5 plus the obvious topology inside the lower stratum induces a topology on $\overline{\mathcal{M}}^{\epsilon}_{l,\underline{l}}(\boldsymbol{H}; X, L)$, which can be proved to be Hausdorff. The main theorem of this paper is, under certain transversality assumption, one can give a chart of (topological) manifold with boundary around a central object in $\mathcal{M}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L)$. To state the main theorem in precise language, we first need to state the transversality assumption.

Remark 5.6 When constructing the local chart, especially when proving the local surjectivity of the gluing map, one has to prove the notion of convergence given by Definition 5.5 (which is weaker) implies the stronger convergence in terms of Banach space norms. The fact that weaker convergence implies stronger convergence will be proved later in this paper.

5.3 Transversality and the Gluing Map

We fix a central singular object $(\boldsymbol{e}_\bullet, \boldsymbol{x}_\bullet, \boldsymbol{v}_\bullet)$ in which we assume the underlying curve $[\boldsymbol{x}_\bullet]$ is the central one we have fixed. Let $\boldsymbol{v}_\bullet$ have components $\boldsymbol{v}_{\bullet,\infty}, \boldsymbol{v}_{\bullet,1}, \ldots, \boldsymbol{v}_{\bullet,m+\underline{m}}$. Let $\boldsymbol{w}_\bullet = (\boldsymbol{e}_\bullet, 0) \in W = W_{\rm per} \times W_{\rm def}$ which parametrizes the perturbation term. Let $\bar{x}_{\bullet,i} \in \bar{X}$ (resp. $\underline{\bar{x}}_{\bullet,j} \in \bar{L}$) be the limit of $\boldsymbol{v}_{\bullet,i}$ (resp. $\underline{\boldsymbol{v}}_{\bullet,j}$) at infinity.

Recall the set-up of Subsection 3.2. For each affine vortex component $\boldsymbol{v}_{\bullet,i}$, there is a Banach manifold $\mathcal{B}_i$ containing $\boldsymbol{v}_{\bullet,i}$, a Banach vector bundle $\mathcal{E}_i \to \mathcal{B}_i$, and a smooth Fredholm section

$$\mathcal{F}_i : \mathcal{B}_i \to \mathcal{E}_i.$$

The perturbation ι given by (5.1) restricts to the i-th component to a smooth map

$$\iota_i : W \times \mathcal{B}_i \to \mathcal{E}_i.$$

Define

$$\tilde{\mathcal{F}}_i : W \times \mathcal{B}_i \to \mathcal{E}_i$$

to be the sum of $\mathcal{F}_i$ and ι_i. Then (5.3) implies that $\tilde{\mathcal{F}}_i(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,i}) = 0$. We have a similar section associated to the ∞ component

$$\tilde{\mathcal{F}}_\infty^H : W \times \mathcal{B}_\infty^H \to \mathcal{E}_\infty^H$$

such that $\tilde{\mathcal{F}}_\infty^H(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\infty}) = 0$. Let the linearizations of $\tilde{\mathcal{F}}_i$ and $\tilde{\mathcal{F}}_\infty^H$ be

$$\tilde{\mathcal{D}}_i : W \oplus \mathcal{B}_i \to \mathcal{E}_i, \quad \tilde{\mathcal{D}}_\infty^H : W \oplus \mathcal{B}_\infty^H \to \mathcal{E}_\infty^H$$

respectively, where we abuse the notations by identifying the Banach manifolds with tangent spaces. There are also smooth evaluation maps

$${\rm ev}_i : \mathcal{B}_i \to \bar{X},\ i = 1, \ldots, m; \quad \underline{{\rm ev}}_j : \underline{\mathcal{B}}_j \to \bar{L},\ j = 1, \ldots, \underline{m},$$

such that ${\rm ev}_i(\boldsymbol{v}_i) = \bar{x}_{\bullet,i}$ for all $i = 1, \ldots, m + \underline{m}$. There are also smooth evaluation maps

$${\rm ev}_\infty^i : \mathcal{B}_\infty^H \to \bar{X},\ i = 1, \ldots, m; \quad \underline{{\rm ev}}_\infty^j : \mathcal{B}_\infty^H \to \bar{L},\ j = 1, \ldots, \underline{m}$$

such that ${\rm ev}_\infty^i(u_{\bullet,\infty}) = \bar{x}_{\bullet,i}$ for $i = 1, \ldots, m + \underline{m}$.

Hypothesis 2

(a) For $i = 1, \ldots, m + \underline{m}$, the linearization of $\tilde{\mathcal{F}}_i$ at $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,i})$, denoted by $\tilde{\mathcal{D}}_i : W \oplus \mathcal{B}_i \to \mathcal{E}_i$, is surjective. In short, each affine vortex plus the perturbation term is regular.

(b) The linearization of $\tilde{\mathcal{F}}_\infty^H$ at $(\boldsymbol{w}_\bullet, u_{\bullet,\infty})$, denoted by $\tilde{D}_\infty^H : W \times \mathcal{B}_\infty^H \to \mathcal{E}_\infty^H$, is surjective. In short, the holomorphic disk in $\bar{X}$ plus the perturbation term is regular in the usual sense.

(c) The map

$$\ker(\tilde{D}_\infty^H) \oplus \prod_{i=1}^{m+\underline{m}} \ker(\tilde{\mathcal{D}}_i) \to W^{m+\underline{m}+1} \oplus \prod_{i=1}^{m} (T_{\bar{x}_i}\bar{X})^2 \oplus \prod_{j=1}^{\underline{m}} (T_{\bar{x}_j}\bar{L})^2 \tag{5.4}$$

defined by

$$\begin{aligned} &((\boldsymbol{h}_\infty, \xi_\infty^H), (\boldsymbol{h}_1, \boldsymbol{\xi}_1), \ldots, (\boldsymbol{h}_{m+\underline{m}}, \boldsymbol{\xi}_{m+\underline{m}})) \\ &\quad \mapsto ((\boldsymbol{h}_\infty, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{m+\underline{m}}), ({\rm dev}_i(\boldsymbol{\xi}_i), {\rm dev}_\infty^i(\xi_\infty^H))_{1\le i\le m}, ({\rm dev}_j(\boldsymbol{\xi}_j), {\rm dev}_\infty^j(\xi_\infty^H))_{1\le j\le \underline{m}}) \end{aligned}$$

is transverse to the diagonal

$$\Delta^{m+\underline{m}+1}(W) \oplus \prod_{i=1}^{m} \Delta(T_{\bar{x}_{\bullet,i}} \bar{X}) \oplus \prod_{j=1}^{\underline{m}} \Delta(T_{\bar{x}_{\bullet,j}} \bar{L}). \tag{5.5}$$

Now we state the main theorem of this paper, which is the precise version of Theorem 1.1.

Theorem 5.7 *Under Hypothesis* 2, *there exist* $\epsilon_1 \in (0, \epsilon_0)$, *a neighborhood* $\mathcal{U}_\clubsuit \subset \mathcal{M}_\clubsuit^{\epsilon_0}(\boldsymbol{H}; X, L)$ *of* $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$, *and a continuous map*

$$\mathrm{Glue} : \mathcal{U}_\clubsuit \times [0, \epsilon_1) \to \overline{\mathcal{M}}_{l,\underline{l}}^{\epsilon_0}(\boldsymbol{H}; X, L)$$

which is a homeomorphism onto an open neighborhood of $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$ *inside the target. Moreover,* $\mathrm{Glue}([\boldsymbol{w}, \boldsymbol{v}], 0) = [\boldsymbol{w}, \boldsymbol{v}]$ *for all* $[\boldsymbol{w}, \boldsymbol{v}] \in \mathcal{U}_\clubsuit$.

Remark 5.8 Theorem 5.7 can be certainly extended and generalized to other situations, including the unbordered case and cases for more complicated strata, or in the case where vortices are combined with other Fredholm problems (for example, in [33] we will consider flow lines of certain Morse function in the setting of "treed disks").

6 Rescaling Holomorphic Disks

In Definition 5.5 we see that the component $\boldsymbol{v}_{\bullet,\infty}$ is the limit after a large rescaling. When we glue, we need to rescale $\boldsymbol{v}_{\bullet,\infty}$ back by the gluing parameter ϵ. The main purpose of this section is then, to construct a right inverse along the singular object where the disk component is rescaled by ϵ, and to show however the norm of the right inverse is uniformly bounded by a constant independent of ϵ. Obviously this also requires a careful choice of a system of ϵ-dependent weighted Sobolev norms.

Since in this section we only consider the central object $(\boldsymbol{e}_\bullet, \boldsymbol{x}_\bullet, \boldsymbol{v}_\bullet)$, we omit the "$\bullet$" from the notations.

6.1 Rescaling of the Disk and Weighted Sobolev Norms

Let $s_\epsilon : \boldsymbol{H} \to \boldsymbol{H}$ be the rescaling map $z \mapsto \epsilon z$. Then define a gauged map $\boldsymbol{v}_{\infty,\epsilon} = (u_{\infty,\epsilon}, \phi_{\infty,\epsilon}, \psi_{\infty,\epsilon})$ on $\boldsymbol{H}$ by

$$u_{\infty,\epsilon} = s_\epsilon^* u_\infty, \quad \phi_{\infty,\epsilon} = \epsilon s_\epsilon^* \phi_\infty, \quad \psi_{\infty,\epsilon} = \epsilon s_\epsilon^* \psi_\infty.$$

In the definition of the Banach space $\mathcal{B}_\infty$ (see (4.8)), if we replace $\boldsymbol{v}_\infty$ by $\boldsymbol{v}_{\infty,\epsilon}$, the Banach space obtained is denoted by $\mathcal{B}_{\infty,\epsilon} := \mathcal{B}_{\boldsymbol{v}_{\infty,\epsilon}}$, which contains infinitesimal deformations of $\boldsymbol{v}_{\infty,\epsilon}$ as gauged maps. We also define the Banach space (bundle) $\mathcal{E}_{\infty,\epsilon} := \mathcal{E}_{\boldsymbol{v}_{\infty,\epsilon}}$ containing sections of $u_{\infty,\epsilon}^* TX \oplus \mathfrak{k} \oplus \mathfrak{k}$ over $\boldsymbol{H}$ which is finite $\tilde{L}^p$-norm. However we will redefine the norms on $\mathcal{B}_{\infty,\epsilon}$ and $\mathcal{E}_{\infty,\epsilon}$ as follows. Define $\rho_{\infty,\epsilon} : \boldsymbol{H} \to [1, +\infty)$ by

$$\rho_{\infty,\epsilon}(z) = \frac{1}{\sqrt{\epsilon}} \rho_\infty(\epsilon z). \tag{6.1}$$

Then for any open subset $U \subset \boldsymbol{H}$ and $f \in L^p_{\mathrm{loc}}(U)$, define

$$\|f\|_{\tilde{L}^p_\epsilon(U)} := \left[\int_U |f(z)|^p [\rho_{\infty,\epsilon}(z)]^{2p-4} ds dt \right]^{\frac{1}{p}}.$$

This induces a norm on the fibres of $\mathcal{E}_{\infty,\epsilon}$. For $\boldsymbol{\xi}_{\infty,\epsilon} = (\xi^H_{\infty,\epsilon}, \boldsymbol{\xi}^G_{\infty,\epsilon}) \in \mathcal{B}_{\infty,\epsilon}$, define

$$\|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(U)} := \|\boldsymbol{\xi}^G_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(U)} + \|\nabla^{a_{\infty,\epsilon}} \boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(U)} + \|\xi^H_{\infty,\epsilon}\|_{L^\infty(U)}. \tag{6.2}$$

Now we look at the relation with the previous norms under rescaling. Define

$$\hat{s}_\epsilon^* : W^{1,p}_{\rm loc}(\Sigma, u_\infty^* H_X \oplus u_\infty^* \boldsymbol{G}_X) \to W^{1,p}_{\rm loc}(\boldsymbol{H}, u_{\infty,\epsilon}^* H_X \oplus u_{\infty,\epsilon}^* \boldsymbol{G}_X),$$
$$\check{s}_\epsilon^* : L^p_{\rm loc}(\Sigma, u_\infty^* H_X \oplus u_\infty^* \boldsymbol{G}_X) \to L^p_{\rm loc}(\Sigma, u_{\infty,\epsilon}^* H_X \oplus u_{\infty,\epsilon}^* \boldsymbol{G}_X)$$

by

$$\hat{s}_\epsilon^*(\xi_\infty^H, \boldsymbol{\xi}_\infty^G) = (s_\epsilon^* \xi_\infty^H, \epsilon s_\epsilon^* \boldsymbol{\xi}_\infty^G); \quad \check{s}_\epsilon^*(\nu_\infty^H, \boldsymbol{\nu}_\infty^G) = (\epsilon s_\epsilon^* \nu_\infty^H, \epsilon s_\epsilon^* \boldsymbol{\nu}_\infty^G).$$

Moreover, for $\boldsymbol{\xi}_\infty = (\xi_\infty^H, \boldsymbol{\xi}_\infty^G) \in W^{1,p}_{\rm loc}(\Sigma, u_\infty^* H_X \oplus u_\infty^* \boldsymbol{G}_X)$, define an auxiliary norm

$$\|\boldsymbol{\xi}_\infty\|_{\tilde{L}^{1,p}_{a;\epsilon}} := \|\xi_\infty^H\|_{\tilde{L}^{1,p}_h} + \|\boldsymbol{\xi}_\infty^G\|_{\tilde{L}^p} + \epsilon \|\nabla^{a_\infty} \boldsymbol{\xi}_\infty^G\|_{\tilde{L}^p}. \tag{6.3}$$

Here the subscript a stands for "auxiliary". Regarding the auxiliary norm, we have the following result. It can be proved via straightforward calculation and we omit the proof.

Lemma 6.1 *For any $\boldsymbol{\xi}_\infty \in \mathcal{B}_\infty$ and $\boldsymbol{\nu}_\infty \in \mathcal{E}_\infty$,*

$$\|\hat{s}_\epsilon^* \boldsymbol{\xi}_\infty\|_{\tilde{L}^{1,p}_{m;\epsilon}} = \|\boldsymbol{\xi}_\infty\|_{\tilde{L}^{1,p}_{a;\epsilon}}, \quad \|\check{s}_\epsilon^* \boldsymbol{\nu}_\infty\|_{\tilde{L}^p_\epsilon} = \|\boldsymbol{\nu}_\infty\|_{\tilde{L}^p}. \tag{6.4}$$

6.2 Right Inverse on the Rescaled Object

We rescale the perturbation by introducing

$$\iota_{\infty,\epsilon} : W \to \mathcal{E}_{\infty,\epsilon}, \quad \iota_{\infty,\epsilon}(\boldsymbol{w}) = \epsilon s_\epsilon^* \iota_\infty(\boldsymbol{w}).$$

Extend the definition of $\hat{s}_\epsilon^*$ to W by $\hat{s}_\epsilon^*(\boldsymbol{w}) = \boldsymbol{w}$. Then

$$\iota_{\infty,\epsilon} \circ \hat{s}_\epsilon^* = \check{s}_\epsilon^* \iota_\infty. \tag{6.5}$$

$\boldsymbol{v}_{\infty,\epsilon}$ satisfy (4.2) after perturbing by $\iota_{\infty,\epsilon}$. Consider the section $\tilde{\mathcal{F}}_{\infty,\epsilon} : W \times \mathcal{B}_{\infty,\epsilon} \to \mathcal{E}_{\infty,\epsilon}$. Then $\tilde{\mathcal{F}}_{\infty,\epsilon}(\boldsymbol{w}, \boldsymbol{v}_{\infty,\epsilon}) = 0$. Let $\mathcal{D}_{\infty,\epsilon}$ be the linearization of the vortex equation along the gauged map $\boldsymbol{v}_{\infty,\epsilon}$, which gives a Fredholm map

$$\mathcal{D}_{\infty,\epsilon} : \mathcal{B}_{\infty,\epsilon} \to \mathcal{E}_{\infty,\epsilon}.$$

Including the contribution of perturbations, we have a total derivative

$$\tilde{\mathcal{D}}_{\infty,\epsilon} := \mathcal{D}_{\infty,\epsilon} + d\iota_{\infty,\epsilon} : W \oplus \mathcal{B}_{\infty,\epsilon} \to \mathcal{E}_{\infty,\epsilon}. \tag{6.6}$$

We intend to prove the following result.

Proposition 6.2 *There exist $\epsilon_2 > 0$ and $c_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2)$, there exists a bounded right inverse $\tilde{Q}_{\infty,\epsilon} : \mathcal{E}_{\infty,\epsilon} \to W \times \mathcal{B}_{\infty,\epsilon}$ to $\tilde{\mathcal{D}}_{\infty,\epsilon}$ such that*

$$\|\tilde{\mathcal{Q}}_{\infty,\epsilon}\| \leq c_2.$$

Proof By Lemma 6.1, it is equivalent to consider the conjugated operator

$$\tilde{\mathcal{D}}^{\rm aux}_{\infty,\epsilon} = (\check{s}_\epsilon^*)^{-1} \circ \tilde{\mathcal{D}}_{\infty,\epsilon} \circ \hat{s}_\epsilon^* : W \oplus \mathcal{B}_\infty \to \mathcal{E}_\infty$$

where on $\mathcal{B}_\infty$ (defined in (4.8)) we use the auxiliary norm $\|\cdot\|_{\tilde{L}^{1,p}_{a;\epsilon}}$. Then for $\boldsymbol{\xi}_\infty = (\xi_\infty^H, \xi_\infty^G, \eta_\infty, \zeta_\infty)$ and $\boldsymbol{h} \in W$, using (6.5) and the fact that $\iota_\infty(\boldsymbol{w})$ is contained in $u_\infty^* H_X$, one has

$$\tilde{\mathcal{D}}^{\rm aux}_{\infty,\epsilon}(\boldsymbol{h}, \boldsymbol{\xi}_\infty) = \begin{bmatrix} D_\infty^H(\xi_\infty^H) & + & E_\infty^1(\xi_\infty^G) & & & + & \iota_\infty(\boldsymbol{h}) \\ \epsilon E_\infty^2(\xi_\infty^H) & + & \epsilon D_\infty^G(\xi_\infty^G) & + & L_\infty(\eta_\infty, \zeta_\infty) & & \\ & & L_\infty^*(\xi_\infty^G) & + & \epsilon \not{D}_\infty^G(\eta_\infty, \zeta_\infty) & & \end{bmatrix}. \tag{6.7}$$

Here D^H_∞, D^G_∞, E^1_∞, E^2_∞ are the entries of (4.14), while

$$L_\infty(\eta_\infty, \zeta_\infty) = \mathcal{X}_{\eta_\infty} + J\mathcal{X}_{\zeta_\infty}, \quad L^*_\infty(\xi^G_\infty) = (d\mu \cdot J\xi^G_\infty, d\mu \cdot \xi^G_\infty).$$

$$\not{D}^G_\infty(\eta_\infty, \zeta_\infty) = (\partial_s \eta_\infty + \partial_t \zeta_\infty + [\phi_\infty, \eta_\infty] + [\psi_\infty, \zeta_\infty],\ \partial_s \zeta_\infty - \partial_t \eta_\infty + [\phi_\infty, \zeta_\infty] - [\psi_\infty, \eta_\infty]).$$

By Lemma 4.7, E^1_∞ and E^2_∞ are bounded operators, and the operator $(\boldsymbol{h}, \xi^H_\infty) \mapsto \tilde{D}_\infty(\boldsymbol{h}, \xi^H_\infty)$ has a bounded right inverse, this proposition follows from Lemma 6.3 below.

Lemma 6.3 *There exist $c_3 > 0$ and $\epsilon_3 > 0$ such that for all $\epsilon \in (0, \epsilon_3)$, the operator*

$$\mathcal{D}^{\mathrm{aux},G}_{\infty,\epsilon} : \tilde{L}^{1,p}_{a;\epsilon}(u^*_\infty \boldsymbol{G}_X) \to \tilde{L}^p(u^*_\infty \boldsymbol{G}_X),$$

defined by

$$\mathcal{D}^{\mathrm{aux},G}_{\infty,\epsilon}(\xi^G_\infty, \eta_\infty, \zeta_\infty) = (\epsilon D^G_\infty(\xi^G_\infty) + L_\infty(\eta_\infty, \zeta_\infty), \epsilon \not{D}^G_\infty(\eta_\infty, \zeta_\infty) + L^*_\infty(\xi^G_\infty))$$

is invertible and for all $\boldsymbol{\xi}^G_\infty = (\xi^G_\infty, \eta_\infty, \zeta_\infty) \in \tilde{L}^{1,p}_{a;\epsilon}(u^*_\infty \boldsymbol{G}_X)$,

$$c_3 \|\mathcal{D}^{\mathrm{aux},G}_{\infty,\epsilon}(\boldsymbol{\xi}^G_\infty)\|_{\tilde{L}^p} \geq \|\boldsymbol{\xi}^G_\infty\|_{\tilde{L}^{1,p}_{a;\epsilon}}.$$

Now we prove Lemma 6.3. Let $\delta \geq 0$ be a small number. Take $k \geq 1$ and abbreviate

$$\mathcal{B}^1 = W^{1,p,\delta}(\boldsymbol{H}, \boldsymbol{C}^k)_R, \quad \mathcal{B}^2 = L^{p,\delta}(\boldsymbol{H}, \boldsymbol{C}^k),$$

where the subscript R means the boundary values are required to be real. Let $\gamma_1, \gamma_2 : \Sigma \to \mathbb{R}^{2k \times 2k}$ be bounded continuous maps, which define zero-th order operators $\gamma_1, \gamma_2 : \mathcal{B}^1 \to \mathcal{B}^2$. Denote

$$\mathcal{D}_\gamma : \mathcal{B}^1 \oplus \mathcal{B}^1 \to \mathcal{B}^2 \oplus \mathcal{B}^2, \quad \mathcal{D}_\gamma(f_1, f_2) = (\partial_{\bar{z}} f_1 + \gamma_1(f_1) + f_2, \partial_z f_2 + \gamma_2(f_2) + f_1).$$

Here ∂_z, $\partial_{\bar{z}}$ are partial derivatives in the standard flat coordinate of $\boldsymbol{H}$.

Lemma 6.4 *When $\gamma_1 = \gamma_2 = 0$, $\mathcal{D}_\gamma : \mathcal{B}^1 \oplus \mathcal{B}^1 \to \mathcal{B}^2 \oplus \mathcal{B}^2$ is an invertible operator.*

Proof In this case denote the operator by $\mathcal{D}$. It is a standard fact that $\mathcal{D}$ is Fredholm, and it needs a bit more effort to check that its index is zero. Hence it remains to show that $\mathcal{D}$ has trivial kernel. Indeed, it is easy to see that $\mathcal{B}^1 \subset L^2(\boldsymbol{H}, \boldsymbol{C}^k)$, and elliptic regularity shows that any $(f_1, f_2) \in \ker \mathcal{D}$ is smooth. Then

$$\mathcal{D}^\dagger \mathcal{D}(f_1, f_2) = (-\Delta f_1 + f_1, -\Delta f_2 + f_2) = (0, 0).$$

However, since $-\Delta + \mathrm{Id}$ is positive on L^2, $f_1 = f_2 = 0$. □

Lemma 6.5 *Consider the operator $\mathcal{D}^{\mathrm{aux}}_{\gamma,\epsilon} : \mathcal{B}^1 \oplus \mathcal{B}^1 \to \mathcal{B}^2 \oplus \mathcal{B}^2$ defined by*

$$\mathcal{D}^{\mathrm{aux}}_{\gamma,\epsilon}(f_1, f_2) = (\epsilon \partial_{\bar{z}} f_1 + \epsilon \gamma_1(f_1) + f_2, \epsilon \partial_z f_2 + \epsilon \gamma_2(f_2) + f_1). \tag{6.8}$$

Then there exist $\epsilon(\gamma) > 0$ and $c(\gamma) > 0$ such that for all $\epsilon \in (0, \epsilon(\gamma)]$, there is a bounded right inverse $\mathcal{Q}^{\mathrm{aux}}_{\gamma,\epsilon}$ of $\mathcal{D}^{\mathrm{aux}}_{\gamma,\epsilon}$ such that

$$\|\mathcal{Q}^{\mathrm{aux}}_{\gamma,\epsilon}(h_1, h_2)\|_{L^{p,\delta}} + \epsilon \|\nabla \mathcal{Q}^{\mathrm{aux}}_{\gamma,\epsilon}(h_1, h_2)\|_{L^{p,\delta}} \leq c(\gamma) \|(h_1, h_2)\|_{L^{p,\delta}}, \quad \forall (h_1, h_2) \in \mathcal{B}^2 \oplus \mathcal{B}^2.$$

Proof For $k = 0, 1$, define the new norms

$$\|f\|_{k,\delta;\epsilon} = \sum_{i=0}^k \epsilon^i \|\nabla^k f\|_{L^{p,\delta}}. \tag{6.9}$$

We first prove the lemma for $\delta = 0$. Denote $f_\epsilon(z) = (s_\epsilon^* f)(z) = f(\epsilon z)$. Then

$$\|f_\epsilon\|_{W^{k,p}} = \epsilon^{-\frac{2}{p}} \|f\|_{k;\epsilon}.$$

Denote $\gamma^\epsilon = (\gamma_1^\epsilon, \gamma_2^\epsilon)$ where $\gamma_1^\epsilon(z) = \gamma_1(\epsilon z)$, $\gamma_2^\epsilon(z) = \gamma_2(\epsilon z)$. Then $s_\epsilon^* \circ \mathcal{D}^{\rm aux}_{\gamma,\epsilon} = \mathcal{D}_{\epsilon\gamma^\epsilon} \circ s_\epsilon^*$. Then by Lemma 6.4, there is a constant $c > 0$ such that

$$\begin{aligned}
\|\mathcal{D}^{\rm aux}_{\gamma,\epsilon}(f_1, f_2)\|_{L^p} &= \|\mathcal{D}^{\rm aux}_{\gamma,\epsilon}(f_1, f_2)\|_{0,0;\epsilon} = \|(s_\epsilon^*)^{-1}\mathcal{D}_{\epsilon\gamma^\epsilon}(f_{1,\epsilon}, f_{2,\epsilon})\|_{0,0;\epsilon} \\
&= \epsilon^{\frac{2}{p}} \|\mathcal{D}_{\epsilon\gamma^\epsilon}(f_{1,\epsilon}, f_{2,\epsilon})\|_{L^p} \geq \epsilon^{\frac{2}{p}} \left[\frac{1}{c} - \epsilon\|\gamma\|_{L^\infty}\right] \|(f_{1,\epsilon}, f_{2,\epsilon})\|_{W^{1,p}} \\
&= \left[\frac{1}{c} - \epsilon\|\gamma\|_{L^\infty}\right] \|(f_1, f_2)\|_{1,0;\epsilon}.
\end{aligned}$$

Then for $\epsilon \leq (2c\|\gamma\|_{L^\infty})^{-1}$, one has

$$\|\mathcal{D}^{\rm aux}_{\gamma,\epsilon}(f_1, f_2)\|_{0,0;\epsilon} \geq \frac{1}{2c} \|(f_1, f_2)\|_{1,0;\epsilon}.$$

Therefore $\mathcal{D}^{\rm aux}_{\gamma,\epsilon}$ has trivial kernel. Moreover, since its index is zero, there is a bounded inverse $\mathcal{Q}^{\rm aux}_{\gamma,\epsilon}$ whose norm is bounded by $2c$. Hence the $\delta = 0$ case is proved.

Now we show that the $\delta = 0$ case implies the general case. Indeed, the map $f \mapsto \rho_\infty^{-\delta} f$ induces isomorphisms between the $W^{k,p}$-norm and the new norm (6.9). Then it is equivalent to consider the operator

$$W^{1,p} \ni f \mapsto \rho_\infty^{\delta} \mathcal{D}^{\rm aux}_{\gamma,\epsilon}(\rho_\infty^{-\delta} f_1, \rho_\infty^{-\delta} f_2) = \mathcal{D}^{\rm aux}_{\gamma,\epsilon}(f_1, f_2) - \epsilon(\rho_\infty^{\delta}(\partial_{\bar z}(\rho_\infty^{-\delta}))f_1,\ \rho_\infty^{\delta}(\partial_z(\rho_\infty^{-\delta}))f_2).$$

Since $\rho_\infty^{\delta} \nabla(\rho_\infty^{-\delta})$ is bounded, the last term is a bounded operator from $W^{1,p}$ to L^p. Hence when ϵ is sufficiently small, the conclusion follows from the $\delta = 0$ case.

Now we are ready to prove Lemma 6.3. Set $\delta = \delta_p$. Since a_∞ is uniformly bounded, the norm $\|\cdot\|_{\tilde L^{1,p}_g} = \|\cdot\|_{W^{1,p,\delta_p}}$ defined using the connection form a_∞ is equivalent to the norm defined for $a_\infty = 0$. Moreover, it is easy to see

$$\begin{aligned}
D^G_\infty(\mathcal{X}_{\eta_1} + J\mathcal{X}_{\zeta_1}) &= L_\infty[\partial_{\bar z}(\eta_1 + \boldsymbol{i}\zeta_1) + \gamma_1(\eta_1 + \boldsymbol{i}\zeta_1)] \\
\not{D}^G_\infty(\eta_2 + \boldsymbol{i}\zeta_2) &= \partial_z(\eta_2 + \boldsymbol{i}\zeta_2) + \gamma_2(\eta_2 + \boldsymbol{i}\zeta_2).
\end{aligned}$$

Here γ_1 and γ_2 are uniformly bounded matrix valued continuous maps. Therefore, $\mathcal{D}^{{\rm aux},G}_{\infty,\epsilon}$ is of the same form as $\mathcal{D}^{\rm aux}_{\gamma,\epsilon}$ in (6.8). Then by Lemma 6.5, for ϵ_3 sufficiently small and $\epsilon \in (0, \epsilon_3)$, the operator $\mathcal{D}^{{\rm aux},G}_{\infty,\epsilon}$ is uniformly invertible w.r.t. the norm $\|\cdot\|_{\tilde L^{1,p}_{a;\epsilon}}$ on the domain and the norm $\|\cdot\|_{\tilde L^p}$ on the target. This proves Lemma 6.3.

6.3 Right Inverses and Matching Condition

Consider the space of infinitesimal deformations of the singular configurations, i.e.,

$$\mathcal{B}_{\#,\epsilon} := \mathcal{B}_{\infty,\epsilon} \times \prod_{i=1}^{\underline{m}+m} \mathcal{B}_i,$$

and the one with the matching condition imposed

$$\mathcal{B}_{\clubsuit,\epsilon} := \{(\xi_\infty^H, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{m+\underline{m}}) \in \mathcal{B}_{\#,\epsilon} \,|\, {\rm ev}^i_\infty(\xi^H_\infty) = {\rm ev}_i(\boldsymbol{\xi}_i),\ i = 1, \ldots, m + \underline{m}\}.$$

Introduce the bundle over $\mathcal{B}_{\clubsuit,\epsilon}$

$$\mathcal{E}_{\clubsuit,\epsilon} := \mathcal{E}_{\infty,\epsilon} \boxtimes \prod_{i=1}^{m+\underline{m}} \mathcal{E}_i,$$

The perturbed equation defines a family of sections parametrized by ϵ

$$\tilde{\mathcal{F}}_{\clubsuit,\epsilon} : W \times \mathcal{B}_{\clubsuit,\epsilon} \to \mathcal{E}_{\clubsuit,\epsilon}.$$

Its linearization at the rescaled object is a linear operator

$$\tilde{\mathcal{D}}_{\clubsuit,\epsilon} : W \oplus \mathcal{B}_{\clubsuit,\epsilon} \to \mathcal{E}_{\clubsuit,\epsilon}.$$

As the last step of the preparation, we prove the existence of a family of uniformly bounded right inverses.

Proposition 6.6 *There exist $c_\clubsuit > 0$, $\epsilon_\clubsuit > 0$ and bounded right operators*

$$\tilde{\mathcal{Q}}_{\clubsuit,\epsilon} : \mathcal{E}_{\clubsuit,\epsilon} \to W \oplus \mathcal{B}_{\clubsuit,\epsilon}, \quad \forall \epsilon \in (0, \epsilon_\clubsuit)$$

which are right inverses to the operators $\tilde{\mathcal{D}}_{\clubsuit,\epsilon}$ and $\|\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}\| \leq c_\clubsuit$. Moreover, as maps between two fixed topological vector spaces, $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$ varies continuously with ϵ.

The operator $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$ will be used to construct right inverses along approximate solutions which we construct in the following section.

To prove Proposition 6.6, we first look at the rescaled disk component. We have three operators associated with the disk component $\boldsymbol{v}_\infty$, which are $\tilde{D}^H_\infty = D^H_\infty + d\iota_\infty : W \oplus \mathcal{B}^H_\infty \to \mathcal{E}^H_\infty$ where D^H_∞ is considered in (4.12) and (4.13), $\tilde{\mathcal{D}}_{\infty,\epsilon} : W \oplus \mathcal{B}_{\infty,\epsilon} \to \mathcal{E}_{\infty,\epsilon}$ (see (6.6)), and $\tilde{\mathcal{D}}^{\rm aux}_{\infty,\epsilon} : W \oplus \tilde{L}^{1,p}_{a;\epsilon}(u^*_\infty H_X \oplus u^*_\infty \boldsymbol{G}_X) \to \tilde{L}^p(u^*_\infty H_X \oplus u^*_\infty \boldsymbol{G}_X)$ defined by (6.7). The transversality assumption Hypothesis 2 assumes that the first operator $\tilde{D}^H_\infty$ is surjective, hence has a bounded right inverse. We need to construct an induced right inverse to $\tilde{\mathcal{D}}_{\infty,\epsilon}$.

Lemma 6.7 *For $\epsilon > 0$ sufficiently small, there is a family of operator $m^*_\epsilon : \ker(\tilde{D}^H_\infty) \to W \oplus \mathcal{B}_{\infty,\epsilon}$ whose norm is uniformly bounded by a constant independent of ϵ, such that*

$$(\hat{s}^*_\epsilon + \epsilon m^*_\epsilon)[\ker(\tilde{D}^H_\infty)] = \ker(\tilde{\mathcal{D}}_{\infty,\epsilon})$$

Proof By (6.7), one can write $\tilde{\mathcal{D}}^{\rm aux}_{\infty,\epsilon} = \check{s}^{-1}_\epsilon \circ \tilde{\mathcal{D}}_{\infty,\epsilon} \circ \hat{s}^*_\epsilon$ in the block matrix form as

$$\tilde{\mathcal{D}}^{\rm aux}_{\infty,\epsilon} = \begin{bmatrix} \tilde{D}^H_\infty & E_1 \\ \epsilon E_2 & \mathcal{D}^{{\rm aux},G}_{\infty,\epsilon} \end{bmatrix}.$$

By Lemma 6.3, $\mathcal{D}^{{\rm aux},G}_{\infty,\epsilon}$ has uniformly bounded inverse $\mathcal{Q}^{{\rm aux},G}_{\infty,\epsilon}$. Moreover, E_1 and E_2 are of zero-th order, and they are also uniformly bounded. Since $\tilde{D}^H_\infty$ is surjective, for ϵ small enough, there exists a bounded right inverse $Q^H_{\infty,\epsilon}$ to the operator $\tilde{D}^H_\infty - \epsilon E_1 \mathcal{Q}^{{\rm aux},G}_{\infty,\epsilon} E_2$ whose norm is uniformly bounded. Denote

$$a^*_\epsilon := \hat{s}^*_\epsilon \circ \begin{bmatrix} {\rm Id} & 0 \\ -\epsilon \mathcal{Q}^{{\rm aux},G}_{\infty,\epsilon} E_2 & {\rm Id} \end{bmatrix} \begin{bmatrix} Q^H_{\infty,\epsilon} \tilde{D}^H_\infty & 0 \\ 0 & {\rm Id} \end{bmatrix}.$$

Then one has

$$\begin{aligned}(\check{s}^*_\epsilon)^{-1} \circ \tilde{\mathcal{D}}_{\infty,\epsilon} \circ a^*_\epsilon &= \begin{bmatrix} \tilde{D}^H_\infty & E_1 \\ \epsilon E_2 & \mathcal{D}^{{\rm aux},G}_{\infty,\epsilon} \end{bmatrix} \begin{bmatrix} {\rm Id} & 0 \\ -\epsilon \mathcal{Q}^{{\rm aux},G}_{\infty,\epsilon} E_2 & {\rm Id} \end{bmatrix} \begin{bmatrix} Q^H_{\infty,\epsilon} \tilde{D}^H_\infty & 0 \\ 0 & {\rm Id} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{D}^H_\infty - \epsilon E_1 \mathcal{Q}^{{\rm aux},G}_{\infty,\epsilon} E_2 & E_1 \\ 0 & \mathcal{D}^{{\rm aux},G}_{\infty,\epsilon} \end{bmatrix} \begin{bmatrix} Q^H_{\infty,\epsilon} \tilde{D}^H_\infty & 0 \\ 0 & {\rm Id} \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} \tilde{D}^H_\infty & E_1 \\ 0 & \mathcal{D}^{\mathrm{aux},G}_{\infty,\epsilon} \end{bmatrix}.$$

So $a^*_\epsilon : \ker[\tilde{D}^H_\infty] \oplus \{0\} \to \ker[\tilde{\mathcal{D}}_{\infty,\epsilon}]$ is an isomorphism. Moreover, $a^*_\epsilon - \hat{s}^*_\epsilon$ is bounded by a multiple of ϵ. Hence the statement of the lemma holds for $m^*_\epsilon = \epsilon^{-1}(a^*_\epsilon - s^*_\epsilon)$.

In Hypothesis 2, the map (5.4) is transversal to the diagonal (5.5). Lemma 6.7 shows that by replacing $\ker(\tilde{D}^H_\infty)$ by $\ker(\tilde{\mathcal{D}}_{\infty,\epsilon})$, we obtain a small perturbation of (5.4)

$$\ker[\tilde{\mathcal{D}}_{\infty,\epsilon}] \oplus \prod_{i=1}^{m+\underline{m}} \ker[\tilde{\mathcal{D}}_i] \to \prod_{i=1}^{m} [T_{\bar{x}_i}\bar{X}]^2 \oplus \prod_{j=1}^{\underline{m}} [T_{\bar{x}_j}\bar{L}]^2 \oplus W^{m+\underline{m}+1}.$$

which is still transversal to the diagonal. Hence for all small ϵ, one can choose a right inverse

$$\mathsf{Q}_\epsilon : \frac{W^{m+\underline{m}+1}}{\Delta^{m+\underline{m}+1}(W)} \oplus \prod_{i=1}^{m} \frac{[T_{\bar{x}_i}\bar{X}]^2}{\Delta(T_{\bar{x}_i}\bar{X})} \oplus \prod_{j=1}^{\underline{m}} \frac{[T_{\underline{\bar{x}}_j}\bar{L}]^2}{\Delta(T_{\underline{\bar{x}}_j}\bar{L})} \to \ker[\tilde{\mathcal{D}}_{\infty,\epsilon}] \oplus \prod_{i=1}^{m+\underline{m}} \ker[\tilde{\mathcal{D}}_i]$$

whose norm is uniformly bounded.

Moreover, Hypothesis 2 tells that there are right inverses

$$\tilde{\mathcal{Q}}_i : \mathcal{E}_i \to W \oplus \mathcal{B}_i.$$

Together with the right inverse $\tilde{\mathcal{Q}}_{\infty,\epsilon}$ of Proposition 6.2, one obtains right inverses

$$\tilde{\mathcal{Q}}_{\#,\epsilon} : \mathcal{E}_{\infty,\epsilon} \oplus \prod_{i=1}^{m+\underline{m}} \mathcal{E}_i \to (W \oplus \mathcal{B}_{\infty,\epsilon}) \oplus \prod_{i=1}^{m+\underline{m}} (W \oplus \mathcal{B}_i)$$

for all ϵ small enough, whose norm is uniformly bounded. The image of $\tilde{\mathcal{Q}}_{\#,\epsilon}$ may not lie in the diagonals, and we use Q_ϵ to correct it. Since the norm of Q_ϵ is also bounded uniformly, it implies Proposition 6.6.

7 Constructing the Gluing Map

In this section we use the standard idea of gluing to construct perturbed $\boldsymbol{H}$-vortices near the singular solution. In Subsection 7.1 we describe how to subdivide the domain and introduce the cut-off functions. In Subsection 7.2 we construct the approximate solutions. In Subsection 7.3 we introduce the ϵ-dependent weighted Sobolev norms along the approximate solutions. In Subsection 7.4 we state the major estimates required to apply the implicit function theorem, which immediately implies the (set-theoretic) construction of the gluing map. In Subsection 7.5 we prove that the gluing map is locally a homeomorphism. Certain technical results will be proved in Section 8 and the appendix.

We will establish several estimates in which we prove the existence of constants $\epsilon_1, \epsilon_2, \ldots > 0$ and $c_1, c_2, \ldots > 0$ (whose values are reset now). We will always assume

$$\epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \cdots, \quad c_1 \leq c_2 \leq c_3 \leq \cdots.$$

7.1 Decomposing the Domain

We use the alternate local model for domain curves discussed in Subsection 2.3, so that the nodal point set $\boldsymbol{z} = (z_1, \ldots, z_m, z_{\underline{1}}, \ldots, z_{\underline{m}})$ is fixed. For $\epsilon > 0$ sufficiently small, define

$$\Sigma_\epsilon = \Sigma_{\epsilon,\boldsymbol{z}} := \boldsymbol{H} \setminus \{z_{1;\epsilon}, \ldots, z_{m+\underline{m};\epsilon}\}.$$

(Recall that $z_{i,\epsilon}$ is defined in (2.4).) Take a number $b > 0$ such that

$$\frac{100 s_p c_\clubsuit}{p-2} \leq \log b \leq \frac{1000 s_p c_\clubsuit}{p-2} \tag{7.1}$$

where s_p is the Sobolev constant given by Definition 3.6 and $c_\clubsuit$ is the bound of the right inverse $\tilde{\mathcal{Q}}_\clubsuit$ given by Proposition 6.6. The reason for this seemingly strange choice will be clear in Proposition 8.1. Then the following sets (in increasing sequence)

$$\check{\check{B}}_\epsilon^i = B\left(z_{i,\epsilon}, \frac{1}{2b\sqrt{\epsilon}}\right), \quad \check{B}_\epsilon^i = B\left(z_{i;\epsilon}, \quad \frac{1}{b\sqrt{\epsilon}}\right), \quad \mathring{B}_\epsilon^i = B\left(z_{i;\epsilon}, \frac{1}{\sqrt{\epsilon}}\right),$$
$$\hat{B}_\epsilon^i = B\left(z_{i;\epsilon}, \frac{b}{\sqrt{\epsilon}}\right), \quad \hat{\hat{B}}_\epsilon^i = B\left(z_{i,\epsilon}, \frac{2b}{\sqrt{\epsilon}}\right).$$

We also denote

$$\check{\check{\Sigma}}_\epsilon := \boldsymbol{H} \setminus \bigcup_{i=1}^{m+\underline{m}} \check{\check{B}}_\epsilon^i, \quad \check{\Sigma}_\epsilon := \boldsymbol{H} \setminus \bigcup_{i=1}^{m+\underline{m}} \check{B}_\epsilon^i, \quad \mathring{\Sigma}_\epsilon := \boldsymbol{H} \setminus \bigcup_{i=1}^{m+\underline{m}} \mathring{B}_\epsilon^i, \text{ and similarly } \hat{\Sigma}_\epsilon,\ \hat{\hat{\Sigma}}_\epsilon;$$

and

$$\dot{A}_\epsilon^i := \hat{B}_\epsilon^i \setminus \check{B}_\epsilon^i, \quad \ddot{A}_\epsilon^i := \hat{\hat{B}}_\epsilon^i \setminus \check{\check{B}}_\epsilon^i.$$

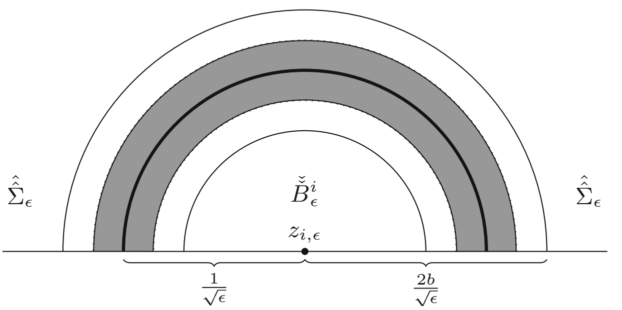

Figure 2 The decomposition of the domain near a node. Over $\dot{A}_\epsilon^i$, the shaded area in the picture, the approximate solution is covariantly constant

Let $\beta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that

$$\beta|_{(-\infty,-1]} \equiv 1, \quad \beta|_{[0,+\infty)} = 0, \quad |\nabla\beta| \leq 2.$$

Then for a given gluing parameter ϵ and i, define $\beta_{i,\epsilon} : \boldsymbol{H} \to [0,1]$ by

$$\beta_{i,\epsilon}(z) = \beta\left(\frac{\log|z - z_{i,\epsilon}| + \log b + \log\sqrt{\epsilon}}{\log 2}\right).$$

Define $\beta_{\infty,\epsilon} : \boldsymbol{H} \to [0,1]$ by

$$\beta_{\infty,\epsilon}(z) = \min_{1 \leq i \leq m+\underline{m}} \beta\left(\frac{-\log|z - z_{i,\epsilon}| + \log b - \log\sqrt{\epsilon}}{\log 2}\right).$$

Then $\beta_{i,\epsilon}$ equals to one inside $\check{\check{B}}_\epsilon^i$ and equals to zero outside $\check{B}_\epsilon^i$. $\beta_{\infty,\epsilon}$ equals to one inside $\hat{\hat{\Sigma}}_\epsilon$ and equals to zero outside $\hat{\Sigma}_\epsilon$. Moreover,

$$\|\nabla\beta_{i,\epsilon}\|_{L^\infty} \leq \frac{2b\sqrt{\epsilon}}{\log 2}\sup|\nabla\beta| \leq 2b\sqrt{\epsilon}, \quad \|\nabla\beta_{\infty,\epsilon}\|_{L^\infty} \leq \frac{\sqrt{\epsilon}}{b\log 2}\sup|\nabla\beta| \leq \frac{\sqrt{\epsilon}}{b}.$$

7.2 The Approximate Solutions

We first consider a small neighborhood of $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$ inside $\mathcal{M}^{\epsilon_0}_{\clubsuit}(\boldsymbol{H}; X, L)$, which is has the structure of a fibre product. If $[\boldsymbol{w}, \boldsymbol{v}] \in \mathcal{M}^{\epsilon_0}_{\clubsuit}(\boldsymbol{H}; X, L)$ is sufficiently close to $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$, then by Theorem 3.7, for $i = 1, \ldots, m + \underline{m}$, the i-th component $\boldsymbol{v}_i$ of $\boldsymbol{v}$ is close to $\boldsymbol{v}_{\bullet,i}$ in the topology of $\mathcal{B}_i$, and can be written uniquely as $\boldsymbol{v}_i = \exp_{\boldsymbol{v}_{\bullet,i}} \boldsymbol{\xi}_i$, where $\boldsymbol{\xi}_i = (\xi_i, \eta_i, \zeta_i) \in \mathcal{B}_i$ satisfying the gauge fixing condition

$$\partial_s \eta_i + [\phi_{\bullet,i}, \eta_i] + \partial_t \zeta_i + [\psi_{\bullet,i}, \zeta_i] + d\mu(u_{\bullet,i}) \cdot J\xi_i = 0.$$

On the other hand, the ∞-th component $\boldsymbol{v}_\infty = (u_\infty, \phi_\infty, \psi_\infty)$ is also sufficiently close to $\boldsymbol{v}_{\bullet,\infty}$, and there is a unique vector $\xi^H_\infty \in \mathcal{B}^H_\infty$ such that $u_\infty = \exp_{u_{\bullet,\infty}} \xi_\infty$. Then for $\epsilon > 0$ sufficiently small, we just use $\mathcal{M}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L)$ to denote the set of gauge equivalence classes of perturbed stable affine vortices which have representatives in the above way such that all $\|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m} < \epsilon$ and $\|\xi_\infty\|_{\tilde{L}^{1,p}_h} < \epsilon$, and every element of $\mathcal{M}^{\epsilon}_{\clubsuit}(\boldsymbol{H}; X, L)$ are provided with canonical representatives where a general one is denoted by $(\boldsymbol{w}, \boldsymbol{v})$.

The purpose of this subsection is, for each ϵ small enough and $(\boldsymbol{w}, \boldsymbol{v})$ in this family of representatives, we would like to define a gauged map $\boldsymbol{v}_\epsilon = (u_\epsilon, \phi_\epsilon, \psi_\epsilon)$ on $\boldsymbol{H}$ from the components $\boldsymbol{v}_i$ and $\boldsymbol{v}_\infty$, and call $(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$ an approximate solution.

We first need to change the gauge of $\boldsymbol{v}_\infty$ (for all $(\boldsymbol{w}, \boldsymbol{v})$ in this family of representatives). Let (r_i, θ_i) be the polar coordinates centered at z_i. Let $g_\infty : \boldsymbol{H} \setminus \{z_1, \ldots, z_m\} \to K$ be a gauge transformation satisfying the following conditions:

(a) For $i = 1, \ldots, m$, $g_\infty(r_i, \theta_i) = \mathrm{e}^{-\lambda_i \theta_i}$ in a small neighborhood of z_i.

(b) g_∞ equals identity outside a compact subset of $\boldsymbol{H}$ and

$$\frac{\partial g_\infty}{\partial t}|_{\partial \boldsymbol{H}} = 0. \tag{7.2}$$

By abuse of notations, we replace $\boldsymbol{v}_\infty$ by $g_\infty \cdot \boldsymbol{v}_\infty = g_\infty \cdot (u_\infty, \phi_\infty, \psi_\infty)$, in which the connection form $a_\infty = \phi_\infty ds + \psi_\infty dt$ has poles at z_i for $i = 1, \ldots, m$. For convenience, set the monodromies at the boundary punctures to be $\lambda_{m+j} = \underline{\lambda}_j = 0$ for $j = 1, \ldots, \underline{m}$. Then it is easy to see

$$\lim_{z \to z_i} \mathrm{e}^{\lambda_i \theta_i} u_\infty(z) = x_i, \quad i = 1, \ldots, m + \underline{m}.$$

Moreover, (7.2) implies the boundary condition

$$u_\infty(\partial \Sigma) \subset L, \quad \psi_\infty|_{\partial \Sigma} = 0.$$

As before, $\boldsymbol{v}_{\infty,\epsilon} = s^*_\epsilon \boldsymbol{v}_\infty$ is the rescaled gauged map defined on Σ_ϵ, which has poles at $z_{i,\epsilon}$. The original (smooth) gauged map is denoted by $\check{\boldsymbol{v}}_\infty = (\check{u}_\infty, \check{\alpha}_\infty)$. From now on symbols with "ˇ" on top indicate that they are associated with objects which have no poles at the nodes.

Now $\boldsymbol{v}_\infty$ and $\boldsymbol{v}_i$ has the same monodromy at nodes and we can form connected sums. More precisely, over the neck region $\ddot{A}^i_\epsilon$ with polar coordinates (r_i, θ_i), denote $\check{\alpha}_i = \mathrm{e}^{\lambda_i \theta_i} \cdot \alpha_i$. Define

$$\alpha_\epsilon = \begin{cases} \alpha_{\infty,\epsilon}(z), & z \in \hat{\hat{\Sigma}}_\epsilon; \\ \mathrm{e}^{-\lambda_i \theta_i} \cdot (\beta_{\infty,\epsilon} \check{\alpha}_{\infty,\epsilon}(r_i, \theta_i) + \beta_{i,\epsilon} \check{\alpha}_i(r_i, \theta_i)), & z \in \ddot{A}^i_\epsilon,\ i = 1, \ldots, m + \underline{m}; \\ \alpha_i(z - z_{i,\epsilon}), & z \in \check{\check{B}}^i_\epsilon. \end{cases} \tag{7.3}$$

Secondly, over the neck region $\ddot{A}^i_\epsilon$, denote $\check{u}_i = e^{\lambda_i\theta_i}u_i$. Using the exponential map, there are $T_{x_i}X$-valued functions $\check{\xi}_{\infty,\epsilon}$ and $\check{\xi}_i$ such that

$$\check{u}_{\infty,\epsilon}(z) = \exp_{x_i}\check{\xi}_{\infty,\epsilon}(z), \quad \check{u}_i(z - z_{i,\epsilon}) = \exp_{x_i}\check{\xi}_i(z),\ z \in \ddot{A}^i_\epsilon.$$

Let (r_i, θ_i) be the polar coordinates defined as $r_i + \boldsymbol{i}\theta_i = \log(z - z_{i,\epsilon})$. Then define

$$u_\epsilon(z) = \begin{cases} u_{\infty,\epsilon}(z), & z \in \hat{\hat{\Sigma}}_\epsilon; \\ e^{-\lambda_i\theta_i}\exp_{x_i}(\beta_{\infty,\epsilon}\check{\xi}_{\infty,\epsilon}(r_i,\theta_i) + \beta_{i,\epsilon}\check{\xi}_i(r_i,\theta_i)), & z \in \ddot{A}^i_\epsilon,\ i = 1,\dots,m+\underline{m}; \\ u_i(z - z_{i,\epsilon}), & z \in \check{\check{B}}^i_\epsilon. \end{cases} \tag{7.4}$$

Finally define $\boldsymbol{v}_\epsilon = (u_\epsilon, \alpha_\epsilon)$ and call the pair $(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$ the **approximate solution**. Notice that they are defined on a fixed presentation of $\boldsymbol{H}$, however with different global holomorphic coordinates $\boldsymbol{z}$ (depending on the conformal class of the underlying marked disk, parametrized by the variable $\boldsymbol{q}$; see Subsection 2.3) and the associated volume form. The central approximate solution $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon})$ is of special role. After we introduce a new, ϵ-dependent weighted norm and a Banach manifold centered $\mathcal{B}_\epsilon$ centered at $\boldsymbol{v}_{\bullet,\epsilon}$, we will see that all gauged maps $\boldsymbol{v}_\epsilon$ belonging to an approximate solution stay close to $\boldsymbol{v}_{\bullet,\epsilon}$ w.r.t. the norm of $\mathcal{B}_\epsilon$.

Notice that over the thinner neck region $\dot{A}^i_\epsilon$ (i.e., the shaded region in Figure 2), $\boldsymbol{v}_{\bullet,\epsilon}$ coincides with the covariantly constant gauged map

$$\boldsymbol{c}_{\bullet,i} := (e^{-\lambda_i\theta_i}x_{\bullet,i}, \lambda_i d\theta_i).$$

For convenience, we also introduce gauged maps

$$\boldsymbol{v}'_{\bullet,i,\epsilon} = \begin{cases} \boldsymbol{v}_{\bullet,\epsilon}, & \text{on } \mathring{B}^i_\epsilon, \\ \boldsymbol{c}_{\bullet,i}, & \text{on } \boldsymbol{A}_i \setminus \mathring{B}^i_\epsilon; \end{cases} \qquad \boldsymbol{v}'_{\bullet,\infty,\epsilon} = \begin{cases} \boldsymbol{v}_{\bullet,\epsilon}, & \text{on } \mathring{\Sigma}_\epsilon, \\ \boldsymbol{c}_{\bullet,i}, & \text{on } \mathring{B}^i_\epsilon \setminus \{z_{i,\epsilon}\}. \end{cases} \tag{7.5}$$

The following two results are proved in Subsection 8.1 by straightforward calculations.

Lemma 7.1 *For each i, we can identify $\boldsymbol{v}'_{\bullet,i,\epsilon}$ with a point $\boldsymbol{\xi}'_{\bullet,i,\epsilon} \in \mathcal{B}_i$ such that*

$$\boldsymbol{v}'_{\bullet,i,\epsilon} = \exp_{\boldsymbol{v}_{\bullet,i}}\boldsymbol{\xi}'_{\bullet,i,\epsilon} \quad \text{and} \quad \lim_{\epsilon\to 0}\|\boldsymbol{\xi}'_{\bullet,i,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}} = 0.$$

Lemma 7.2 *We can identify $\boldsymbol{v}'_{\infty,\epsilon}$ with a point $\boldsymbol{\xi}'_{\infty,\epsilon} \in \mathcal{B}_{\infty,\epsilon}$ such that*

$$\boldsymbol{v}'_{\bullet,\infty,\epsilon} = \exp_{\boldsymbol{v}_{\bullet,\infty,\epsilon}}\boldsymbol{\xi}'_{\bullet,\infty,\epsilon} \quad \text{and} \quad \lim_{\epsilon\to 0}\|\boldsymbol{\xi}'_{\bullet,\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}} = 0.$$

Remark 7.3 We only need the above two convergence results for the central one $(\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet)$ in the family of stable solutions, in order to estimate the approximate right inverses.

7.3 Weighted Sobolev Norms

To save notations, in this subsection we abbreviate the central approximate solution $\boldsymbol{v}_{\bullet,\epsilon}$ simply as $\boldsymbol{v}_\epsilon$. One use formula (3.5) to define a Banach space $\mathcal{B}_{\boldsymbol{v}_\epsilon}$ of infinitesimal deformations of $\boldsymbol{v}_\epsilon$, where we replace the gauge field a by a_ϵ. Using the exponential map of the metric h_X one can identify a small ball of $\mathcal{B}_\epsilon$ with a Banach manifold of triples $\boldsymbol{v}_\epsilon = (u'_\epsilon, \phi'_\epsilon, \psi'_\epsilon)$ near $\boldsymbol{v}_\epsilon$. As always we still denote this Banach manifold by the same symbol $\mathcal{B}_{\boldsymbol{v}_\epsilon}$ representing the tangent space. We can also define the Banach vector bundle $\mathcal{E}_{\boldsymbol{v}_\epsilon} \to \mathcal{B}_\epsilon$ whose fibre over $\boldsymbol{v}'_\epsilon$ is the space $\tilde{L}^p(\boldsymbol{H}, (u'_\epsilon)^*TX \oplus \mathfrak{k} \oplus \mathfrak{k})$. However, to obtain uniform estimates needed for the gluing, one has to modify the norms on $\mathcal{B}_{\boldsymbol{v}_\epsilon}$ and $\mathcal{E}_{\boldsymbol{v}_\epsilon}$. Define $\rho_\epsilon : \boldsymbol{H} \to \mathbb{R}$ by

$$\rho_\epsilon(z) = \begin{cases} \rho_{\boldsymbol{A}_i}(z), & z \in \mathring{B}_\epsilon^i; \\ \rho_{\infty,\epsilon}(z), & z \in \mathring{\Sigma}_\epsilon. \end{cases} \tag{7.6}$$

By definition (6.1), ρ_ϵ is continuous and has value $\frac{1}{\sqrt{\epsilon}}$ on $\partial \mathring{B}_\epsilon^i$. Then for $U \subset \boldsymbol{H}$ and a section $f \in L^p_{\mathrm{loc}}(U, E)$ of an Euclidean bundle $E \to U$, define

$$\|f\|_{\tilde{L}^p_\epsilon} = \left[\int_U |f(z)|^p [\rho_\epsilon(z)]^{2p-4} ds dt \right]^{\frac{1}{p}}.$$

This induces a norm on the fibres of $\mathcal{E}_{\boldsymbol{v}_\epsilon}$, which we abbreviate as $\mathcal{E}_\epsilon$. On the other hand, for $\boldsymbol{\xi}_\epsilon = (\xi_\epsilon, \eta_\epsilon, \zeta_\epsilon) \in \mathcal{B}_{\boldsymbol{v}_\epsilon}$ define

$$\|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} := \|\eta_\epsilon\|_{\tilde{L}^p_\epsilon} + \|\zeta_\epsilon\|_{\tilde{L}^p_\epsilon} + \|d\mu \cdot \xi_\epsilon\|_{\tilde{L}^p_\epsilon} + \|d\mu \cdot J\xi_\epsilon\|_{\tilde{L}^p_\epsilon} + \|\nabla^{a_\epsilon} \boldsymbol{\xi}_\epsilon\|_{\tilde{L}^p_\epsilon} + \|\xi_\epsilon\|_{L^\infty}. \tag{7.7}$$

The space $\mathcal{B}_{\boldsymbol{v}_\epsilon}$ with the new norm is denoted by $\mathcal{B}_\epsilon$. The norm $\tilde{L}^{1,p}_{m;\epsilon}(U)$ for a subset $U \subset \boldsymbol{H}$ can be understood and we do not bother to define it explicitly.

We need the following uniform Sobolev estimate.

Lemma 7.4 (Sobolev Embedding) *There exist c_2 and ϵ_2 such that for all $\epsilon \in (0, \epsilon_2)$ and $\boldsymbol{\xi}_\epsilon \in \mathcal{B}_\epsilon$, one has $\|\boldsymbol{\xi}_\epsilon\|_{L^\infty} \le c_2 \|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}$.*

Proof By the definition (7.7), $\|\xi_\epsilon\|_{L^\infty}$ is already contained in the norm $\|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}$. Hence it suffices to bound the L^∞ norm of ϕ_ϵ and ψ_ϵ. Notice that ∇^{a_ϵ} is a connection that preserves the metric on $\mathfrak{k}$. Hence by [12, Remark 3.5.1], the Sobolev embedding for ϕ_ϵ and ψ_ϵ follows from the standard Sobolev embedding if the weight function is uniformly bounded from below in the standard coordinate. This is indeed true from the definition. □

When the moduli space in the lower stratum has positive dimensions, we also need to show that all approximate solutions constructed from nearby singular solutions are in a neighborhood of $\boldsymbol{v}_{\bullet,\epsilon}$ w.r.t. the distance of $\mathcal{B}_\epsilon$. More precisely, we aim at proving the following lemma.

Lemma 7.5 *Given $r > 0$, there exists $\epsilon(r) > 0$ satisfying the following conditions. Given an element $\boldsymbol{v}'$ of the same combinatorial type as $\boldsymbol{v}_\bullet$ with components*

$$\boldsymbol{v}_i' = \exp_{\boldsymbol{v}_i} \boldsymbol{\xi}_i \in \mathcal{B}_i, \ i = 1, \ldots, m + \underline{m}, \quad u_\infty' \in \exp_{u_\infty} \xi_\infty^H \in \mathcal{B}_\infty^H$$

which satisfies the matching condition, and which satisfies

$$\|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m} \le \epsilon(r), \quad \|\xi_\infty^H\|_{\tilde{L}^{1,p}_h} \le \epsilon(r).$$

Given $\epsilon \in (0, \epsilon(r)]$ and let $\boldsymbol{v}_\epsilon'$ be the object obtained from $\boldsymbol{v}'$ by the pregluing construction, then $\boldsymbol{v}_\epsilon' \in \mathcal{B}_\epsilon$, and we can write

$$\boldsymbol{v}_\epsilon' = \exp_{\boldsymbol{v}_{\bullet,\epsilon}} \boldsymbol{\xi}_\epsilon, \quad \boldsymbol{\xi}_\epsilon \in T_{\boldsymbol{v}_{\bullet,\epsilon}} \mathcal{B}_\epsilon, \quad \text{with } \|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \le r. \tag{7.8}$$

Proof One can write (7.8) formally for some $\boldsymbol{\xi}_\epsilon$ which has local regularity $W^{1,p}$. On the other hand, over the region where $\boldsymbol{v}_\epsilon' = \boldsymbol{v}_i$ (up to a translation), i.e., over $\check{\check{B}}_\epsilon^i$, we have

$$\|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}(\check{\check{B}}_\epsilon^i)} = \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_{m;\epsilon}(\check{\check{B}}_\epsilon^i - z_{i,\epsilon})}.$$

Over $\hat{\hat{\Sigma}}_\epsilon$ where $\boldsymbol{v}_\epsilon' = \boldsymbol{v}_{\infty,\epsilon}$, we have

$$\|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}(\hat{\hat{\Sigma}}_\epsilon)} = \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\hat{\hat{\Sigma}}_\epsilon)} \lesssim \|\boldsymbol{\xi}_\infty^H\|_{\tilde{L}^{1,p}_h(\boldsymbol{H})}.$$

The last inequality follows from Lemma 4.8. Then by taking $\epsilon(r)$ sufficiently small, the norm of $\boldsymbol{\xi}_\epsilon$ away from all neck regions $\ddot{A}_\epsilon^i$ can be made very small.

It remains to estimate the difference over the neck regions. Indeed, let $x_i, x_i' \in \mu^{-1}(0)$ be the image of the i-th nodal point of $\boldsymbol{v}_\bullet$ and $\boldsymbol{v}'$ respectively.

Claim The monodromy at ∞ of $\boldsymbol{v}_i'$ is the same as that of $\boldsymbol{v}_{\bullet,i}$, denoted by λ_i. Moreover, if $\boldsymbol{v}_i'$ converges to the loop $\mathrm{e}^{\lambda_i\theta} x_i'$ for $x_i' \in \mu^{-1}(0)$, then $x_i' \in \exp_{x_{\bullet,i}}(H_{X,x_{\bullet,i}})$.

Proof of the Claim The conclusion basically follows from the fact that $\boldsymbol{v}_i'$ is in the Banach manifold centered at $\boldsymbol{v}_{\bullet,i}$. Near infinity, we write the gauge fields $a_i' = \phi_i' ds + \psi_i' dt$ of $\boldsymbol{v}_i'$ as $\Phi_i' d\tau + \Psi_i' d\theta$ in the cylindrical coordinate (τ,θ). Since $a_i' - a_{\bullet,i}$ is of class $\tilde{L}^p$, while $|d\theta| = \mathrm{e}^{-\tau}$ is not of this class, we see Ψ_i' converges to λ_i at ∞. On the other hand, we can write $u_i' = \exp_{u_{\bullet,i}} \xi_i'$ with $\xi_i' \in \tilde{L}_m^{1,p}(u_{\bullet,i}^* TX)$. This implies that the limit of ξ_i' at ∞ exists and is in the horizontal distribution. Hence $x_i' = \exp_{x_{\bullet,i}} \xi_i'(\infty)$. □

Then it is routine to estimate the difference between $\boldsymbol{v}_{\bullet,\epsilon}$ and $\boldsymbol{v}_\epsilon'$ over the neck region. They are close to the covariantly constant gauged maps $\boldsymbol{c}_{\bullet,i}$ and $\boldsymbol{c}_i'$. Moreover, the difference between $\boldsymbol{c}_{\bullet,i}$ and $\boldsymbol{c}_i'$, which is essentially the size of the vector v_i, is small.

Putting the estimates in the three types of regions together, we finish the proof. □

7.4 Implicit Function Theorem

7.4.1 The Failure

Consider the perturbed vortex equation over $\boldsymbol{H}$, where the perturbation term ι also depends on the gluing parameter ϵ,

$$\partial_s u + \mathcal{X}_\phi + J(\partial_t u + \mathcal{X}_\psi) + \iota_\epsilon(\boldsymbol{w},\boldsymbol{v}) = 0, \quad \partial_s \psi - \partial_t \phi + [\phi,\psi] + \sigma_{\boldsymbol{q}}^\epsilon \mu(u) = 0.$$

Recall that $\sigma_{\boldsymbol{q}}^\epsilon dsdt$ is the volume form on $\boldsymbol{H}$ depending on the gluing parameter and $\boldsymbol{q}$ which parametrizes the marked disk, such that $\sigma_\bullet^\epsilon \equiv 1$. Moreover, we use $\boldsymbol{v}_{\bullet,\epsilon}$ as a reference to define the local gauge fixing condition, which reads

$$\partial_s(\phi - \phi_{\bullet,\epsilon}) + [\phi_{\bullet,\epsilon}, \phi - \phi_{\bullet,\epsilon}] + \partial_t(\psi - \psi_{\bullet,\epsilon}) + [\psi_{\bullet,\epsilon}, \psi - \psi_{\bullet,\epsilon}] + \sigma_{\boldsymbol{q}}^\epsilon d\mu(u_{\bullet,\epsilon}) \cdot \xi = 0.$$

Here ξ is defined by $u = \exp_{u_{\bullet,\epsilon}} \xi$. Then we obtain a section

$$\tilde{\mathcal{F}}_\epsilon : W \times \mathcal{B}_\epsilon \to \mathcal{E}_\epsilon.$$

The first major estimate is for the norm of $\tilde{\mathcal{F}}_\epsilon(\boldsymbol{w},\boldsymbol{v}_\epsilon)$ for all approximate solutions we constructed.

Proposition 7.6 *There exists $\epsilon_3 > 0$ such that for all $\epsilon \in (0,\epsilon_3)$ and all $\gamma \in (0, 1 - \frac{2}{p})$, there exists $c_3(\gamma) > 0$ such that for all $(\boldsymbol{w},\boldsymbol{v})$ in a neighborhood of $(\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet)$,*

$$\|\tilde{\mathcal{F}}_\epsilon(\boldsymbol{w},\boldsymbol{v}_\epsilon)\|_{\tilde{L}_\epsilon^p} \leq c_3(\gamma)(\sqrt{\epsilon})^\gamma.$$

It is proved in Subsection 8.2. Now fix the value of γ and abbreviate $c_3(\gamma) = c_3$.

7.4.2 Variation of Derivatives

Let $\tilde{\mathcal{D}}_\epsilon : W \oplus \mathcal{B}_\epsilon \to \mathcal{E}_\epsilon|_{\boldsymbol{v}_\epsilon}$ be the linearization of $\tilde{\mathcal{F}}_\epsilon$ at $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon})$. To apply the implicit function theorem, we also need to bound the variation of $\tilde{\mathcal{D}}_\epsilon$.

Consider a small different perturbation parameter $\boldsymbol{e}$, a small deformation parameter $\boldsymbol{a}$, and a deformation of the gauged map $\boldsymbol{\xi}_\epsilon = (\xi_\epsilon, \eta_\epsilon, \zeta_\epsilon) \in \mathcal{B}_\epsilon$. Denote $\boldsymbol{h}' = (\boldsymbol{e}', \boldsymbol{a}')$ and denote $\boldsymbol{w}'' = \boldsymbol{w}_\bullet + \boldsymbol{h}'$. Also define

$$\boldsymbol{v}''_\epsilon = (u''_\epsilon, a'_\epsilon) := (\exp_{u_{\bullet,\epsilon}} \xi_\epsilon, \phi_{\bullet,\epsilon} + \eta_\epsilon, \psi_{\bullet,\epsilon} + \zeta_\epsilon).$$

The linearization of $\tilde{\mathcal{F}}_\epsilon$ at $(\boldsymbol{w}'', \boldsymbol{v}''_\epsilon)$ is a linear operator

$$\tilde{\mathcal{D}}''_\epsilon : W \oplus T_{\boldsymbol{v}''_\epsilon}\mathcal{B}_\epsilon \to \mathcal{E}_\epsilon|_{\boldsymbol{v}''_\epsilon}.$$

The quadratic estimate asks to control the variation $\tilde{\mathcal{D}}''_\epsilon - \tilde{\mathcal{D}}_\epsilon$. To compare, one identifies $(u''_\epsilon)^* TX$ with $u^*_{\bullet,\epsilon} TX$ using the parallel transport of ∇ along shortest geodesics[4)]

$$Pl_\epsilon : u^*_{\bullet,\epsilon} TX \to (u''_\epsilon)^* TX.$$

By abuse of notation, it induces the identifications

$$Pl_\epsilon : T_{\boldsymbol{v}_\epsilon}\mathcal{B}_\epsilon \to T_{\boldsymbol{v}''_\epsilon}\mathcal{B}_\epsilon, \quad Pl_\epsilon : \mathcal{E}_\epsilon|_{\boldsymbol{v}_\epsilon} \to \mathcal{E}_\epsilon|_{\boldsymbol{v}''_\epsilon}.$$

Then we would like to estimate the difference

$$Pl_\epsilon^{-1} \circ \tilde{\mathcal{D}}''_\epsilon \circ Pl_\epsilon - \tilde{\mathcal{D}}_\epsilon : W \oplus \mathcal{B}_\epsilon \to \mathcal{E}_\epsilon|_{\boldsymbol{v}_{\bullet,\epsilon}}.$$

For convenience, for $\boldsymbol{\xi}_\epsilon = (\xi_\epsilon, \eta_\epsilon, \zeta_\epsilon) \in T_{\boldsymbol{v}_\epsilon}\mathcal{B}_\epsilon$, denote $\boldsymbol{\xi}''_\epsilon := (\xi''_\epsilon, \eta_\epsilon, \zeta_\epsilon) = (Pl_\epsilon(\xi_\epsilon), \eta_\epsilon, \zeta_\epsilon) \in T_{\boldsymbol{v}''_\epsilon}\mathcal{B}_\epsilon$. We will also omit Pl_ϵ whenever no confusion is caused.

First of all, by the smoothness of the perturbation term and the uniform Sobolev estimate (Lemma 7.4), we have

Lemma 7.7 *There exist $\epsilon_4 > 0$ and $c_4 > 0$ such that, for all $\epsilon \in (0, \epsilon_4)$, $\boldsymbol{h}' \in W$ with $|\boldsymbol{h}'| \leq \epsilon_4$ and all $\boldsymbol{\xi}'_\epsilon \in \mathcal{B}_\epsilon$ with $\|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \leq \epsilon_4$, denoting $\boldsymbol{v}''_\epsilon = \exp_{\boldsymbol{v}_{\bullet,\epsilon}} \boldsymbol{\xi}'_\epsilon$ and $\boldsymbol{w}'' = \boldsymbol{w}_\bullet + \boldsymbol{h}'$, one has*

$$\|Pl_\epsilon^{-1} d\iota^\epsilon_{\boldsymbol{w}'', \boldsymbol{v}''_\epsilon}(\boldsymbol{h}, \boldsymbol{\xi}''_\epsilon) - d\iota^\epsilon_{\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon}}(\boldsymbol{h}, \boldsymbol{\xi}_\epsilon)\|_{\tilde{L}^p_\epsilon} \leq c_4(\|\boldsymbol{h}'\| + \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}})(\|\boldsymbol{h}\| + \|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}).$$

Therefore, to prove the quadratic estimate it remains to bound the variation of $d\mathcal{F}$.

Proposition 7.8 *There exist $\epsilon_5 > 0$ and $c_5 > 0$ such that for all $\epsilon \in (0, \epsilon_5)$ and all $\boldsymbol{\xi}'_\epsilon \in T_{\boldsymbol{v}_\epsilon}\mathcal{B}_\epsilon$ with $\|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \leq \epsilon_5$, using the notations above, we have*

$$\|\mathcal{D}''_\epsilon(\boldsymbol{\xi}''_\epsilon) - \mathcal{D}_\epsilon(\boldsymbol{\xi}_\epsilon)\|_{\tilde{L}^p_\epsilon} \leq c_5 \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}. \tag{7.9}$$

This proposition is proved in Subsection 8.3. Lemma 7.7 and Proposition 7.8 together imply the following estimate.

Corollary 7.9 *There exist ϵ_6 and c_6 such that, for all $\epsilon \in (0, \epsilon_6)$ and all $(\boldsymbol{h}', \boldsymbol{\xi}'_\epsilon) \in W \oplus T_{\boldsymbol{v}_\epsilon}\mathcal{B}_\epsilon$ with $\|\boldsymbol{h}'\| + \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \leq \epsilon_6$, denoting $\boldsymbol{v}''_\epsilon = \exp_{\boldsymbol{v}_\epsilon} \boldsymbol{\xi}'_\epsilon$ and $\boldsymbol{w}'' = \boldsymbol{w}_\bullet + \boldsymbol{h}'$, one has*

$$\|\tilde{\mathcal{D}}''_\epsilon(\boldsymbol{h}, \boldsymbol{\xi}''_\epsilon) - \tilde{\mathcal{D}}_\epsilon(\boldsymbol{h}, \boldsymbol{\xi})\|_{\tilde{L}^p_\epsilon} \leq c_6(\|\boldsymbol{h}'\| + \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}})(\|\boldsymbol{h}\| + \|\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}).$$

7.4.3 The Right Inverse

We use the family of right inverses $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$ constructed in Section 6 to construct the right inverses along $\boldsymbol{v}_\epsilon$.

Proposition 7.10 *There exist ϵ_7, c_7, and, for each $\epsilon \in (0, \epsilon_7)$, a bounded right inverse $\tilde{\mathcal{Q}}_\epsilon : \mathcal{E}_\epsilon|_{\boldsymbol{v}_\epsilon} \to W \oplus T_{\boldsymbol{v}_\epsilon}\mathcal{B}_\epsilon$ to the operator $\tilde{\mathcal{D}}_\epsilon$ such that $\|\tilde{\mathcal{Q}}_\epsilon\| \leq c_7$.*

The construction of $\tilde{\mathcal{Q}}_\epsilon$ and the proof of this proposition are given in Subsection 8.3.

4) Notice ∇ is not diagonal w.r.t. the splitting $H_X \oplus G_X$ near $\mu^{-1}(0)$.

7.4.4 The Gluing Map

Now we are ready to apply the implicit function theorem. Let us first cite a precise version of it.

Proposition 7.11 ([12, Proposition A.3.4]) *Let X, Y be Banach spaces, $U \subset X$ be an open set and $f : U \to Y$ be a continuously differentiable map. Let $x_\bullet \in U$ be such that $df(x_\bullet) : X \to Y$ is surjective and has a bounded right inverse $Q : Y \to X$.*

Assume there are constants $\varepsilon, c > 0$ such that

$$\|Q\| \leq c; \tag{7.10}$$

$$B_\varepsilon(x_\bullet) \subset U \text{ and } x \in B_\varepsilon(x_\bullet) \Longrightarrow \|df(x) - df(x_\bullet)\| \leq \frac{1}{2c}. \tag{7.11}$$

Suppose $x' \in X$ satisfies

$$\|f(x')\| < \frac{\varepsilon}{4c}, \quad \|x' - x_\bullet\| < \frac{\varepsilon}{8}, \tag{7.12}$$

then there exists a unique $x \in X$ satisfying

$$f(x) = 0, \quad x - x' \in \operatorname{Im} Q, \quad \|x - x_\bullet\| \leq \varepsilon. \tag{7.13}$$

Moreover,

$$\|x - x'\| \leq 2c\|f(x')\|. \tag{7.14}$$

Now let $X = W \oplus \mathcal{B}_\epsilon$, $Y = \mathcal{E}_\epsilon|_{\boldsymbol{v}_\epsilon}$, $f = Pl_\epsilon^{-1} \circ \tilde{\mathcal{F}}_\epsilon \circ Pl_\epsilon$, $x_\bullet = (\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon})$. Then (7.10) holds for $c = c_7$. Corollary 7.9 implies (7.11) holds for

$$\varepsilon = \frac{1}{2c_6c_7}.$$

By Lemma 7.5, for any approximate solution $(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$ constructed from a nearby singular solution $(\boldsymbol{w}, \boldsymbol{v})$ sufficiently close to $(\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet)$, and for sufficiently small ϵ, there is a corresponding point $x' \in X$ whose distance from $x_\bullet$ is less than $\varepsilon/8$. Proposition 7.6 implies that, when $\epsilon < (8c_3c_6c_7^2)^{-2/\gamma}$, (7.12) is satisfied. Therefore by Proposition 7.11 there exists a unique x satisfying (7.13), which we denote by

$$(\boldsymbol{w}_\epsilon', \boldsymbol{v}_\epsilon') = (\boldsymbol{w} + \boldsymbol{h}_\epsilon', \exp_{\boldsymbol{v}_\epsilon} \boldsymbol{\xi}_\epsilon') \in W \times \mathcal{B}_\epsilon.$$

Then we denote the gluing map by

$$\mathrm{Glue}([\boldsymbol{w}, \boldsymbol{v}], \epsilon) = [\boldsymbol{w}_\epsilon', \boldsymbol{v}_\epsilon'], \quad \forall \epsilon \in (0, \epsilon_7), \quad \mathrm{Glue}([\boldsymbol{w}, \boldsymbol{v}], 0) = [\boldsymbol{w}, \boldsymbol{v}].$$

It is not hard to check that Glue is a continuous map from $[0, \epsilon_7) \times \mathcal{M}_{\clubsuit}^{\epsilon_7}(\boldsymbol{H}; X, L)$ to $\overline{\mathcal{M}}_{l,\underline{l}}^{\epsilon_0}(\boldsymbol{H}; X, L)$. To prove our main theorem (Theorem 5.7), it remains to show that it is a local homeomorphism, hence a local chart of topological manifold with boundary. Notice that by the transversality assumption, the domain of Glue is identified with an open subset of an Euclidean space. A standard result in general topology tells that a one-to-one and onto continuous map from a compact space to a Hausdorff space is necessarily a diffeomorphism. Then to prove Glue is a homeomorphism, it suffices to show that in a small neighborhood of $(0, [\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet])$, Glue is injective and surjective onto a neighborhood of $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$ in $\overline{\mathcal{M}}_{l,\underline{l}}^{\epsilon_0}(\boldsymbol{H}; X, L)$.

7.5 Injectivity and Surjectivity

For injectivity, suppose on the contrary that there are sequences $\boldsymbol{w}_{n,a}, \boldsymbol{w}_{n,b} \in W$ converging to $\boldsymbol{w}_\bullet$, sequences of stable perturbed affine vortices $\boldsymbol{v}_{n,a}, \boldsymbol{v}_{n,b}$ of type ♣ on the gauge slice through $\boldsymbol{v}_\bullet$ converging to $\boldsymbol{v}_\bullet$, and sequences of gluing parameters $\epsilon_{n,a}, \epsilon_{n,b}$ converging to zero, such that

$$\mathrm{Glue}([\boldsymbol{w}_{n,a}, \boldsymbol{v}_{n,a}], \epsilon_{n,a}) = \mathrm{Glue}([\boldsymbol{w}_{n,b}, \boldsymbol{v}_{n,b}], \epsilon_{n,b}).$$

By definition, for all n, $\boldsymbol{w}_{n,a} = \boldsymbol{w}_{n,b}$ and $\epsilon_{n,a} = \epsilon_{n,b}$. We would like to show that for large n, $\boldsymbol{v}_{n,a} = \boldsymbol{v}_{n,b}$. Suppose it is not the case. Let the corresponding approximate solutions be $\boldsymbol{v}_{\epsilon_n,a}$, $\boldsymbol{v}_{\epsilon_n,b}$ respectively, which are in small neighborhoods of the central approximate solution $\boldsymbol{v}_{\bullet,\epsilon_n}$. By construction, $\boldsymbol{v}_{\epsilon_n,a}$ and $\boldsymbol{v}_{\epsilon_n,b}$ can be identified with different elements $\boldsymbol{\xi}_{n,a}^{\mathrm{app}}, \boldsymbol{\xi}_{n,b}^{\mathrm{app}}$ in the tangent space $\mathcal{B}_{\epsilon_n}$, while the exact solutions are identified with $\boldsymbol{\xi}_{n,a}^{\mathrm{exact}}, \boldsymbol{\xi}_{n,b}^{\mathrm{exact}}$. Since the exact solutions are both in the Coulomb slice through $\boldsymbol{v}_{\bullet,\epsilon_n}$ and represent the same point in the moduli space, the two exact solutions are identical. Moreover, since exact solutions differ from the approximate solutions by elements in the image of the right inverse $\tilde{\mathcal{Q}}_{\epsilon_n}$, we know that

$$\boldsymbol{\xi}_{n,a}^{\mathrm{app}} - \boldsymbol{\xi}_{n,b}^{\mathrm{app}} \in \mathrm{Im}\tilde{\mathcal{Q}}_{\epsilon_n}.$$

Therefore, by the uniform boundedness of the right inverse, there is a number $c > 0$ independent of n such that

$$\|\tilde{\mathcal{D}}_{\epsilon_n}(\boldsymbol{\xi}_{n,a}^{\mathrm{app}} - \boldsymbol{\xi}_{n,b}^{\mathrm{app}})\|_{\tilde{L}^p_{\epsilon_n}} \geq \frac{1}{c}\|\boldsymbol{\xi}_{n,a}^{\mathrm{app}} - \boldsymbol{\xi}_{n,b}^{\mathrm{app}}\|_{\tilde{L}^{1,p}_{m;\epsilon_n}}.$$

However, by straightforward estimate, for ϵ_n sufficiently small, we have

$$\lim_{n\to\infty} \|\tilde{\mathcal{D}}_{\epsilon_n}(\boldsymbol{\xi}_{n,a}^{\mathrm{app}} - \boldsymbol{\xi}_{n,b}^{\mathrm{app}})\|_{\tilde{L}^p_{\epsilon_n}} / \|\boldsymbol{\xi}_{n,a}^{\mathrm{app}} - \boldsymbol{\xi}_{n,b}^{\mathrm{app}}\|_{\tilde{L}^{1,p}_{m;\epsilon_n}} = 0.$$

We omit the rather tedious estimate. This contradicts the previous inequality. Hence for n large, $\boldsymbol{v}_{n,a} = \boldsymbol{v}_{n,b}$ and we proved the injectivity of the gluing map.

Now we prove the surjectivity. The main difficulty is that the topology of $\overline{\mathcal{M}}_{l,\underline{l}}^{\epsilon_0}(\boldsymbol{H}; X, L)$ is defined in terms of a very weak notion of convergence. In particular, for a sequence of perturbed vortices $\boldsymbol{v}_n$, for each $R > 0$, we only require that a C^0_{loc} convergence for the rescaled one $(s_{\epsilon_n}^{-1})^*(\boldsymbol{v}_n|_{C_{R/\epsilon_n}})$. However, we need to show that every nearby smooth vortex $\boldsymbol{v}_n$ is actually close to the approximate solution $\boldsymbol{v}_{\epsilon_n}$ w.r.t. our norm in $\mathcal{B}_{\epsilon_n}$. This is similar to the case of Gaio–Salamon [9], where the surjectivity, based on their *a priori* estimates in their Section 9 and 10, was the most difficult part. More precisely, in Subsection 8.5 we prove the following theorem.

Proposition 7.12 *Suppose a sequence* $[\boldsymbol{w}_n, \boldsymbol{v}_n] \in \overline{\mathcal{M}}_{l,\underline{l}}^{\epsilon_0}(\boldsymbol{H}; X, L)$ *converges to* $[\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet]$, *namely, up to gauge transformation,* $(\boldsymbol{w}_n, \boldsymbol{v}_n)$ *and* $(\boldsymbol{w}_\bullet, \boldsymbol{v}_\bullet)$ *satisfy the conditions of Definition* 5.5 *for a sequence of gluing parameters* $\epsilon_n \to 0$. *Then for i sufficiently large, we can gauge transform* $\boldsymbol{v}_n$ *to a sequence of smooth vortices* (*which we still denote by* $\boldsymbol{v}_n$) *such that* $(\boldsymbol{w}_n, \boldsymbol{v}_n) = (\boldsymbol{w}_\bullet + \boldsymbol{h}_n, \exp_{\boldsymbol{v}_{\bullet,\epsilon_n}} \boldsymbol{\xi}_n)$ *where* $\boldsymbol{\xi}_n \in \mathcal{B}_{\epsilon_n}$ *and*

$$\lim_{i\to\infty} (\|\boldsymbol{h}_n\| + \|\boldsymbol{\xi}_n\|_{\tilde{L}^{1,p}_{m;\epsilon_n}}) = 0.$$

The surjectivity of the gluing map follows from this proposition in a rather abstract way. By this proposition, it suffices to show that there exists $r > 0$ such that for all ϵ sufficiently small, the zero locus $\tilde{\mathcal{F}}_{\epsilon_n}^{-1}(0)$ intersecting the radius r ball centered at $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon})$ is contained in the image of the gluing map. If this is not true, then there exists a sequence $\epsilon_n \to 0$ and a sequence of elements $(\boldsymbol{w}_n, \boldsymbol{v}_n) \in \tilde{\mathcal{F}}_{\epsilon_n}^{-1}(0)$ whose distances from $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon_n})$ converge to zero, but do not coincide with any exact solution. Identify $(\boldsymbol{w}_n, \boldsymbol{v}_n)$ with a sequence of tangent vectors $\tilde{\boldsymbol{\xi}}_n \in W \oplus \mathcal{B}_{\epsilon_n}$. Consider the path $t\tilde{\boldsymbol{\xi}}_n$. Then by Proposition 7.8, we know that

$$\lim_{n\to\infty} \|\tilde{\mathcal{F}}_{\epsilon_n}(t\tilde{\boldsymbol{\xi}}_n)\| = 0, \quad \text{uniformly in } t \in [0,1].$$

Then by the implicit function theorem, by correcting $t\tilde{\boldsymbol{\xi}}_n$ there exists a sequence of exact solutions $\tilde{\boldsymbol{\xi}}_n(t)$. Notice that $\tilde{\boldsymbol{\xi}}_n(0)$ is the correction of $(\boldsymbol{w}_\bullet, \boldsymbol{v}_{\bullet,\epsilon_n})$ which lies in the image of the gluing map. Let $t_n \in [0,1)$ be the largest number such that $\tilde{\boldsymbol{\xi}}_n(t)$ is in the image of the gluing map for all $t \leq t_n$. Then $\tilde{\boldsymbol{\xi}}_n(t_n)$ is on the boundary of the image of the gluing map (for the fixed value $t = t_n$) and its distance from the origin of $\mathcal{B}_{\epsilon_n}$ is uniformly bounded away from zero. However, by the implicit function theorem (more precisely by (7.14)) we have

$$\|t_n\tilde{\boldsymbol{\xi}}_n - \tilde{\boldsymbol{\xi}}_n(t_n)\| \lesssim \|\tilde{\mathcal{F}}_{\epsilon_n}(t\tilde{\boldsymbol{\xi}}_n)\| \to 0.$$

We also have $\|t_n\tilde{\boldsymbol{\xi}}_n\| \to 0$, which is a contradiction. This finishes the proof of surjectivity of the gluing map and hence finishes the proof of our main theorem (Theorem 5.7).

8 Technical Results for Gluing

We prove certain technical results which have been stated earlier in the gluing construction.

8.1 Proof of Lemmas 7.1, 7.2

Since we only care the central object for these two lemmata, we remove "$\bullet$" from the notations.

Proof of Lemma 7.1 The C^0 distance between $\boldsymbol{v}'_{i,\epsilon}$ and $\boldsymbol{v}_i$ is small. Hence we obtain $\boldsymbol{\xi}'_{i,\epsilon}$ satisfying $\boldsymbol{v}'_{i,\epsilon} = \exp_{\boldsymbol{v}_i} \boldsymbol{\xi}'_{i,\epsilon}$ pointwise. Without loss of generality, assume the base point $z_{i,\epsilon} = 0$. Hence $\boldsymbol{\xi}'_{i,\epsilon}$ is supported in $\boldsymbol{A}_i \setminus \check{\tilde{B}}{}^i_\epsilon$. In this region we have

$$\check{\boldsymbol{v}}_i = (\exp_{x_i} \check{\xi}_i, \check{\phi}_i, \check{\psi}_i), \quad \check{\boldsymbol{v}}'_{i,\epsilon} = (\exp_{x_i} \beta_{i,\epsilon}\check{\xi}_i, \beta_{i,\epsilon}\check{\phi}_i, \beta_{i,\epsilon}\check{\psi}_i).$$

Then (after identifying $\check{\xi}_i$ as a tangent vector along $\check{u}_i$) we have

$$\check{\boldsymbol{\xi}}'_{i,\epsilon} := \mathrm{e}^{\lambda_i\theta}\boldsymbol{\xi}'_{i,\epsilon} = (\beta_{i,\epsilon} - 1)(\check{\xi}_i, \check{\phi}_i, \check{\psi}_i).$$

It suffices to estimate $\|\check{\boldsymbol{\xi}}'_{i,\epsilon}\|_{\tilde{L}^{1,p}_m}$ using the trivial connection. By Lemma 3.4, we know that over $\boldsymbol{A}_i \setminus \check{\tilde{B}}{}^i_\epsilon$, for all $\delta \in (1 - \frac{2}{p}, 1)$,

$$\|\nabla\check{\xi}_i\|_{L^{p,\delta}} + \|d\mu \cdot \check{\xi}_i\|_{L^{p,\delta}} + \|d\mu \cdot J\check{\xi}_i\|_{L^{p,\delta}} + \|\check{\phi}_i\|_{W^{1,p,\delta}} + \|\check{\psi}_i\|_{W^{1,p,\delta}} < \infty.$$

Choose $\delta \in (\delta_p, 1)$. Then

$$\begin{aligned}\|\mathrm{e}^{\lambda_i\theta}\boldsymbol{\xi}'_i(z)\|_{\tilde{L}^{1,p}_m} &\leq \|\check{\boldsymbol{\xi}}_i\|_{\tilde{L}^{1,p}_m(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)} + \|\nabla\beta_{i,\epsilon}\|_{L^\infty}\|\check{\boldsymbol{\xi}}^G_i\|_{\tilde{L}^p(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)} + \|(\nabla\beta_{i,\epsilon})\check{\xi}^H_i\|_{\tilde{L}^p(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)}\\ &\lesssim \sqrt{\epsilon}^{\delta-\delta_p} + \sqrt{\epsilon}\|\check{\boldsymbol{\xi}}_i\|_{\tilde{L}^p(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)} + \|(\nabla\beta_{i,\epsilon})\check{\xi}^H_i\|_{\tilde{L}^p(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)}.\end{aligned}$$

The first two terms converge to zero as $\epsilon \to 0$; the third term is estimated as follows.

$$\begin{aligned}\|(\nabla\beta_{i,\epsilon})\check{\xi}^H_i\|_{\tilde{L}^p(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)} &= \left[\int_{\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon} |\nabla\beta_{i,\epsilon}|^p|\check{\xi}^H_i|^p[\rho_\epsilon(z)]^{2p-4}dsdt\right]^{\frac{1}{p}}\\ &\lesssim \left[\int_{\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon} |\check{\xi}^H_i|^p[\rho_\epsilon(z)]^{p-4}dsdt\right]^{\frac{1}{p}} = \|\check{\xi}^H_i\|_{L^{p,\delta_p-1}(\boldsymbol{A}_i\setminus\check{\tilde{B}}{}^i_\epsilon)}\\ &\lesssim \sqrt{\epsilon}^{\delta-\delta_p}\|\check{\xi}^H_i\|_{L^{p,\delta-1}} \lesssim \sqrt{\epsilon}^{\delta-\delta_p}(\|\check{\xi}^H_i\|_{L^\infty} + \|\nabla\check{\xi}^H_i\|_{L^{p,\delta}}).\end{aligned}$$

Here the last inequality follows from (3.7) and the fact that the limit of $\check{\xi}_i^H$ at ∞ is zero. Hence this also converges to zero as $\epsilon \to 0$. This finishes the proof of Lemma 7.1. □

Proof of Lemma 7.2 Similar to the previous proof, we can assume $z_{i,\epsilon}$. Inside $\hat{\hat{B}}_\epsilon^i$, we can write

$$\check{\boldsymbol{\xi}}'_{\infty,\epsilon} := \mathrm{e}^{\lambda_i\theta}\boldsymbol{\xi}'_{\infty,\epsilon}(z) = (\beta_{\infty,\epsilon} - 1)(\check{\xi}_{\infty,\epsilon}, \check{\phi}_{\infty,\epsilon}, \check{\psi}_{\infty,\epsilon})$$

Denote $V_\epsilon^i = B(2b\sqrt{\epsilon}) = s_\epsilon(\hat{\hat{B}}_\epsilon^i)$. Then since $\boldsymbol{v}_{\infty,\epsilon}$ is the rescaling of $\boldsymbol{v}_\infty$, we can write $\check{\xi}_{\infty,\epsilon} = s_\epsilon^*\check{\xi}_\infty$, $\check{\phi}_{\infty,\epsilon} = \epsilon s_\epsilon^*\check{\phi}_\infty$, and $\check{\psi}_{\infty,\epsilon} = \epsilon s_\epsilon^*\check{\psi}_\infty$. Since $\boldsymbol{v}_\infty$ comes from a smooth holomorphic disk in $\bar{X}$, we know that $|\check{\xi}_\infty| \lesssim |z|$ and $\check{\phi}_\infty, \check{\psi}_\infty$ are bounded. Moreover, by the special gauge we choose for $\boldsymbol{v}_\infty$, over V_ϵ^i we have $\check{\xi}_\infty = \check{\xi}_\infty^H$. Hence we have

$$\|\check{\xi}_\infty^H\|_{\tilde{L}^p(V_\epsilon^i)} + \epsilon\|\check{\boldsymbol{\xi}}_\infty^G\|_{\tilde{L}^p(V_\epsilon^i)} \lesssim \sqrt{\epsilon}^{1+\frac{2}{p}}, \quad \|\nabla^{a_\infty}\check{\boldsymbol{\xi}}_\infty\|_{\tilde{L}^p(V_\epsilon)} \lesssim \sqrt{\epsilon}^{\frac{2}{p}}. \tag{8.1}$$

Then we have (recall the definition of the $\tilde{L}^{1,p}_{m;\epsilon}$-norm given by (6.2))

$$\begin{aligned}
\|\check{\boldsymbol{\xi}}'_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\hat{B}_\epsilon^i)} &\leq \|\nabla\beta_{\infty,\epsilon}\|_{L^\infty}\|\check{\boldsymbol{\xi}}^G_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)} + \|\nabla\beta_{\infty,\epsilon}\|_{L^\infty}\|\check{\xi}^H_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)} + \|\check{\boldsymbol{\xi}}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\hat{B}_\epsilon^i)} \\
&\lesssim \sqrt{\epsilon}(\epsilon\|s_\epsilon^*\check{\boldsymbol{\xi}}^G_\infty\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)} + \|s_\epsilon^*\xi^H_\infty\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)}) + \epsilon\|s_\epsilon^*\nabla^{a_\infty}\boldsymbol{\xi}_\infty\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)} \\
&\quad + \epsilon\|s_\epsilon^*\boldsymbol{\xi}^G_\infty\|_{\tilde{L}^p_\epsilon(\hat{B}_\epsilon^i)} + \|\check{\xi}^H_{\infty,\epsilon}\|_{L^\infty(\hat{B}_\epsilon^i)} \\
&\lesssim \frac{1}{\sqrt{\epsilon}}(\epsilon\|\check{\boldsymbol{\xi}}^G_\infty\|_{\tilde{L}^p(V_\epsilon^i)} + \|\check{\xi}^H_\infty\|_{\tilde{L}^p(V_\epsilon^i)}) + \|\nabla^{a_\infty}\boldsymbol{\xi}_\infty\|_{\tilde{L}^p(V_\epsilon^i)} + \|\boldsymbol{\xi}^G_\infty\|_{\tilde{L}^p(V_\epsilon^i)} + \|\check{\xi}^H_\infty\|_{L^\infty(V_\epsilon^i)}.
\end{aligned}$$

Here for the last line we used Lemma 6.1. Then by (8.1) we see that $\lim_{\epsilon\to 0}\|\check{\boldsymbol{\xi}}'_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}} \to 0$. □

8.2 Proof of Proposition 7.6

We denote the three components of $\tilde{\mathcal{F}}_\epsilon$ by $\tilde{\mathcal{F}}_1$, $\tilde{\mathcal{F}}_2$ and $\tilde{\mathcal{F}}_3$ respectively, where only $\tilde{\mathcal{F}}_1$ contains on the perturbation term. We first estimate $\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$

(a) Inside each $\check{\check{B}}_\epsilon^i$, i.e., a region with radius $\approx \frac{1}{\sqrt{\epsilon}}$ corresponding to the affine vortex $\boldsymbol{v}_i$,

$$\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon) = \overline{\partial}_{A_i}u_i + \iota_\epsilon(\boldsymbol{w}, u_i) = \iota_\epsilon(\boldsymbol{w}, u_i) - \iota_0(\boldsymbol{w}, u_i).$$

Here we used the equation $\overline{\partial}_{A_i}u_i + \iota_0(\boldsymbol{w}, u_i) = 0$. Then by the definition of the norm the definition of the perturbation ι_ϵ (which is supported on a compact subset and depends on ϵ smoothly), one has

$$\|\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)\|_{\tilde{L}^p_\epsilon(\check{\check{B}}_\epsilon^i)} \lesssim \epsilon. \tag{8.2}$$

(b) Over $\hat{\hat{\Sigma}}_\epsilon$, i.e., a region corresponding to a large part of the disk $\boldsymbol{v}_\infty$, one has

$$\begin{aligned}
\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon) &= \overline{\partial}_{A_{\infty,\epsilon}}u_{\infty,\epsilon} + \iota_{\infty,\epsilon}(\boldsymbol{w}, u_{\infty,\epsilon}) \\
&= \iota_{\infty,\epsilon}(\boldsymbol{w}, u_{\infty,\epsilon}) - \iota_{\infty,0}(\boldsymbol{w}, u_{\infty,\epsilon}) = \epsilon s_\epsilon^*(\iota_\epsilon(\boldsymbol{w}, u_\infty) - \iota_0(\boldsymbol{w}, u_\infty)).
\end{aligned}$$

Then by the property of ι_ϵ, and Lemma 6.1, one has

$$\|\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)\|_{\tilde{L}^p_\epsilon(\hat{\hat{\Sigma}}_\epsilon)} \lesssim \epsilon. \tag{8.3}$$

(c) In the interior part of the neck region $\mathring{B}_\epsilon^i \setminus \check{\check{B}}_\epsilon^i \subset \ddot{A}_\epsilon^i$, one has

$$u_\epsilon(z) = \mathrm{e}^{-\lambda_i\theta_i}\exp_{x_i}(\beta_{i,\epsilon}\check{\xi}_i(z)), \quad a_\epsilon = \mathrm{e}^{-\lambda_i\theta_i}\cdot(\beta_{i,\epsilon}\check{\alpha}_i).$$

Recall that the perturbation term vanishes in the neck region. Using the normal coordinate centered at x_i, we can write

$$\begin{aligned} e^{\lambda_i\theta_i}(\overline{\partial}_{A_\epsilon} u_\epsilon) &= \frac{\partial \check{u}_\epsilon}{\partial s} + J(\check{u}_\epsilon)\frac{\partial \check{u}_\epsilon}{\partial t} + \beta_{i,\epsilon}(\mathcal{X}_{\check{\phi}_i} + J(\check{u}_\epsilon)\mathcal{X}_{\check{\psi}_i}) \\ &= \beta_{i,\epsilon}\left[\frac{\partial \check{\xi}_i}{\partial s} + \mathcal{X}_{\check{\phi}_i}(\check{u}_\epsilon) + J(\check{u}_\epsilon)\left(\frac{\partial \check{\xi}_i}{\partial t} + \mathcal{X}_{\check{\psi}_i}(\check{u}_\epsilon)\right)\right] + \frac{\partial \beta_{i,\epsilon}}{\partial s}\check{\xi}_i + J(\check{u}_\epsilon)\frac{\partial \beta_{i,\epsilon}}{\partial t}\check{\xi}_i \\ &= \beta_{i,\epsilon}[\mathcal{X}_{\check{\phi}_i}(\check{u}_\epsilon) - \mathcal{X}_{\check{\phi}_i}(\check{u}_i) + J(\check{u}_\epsilon)\mathcal{X}_{\check{\psi}_i}(\check{u}_\epsilon) - J(\check{\xi}_i)\mathcal{X}_{\check{\psi}_i}(\check{\xi}_i)] + \frac{\partial \beta_{i,\epsilon}}{\partial s}\check{\xi}_i + J(\check{u}_\epsilon)\frac{\partial \beta_{i,\epsilon}}{\partial t}\check{\xi}_i. \end{aligned}$$

In the last line, let F_1 be the sum of the terms involving the derivatives of $\beta_{i,\epsilon}$, and F_2 be the sum of other terms. To estimate them, firstly recall that, by the main result of [38] and [25, Proposition A.4], the energy density of $\boldsymbol{v}_i$ decays in the following way:

$$|d_{a_i}\boldsymbol{v}_i(w_i)|^2 + |F_{A_i}(w_i)|^2 + |\mu(u_i(w_i))|^2 \lesssim_\alpha |w_i|^{-4+\alpha}, \quad \forall \alpha > 0. \tag{8.4}$$

Here $w_i = z - z_{i,\epsilon}$ is the shifted affine coordinate. Then if $z \in A^{i,-}_{\epsilon;2}$,

$$\mathrm{dist}(\check{u}_i(w_i), x_i) \lesssim_\alpha (\sqrt{\epsilon})^{1-\alpha} \Longrightarrow \sup_{\ddot{A}^i_\epsilon} |u_i - u_\epsilon| \lesssim_\alpha (\sqrt{\epsilon})^{1-\alpha}. \tag{8.5}$$

So using the fact that $|\nabla \beta_{i,\epsilon}| \lesssim \sqrt{\epsilon}$, one has

$$\begin{aligned} \|(\partial_s\beta_{i,\epsilon})\check{\xi}_i + J(\check{u}_\epsilon)(\partial_t\beta_{i,\epsilon})\check{\xi}_i\|_{\tilde{L}^p_\epsilon} &\lesssim \left[\int_{\ddot{A}^i_\epsilon} |\nabla\beta_{i,\epsilon}(z)|^p \Big[\sup_{z\in\ddot{A}^i_\epsilon} \mathrm{dist}(\check{u}_i, x_i)\Big]^p \epsilon^{2-p} ds dt\right]^{\frac{1}{p}} \\ &\lesssim_\alpha (\sqrt{\epsilon})^{\frac{4}{p}-\alpha}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}} \lesssim_\alpha (\sqrt{\epsilon})^{\frac{2}{p}-\alpha}, \quad \forall \alpha > 0. \end{aligned} \tag{8.6}$$

On the other hand, by Sobolev embedding, one has

$$\|\check{\alpha}_i\|_{\tilde{L}^{1,p}_g} \leq c_i \Longrightarrow \sup_{\ddot{A}^i_\epsilon}(|\check{\phi}_i| + |\check{\psi}_i|) \lesssim \epsilon^{1-\frac{2}{p}}. \tag{8.7}$$

Then by (8.5) and (8.7) one has

$$\begin{aligned} &\|\beta_{i,\epsilon}(\mathcal{X}_{\check{\phi}_i}(u_\epsilon) - \mathcal{X}_{\check{\phi}_i}(u_i) + J(u_\epsilon)\mathcal{X}_{\check{\psi}_i}(u_\epsilon) - J(u_i)\mathcal{X}_{\check{\psi}_i}(u_i))\|_{\tilde{L}^p_\epsilon(\ddot{A}^i_\epsilon)} \\ &\lesssim \left[\int_{\ddot{A}^i_\epsilon} \Big[\sup_{\ddot{A}^i_\epsilon} |\check{\alpha}_i|\Big]^p \Big[\sup_{\ddot{A}^i_\epsilon} \mathrm{dist}(u_\epsilon, u_i)\Big]^p \epsilon^{2-p} ds dt\right]^{\frac{1}{p}} \lesssim_\alpha (\sqrt{\epsilon})^{1-\alpha}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}} \lesssim_\alpha (\sqrt{\epsilon})^{1-\alpha-\frac{2}{p}}. \end{aligned}$$

Together with (8.6), we find that

$$\|\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)\|_{\tilde{L}^p_\epsilon(\mathring{B}^i_\epsilon \setminus \check{B}^i_\epsilon)} \lesssim_\alpha (\sqrt{\epsilon})^{1-\alpha-\frac{2}{p}}, \quad \forall \alpha > 0. \tag{8.8}$$

(d) In the outer part of the neck region $\hat{\mathring{B}}^i_\epsilon \setminus \mathring{B}^i_\epsilon \subset \ddot{A}^i_\epsilon$ we can derive similar estimate. We remark that we have slightly better bound than (8.5) and (8.7) if replace $\check{u}_i$ by $\check{u}_{\infty,\epsilon}$ and $\check{\alpha}_i$ by $\check{\alpha}_{\infty,\epsilon}$, i.e.,

$$\sup_{\ddot{A}^i_\epsilon} \mathrm{dist}(u_{\infty,\epsilon} - u_\epsilon) \lesssim \sqrt{\epsilon}; \quad \sup_{\ddot{A}^i_\epsilon} |\check{\alpha}_{\infty,\epsilon}| \lesssim \epsilon. \tag{8.9}$$

So we obtain a bound of $\|\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)\|_{\tilde{L}^p_\epsilon(\hat{\mathring{B}}^i_\epsilon \setminus \mathring{B}^i_\epsilon)}$ as in (8.8) for all $\alpha > 0$. In summary, we have obtained desired bound for the component $\tilde{\mathcal{F}}_1(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$.

To estimate $\tilde{\mathcal{F}}_2(\boldsymbol{w}, \boldsymbol{v}_\epsilon)$ we cut $\boldsymbol{H}$ as the union of $\mathring{\Sigma}_\epsilon$ and all $\mathring{B}^i_\epsilon$. Over $\mathring{\Sigma}_\epsilon$, one has

$$*F_{A_\epsilon} + \sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon) = *d(\beta_{\infty,\epsilon}\check{\alpha}_{\infty,\epsilon}) + \sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon) = \beta_{\infty,\epsilon} * F_{A_{\infty,\epsilon}} + *(d\beta_{\infty,\epsilon} \wedge \check{\alpha}_{\infty,\epsilon}) + \sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon).$$

Over $\mathring{B}^i_\epsilon$, using the vortex equation $*F_{A_i} + \mu(u_i) = 0$ and $\sigma^\epsilon_{\boldsymbol{q}} = 1$, we obtain

$$*F_{A_\epsilon} + \sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon) = \beta_{i,\epsilon}\sigma^\epsilon_{\boldsymbol{q}}[\mu(u_\epsilon) - \mu(u_i)] + (1-\beta_{i,\epsilon})\sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon) + *(d\beta_{i,\epsilon}\wedge\check{\alpha}_{i,\epsilon}).$$

So as a unifying expression,

$$\tilde{\mathcal{F}}_2(\boldsymbol{w},\boldsymbol{v}_\epsilon) = \beta_{\infty,\epsilon}*F_{A_{\infty,\epsilon}} + *(d\beta_{\infty,\epsilon}\wedge\check{\alpha}_{\infty,\epsilon}) + *(d\beta_{i,\epsilon}\wedge\check{\alpha}_i) + (1-\beta_{i,\epsilon})\sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon) + \beta_{i,\epsilon}\sigma^\epsilon_{\boldsymbol{q}}(\mu(u_\epsilon) - \mu(u_i)).$$

We estimate the terms one by one (all norms below without labelling are $\|\cdot\|_{\tilde{L}^p_\epsilon}$).

(a) One has $F_{A_{\infty,\epsilon}} = s_\epsilon^* F_{A_\infty}$, so by Lemma 6.1 (extending to two-forms),

$$\|\beta_{\infty,\epsilon} * F_{A_{\infty,\epsilon}}\| \leq \|s_\epsilon^* F_{A_\infty}\| = \epsilon\|F_{A_\infty}\|_{\tilde{L}^p} \lesssim \epsilon.$$

(b) By (8.9), and the fact that $|\nabla\beta_{\infty,\epsilon}| \lesssim \sqrt{\epsilon}$,

$$\|*(d\beta_{\infty,\epsilon}\wedge\check{\alpha}_{\infty,\epsilon})\| \lesssim \sqrt{\epsilon}\bigg[\int_{\ddot{A}^i_\epsilon} |\check{\alpha}_{\infty,\epsilon}(z)|^p[\rho_{\infty,\epsilon}(z)]^{2p-4}dsdt\bigg]^{\frac{1}{p}} \lesssim \epsilon^{\frac{1}{2}+\frac{2}{p}}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}} \lesssim (\sqrt{\epsilon})^{1+\frac{2}{p}}.$$

(c) By (8.7) and $|\nabla\beta_{i,\epsilon}| \lesssim \sqrt{\epsilon}$, one has

$$\|*(d\beta_{i,\epsilon}\wedge\check{\alpha}_i)\| \lesssim \sqrt{\epsilon}\bigg[\int_{\ddot{A}^i_\epsilon} |\check{\alpha}_i(z)|^p[\rho_\epsilon(z)]^{2p-4}dsdt\bigg]^{\frac{1}{p}} \lesssim \sqrt{\epsilon}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}} \lesssim (\sqrt{\epsilon})^{1-\frac{2}{p}}.$$

(d) By (8.4) we know that in the neck region $|\mu(u_\epsilon)| \lesssim_\alpha (\sqrt{\epsilon})^{2-\alpha}$, so

$$\begin{aligned}\|(1-\beta_{i,\epsilon})\sigma^\epsilon_{\boldsymbol{q}}\mu(u_\epsilon)\| &\leq \bigg[\int_{\ddot{A}^i_\epsilon}\Big[\sup_{\ddot{A}^i_\epsilon}|\mu(u_\epsilon)|\Big]^p[\rho_\epsilon(z)]^{2p-4}dsdt\bigg]^{\frac{1}{p}}\\ &\lesssim_\alpha (\sqrt{\epsilon})^{\frac{4}{p}-\alpha}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}}\\ &\lesssim (\sqrt{\epsilon})^{\frac{2}{p}-\alpha}.\end{aligned}$$

(e) By the energy decay we also have $\sigma^\epsilon_{\boldsymbol{q}}|\mu(u_\epsilon) - \mu(u_i)| \lesssim_\alpha (\sqrt{\epsilon})^{2-\alpha}$. So

$$\|\beta_{i,\epsilon}\sigma^\epsilon_{\boldsymbol{q}}(\mu(u_\epsilon) - \mu(u_i))\| \lesssim (\sqrt{\epsilon})^{2-\alpha}[\mathrm{Area}(\ddot{A}^i_\epsilon)]^{\frac{1}{p}} \lesssim_\alpha (\sqrt{\epsilon})^{2-\frac{2}{p}-\alpha}.$$

By (a)–(e) above, for any $\gamma < 1 - \frac{2}{p}$, by taking α appropriately, one obtains

$$\|\tilde{\mathcal{F}}_2(\boldsymbol{w},\boldsymbol{v}_\epsilon)\|_{\tilde{L}^p_\epsilon} \lesssim (\sqrt{\epsilon})^\gamma.$$

Now we estimate $\tilde{\mathcal{F}}_3(\boldsymbol{w},\boldsymbol{v}_\epsilon)$. Introduce $\boldsymbol{\xi}_\epsilon = (\xi_\epsilon, \eta_\epsilon, \zeta_\epsilon)$ by

$$u_\epsilon = \exp_{u_{\bullet,\epsilon}}\xi_\epsilon, \quad \eta_\epsilon = \phi_\epsilon - \phi_{\bullet,\epsilon}, \quad \zeta_\epsilon = \psi_\epsilon - \psi_{\bullet,\epsilon}.$$

Then

$$\tilde{\mathcal{F}}_3(\boldsymbol{w},\boldsymbol{v}_\epsilon) = \partial_s\eta_\epsilon + [\phi_{\bullet,\epsilon},\eta_\epsilon] + \partial_t\zeta_\epsilon + [\psi_{\bullet,\epsilon},\zeta_\epsilon] + d\mu(u_{\bullet,\epsilon})\cdot J\xi_\epsilon.$$

It vanishes on all $\check{\check{B}}^i_\epsilon$ since $\boldsymbol{v}_i$ is already in the Coulomb slice through $\boldsymbol{v}_{\bullet,i}$. On the other hand, over $\hat{\hat{\Sigma}}_\epsilon$, $J\xi_\epsilon$ is in the horizontal distribution. Since in this region $\boldsymbol{v}_\epsilon$ and $\boldsymbol{v}_{\bullet,\epsilon}$ are both obtained by pulling back disks in $\bar{X}$, $\tilde{\mathcal{F}}_3(\boldsymbol{w},\boldsymbol{v}_\epsilon)$ is $\epsilon^2 s_\epsilon^* F$ for some function F on $s_\epsilon(\hat{\hat{\Sigma}}_\epsilon)$, and by Lemma 4.8, the $\tilde{L}^p$-norm of F is finite. Combining with Lemma 6.1 we see the size of $\tilde{\mathcal{F}}_3(\boldsymbol{w},\boldsymbol{v}_\epsilon)$ over $\hat{\hat{\Sigma}}_\epsilon$ is bounded by a constant multiple of ϵ. Lastly, the estimate of $\tilde{\mathcal{F}}_3(\boldsymbol{w},\boldsymbol{v}_\epsilon)$ over the neck regions $\ddot{A}^\epsilon_i$ can be estimated similarly as $\tilde{\mathcal{F}}_2(\boldsymbol{w},\boldsymbol{v}_\epsilon)$ and we omit it. This finishes proving Proposition 7.6.

8.3 Proof of Proposition 7.8

Consider an intermediate object $\boldsymbol{v}_\epsilon''' = (u_\epsilon, \phi_\epsilon', \psi_\epsilon')$ with linearized operator $\mathcal{D}_\epsilon''' = d\mathcal{F}_{\boldsymbol{v}_\epsilon'''} : \mathcal{B}_{\boldsymbol{v}_\epsilon'''} \to \mathcal{E}_{\boldsymbol{v}_\epsilon'''}$. Namely, $\boldsymbol{v}_\epsilon'''$ and $\boldsymbol{v}_\epsilon$ only differ in their gauge fields.

We first compare $\mathcal{D}_\epsilon'''$ and $\mathcal{D}_\epsilon$, whose domains and targets are identified without using parallel transport. Suppose $\phi_\epsilon' = \phi_\epsilon + \eta_\epsilon'$, $\psi_\epsilon' = \psi_\epsilon + \zeta_\epsilon'$. Then for an infinitesimal deformation $\boldsymbol{\xi} = (\xi, \eta, \zeta)$,

$$(\mathcal{D}_\epsilon''' - \mathcal{D}_\epsilon)(\xi,\eta,\zeta) = \begin{bmatrix} \nabla_\xi \mathcal{X}_{\eta_\epsilon'}(u_\epsilon) + \nabla_\xi (J\mathcal{X}_{\zeta_\epsilon'})(u_\epsilon) \\ [\eta_\epsilon', \eta] + [\zeta_\epsilon', \zeta] \\ [\eta_\epsilon', \zeta] - [\zeta_\epsilon', \eta] \end{bmatrix}.$$

By Lemma 7.4 and the definition of the norm (7.7), one has

$$\|(\mathcal{D}_\epsilon''' - \mathcal{D}_\epsilon)(\boldsymbol{\xi})\|_{\tilde{L}^p_\epsilon} \lesssim \|\boldsymbol{\xi}\|_{L^\infty} (\|\eta_\epsilon'\|_{\tilde{L}^p_\epsilon} + \|\zeta_\epsilon'\|_{\tilde{L}^p_\epsilon}) \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}} (\|\eta_\epsilon'\|_{\tilde{L}^{1,p}_{g;\epsilon}} + \|\zeta_\epsilon'\|_{\tilde{L}^{1,p}_{g;\epsilon}}). \tag{8.10}$$

Now we compare $\mathcal{D}_\epsilon'''$ with $\mathcal{D}_\epsilon''$ whose domains and targets are identified via the parallel transport Pl_ϵ between u_ϵ and $u_\epsilon' = \exp_{u_\epsilon} \xi_\epsilon'$. Firstly, given an infinitesimal deformations of the type $\boldsymbol{\xi} = (0, \eta, \zeta)$, only the first component of $(\mathcal{D}_\epsilon'' - \mathcal{D}_\epsilon''')(\boldsymbol{\xi})$ is nonzero, which reads

$$Pl_\epsilon^{-1}[\mathcal{X}_\eta(u_\epsilon') + J(u_\epsilon')\mathcal{X}_\zeta(u_\epsilon')] - \mathcal{X}_\eta(u_\epsilon) - J(u_\epsilon)\mathcal{X}_\zeta(u_\epsilon).$$

Hence similar to (8.10),

$$\|(\mathcal{D}_\epsilon'' - \mathcal{D}_\epsilon''')(\boldsymbol{\xi})\|_{\tilde{L}^p_\epsilon} \lesssim (\|\eta\|_{\tilde{L}^p_\epsilon} + \|\zeta\|_{\tilde{L}^p_\epsilon}) \|\xi_\epsilon'\|_{L^\infty} \lesssim (\|\eta\|_{\tilde{L}^{1,p}_{m;\epsilon}} + \|\zeta\|_{\tilde{L}^{1,p}_{m;\epsilon}}) \|\boldsymbol{\xi}_\epsilon'\|_{\tilde{L}^{1,p}_{m;\epsilon}}. \tag{8.11}$$

Further, consider an infinitesimal deformation of the form $\boldsymbol{\xi} = (\xi, 0, 0)$ and denote $\xi'' = Pl_\epsilon(\xi)$. Then we have

$$(\mathcal{D}_\epsilon'' - \mathcal{D}_\epsilon''')(\boldsymbol{\xi}) = \begin{bmatrix} Pl_\epsilon^{-1}[I(u_\epsilon', \phi_\epsilon', \psi_\epsilon')(\xi'')] - I(u_\epsilon, \phi_\epsilon', \psi_\epsilon')(\xi) \\ d\mu(u_\epsilon') \cdot \xi'' - d\mu(u_\epsilon) \cdot \xi \\ d\mu(u_\epsilon') \cdot J\xi'' - d\mu(u_\epsilon) \cdot J\xi \end{bmatrix}. \tag{8.12}$$

Here the term I in the first entry is defined by

$$I(u,\phi,\psi)(\xi) = \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi + J(\nabla_t \xi + \nabla_\xi \mathcal{X}_\psi) + (\nabla_\xi J)(\partial_t u + \mathcal{X}_\psi(u)).$$

Now we estimate (8.12). The last two entries are easy to bound. Indeed, by the Sobolev estimate (Lemma 7.4) and the definition of the norm, also the fact that Pl_ϵ preserves the splitting $H_X \oplus G_X$ in a neighborhood of $\mu^{-1}(0)$, one has

$$\|d\mu(u_\epsilon') \cdot \xi'' - d\mu(u_\epsilon) \cdot \xi\|_{\tilde{L}^p_\epsilon} + \|d\mu(u_\epsilon') \cdot J\xi'' - d\mu(u_\epsilon) \cdot J\xi\|_{\tilde{L}^p_\epsilon} \lesssim \|\boldsymbol{\xi}_\epsilon'\|_{\tilde{L}^{1,p}_{m;\epsilon}} \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}. \tag{8.13}$$

It remains to estimate the first entry of (8.12). For convenience, introduce

$$I_1(u)(\xi) = \nabla_s \xi + J\nabla_t \xi + (\nabla_\xi J)(\partial_t u), \quad I_2(u,\phi,\psi)(\xi) = \nabla_\xi \mathcal{X}_\phi + J\nabla_\xi \mathcal{X}_\psi + (\nabla_\xi J)\mathcal{X}_\psi,$$

and write the first entry of (8.12) as $(\delta I_1)(\xi) + (\delta I_2)(\xi)$. Take $R > 0$ and define

$$U_1 := \bigcup_{i=1}^{m+\underline{m}} B_R^i := \bigcup_{i=1}^{m+\underline{m}} B_R(z_{i,\epsilon}), \quad U_2 := \boldsymbol{H} \setminus \bigcup_{i=1}^{m+\underline{m}} B_{1/(\epsilon R)}(z_{i,\epsilon}), \quad U_3 := \boldsymbol{H} \setminus (U_1 \cup U_2).$$

In the following we estimate the variation of I_1 and I_2 in the above regions. We will use the fact that $\|\boldsymbol{\xi}_\epsilon'\|_{\tilde{L}^{1,p}_{m;\epsilon}} \leq 1$ and the uniform Sobolev embedding $\tilde{L}^{1,p}_{m;\epsilon} \hookrightarrow C^0$ (Lemma 7.4) frequently without explicitly mentioning them.

(a) Inside each B_R^i, the variation of I_1 can be bounded pointwise as

$$|(\delta I_1)(\xi)| \lesssim |du_\epsilon||\xi||\xi'_\epsilon| + |\nabla\xi||\xi||\xi'_\epsilon| + |\nabla\xi||\xi'_\epsilon| + |\nabla\xi'_\epsilon||\xi|. \tag{8.14}$$

(See [12, Proof of Proposition 3.5.3] for details.) One has

$$|\nabla\xi| \lesssim |du_\epsilon||\xi| + |\nabla^{a_\epsilon}\xi| + |\phi_\epsilon||\xi| + |\psi_\epsilon||\xi|, \tag{8.15}$$

$$|\nabla\xi'| \lesssim |du_\epsilon||\xi'| + |\nabla^{a_\epsilon}\xi'| + |\phi_\epsilon||\xi'| + |\psi_\epsilon||\xi'|. \tag{8.16}$$

Then since $\boldsymbol{v}_\epsilon|_{B_R^i} = \boldsymbol{v}_i$ which is independent of ϵ, one has

$$\|\nabla\xi\|_{\tilde{L}^p_\epsilon(B_R^i)} \lesssim \|\nabla^{a_\epsilon}\xi\|_{\tilde{L}^p_\epsilon(B_R^i)} + [\|du_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} + \|a_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)}]\|\xi\|_{L^\infty} \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}};$$

$$\|\nabla\xi'_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} \lesssim \|\nabla^{a_\epsilon}\xi'_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} + [\|du_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} + \|a_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)}]\|\xi'_\epsilon\|_{L^\infty} \lesssim \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}.$$

Hence by (8.14),

$$\begin{aligned}\|(\delta I_1)(\xi)\|_{\tilde{L}^p_\epsilon(B_R^i)} &\lesssim [\|du_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} + \|\nabla\xi\|_{\tilde{L}^p_\epsilon(B_R^i)}]\|\xi\|_{L^\infty}\|\xi'_\epsilon\|_{L^\infty} \\ &\quad + \|\xi'_\epsilon\|_{L^\infty}\|\nabla\xi\|_{\tilde{L}^p_\epsilon(B_R^i)} + \|\nabla\xi'_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)}\|\xi\|_{L^\infty} \lesssim \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}\|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}.\end{aligned} \tag{8.17}$$

(b) Since I_2 is a tensor in ξ, ϕ'_ϵ and ψ'_ϵ, one has

$$|(\delta I_2)(\xi)| \lesssim |\xi||\xi'_\epsilon||a'_\epsilon| \le |\xi||\xi'_\epsilon|[|a_\epsilon| + |\alpha'_\epsilon|].$$

Therefore

$$\|(\delta I_2)(\xi)\|_{\tilde{L}^p_\epsilon(B_R^i)} \lesssim \|\xi\|_{L^\infty}\|\xi'_\epsilon\|_{L^\infty}\|a'_\epsilon\|_{\tilde{L}^p_\epsilon(B_R^i)} \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}\|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}. \tag{8.18}$$

(c) Now we estimate the variation of I_1, I_2 over U_2. Using the same argument as in Step (a), similar to (8.15), one has

$$\begin{aligned}\|\nabla\xi\|_{\tilde{L}^p_\epsilon(U_2)} &\lesssim [\|du_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(U_2)} + \|a_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(U_2)}]\|\xi\|_{L^\infty} + \|\nabla^{a_{\infty,\epsilon}}\xi\|_{\tilde{L}^p_\epsilon(U_2)} \\ &\le \epsilon[\|s_\epsilon^* du_\infty\|_{\tilde{L}^p_\epsilon} + \|s_\epsilon^*\phi_\infty\|_{\tilde{L}^p_\epsilon} + \|s_\epsilon^*\psi_\infty\|_{\tilde{L}^p_\epsilon}]\|\xi\|_{L^\infty} + \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}} \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}.\end{aligned}$$

Here to obtain the last inequality we used Lemma 6.1. Similarly

$$\|\nabla\xi'_\epsilon\|_{\tilde{L}^p_\epsilon(U_2)} \lesssim \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}.$$

Hence (8.14) implies that

$$\begin{aligned}\|(\delta I_1)(\xi)\|_{\tilde{L}^p_\epsilon(U_2)} &\lesssim [\|du_\epsilon\|_{\tilde{L}^p_\epsilon(U_2)} + \|\nabla\xi\|_{\tilde{L}^p_\epsilon(U_2)}]\|\xi\|_{L^\infty}\|\xi'_\epsilon\|_{L^\infty} \\ &\quad + \|\xi'_\epsilon\|_{L^\infty}\|\nabla\xi\|_{\tilde{L}^p_\epsilon(U_2)} + \|\nabla\xi'_\epsilon\|_{\tilde{L}^p_\epsilon(U_2)}\|\xi\|_{L^\infty} \lesssim \|\boldsymbol{\xi}\|_{1;\epsilon}\|\boldsymbol{\xi}'_\epsilon\|_{1;\epsilon}.\end{aligned}$$

Further, as in Step (b), one has

$$\|(\delta I_2)(\xi)\|_{\tilde{L}^p_\epsilon(U_2)} \lesssim \|a'_\epsilon\|_{\tilde{L}^p_\epsilon(U_2)}\|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}\|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}} \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}}\|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}.$$

Here the uniform boundedness of the norm $\|\alpha'_\epsilon\|_{\tilde{L}^p_\epsilon(U_2)}$ follows from Lemma 6.1 and the relation $(\phi_\epsilon, \psi_\epsilon)|_{U_2} = \epsilon(s_\epsilon^*\phi_\infty, s_\epsilon^*\psi_\infty)$.

(d) Lastly, we estimate over the neck region U_3, which has connected components U_3^i for $i = 1, \dots, m + \underline{m}$. For each i, denote

$$\check{\alpha}'_\epsilon = \check{\phi}'_\epsilon ds + \check{\psi}'_\epsilon dt = \mathrm{e}^{-\lambda_i\theta_i}\alpha'_\epsilon\mathrm{e}^{\lambda_i\theta_i} - \lambda_i d\theta_i,$$

$$\check{\xi} = \mathrm{e}^{-\lambda_i \theta_i} \xi, \quad \check{\xi}'' = \mathrm{e}^{-\lambda_i \theta_i} \xi'', \quad \check{\xi}'_\epsilon = \mathrm{e}^{-\lambda_i \theta_i} \xi'_\epsilon.$$

Recall that

$$\check{\alpha}_\epsilon = \check{\phi}_\epsilon ds + \check{\psi}_\epsilon dt = \beta_{i,\epsilon} \check{\alpha}_i + \beta_{\infty,\epsilon} \check{\alpha}_{\infty,\epsilon}.$$

Denote $U_3^{i,-} = U_3^i \cap \mathring{B}_\epsilon^i$, $U_3^{i,+} = U_3^i \cap \mathring{\Sigma}_\epsilon$. So

$$\|\check{\alpha}_\epsilon\|_{\tilde{L}_\epsilon^p(U_3^i)} \leq \|\check{\alpha}_i\|_{\tilde{L}^p(U_2^{i,-})} + \|\check{\alpha}_{\infty,\epsilon}\|_{\tilde{L}_\epsilon^p(U_2^{i,+})} \leq c$$

for some constant $c > 0$ independent of ϵ. One can also obtain a uniform bound on $\|d\check{u}_\epsilon\|_{\tilde{L}_\epsilon^p(U_3^i)}$.

Using the above notation, $\mathrm{e}^{-\lambda_i \theta_i}(\delta I)(\xi)$ is the sum of the following two parts.

$$(\delta \check{I}_1)(\check{\xi}) = \check{P}l_\epsilon [\nabla_s \check{\xi}'' + J \nabla_t \check{\xi}'' + (\nabla_{\check{\xi}''} J)(\partial_t \check{u}_\epsilon)] - [\nabla_s \check{\xi} + J \nabla_t \check{\xi} + (\nabla_{\check{\xi}} J) \partial_t \check{u}_\epsilon];$$
$$(\delta \check{I}_2)(\check{\xi}) = \check{P}l_\epsilon [\nabla_{\check{\xi}''} \mathcal{X}_{\check{\phi}'_\epsilon} + \nabla_{\check{\xi}''}(J \mathcal{X}_{\check{\psi}'_\epsilon})] - [\nabla_{\check{\xi}} \mathcal{X}_{\check{\phi}'_\epsilon} + \nabla_{\check{\xi}}(J \mathcal{X}_{\check{\psi}'_\epsilon})].$$

Here $\check{P}l_\epsilon$ is the conjugation of Pl_ϵ by $\mathrm{e}^{-\lambda_i \theta_i}$. One can reproduce similar estimates as in previous steps using the uniform bounds on $\check{\phi}_\epsilon$, $\check{\psi}_\epsilon$ and $d\check{u}_\epsilon$. The detail is omitted.

The above steps (a)–(d) provides the bound

$$\|(\delta I)(\xi)\|_{\tilde{L}_\epsilon^p} \lesssim \|\boldsymbol{\xi}\|_{\tilde{L}^{1,p}_{m;\epsilon}} \|\boldsymbol{\xi}'_\epsilon\|_{\tilde{L}^{1,p}_{m;\epsilon}}.$$

This completes the proof of Proposition 7.8.

8.4 Proof of Proposition 7.10

Now we construct the approximate right inverse along the gauged map $\boldsymbol{v}_\epsilon = (u_\epsilon, a_\epsilon)$. We reset the values of the constants $c_1 \leq c_2 \leq \cdots$ and $\epsilon_1 \geq \epsilon_2 \geq \cdots$. Again, since the construction only involves the central object $\boldsymbol{v}_{\bullet,\epsilon}$, we abbreviate it as $\boldsymbol{v}_\epsilon$ and similarly for other relevant objects.

Since u_ϵ is close to $u_{\infty,\epsilon}$ over $\mathring{\Sigma}_\epsilon$, there is a parallel transport using certain connection on TX

$$Pl_\infty : u_{\infty,\epsilon}^* TX|_{\mathring{\Sigma}_\epsilon} \to u_\epsilon^* TX|_{\mathring{\Sigma}_\epsilon}.$$

We require that this connection respects the metric as well as the splitting $H_X \oplus G_X$ near $\mu^{-1}(0)$. Using the same connection there are also parallel transports

$$Pl_i : u_i^* TX|_{\mathring{B}_\epsilon^i} \to u_\epsilon^* TX|_{\mathring{B}_\epsilon^i}, \quad i = 1, \ldots, m + \underline{m}.$$

Recall that b is chosen by (7.1). Choose $e < b$ and introduce cut-off functions $\chi_\infty^\epsilon, \chi_i^\epsilon : \boldsymbol{H} \to [0,1]$ satisfying the following conditions.

$$\mathrm{supp} \chi_\infty^\epsilon \subset \boldsymbol{H} \setminus \bigcup_{i=1}^{\underline{s}+s} B\left(z_{i,\epsilon}, \frac{1}{e\sqrt{\epsilon}}\right), \quad \chi_\infty^\epsilon|_{\Sigma_{\epsilon;0}} \equiv 1, \tag{8.19}$$

and for $z \in B(z_{i,\epsilon}, \frac{1}{\sqrt{\epsilon}}) \setminus B(z_{i,\epsilon}, \frac{1}{e\sqrt{\epsilon}})$,

$$|\nabla \chi_\infty^\epsilon(z)| \leq \frac{2}{\log e} \frac{1}{|z - z_{i,\epsilon}|}. \tag{8.20}$$

Similarly we require

$$\mathrm{supp} \chi_i^\epsilon \subset B\left(z_{i,\epsilon}, \frac{e}{\sqrt{\epsilon}}\right), \quad \chi_i^\epsilon|_{B_{\epsilon;0}^i} \equiv 1$$

and for all $z \in B(z_{i,\epsilon}, \frac{e}{\sqrt{\epsilon}}) \setminus B(z_{i,\epsilon}, \frac{1}{\sqrt{\epsilon}})$,

$$|\nabla \chi_i^\epsilon(z)| \leq \frac{2}{\log e} \frac{1}{|z - z_{i,\epsilon}|}. \tag{8.21}$$

Notice that when $\nabla\chi_\infty^\epsilon \neq 0$ or $\nabla\chi_i^\epsilon \neq 0$, $\boldsymbol{v}_\epsilon = \boldsymbol{c}_i = (\mathrm{e}^{\lambda_i\theta_i}x_i, \lambda_i d\theta_i)$.

Using Pl_∞, Pl_i and χ_∞^ϵ, χ_i^ϵ, we define the maps

$$\mathrm{Cut}: \mathcal{E}_\epsilon \to \mathcal{E}_{\infty,\epsilon} \oplus \bigoplus_{i=1}^{m+\underline{m}} \mathcal{E}_i, \quad \mathrm{Paste}: \mathcal{B}_{\clubsuit,\epsilon} \to \mathcal{B}_\epsilon.$$

as follows. For $\boldsymbol{\nu} \in \mathcal{E}_\epsilon$, define $\mathrm{Cut}(\boldsymbol{\nu}) = (\boldsymbol{\nu}_{\infty,\epsilon}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{m+\underline{m}})$ where

$$\boldsymbol{\nu}_{\infty,\epsilon}(z) = \begin{cases} Pl_\infty^{-1}[\boldsymbol{\nu}(z)], & z \in \mathring{\Sigma}_\epsilon; \\ 0, & z \notin \mathring{\Sigma}_\epsilon, \end{cases} \qquad \boldsymbol{\nu}_i(z) = \begin{cases} Pl_i^{-1}[\boldsymbol{\nu}(z + z_{i,\epsilon})], & z \in \mathring{B}_\epsilon^i; \\ 0, & z \notin \mathring{B}_\epsilon^i. \end{cases}$$

On the other hand, take

$$\vec{\boldsymbol{\xi}} = (\boldsymbol{\xi}_{\infty,\epsilon}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{m+\underline{m}}) \in \mathcal{B}_{\clubsuit,\epsilon} \subset \mathcal{B}_{\infty,\epsilon} \oplus \bigoplus_{i=1}^{m+\underline{m}} \mathcal{B}_i.$$

Then $u_i(\boldsymbol{A}_i \setminus \check{\check{B}}_\epsilon^i) \subset U_X$ and we can decompose $\xi_i = \xi_i^H + \xi_i^G$ w.r.t. the splitting $H_X \oplus G_X$. Then by the matching condition, there exist $\xi^{H,i} \in H_{X,x_i}$ for $i = 1, \dots, m$ and $\underline{\xi}^{H,j} \in H_{L,x_j}$ for $j = 1, \dots, \underline{m}$, such that

$$\lim_{z \to z_{i,\epsilon}} \mathrm{e}^{\lambda_i\theta_i}\xi_{\infty,\epsilon}^H(z) = \lim_{z\to\infty} \mathrm{e}^{\lambda_i\theta_i}\xi_i^H(z) = \xi^{H,i}, \quad i = 1, \dots, m + \underline{m}.$$

Then define

$$\boldsymbol{\xi}_\epsilon(z) := \mathrm{Paste}(\boldsymbol{\xi}_{\infty,\epsilon}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{m+\underline{m}})(z) = \begin{cases} \boldsymbol{\xi}_i(z - z_{i,\epsilon}), & z \in \check{\check{B}}_\epsilon^i; \\ [\xi_\epsilon^H(z), \boldsymbol{\xi}_\epsilon^G(z)], & z \in \check{\check{\Sigma}}_\epsilon. \end{cases} \tag{8.22}$$

where

$$\xi_\epsilon^H(z) = \begin{cases} Pl_i[\xi_i^H(z - z_{i,\epsilon})] + \chi_\infty^\epsilon[Pl_\infty[\xi_{\infty,\epsilon}^H(z)] - \mathrm{e}^{-\lambda_i\theta_i}\xi^{H,i}], & \substack{z\in\mathring{B}_\epsilon^i\setminus\check{\check{B}}_\epsilon^i,\\ i=1,\dots,m+\underline{m};} \\ Pl_\infty[\xi_{\infty,\epsilon}^H(z)] + \sum_{i=1}^{m+\underline{m}} \chi_i^\epsilon[Pl_i[\xi_i^H(z - z_{i,\epsilon})] - \mathrm{e}^{-\lambda_i\theta_i}\xi^{H,i}], & z \in \mathring{\Sigma}_\epsilon. \end{cases} \tag{8.23}$$

$$\boldsymbol{\xi}_\epsilon^G(z) = \sum_{i=1}^{m+\underline{m}} \chi_i^\epsilon Pl_i[\boldsymbol{\xi}_i^G] + \chi_\infty^\epsilon Pl_\infty[\boldsymbol{\xi}_{\infty,\epsilon}^G]. \tag{8.24}$$

With abuse of notation, use $\mathrm{Paste}: W \oplus \mathcal{B}_{\clubsuit,\epsilon} \to W \oplus \mathcal{B}_\epsilon$ to denote the induced map which is the identity on the factor W.

Finally, define the "approximate right inverse"

$$\tilde{\mathcal{Q}}_\epsilon^{\mathrm{app}} = \mathrm{Paste} \circ \tilde{\mathcal{Q}}_{\clubsuit,\epsilon} \circ \mathrm{Cut}: \mathcal{E}_\epsilon \to W \times \mathcal{B}_\epsilon. \tag{8.25}$$

Here $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$ is the operator given by Proposition 6.6.

Proposition 8.1 *Suppose* $\log e \geq \frac{8\pi s_p c_\clubsuit}{p-2}$ (*which is consistent with* (7.1)). *Then there exist* ϵ_1 *and* c_1 (*which depend on* b *and* e) *such that for* $\epsilon \in (0, \epsilon_1]$

$$\|\tilde{\mathcal{Q}}_\epsilon^{\mathrm{app}}\| \leq c_1, \quad \|\tilde{\mathcal{D}}_\epsilon \circ \tilde{\mathcal{Q}}_\epsilon^{\mathrm{app}} - \mathrm{Id}\| \leq \frac{1}{2}. \tag{8.26}$$

The operator norms are taken w.r.t. the norm $\|\cdot\|_{\tilde{L}_\epsilon^p}$ *on* $\mathcal{E}_\epsilon$ *and the norm* $\|\cdot\|_{\tilde{L}_{m;\epsilon}^{1,p}}$ *on* $\mathcal{B}_\epsilon$.

Proposition 7.10 then follows from Proposition 8.1 by setting $\tilde{\mathcal{Q}}_\epsilon = \tilde{\mathcal{Q}}_\epsilon^{\mathrm{app}} \circ (\tilde{\mathcal{D}}_\epsilon \circ \tilde{\mathcal{Q}}_\epsilon^{\mathrm{app}})^{-1}$. It remains to prove Proposition 8.1. By straightforward estimate and comparison between different norms, one has the following bounds.

Lemma 8.2 *There exist $c_2 > 0$ and $\epsilon_2 \in (0, \epsilon_1]$ such that for all $\epsilon \in (0, \epsilon_2)$, one has*

$$\|\mathrm{Cut}(\boldsymbol{\nu})\| = \|\boldsymbol{\nu}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} + \sum_{i=1}^{m+\underline{m}} \|\boldsymbol{\nu}_i\|_{\tilde{L}^p} \leq \|\boldsymbol{\nu}\|_{\tilde{L}^p_\epsilon}, \tag{8.27}$$

$$\|\mathrm{Paste}(\boldsymbol{\xi}_{\infty,\epsilon}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{m+\underline{m}})\|_{\tilde{L}^{1,p}_{m;\epsilon}} \leq c_2 \left[\|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}} + \sum_{i=1}^{m+\underline{m}} \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m} \right]. \tag{8.28}$$

Proof (8.27) is an easy consequence of the definition of the weight functions, the definition of Cut, and the fact that the parallel transport is isometric fibrewise.

To prove the estimate about Paste, one needs to pay extra attention to the cut-off functions χ^ϵ_∞ and χ^ϵ_i. By the definition of Paste (see (8.22)), we have

$$\|\mathrm{Paste}(\boldsymbol{\xi}^\star)\|_{\tilde{L}^{1,p}_{m;\epsilon}(\check{B}^i_\epsilon)} \lesssim \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m(\boldsymbol{A}_i)}, \quad \|\mathrm{Paste}(\boldsymbol{\xi}^\star)\|_{\tilde{L}^{1,p}_{m;\epsilon}(\hat{\Sigma}_\epsilon)} \lesssim \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\Sigma_\epsilon)}.$$

Hence it remains to bound the norm of $\mathrm{Paste}(\boldsymbol{\xi}^\star)$ over the neck regions $\ddot{A}^i_\epsilon$.

By the definition of the norm $\|\cdot\|_{\tilde{L}^{1,p}_{m;\epsilon}}$ (see (7.7)) and the definition of Paste (see (8.22)–(8.24)), and the norm along $\boldsymbol{v}_{\infty,\epsilon}$ (see (6.2)),

$$\|\mathrm{Paste}(\boldsymbol{\xi}^\star)\|_{\tilde{L}^{1,p}_{m;\epsilon}(\ddot{A}^i_\epsilon)} \leq \|\xi^H_\epsilon\|_{L^\infty(\ddot{A}^i_\epsilon)} + \|\xi^G_\epsilon\|_{L^\infty(A^i_\epsilon)} + \|\boldsymbol{\xi}^G_\epsilon\|_{\tilde{L}^p_\epsilon(A^i_\epsilon)} + \|\nabla^{a_\epsilon}\boldsymbol{\xi}_\epsilon\|_{\tilde{L}^p_\epsilon(\ddot{A}^i_\epsilon)}.$$

The first two terms are bounded by $\|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}}$ and $\|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m}$ by Sobolev embeddings. The third term is also easy to bound. It remains to bound the last term which involves the derivatives of the cut-off functions χ^ϵ_∞ and χ^ϵ_i. By (8.23) and (8.24), we have

$$\begin{aligned}\nabla^{a_\epsilon}\boldsymbol{\xi}_\epsilon &= \chi^\epsilon_\infty[\nabla^{a_\epsilon} Pl_\infty[\boldsymbol{\xi}_{\infty,\epsilon}(z)]] + \chi^\epsilon_i[\nabla^{a_\epsilon} Pl_i[\boldsymbol{\xi}_i(z - z_{i,\epsilon})]] \\ &\quad + (\nabla\chi^\epsilon_\infty)[Pl_\infty \boldsymbol{\xi}^G_{\infty,\epsilon}(z)] + (\nabla\chi^\epsilon_i)[Pl_i\boldsymbol{\xi}^G_i(z - z_{i,\epsilon})] \\ &\quad + (\nabla\chi^\epsilon_\infty)[Pl_\infty[\xi^H_{\infty,\epsilon}(z)] - \check{\xi}^{H,i}] + (\nabla\chi^\epsilon_i)[Pl_i[\xi^H_i(z - z_{i,\epsilon})] - \check{\xi}^{H,i}].\end{aligned} \tag{8.29}$$

Here we used the fact that over the intersection of the supports of χ^ϵ_∞ and χ^ϵ_i, $\boldsymbol{v}_\epsilon = \boldsymbol{c}_i$, the "constant" object, and the covariant derivative of the "constant" $\check{\xi}^{H,i}$ is zero.

(a) For the first term of the right hand side of (8.29),

$$\begin{aligned}\|\chi^\epsilon_\infty \nabla^{a_\epsilon} Pl_\infty \boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon(\ddot{A}^i_\epsilon)} &\lesssim \|\nabla^{a_{\infty,\epsilon}} Pl_\infty\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} + \|a_\epsilon - a_{\infty,\epsilon}\|_{L^\infty}\|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} \\ &\lesssim \|Pl_\infty\nabla^{a_{\infty,\epsilon}}\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} + \|Pl_\infty\nabla^{a_{\infty,\epsilon}}\boldsymbol{\xi}_{\infty,\epsilon} - \nabla^{a_{\infty,\epsilon}} Pl_\infty\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} + \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} \lesssim \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}}.\end{aligned}$$

Similarly one can estimate

$$\|\chi^\epsilon_i\nabla^{a_\epsilon} Pl_i[\boldsymbol{\xi}_i(z - z_{i,\epsilon})]\|_{\tilde{L}^p_\epsilon(\ddot{A}^i_\epsilon)} \lesssim \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m}.$$

(b) For the third term on the right hand side of (8.29), notice that over the support of $\nabla\chi^\epsilon_\infty$, $\rho_\epsilon(z) \leq \rho_{\infty,\epsilon}(z)$ and $|\nabla\chi^\epsilon_\infty| \lesssim \sqrt{\epsilon}$. Therefore, by the definition of the norm,

$$\begin{aligned}\|(\nabla\chi^\epsilon_\infty)Pl_\infty\boldsymbol{\xi}^G_{\infty,\epsilon}\|_{\tilde{L}^p_\epsilon} &= \left[\int_{\boldsymbol{H}} |\nabla\chi^\epsilon_\infty(z)|^p|\boldsymbol{\xi}^G_{\infty,\epsilon}(z)|^p[\rho_\epsilon(z)]^{2p-4}dsdt\right]^{\frac{1}{p}} \\ &\lesssim \sqrt{\epsilon}\left[\int_{\mathrm{supp}\nabla\chi^\epsilon_\infty} |\boldsymbol{\xi}^G_{\infty,\epsilon}(z)|^p[\rho_{\infty,\epsilon}(z)]^{2p-4}dsdt\right]^{\frac{1}{p}} \leq \sqrt{\epsilon}\|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}}.\end{aligned} \tag{8.30}$$

Similarly, over the support of $\nabla\chi^\epsilon_i$ one has $\rho_\epsilon(z) \leq \rho_{\boldsymbol{A}_i}(z - z_{i,\epsilon})$, so

$$\|(\nabla\chi^\epsilon_i)Pl_i\boldsymbol{\xi}^G_i\|_{\tilde{L}^p_\epsilon} \lesssim \sqrt{\epsilon}\|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m}. \tag{8.31}$$

(c) The estimate of the fifth term on the right hand side of (8.29) is one for which we prefer a sharper bound. Notice that over supp $\nabla\chi^\epsilon_\infty$, $\rho_\epsilon(z) = |w_i| = |z - z_{i,\epsilon}|$. So

$$\begin{aligned}
&\sum_{i=1}^{m+\underline{m}} \|(\nabla\chi^\epsilon_\infty)(\xi^H_{\infty,\epsilon} - \check{\xi}^{H,i})\|_{\tilde{L}^p_\epsilon(\hat{\hat{B}}^i_\epsilon)} \\
&= \sum_{i=1}^{m+\underline{m}} \left[\int_{\hat{\hat{B}}^i_\epsilon} |\nabla\chi^\epsilon_\infty(z)|^p |\xi^H_{\infty,\epsilon}(z) - \check{\xi}^{H,i}|^p [\rho_\epsilon(z)]^{2p-4} dsdt\right]^{\frac{1}{p}} \\
&\le \frac{2s_p}{\log e} \sum_{i=1}^{m+\underline{m}} \|\xi^H_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{h;\epsilon}(\hat{\hat{B}}^i_\epsilon)} \left[\int_{\hat{\hat{B}}^i_\epsilon \cap \operatorname{supp}\nabla\chi^\epsilon_\infty} |\epsilon w_i|^{p-2} |w_i|^{p-4} dsdt\right]^{\frac{1}{p}} \\
&\le \frac{2s_p \epsilon^{1-\frac{2}{p}}}{\log e} \sum_{i=1}^{m+\underline{m}} \|\xi^H_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{h;\epsilon}(\hat{\hat{B}}^i_\epsilon)} \left[\int_{\hat{\hat{B}}^i_\epsilon \cap \operatorname{supp}\nabla\chi^\epsilon_\infty} |w_i|^{2p-6} dsdt\right]^{\frac{1}{p}} \\
&\le \frac{2\pi s_p}{(p-2)\log e} \|\xi^H_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{h;\epsilon}}. \qquad (8.32)
\end{aligned}$$

Here for the second line we used (8.21) and for the third line we used the Sobolev embedding $W^{1,p} \hookrightarrow C^{0,1-\frac{2}{p}}$. Notice that (8.32) is a precise estimate, which will be used in a minute.

(d) For the sixth term, over the support of $\nabla\chi^\epsilon_i$, one has $\rho_\epsilon(z) \le \rho_{\boldsymbol{A}_i}(w_i) = |w_i|$ and using cylindrical coordinates, one has

$$\begin{aligned}
\|(\nabla\chi^\epsilon_i)(\xi^H_i - \check{\xi}^{H,i})\|_{\tilde{L}^p_\epsilon} &= \left[\int_{\hat{B}^i_\epsilon} |\nabla\chi^\epsilon_i(z)|^p |\xi^H_i(z) - \check{\xi}^{H,i}|^p [\rho_\epsilon(z)]^{2p-4} dsdt\right]^{\frac{1}{p}} \\
&\le \frac{2}{\log e}\left[\int_{\operatorname{supp}\nabla\chi^\epsilon_i} |\xi^H_i(z) - \check{\xi}^{H,i}|^p |w_i|^{p-4} dsdt\right]^{\frac{1}{p}} \\
&\le \frac{2}{\log e} \|\xi^H_i - \check{\xi}^{H,i}\|_{L^{p,1-\frac{2}{p}}_{\mathrm{cyl}}} \\
&\le \frac{2s_p}{\log e} \|\xi^H_i\|_{\tilde{L}^{1,p}_h}. \qquad (8.33)
\end{aligned}$$

For the last inequality we used Lemma 3.5. □

Proof of Proposition 8.1 By Lemma 8.2 and the definition of $\tilde{\mathcal{Q}}^{\mathrm{app}}_\epsilon$ (8.25), the bound on $\tilde{\mathcal{Q}}^{\mathrm{app}}_\epsilon$ is equivalent to that of $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$, which is given by Proposition 6.6. So it remains to prove the second inequality of (8.26). Given $\boldsymbol{\nu} \in \mathcal{E}_\epsilon$, denote

$$(\boldsymbol{h}, \boldsymbol{\xi}^\star) = (\boldsymbol{h}, \boldsymbol{\xi}_{\infty,\epsilon}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{m+\underline{m}}) = \tilde{\mathcal{Q}}_{\clubsuit,\epsilon}(\mathrm{Cut}(\boldsymbol{\nu})).$$

Then by definition of $\tilde{\mathcal{Q}}_{\clubsuit,\epsilon}$ and Cut, we have

$$\boldsymbol{\nu} = Pl_\infty[\tilde{\mathcal{D}}_{\infty,\epsilon}(\boldsymbol{h}, \boldsymbol{\xi}_{\infty,\epsilon})] + \sum_{i=1}^{s} Pl_i[\tilde{\mathcal{D}}_i(\boldsymbol{h}, \boldsymbol{\xi}_i)].$$

Therefore

$$\begin{aligned}
\tilde{\mathcal{D}}_\epsilon \circ \tilde{\mathcal{Q}}^{\mathrm{app}}_\epsilon(\boldsymbol{\nu}) - \boldsymbol{\nu} &= \tilde{\mathcal{D}}_\epsilon[\boldsymbol{h}, \mathrm{Paste}(\boldsymbol{\xi}^\star)] - \left[Pl_\infty\tilde{\mathcal{D}}_{\infty,\epsilon}(\boldsymbol{h}, \boldsymbol{\xi}_{\infty,\epsilon}) + \sum_{i=1}^{m+\underline{m}} Pl_i\tilde{\mathcal{D}}_i(\boldsymbol{h}, \boldsymbol{\xi}_i)\right] \\
&= \mathcal{D}_\epsilon[\mathrm{Paste}(\boldsymbol{\xi}^\star)] - \left[Pl_\infty\mathcal{D}_{\infty,\epsilon}(\boldsymbol{\xi}_{\infty,\epsilon}) + \sum_{i=1}^{m+\underline{m}} Pl_i\mathcal{D}_i(\boldsymbol{\xi}_i)\right].
\end{aligned}$$

Here the last inequality follows from the property of the perturbation term. The last line is estimated in different regions as follows.

(a) Inside $B^i_{\epsilon;e} := \mathring{B}^i_\epsilon \setminus \operatorname{supp} \chi^\epsilon_\infty$, we have $\boldsymbol{v}_\epsilon = \boldsymbol{v}'_{i,\epsilon}$ where the latter is defined by (7.5). So

$$\mathcal{D}_\epsilon[\operatorname{Paste}(\boldsymbol{\xi}^\star)] - \boldsymbol{\nu} = \mathcal{D}_{\boldsymbol{v}'_{i,\epsilon}}[Pl_i(\boldsymbol{\xi}_i)] - Pl_i[\mathcal{D}_i(\boldsymbol{\xi}_i)]$$

while we can write $\boldsymbol{v}'_{i,\epsilon} = \exp_{\boldsymbol{v}_i} \boldsymbol{\xi}'_{i,\epsilon}$ (see Lemma 7.1). Then by using the same method as proving Proposition 7.8, we have

$$\|\mathcal{D}_{\boldsymbol{v}'_{i,\epsilon}}[Pl_i(\boldsymbol{\xi}_i)] - Pl_i[\mathcal{D}_i(\boldsymbol{\xi}_i)]\|_{\tilde{L}^p_\epsilon(B^i_{\epsilon;e})} \leq c_{i,e} \|\boldsymbol{\xi}'_{i,\epsilon}\|_{\tilde{L}^{1,p}_m(\mathring{B}^i_\epsilon)} \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m(\boldsymbol{A}_i)}. \tag{8.34}$$

(b) Similarly, inside $\Sigma_{\epsilon;e} := \mathring{\Sigma}_\epsilon \setminus \bigcup_{i=1}^{m+\underline{m}} \operatorname{supp} \chi^\epsilon_i$, we have

$$\mathcal{D}_\epsilon[\operatorname{Paste}(\boldsymbol{\xi}^\star)] - \boldsymbol{\nu} = \mathcal{D}_{\boldsymbol{v}'_{\infty,\epsilon}}[Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon})] - Pl_\infty[\mathcal{D}_\infty(\boldsymbol{\xi}_{\infty,\epsilon})]$$

while we can write $\boldsymbol{v}'_{\infty,\epsilon} = \exp_{\boldsymbol{v}_{\infty,\epsilon}} \boldsymbol{\xi}'_{\infty,\epsilon}$ (see Lemma 7.2). Similar to the above case, for some $c_{\infty,e} > 0$ and ϵ sufficiently small,

$$\|\mathcal{D}_\epsilon[\operatorname{Paste}(\boldsymbol{\xi}^\star)] - \boldsymbol{\nu}\|_{\tilde{L}^p_\epsilon(\Sigma_{\epsilon;e})} \leq c_{\infty,e} \|\boldsymbol{\xi}'_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\Sigma_{\epsilon;e})} \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\Sigma_\epsilon)}. \tag{8.35}$$

(c) In one of remaining the neck regions, i.e., $N^i_\epsilon := \operatorname{supp} \chi^\epsilon_\infty \cap \operatorname{supp} \chi^\epsilon_i$, by our construction of the approximate solution, $\boldsymbol{v}_\epsilon = \boldsymbol{c}_i = (d + \lambda_i d\theta_i, \mathrm{e}^{-\lambda_i \theta_i} x_i)$. By the definition of Paste (see (8.22)–(8.24)) and the fact that

$$\mathcal{D}_\epsilon(\mathrm{e}^{-\lambda_i \theta_i} \xi^H_i) = 0,$$

one sees that over N^i_ϵ (all norms below are $\|\cdot\|_{\tilde{L}^p_\epsilon(N^i_\epsilon)}$)

$$\begin{aligned}
&\|\mathcal{D}_\epsilon \operatorname{Paste}(\boldsymbol{\xi}^\star) - \boldsymbol{\nu}\| \\
&\quad = \|\mathcal{D}_\epsilon[\chi^\epsilon_\infty(Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - \check{\xi}^{H,i}) + \chi^\epsilon_i(Pl_i(\boldsymbol{\xi}_i) - \check{\xi}^{H,i})] - Pl_\infty \mathcal{D}_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - Pl_i \mathcal{D}_i(\boldsymbol{\xi}_i)\| \\
&\quad = \|\mathcal{D}_\epsilon[\chi^\epsilon_\infty(Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - \check{\xi}^{H,i}) + \chi^\epsilon_i(Pl_i(\boldsymbol{\xi}_i) - \check{\xi}^{H,i})] - \chi^\epsilon_\infty Pl_\infty[\mathcal{D}_\infty(\boldsymbol{\xi}_{\infty,\epsilon})] - \chi^\epsilon_i Pl_i[\mathcal{D}_i(\boldsymbol{\xi}_i)]\| \\
&\quad \leq \|\chi^\epsilon_\infty(\mathcal{D}_\epsilon Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - Pl_\infty(\mathcal{D}_\infty \boldsymbol{\xi}_{\infty,\epsilon}))\| + \|\chi^\epsilon_i(\mathcal{D}_\epsilon Pl_i(\boldsymbol{\xi}_i) - Pl_i(\mathcal{D}_i \boldsymbol{\xi}_i))\| \\
&\qquad + \|[\mathcal{D}_\epsilon, \chi^\epsilon_\infty](Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - \check{\xi}^{H,i})\| + \|[\mathcal{D}_\epsilon, \chi^\epsilon_i](Pl_i(\boldsymbol{\xi}_i) - \check{\xi}^{H,i})\|.
\end{aligned}$$

Here the third line follows from the fact that over the region where $\chi^\epsilon_\infty \neq 0$ (resp. $\chi^\epsilon_i \neq 0$), $\mathcal{D}_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) = 0$ (resp. $\mathcal{D}_i(\boldsymbol{\xi}_i) = 0$). The first and the second terms of the last line can be bounded by the same method of deriving (8.34) and (8.35), which gives

$$\begin{aligned}
&\|\chi^\epsilon_\infty(\mathcal{D}_\epsilon Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - Pl_\infty(\mathcal{D}_{\infty,\epsilon} \boldsymbol{\xi}_{\infty,\epsilon}))\| \lesssim \|\boldsymbol{\xi}'_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(N^i_\epsilon)} \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}(\Sigma_\epsilon)}; \\
&\|\chi^\epsilon_i(\mathcal{D}_\epsilon Pl_i(\boldsymbol{\xi}_i) - Pl_i(\mathcal{D}_i \boldsymbol{\xi}_i))\| \lesssim \|\boldsymbol{\xi}'_{i,\epsilon}\|_{\tilde{L}^{1,p}_m(N^i_\epsilon)} \|\boldsymbol{\xi}_i\|_{\tilde{L}^{1,p}_m(\boldsymbol{A}_i)}.
\end{aligned} \tag{8.36}$$

On the other hand, the third and the fourth terms can be estimated similarly to the estimates we had in the proof of Lemma 8.2. More precisely, using (8.30), (8.32) and (8.27), for some constant $C > 0$, we have

$$\begin{aligned}
\|[\mathcal{D}_\epsilon, \chi^\epsilon_\infty](Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - \check{\xi}^{H,i})\| &\leq \left(C\sqrt{\epsilon} + \frac{2\pi s_p}{(p-2)\log e}\right) \|\boldsymbol{\xi}_{\infty,\epsilon}\|_{\tilde{L}^{1,p}_{m;\epsilon}} \\
&\leq \left(C c_\clubsuit \sqrt{\epsilon} + \frac{2\pi s_p c_\clubsuit}{(p-2)\log e}\right) \|\operatorname{Cut}(\boldsymbol{\nu})\|_{\tilde{L}^p_\epsilon} \\
&\leq \left(C c_\clubsuit \sqrt{\epsilon} + \frac{2\pi s_p c_\clubsuit}{(p-2)\log e}\right) \|\boldsymbol{\nu}\|_{\tilde{L}^p_\epsilon}.
\end{aligned}$$

By the choice of the value of b (see (7.1)), it is possible to choose $e < b$ such that for ϵ sufficiently small, we have

$$\|[\mathcal{D}_\epsilon, \chi_\infty^\epsilon](Pl_\infty(\boldsymbol{\xi}_{\infty,\epsilon}) - \check{\xi}^{H,i})\| \leq \frac{1}{8}\|\boldsymbol{\nu}\|_{\tilde{L}_\epsilon^p}. \tag{8.37}$$

Similarly, by utilizing (8.31) and (8.33), we obtain

$$\sum_{i=1}^{m+\underline{m}} \|[\mathcal{D}_\epsilon, \chi_i^\epsilon](Pl_i(\boldsymbol{\xi}_i) - \check{\xi}^{H,i})\| \leq \frac{1}{8}\|\boldsymbol{\nu}\|_{\tilde{L}_\epsilon^p} \tag{8.38}$$

for appropriate value of $e < b$ and sufficiently small ϵ.

Therefore, for appropriate value of e and sufficiently small ϵ, (8.34)–(8.40) imply the second bound in (8.26). □

8.5 Proof of Proposition 7.12

The convergence $\|\boldsymbol{h}_n\| \to 0$ follows from the convergence of the underlying curve and the continuity of perturbations. The sequence of underlying domains of $\boldsymbol{v}_n$ uniquely determines a sequence of deformation parameters $\boldsymbol{a}_n \in W_{\text{per}}$ and a sequence of gluing parameters ϵ_n. Hence we only consider the difference between $\boldsymbol{v}_n$ and certain sequence of approximate solutions $\boldsymbol{v}_{\bullet,\epsilon_n}$. Here we abbreviate $\boldsymbol{v}_{\bullet,\epsilon_n}$ by $\boldsymbol{v}_{\epsilon_n}$.

For simplicity, we assume that the stable affine vortex $\boldsymbol{v}_\bullet$ has only two components, the disk component $\boldsymbol{v}_\infty = (u_\infty, a_\infty)$ and an $\boldsymbol{H}$-vortex component $\underline{\boldsymbol{v}} = (\underline{u}, \underline{\phi}, \underline{\psi})$ attached at the origin of the disk component. The case that $\boldsymbol{v}_\bullet$ has more components (including $\boldsymbol{C}$-vortices) can be proved in the same way with only more complicated symbol manipulations. We assume that

$$\lim_{z\to\infty} \underline{u}(z) = \lim_{z\to 0} u_\infty(z) = x_0 \in L, \quad \lim_{z\to\infty} u_\infty(z) = x_\infty \in L.$$

8.5.1 Estimates over the Neck Region and Near Infinity

Recall that when constructing the gluing map, we have already fixed certain gauge of $\underline{\boldsymbol{v}}$ and $\boldsymbol{v}_\infty$. Then $\underline{\boldsymbol{v}}$ satisfies the conditions of Lemma 3.4. Let $R > 0$ be sufficiently big so that $u(C_R)$ and $u_\infty(B_{1/R})$ are contained in a very small neighborhood of x_0. Then we can write

$$\underline{u}(z) = \exp_{x_0} \underline{\xi}, \ \forall z \in C_R; \quad u_\infty(z) = \exp_{x_0} \xi_\infty, \ \forall z \in B_{1/R}.$$

The gauge for $\boldsymbol{v}_\infty$ implies that ξ_∞ takes value in the horizontal distribution.

For the sequence of gluing parameters ϵ_n, define the neck region $N_{n,R} = N_{n,R}^- \cup N_{n,R}^+$ by

$$N_{n,R} := B_{1/(\epsilon_n R)} \setminus B_R, \text{ and } N_{n,R}^- := B_{1/\sqrt{\epsilon_n}} \setminus B_R, \ N_{n,R}^+ := B_{1/(\epsilon_n R)} \setminus B_{1/\sqrt{\epsilon_n}}.$$

We transform $\boldsymbol{v}_n$ on $N_{n,R}$ into temporal gauge, i.e., in the polar coordinates (r, θ), the gauge field a_n of $\boldsymbol{v}_n$ is written as

$$a_n = \psi_n(r, \theta) d\theta, \quad (r, \theta) \in N_{n,R}.$$

Moreover, we require that for all θ, $\psi_n(\frac{1}{\sqrt{\epsilon_n}}, \theta) \equiv 0$. Then the vortex equation in this gauge, written in terms of cylindrical coordinates (τ, θ), reads

$$\partial_\tau u_n + J(\partial_\theta u_n + \mathcal{X}_{\psi_n}) = 0, \quad \partial_s \psi_n + \mathrm{e}^{2\tau} \mu(u_n) = 0.$$

Denote temporarily its energy density function as

$$e_n(\tau, \theta) = |\partial_\tau u_n(\tau, \theta)|^2 + \mathrm{e}^{2\tau} |\mu(u_n(\tau, \theta))|^2.$$

When R is large, after this gauge transformation u_n is still contained in a small neighborhood of x_0. Then by the annulus lemma (proved as [39, Proposition 45] for $\boldsymbol{C}$-vortices and as [26, Proposition A.11] for $\boldsymbol{H}$-vortices), for all $\gamma > 0$, one has

$$e_n(\tau, \theta) \lesssim \max \left\{ R e^{-\tau}, \frac{e^{\tau}}{\epsilon_n R} \right\}^{2-\gamma}, \quad \log R \leq \tau \leq -\log(\epsilon_n R). \tag{8.39}$$

Here the constant absorbed by $\lesssim$ only depends on γ, R, the total energy and the local geometry, and hence uniform for all large n. The energy density of $\boldsymbol{v}_\infty$ and $\underline{\boldsymbol{v}}$ decays similarly. Hence for R sufficiently large, $u_n(N_{n,R})$ is contained in a small neighborhood of x_0, so we can write

$$\boldsymbol{v}_n|_{N_{n,R}} = \exp_{\boldsymbol{v}_{\epsilon_n}} \boldsymbol{\xi}_n, \quad \boldsymbol{\xi}_n = (\xi_n, \eta_n, \zeta_n) \in W^{1,p}(N_{n,R}).$$

Lemma 8.3 *For any $\alpha > 0$, there exists $R_\alpha > R$ and $n_\alpha > 0$ such that for $n \geq n_\alpha$,*

$$\|\boldsymbol{\xi}_n\|_{\tilde{L}^{1,p}_{m;\epsilon}(N_{n,R_\alpha})} \leq \alpha.$$

Proof It is not hard to see (via a simple estimate) that it suffices to compare both $\boldsymbol{v}_n|_{N_{n,R}}$ and $\boldsymbol{v}_{\epsilon_n}|_{N_{n,R}}$ with the covariantly constant gauged map $\boldsymbol{c}_0 = (x_0, 0, 0)$ with higher Sobolev norms defined using the trivial connection. Since $\boldsymbol{v}_{\epsilon_n}|_{N_{n,R}}$ is defined in a straightforward way and it is difference from $\boldsymbol{c}_0$ is easy to estimate, we only estimate the difference between $\boldsymbol{v}_n$ and $\boldsymbol{c}_0$.

We first estimate the gauge fields $a_n = \psi_n d\theta$. Let $Z_\tau \subset [\log R, -\log \sqrt{\epsilon}_n] \times [0, \pi]$ be a closed region containing $[\tau, \tau+1] \times [0, \pi]$ such that for different values of τ, Z_τ differs by a translation in the τ-direction. Let $\Omega_\tau \supset Z_\tau$ be a bigger open set containing Z_τ. Then given τ_0, over Ω_{τ_0} we have

$$\boldsymbol{v}_\tau + J\boldsymbol{v}_\theta = 0, \quad \kappa + e^{2\tau_0} \cdot e^{2\tau - 2\tau_0} \mu(u) = 0.$$

Applying Lemma A.4 for this pair of sets $Z_{\tau_0} \subset \Omega_{\tau_0}$ with $\epsilon = e^{-\tau_0}$, $\sigma = e^{2\tau - 2\tau_0}$, we obtain

$$\|\mu(u_n)\|_{L^p(Z_{\tau_0})} \lesssim e^{-(1+\frac{2}{p})\tau_0} (\|\boldsymbol{v}_\tau\|_{L^2(\Omega_{\tau_0})} + e^{2\tau_0} \|\mu(u_n)\|_{L^2(\Omega_{\tau_0})}) \lesssim e^{-(2+\frac{2}{p}-\gamma)\tau_0}.$$

The last inequality follows from the exponential decay of energy ((8.39)).

On the other hand, by the temporal gauge condition and the vortex equation, we have

$$\begin{aligned} |\psi_n(\tau_2, \theta) - \psi_n(\tau_1, \theta)|^p &= \left| \int_{\tau_1}^{\tau_2} \frac{\partial \psi_n}{\partial \tau} d\tau \right|^p \\ &\leq \left[\int_{\tau_1}^{\tau_2} e^{-a\tau} e^{a\tau} \left| \frac{\partial \psi_n}{\partial \tau} \right| d\tau \right]^p \\ &\leq \left[\int_{\tau_1}^{\tau_2} e^{-aq\tau} d\tau \right]^{p-1} \int_{\tau}^{\tau_n} e^{ap\tau} \left| \frac{\partial \psi_n}{\partial \tau} \right|^p d\tau \\ &\lesssim e^{-ap\tau_1} \int_{\tau_1}^{\tau_2} e^{(2+a)p\tau} |\mu(u_n)|^p d\tau. \end{aligned}$$

Here $a > 0$ is a small positive number and $q = p/(p-1)$; the second line uses Hölder inequality. Set $\tau_2 = \tau_n := -\log \sqrt{\epsilon}_n$ and $\tau_1 \in [\tau_n - k, \tau_n - k + 1)$, k being a positive integer. Then we have

$$\begin{aligned} \int_0^\pi |\psi_n(\tau_1, \theta)|^p d\theta &\lesssim e^{-ap(\tau_n - k)} \int_{\tau_n - k}^{\tau_n} \int_0^\pi e^{(2+a)p\tau} |\mu(u_n)|^p d\tau d\theta \\ &\lesssim e^{-ap(\tau_n - k)} \sum_{i=1}^k \int_{Z_{\tau_n - i}} e^{(2+a)p\tau} |\mu(u_n)|^p d\tau d\theta \end{aligned}$$

$$\leq \mathrm{e}^{-ap(\tau_n-k)} \sum_{i=1}^{k} \mathrm{e}^{(2+a)p(\tau_n-i+1)} \|\mu(u_n)\|^p_{L^p(Z_{\tau_n-i})}$$
$$\lesssim \mathrm{e}^{-ap(\tau_n-k)} \sum_{i=1}^{k} \mathrm{e}^{-(2-ap-p\gamma)(\tau_n-i)}$$
$$\lesssim \mathrm{e}^{-(2-p\gamma)\tau_1}. \tag{8.40}$$

Then the norm of the gauge field $\psi_n d\theta$ can be estimated as

$$\begin{aligned}\|\psi_n d\theta\|_{\tilde{L}^p_{\epsilon_n}(N^-_{n,R})} &:= \int_R^{\tau_n}\int_0^{\pi} |\psi_n d\theta|^p [\rho_{\epsilon_n}(z)]^{2p-2} d\tau d\theta \\ &= \int_R^{\tau_n}\int_0^{\pi} |\psi_n(\tau,\theta)|^p \mathrm{e}^{(p-2)\tau} d\tau d\theta \\ &\lesssim \int_R^{\tau_n} \mathrm{e}^{(p-4+p\gamma)\tau} d\tau \\ &\lesssim \mathrm{e}^{(p-4+p\gamma)R}.\end{aligned}$$

Since $p < 4$, choosing an appropriate small value of γ, for R sufficiently large, the above integral can be as small as we need. A bound similar to (8.40) can be derived in the same way for $\tau_1 = \tau_n$, $\tau_2 \in [\tau_n + k, \tau_n + k + 1)$ using (8.39). We omit the details. Therefore for R and n sufficiently large, we have

$$\|\psi_n d\theta\|_{\tilde{L}^p_{\epsilon_n}(N_{n,R})} \leq \alpha.$$

The estimates of derivatives of $\psi_n d\theta$ can be done similarly. Here we only sketch it. For the τ-derivative of $\psi_n d\theta$, we have

$$|\nabla_\tau(\psi_n d\theta) \otimes d\tau| = \mathrm{e}^{-2\tau}|\mu(u_n)|.$$

It has higher order than the above estimate for $\psi_n d\theta$ itself, so the estimate of the τ-derivative follows. For the θ-derivative, we have

$$\left|\frac{\partial \psi_n}{\partial\theta}(\tau_1,\theta)d\theta\right| = \mathrm{e}^{-\tau_1}\int_{\tau_1}^{\tau_n} \mathrm{e}^{2\tau}\left|\frac{\partial\mu(u_n)}{\partial\theta}\right| d\tau \lesssim \mathrm{e}^{-\tau_1}\int_{\tau_1}^{\tau_n} \mathrm{e}^{2\tau}(|d\mu(u_n)\cdot \boldsymbol{v}_{n,\theta}| + |\mu(u_n)||\psi_n|)d\tau.$$

The second integrand is easy to estimate. For the first integrand $d\mu(u_n)\cdot \boldsymbol{v}_{n,\theta}$, by Lemma A.4 we know it has one order lower in $\mathrm{e}^{-\tau}$ than $|\mu(u)|$. However the factor $\mathrm{e}^{-\tau_1}$ in front of the above integral (which comes from $|d\theta|$) compensates this drop of order so we can derive similar bound as the case for $|\psi_n d\theta|$.

Now we estimate the difference in the matter fields u_n and the constant x_0. We can write $u_n(\tau,\theta) = \exp_{x_0}\xi_n$. Recall the definition of the norm $\|\cdot\|_{\tilde{L}^{1,p}_{m;\epsilon_n}}$ (see (7.7)) that

$$\|\xi_n\|_{\tilde{L}^{1,p}_{m;\epsilon_n}(N_{n,R})} \approx \|\xi_n\|_{L^\infty(N_{n,R})} + \|\xi_n^G\|_{\tilde{L}^p_{\epsilon_n}(N_{n,R})} + \|\nabla\xi_n\|_{\tilde{L}^p_{\epsilon_n}(N_{n,R})}.$$

First by the convergence of $\boldsymbol{v}_n$ towards the stable affine vortex, we have convergence

$$\lim_{R\to\infty}\|\xi_n\|_{L^\infty(N_{n,R})} = 0, \quad \text{uniform in } n.$$

Moreover, we know $\boldsymbol{v}_{n,\tau} = E_2\nabla_\tau\xi_n$, $\boldsymbol{v}_{n,\theta} = E_2\nabla_\theta\xi_n + \mathcal{X}_{\psi_n}$ (where we use the notations introduced at the beginning of the appendix). Hence

$$\|\nabla_r\xi_n\|_{\tilde{L}^p_{\epsilon_n}(N_{n,R})} \approx \left[\int_{N_{n,R}} |\boldsymbol{v}_{n,r}|^p[\rho_{\epsilon_n}(z)]^{2p-4} dsdt\right]^{\frac{1}{p}} = \left[\int_{N_{n,R}} |\boldsymbol{v}_{n,\tau}|^p \mathrm{e}^{(p-2)\tau} d\tau d\theta\right]^{\frac{1}{p}}.$$

By the energy decay property ((8.39)), one can find R_α so that $\|\nabla_r \xi\|_{\tilde{L}^p_{\epsilon_n}(N_{n,R_\alpha})} \leq \alpha$ for all n. Using the bound on ψ_n achieved previously, one can also obtain a similar bound for $\nabla_\theta \xi$.

Now we turn to the estimate of ξ_n^G, which can be written as $\mathcal{X}_{h_n'} + J\mathcal{X}_{h_n''}$. Notice that $|h_n''| \approx |\mu(u_n)|$ whose $\tilde{L}^p_\epsilon$-norm has been estimated. On the other hand, we know that h_n' is comparable to $L_{x_0}^* \xi_n$; by Lemma A.1, we have $\partial_\tau L_{x_0}^* \xi_n = L_{x_0}^* \nabla_\tau \xi_n$; also $\nabla_\tau \xi_n$ is comparable to $\boldsymbol{v}_{n,\tau}$ and $L_{x_0}^* \nabla_s \xi_n$ is then comparable to $d\mu(x_0) \cdot J\boldsymbol{v}_{n,\tau}$. Therefore we have

$$|\partial_\tau h_n'| \approx |d\mu(x_0) \cdot J\boldsymbol{v}_{n,\tau}|.$$

Notice that we have an estimate for the L^p-norm of the right hand side by Lemma A.4. Hence using the same method as estimating $\psi_n d\theta$, we can ask $\|h_n' - h_n'(\tau_n, \cdot)\|_{\tilde{L}^p_\epsilon(N_{n,R})}$ as small as possible (by increasing R). However, the difference from the case for ψ_n is that we do not have $h_n'(\tau_n, \cdot) \equiv 0$. Hence we have to estimate the norm of the function $h_n'(\tau_n, \cdot)$ (which is independent of the τ variable).

To estimate the value of $h_n'(\tau_n, \cdot)$, we choose a bounded region Z containing the segment $\tau = \tau_n$ in its interior. Recall that the gauge fields $a_n = \psi_n d\theta$ is in temporal gauge in $N_{n,R}$ with $\psi_n(\tau_n, \theta) \equiv 0$; the remaining degree of freedom of gauging $\boldsymbol{v}_n$ is by using a constant gauge transformation. Therefore by using a constant gauge transformation we can ask the average of h_n' over Z to be zero. Then by the Poincaré inequality and Sobolev embedding, we obtain that

$$\|h_n'\|_{L^\infty(Z)} \lesssim \|\nabla h_n'\|_{L^p(Z)}.$$

Notice that $\partial_\tau h_n'$ (resp. $\partial_\theta h_n'$) is comparable to $d\mu(x_0) \cdot \boldsymbol{v}_{n,\tau}$ (resp. $d\mu(x_0) \cdot \boldsymbol{v}_{n,\theta}$) plus terms which are of even lower order. Then by Lemma A.4 and the energy decay property, for any $\gamma > 0$ and a region Ω containing Z in its interior, we have

$$\sup_{\theta \in [0,\pi]} h_n'(\tau_n, \theta) \lesssim \|\nabla h_n'\|_{L^p(Z)} \lesssim (\sqrt{\epsilon}_n)^{\frac{2}{p}} (\|\mu(u_n)\|_{L^2(\Omega)} + \epsilon_n^{-1} \|\boldsymbol{v}_{n,\tau}\|_{L^2(\Omega)}) \lesssim (\sqrt{\epsilon}_n)^{1+\frac{2}{p}-\gamma}.$$

Notice that the $\tilde{L}^p_\epsilon$-norm over $N_{n,R}$ of the constant $(\sqrt{\epsilon}_n)^{1+\frac{2}{p}-\gamma}$ can be as small as we want. Hence we can take $R_\alpha > 0$ such that for n sufficiently large, $\|h_n'\|_{\tilde{L}^p_\epsilon(N_{n,R_\alpha})} \leq \alpha$ □

Notice that in estimating the distance between $\boldsymbol{v}_n$ and $\boldsymbol{v}_{\epsilon_n}$ over the neck region, we do not compare them directly but only compare with a constant object. Using the same method, we can estimate the distance between $\boldsymbol{v}_\infty$ and $\boldsymbol{v}_n^{rs} = (u_n^{rs}, \phi_n^{rs}, \psi_n^{rs})$ near infinity. Here the latter is the rescaling of $\boldsymbol{v}_n$ by ϵ_n which satisfies

$$\partial_s u_n^{rs} + \mathcal{X}_{\phi_n^{rs}} + J(\partial_t u_n^{rs} + \mathcal{X}_{\psi_n^{rs}}) = 0, \quad \partial_s \psi_n^{rs} - \partial_t \phi_n^{rs} + [\phi_n^{rs}, \psi_n^{rs}] + \epsilon_n^{-2} \mu(u_n^{rs}) = 0.$$

To carry out the estimate, first we need a uniform energy decay.

Lemma 8.4 *For all $\gamma > 0$, we have*

$$\limsup_{n \to \infty} \limsup_{z \to \infty} |z|^{4-\gamma} [|\boldsymbol{v}_{n,s}^{rs}|^2 + \epsilon_n^{-2} |\mu(u_n^{rs})|^2] < +\infty.$$

This can be proved by utilizing the annulus lemma ([39, Proposition 45] for $\boldsymbol{A} = \boldsymbol{C}$ and [26, Proposition A.11] for $\boldsymbol{A} = \boldsymbol{H}$). It implies a uniform C^0 convergence of $u_n^{rs} \to u_\infty$ near ∞.

Lemma 8.5 *For any $R > 0$, up to gauge transformations, u_n^{rs} converges to u_∞ uniformly on C_R. Moreover, the evaluations $\mathrm{ev}_\infty(\boldsymbol{v}_n^{rs}) \in \bar{L}$ converges to $\mathrm{ev}_\infty(\boldsymbol{v}_\infty)$.*

Therefore, by the same method of proving Lemma 8.3, we have

Lemma 8.6 *For any $\alpha > 0$, there exist $R'_\alpha > 0$ and n'_α such that, for $n \geq n'_\alpha$, we can write $\boldsymbol{v}_n^{rs} = \exp_{\boldsymbol{v}_\infty} \boldsymbol{\xi}_n^{rs}$ and $\|\boldsymbol{\xi}_n^{rs}\|_{\tilde{L}^{1,p}_{a;\epsilon_n}(C_{R'_\alpha})} \leq \alpha$.*

We omit the proof since it can still be checked by straightforward calculations. Recall that $\|\cdot\|_{\tilde{L}^{1,p}_{a;\epsilon_n}}$ is the auxiliary norm defined in (6.3). Define $\boldsymbol{\xi}_n = s_\epsilon^* \boldsymbol{\xi}_n^{rs}$ which is an infinitesimal deformation of $\boldsymbol{v}_{\epsilon_n}$ over C_{R/ϵ_n}. Then $\exp_{\boldsymbol{v}_{\epsilon_n}} \boldsymbol{\xi}_n = \boldsymbol{v}_n$ over C_{R/ϵ_n} and by Lemma 6.1,

$$\|\boldsymbol{\xi}_n\|_{\tilde{L}^{1,p}_{m;\epsilon_n}(C_{R/\epsilon_n})} \leq \alpha.$$

8.5.2 In the Compact Region

Without loss of generality, assume that $n_\alpha = n'_\alpha$ and $R_\alpha = R'_\alpha$. Consider the convergence over the compact region B_{R_α}, which is part of the domain of $\underline{\boldsymbol{v}}$. Since the expression of the norm $\|\cdot\|_{\tilde{L}^{1,p}_{m;\epsilon_n}}$ restricted to B_{R_α} has no dependence on ϵ_n and is equivalent to the unweighted norm $\|\cdot\|_{W^{1,p}(B_{R_\alpha})}$, using standard elliptic theory, we can show that (up to gauge transformations)

$$\lim_{n\to\infty} \|\boldsymbol{\xi}_n\|_{\tilde{L}^{1,p}_{m;\epsilon_n}(B_{R_\alpha})} = 0.$$

On the other hand, we compare $\boldsymbol{v}_n^{rs}$ with $\boldsymbol{v}_\infty$ over $\Omega = B_{R_\alpha} \setminus B_{\frac{1}{R_\alpha}}$. Using the graph construction (see [9, Appendix A]), we may regard all the perturbed vortices as unperturbed vortices with target $X \times \boldsymbol{C}$ with the Lagrangian $L \times \mathbb{R}$. So without loss of generality, assume that the perturbation term vanishes. We write $\boldsymbol{v}_n^{rs} = \exp_{\boldsymbol{v}_\infty} \boldsymbol{\xi}_n^{rs}$ where $\boldsymbol{\xi}_n^{rs} = (\xi_n^{rs}, \eta_n^{rs}, \zeta_n^{rs})$ and $\alpha_n^{rs} = \eta_n^{rs} ds + \zeta_n^{rs} dt$. Then Theorem A.11 implies that after suitable gauge transformation,

$$\lim_{n\to\infty} [\|\xi_n^{rs;H}\|_{W^{1,p}(\Omega)} + \epsilon_n^{-1}\|\xi_n^{rs;G}\|_{L^p(\Omega)} + \|\nabla\xi_n^{rs;G}\|_{L^p(\Omega)} + \|\alpha_n^{rs}\|_{L^p(\Omega)} + \epsilon_n \|\nabla^{a_\infty}\alpha_n^{rs}\|_{L^p(\Omega)}] = 0.$$

Rescale back, it implies that the distance between $\boldsymbol{v}_n$ and $\boldsymbol{v}_{\infty,\epsilon}$ over Ω converges to zero. This finishes the proof of Proposition 7.12.

Appendix

A Technical Results about Vortices and Adiabatic Limits

A.1 Derivatives of the Exponential Map

One uses the exponential map to compare nonlinear objects and uses derivatives of the exponential map to compare derivatives. Many discussions below are identical to part of [9, Appendix C].

Let M be a Riemannian manifold. For $v \in T_x M$ and $i, j \in \{1, 2\}$ there are linear maps

$$E_i(x,v) : T_x M \to T_{\exp_x v} M, \quad E_{ij}(x,v) : T_x M \oplus T_x M \to T_{\exp_x v} M$$

defined by the following identities

$$\begin{aligned} d\exp_x v &= E_1(x,v)dx + E_2(x,v)\nabla v, \\ \nabla E_1(x,v)w &= E_{11}(x,v)(w,dx) + E_{12}(x,v)(w,\nabla v) + E_1(x,v)\nabla w, \\ \nabla E_2(x,v)w &= E_{21}(x,v)(w,dx) + E_{22}(x,v)(w,\nabla v) + E_2(x,v)\nabla w. \end{aligned}$$

To save space, we often omit the variables (x,v) of E_i or E_{ij}.

Now let $M = X$ and let g be a K-invariant Riemannian metric. Let ∇ be the Levi–Civita connection of g. By the K-invariance of the metric, one has

$$\mathcal{X}_\eta(\exp_x \xi) = E_1(x,\xi)\mathcal{X}_\eta(x) + E_2(x,\xi)\nabla_\xi \mathcal{X}_\eta, \quad \forall x \in X,\ \xi \in T_x X,\ \eta \in \mathfrak{k}.$$

To continue, define (recall) the following notations.

- Abbreviate the map $\eta \mapsto \mathcal{X}_\eta(x)$ by L_x and its dual $L_x^*(\xi) = d\mu(x) \cdot J\xi$.
- Denote $X^* = \{x \in X \mid \ker L_x = \{0\}\}$.
- Define a 2-form $\rho \in \Omega^2(X, \mathfrak{k})$ by $\langle \rho(\xi_1, \xi_2), \eta \rangle = \langle \nabla_{\xi_1} \mathcal{X}_\eta, \xi_2 \rangle$.
- Let $\boldsymbol{v} = (u, \phi, \psi) : \mathbb{R}^2 \to X \times \mathfrak{k} \times \mathfrak{k}$ be a smooth map. Denote

$$a = \phi ds + \psi dt, \quad \boldsymbol{v}_s = \partial_s u + \mathcal{X}_\phi, \quad \boldsymbol{v}_t = \partial_t u + \mathcal{X}_\psi.$$

Moreover, for $\xi \in \Gamma(u^*TX)$ and $\eta : \mathbb{R}^2 \to \mathfrak{k}$, define

$$\nabla_s^a \xi = \nabla_s \xi + \nabla_\xi \mathcal{X}_\phi, \quad \nabla_t^a \xi = \nabla_t \xi + \nabla_\xi \mathcal{X}_\psi.$$
$$\nabla_s^a \eta = \partial_s \eta + [\phi, \eta], \quad \nabla_t^a \eta = \partial_t \eta + [\psi, \eta].$$

Then ∇^a induces a covariant derivative along u of all tensor fields. Moreover, it is easy to see that if T is a K-invariant tensor field, we have

$$\nabla_s^a T = \nabla_{\boldsymbol{v}_s} T, \quad \nabla_t^a T = \nabla_{\boldsymbol{v}_t} T.$$

In particular, we can obtain the following useful formula.

Lemma A.1 ([9, Lemma C.2], [8]) *For $\eta : \mathbb{R}^2 \to \mathfrak{k}$, $\xi \in \Gamma(u^*TX)$, one has*

$$[\nabla_{s/t}^a,\ L_u](\eta) = \nabla_{\boldsymbol{v}_{s/t}} \mathcal{X}_\eta(u), \quad [\nabla_{s/t}^a,\ L_u^*](\xi) = \rho(\boldsymbol{v}_{s/t}, \xi). \tag{A.1}$$

Let $X^* \subset X$ be the open subset consisting of $x \in X$ for which $a \mapsto \mathcal{X}_a(x)$ is injective. So we have the distribution $H_X \subset TX|_{X^*}$ which is defined to be the h_X-orthogonal complement of $K_X \oplus JK_X$. Let $\Omega \subset \boldsymbol{H}$ be an open subset and let $u_0 : \Omega \to X^*$ be a C^1-map. Then u_0 pulls back a connection form $a_0 = \phi_0 ds + \psi_0 dt$ in such a way that if we denote by $\boldsymbol{v}_0 = (u_0, \phi_0, \psi_0)$, then

$$\boldsymbol{v}_{0,s} = \partial_s u_0 + \mathcal{X}_{\phi_0} \in H_X \oplus JK_X, \quad \boldsymbol{v}_{0,t} = \partial_t u_0 + \mathcal{X}_{\psi_0} \in H_X \oplus JK_X.$$

Suppose $\xi_0 \in \Gamma(\Omega, u_0^*TX)$ and denote $u = \exp_{u_0} \xi_0$; $\eta_0, \zeta_0 : \Omega \to \mathfrak{k}$ and denote $\phi = \phi_0 + \eta_0$, $\psi = \psi_0 + \zeta_0$ and denote $a = \phi ds + \psi dt$.

Lemma A.2 ([9, Lemma C.3])

$$\mathcal{X}_{\eta_0}(u) = \boldsymbol{v}_s - E_1(u_0, \xi_0)\boldsymbol{v}_{0,s} - E_2(u_0, \xi_0)\nabla_s^{a_0}\xi_0,$$
$$\mathcal{X}_{\zeta_0} = \boldsymbol{v}_t - E_1(u_0, \xi_0)\boldsymbol{v}_{0,t} - E_2(u_0, \xi_0)\nabla^{a_0}\xi_0.$$

Lemma A.3 ([9, Lemma C.5]) *Suppose $L_{u_0}\xi_0 = 0$, then*

$$\begin{aligned} L_u \nabla_t^a \eta_0 = {} & \nabla_t^a \boldsymbol{v}_s + \nabla_{\mathcal{X}_{\eta_0}} \mathcal{X}_{\zeta_0} - \nabla_{\boldsymbol{v}_t} \mathcal{X}_{\eta_0} - \nabla_{\boldsymbol{v}_s} \mathcal{X}_{\zeta_0} - E_{11}(\boldsymbol{v}_{0,s}, \boldsymbol{v}_{0,t}) - E_{12}(\boldsymbol{v}_{0,s}, \nabla_t^{a_0}\xi_0) \\ & - E_{21}(\nabla_s^{a_0}\xi_0, \boldsymbol{v}_{0,t}) - E_{22}(\nabla_s^{a_0}\xi_0, \nabla_t^{a_0}\xi_0) - E_1 \nabla_t^{a_0} \boldsymbol{v}_{0,s} - E_2 \nabla_t^{a_0} \nabla_s^{a_0} \xi_0; \\ L_u \nabla_s^a \eta_0 = {} & \nabla_s^a \boldsymbol{v}_s + \nabla_{\mathcal{X}_{\eta_0}} \mathcal{X}_{\eta_0} - 2\nabla_{\boldsymbol{v}_s} \mathcal{X}_{\eta_0} - E_{11}(\boldsymbol{v}_{0,s}, \boldsymbol{v}_{0,s}) - E_{12}(\boldsymbol{v}_{0,s}, \nabla_s^{a_0}\xi_0) \\ & - E_{21}(\nabla_s^{a_0}\xi_0, \boldsymbol{v}_{0,s}) - E_{22}(\nabla_s^{a_0}\xi_0, \nabla_s^{a_0}\xi_0) - E_1 \nabla_s^{a_0} \boldsymbol{v}_{0,s} - E_2 \nabla_s^{a_0} \nabla_s^{a_0} \xi_0. \end{aligned}$$

A.2 *A Priori* Estimates

Let $\Omega \subset \boldsymbol{H}$ be an open subset and let $\sigma : \Omega \to (0, +\infty)$ be a smooth function which is bounded from below and from above, and which has bounded derivatives in every order. For any gauged map $\boldsymbol{v} = (u, \phi, \psi)$ from Ω to X, denote

$$\boldsymbol{v}_s = \partial_s u + \mathcal{X}_\phi, \quad \boldsymbol{v}_t = \partial_t u + \mathcal{X}_\psi, \quad \kappa = \partial_s \psi - \partial_t \phi + [\phi, \psi].$$

Let $\epsilon > 0$ be a (small) constant. Consider the local form of the perturbed vortex equation

$$\boldsymbol{v}_s + J\boldsymbol{v}_t = 0, \quad \kappa + \epsilon^{-2}\sigma\mu(u) = 0, \quad u(\partial\Omega) \subset L. \tag{A.2}$$

Let $\tilde{\mathcal{M}}(\Omega, \epsilon^{-2}\sigma; X, L)$ be the set of smooth solutions to (A.2). On the other hand, let $\tilde{\mathcal{M}}(\Omega, \infty; X, L)$ be the set of solutions to

$$\boldsymbol{v}_s + J\boldsymbol{v}_t = 0, \quad \mu(u) \equiv 0, \quad u(\partial\Omega) \subset L.$$

For each $M > 0$, define

$$U_M := \left\{ x \in X^* \;\middle|\; \frac{1}{M}|a| \leq |\mathcal{X}_a(x)|, \ \forall a \in \mathfrak{k} \right\}.$$

Lemma A.4 (cf. [9, Lemma 9.3]) *Assume $2 \leq p \leq \infty$. For any $M > 0$ and any compact subset $Z \subset \Omega$, there exist $c(M, \Omega, Z) > 0$ and $\epsilon(M)$ (both of which also depend on σ) that satisfy the following condition. Suppose $\epsilon \in (0, \epsilon(M)]$ and $\boldsymbol{v}$ is a solution to* (A.2) *over Ω such that $u(\Omega) \subset U_M$ and*

$$\sup_{z \in \Omega} \left[|\boldsymbol{v}_s(z)| + \frac{\sqrt{\sigma(z)}}{\epsilon} |\mu(u(z))| \right] \leq M. \tag{A.3}$$

Then

$$\frac{1}{\epsilon}\|\mu(u)\|_{L^p(Z)} + \|d\mu(u) \cdot \boldsymbol{v}_s\|_{L^p(Z)} + \|d\mu(u) \cdot J\boldsymbol{v}_s\|_{L^p(Z)} + \epsilon\|\nabla_s^a \boldsymbol{v}_s\|_{L^p(Z)} + \epsilon\|\nabla_t^a \boldsymbol{v}_s\|_{L^p(Z)}$$
$$\leq c(M, \Omega, Z)\epsilon^{\frac{2}{p}}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}]. \tag{A.4}$$

Remark A.5 Lemma A.4 was proved as [9, Lemma 9.3] for Ω with empty boundary. The proof in the case that $\partial\Omega \neq \emptyset$ is essentially the same. However, in order to use reflection across the boundary, one has to use a (J, L, μ)-admissible metric and the associated Levi–Civita connection, instead of any K-invariant metric and its Levi–Civita connection.

A.3 Projection to $\mu^{-1}(0)$

Now suppose one has a gauged map $\boldsymbol{v}$ from Ω to X with $\sup_\Omega |\mu(u)|$ being sufficiently small with image having compact closure. So its image is contained in U_M for some M. Then there is a unique function $h : \Omega \to \mathfrak{k}$ such that

$$\mu(\exp_u J\mathcal{X}_h) \equiv 0.$$

Define $u' = \exp_u J\mathcal{X}_h$ and $\boldsymbol{v}' = (u', a')$ where $a' = \phi' ds + \psi' dt$ is the connection pulled back by u'. Then it is easy to see that for all $Z \subset \Omega$,

$$\|J\mathcal{X}_h\|_{L^p(Z)} \lesssim \|\mu(u)\|_{L^p(Z)}. \tag{A.5}$$

Lemma A.6 *Under the hypothesis of Lemma* A.4*, by possibly decreasing* $\epsilon(M)$ *and increasing* $c(M,\Omega,Z)$*, one has*

$$\|\nabla^a(J\mathcal{X}_h)\|_{L^p(Z)} \leq c(M,\Omega,Z)[\|\mu(u)\|_{L^p(Z)} + \|d\mu(u)\cdot \boldsymbol{v}_s\|_{L^p(Z)} + \|d\mu(u)\cdot \boldsymbol{v}_t\|_{L^p(Z)}].$$

Proof Denote $\eta' ds + \zeta' dt = a - a'$. Then by Lemma A.2, along u' one has

$$\mathcal{X}_{\eta'} + \boldsymbol{v}_s' - E_1(\boldsymbol{v}_s) - E_2\nabla_s^a(J\mathcal{X}_h) = \mathcal{X}_{\zeta'} + \boldsymbol{v}_t' - E_1(\boldsymbol{v}_t) - E_2\nabla_t^a(J\mathcal{X}_h) = 0. \tag{A.6}$$

Since u' is contained in $\mu^{-1}(0)$, applying $d\mu(u')$ to the above identities gives

$$d\mu(u')\cdot E_2\nabla_s^a(J\mathcal{X}_h) = -d\mu(u')\cdot E_1(\boldsymbol{v}_s), \quad d\mu(u')\cdot E_2\nabla_t^a(J\mathcal{X}_h) = -d\mu(u')\cdot E_1(\boldsymbol{v}_t).$$

By the smoothness of $d\mu$, one has

$$\begin{aligned}
&|d\mu(u')\cdot E_2\nabla_s^a(J\mathcal{X}_h) - d\mu(u)\cdot\nabla_s^a(J\mathcal{X}_h)| \lesssim |J\mathcal{X}_h||\nabla_s^a(J\mathcal{X}_h)|.\\
&|d\mu(u')\cdot E_1(\boldsymbol{v}_s) - d\mu(u)\cdot \boldsymbol{v}_s| \lesssim |J\mathcal{X}_h||\boldsymbol{v}_s|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\nabla_s^a J\mathcal{X}_h| &\lesssim |d\mu(u)\cdot\nabla_s^a J\mathcal{X}_h| + |\boldsymbol{v}_s||J\mathcal{X}_h|\\
&\lesssim |d\mu(u')\cdot E_2(\nabla_s^a J\mathcal{X}_h)| + |J\mathcal{X}_h||\nabla_s^a J\mathcal{X}_h| + |\boldsymbol{v}_s||J\mathcal{X}_h|\\
&\lesssim |d\mu(u')\cdot E_1(\boldsymbol{v}_s)| + |J\mathcal{X}_h||\nabla_s^a J\mathcal{X}_h| + |J\mathcal{X}_h||\boldsymbol{v}_s|\\
&\lesssim |d\mu(u)\cdot \boldsymbol{v}_s| + |J\mathcal{X}_h||\nabla_s^a J\mathcal{X}_h| + |J\mathcal{X}_h||\boldsymbol{v}_s|.
\end{aligned}$$

Therefore if the ϵ in (A.3) is sufficiently small, which implies $|J\mathcal{X}_h|$ is sufficiently small, one has

$$|\nabla_s^a J\mathcal{X}_h| \lesssim |d\mu(u)\cdot\boldsymbol{v}_s| + |\boldsymbol{v}_s||J\mathcal{X}_h|. \tag{A.7}$$

Then by the bound on $|\boldsymbol{v}_s|$ given by (A.3) and (A.5),

$$\begin{aligned}
\|\nabla_s^a(J\mathcal{X}_h)\|_{L^p(Z)} &\lesssim \|\boldsymbol{v}_s\|_{L^\infty(\Omega)}\|J\mathcal{X}_h\|_{L^p(Z)} + \|d\mu(u)\cdot\boldsymbol{v}_s\|_{L^p(Z)}\\
&\lesssim \|\mu(u)\|_{L^p(Z)} + \|d\mu(u)\cdot\boldsymbol{v}_s\|_{L^p(Z)}.
\end{aligned}$$

The bound of $\nabla_t^a(J\mathcal{X}_h)$ can be proved similarly. □

In the rest of this appendix, the values of $C(M,\Omega,Z)$ will be modified to satisfy various estimates without mentioning.

The gauged map $\boldsymbol{v}'$ is nearly holomorphic and one needs to estimate its failure.

Corollary A.7 *Under the hypothesis of Lemma* A.4*, one has*

$$\|\boldsymbol{v}_s' + J\boldsymbol{v}_t'\|_{L^p(Z)} \lesssim \|\mu(u)\|_{L^p(Z)} + \|d\mu(u)\cdot\boldsymbol{v}_s\|_{L^p(Z)} + \|d\mu(u)\cdot\boldsymbol{v}_t\|_{L^p(Z)}.$$

Proof Since $\boldsymbol{v}_s'$ and $\boldsymbol{v}_t'$ are contained in H_X, (A.6) implies that

$$\begin{aligned}
\boldsymbol{v}_s' + J\boldsymbol{v}_t' &= P_H[E_1(\boldsymbol{v}_s) + E_2(\nabla_s^a J\mathcal{X}_h) + JE_1(\boldsymbol{v}_t) + JE_2(\nabla_t^a J\mathcal{X}_h)]\\
&= P_H[(JE_1 - E_1 J)(\boldsymbol{v}_t) + E_2(\nabla_s^a J\mathcal{X}_h) + JE_2(\nabla_t^a J\mathcal{X}_h)].
\end{aligned}$$

Since $JE_1 - E_1J$ is bounded by a multiple of $|J\mathcal{X}_h|$, one has

$$\begin{aligned}\|\boldsymbol{v}_s' + J\boldsymbol{v}_t'\|_{L^p(K)} &\lesssim \|J\mathcal{X}_h\|_{L^p(K)}\|\boldsymbol{v}_t\|_{L^\infty(K)} + \|\nabla_s^a J\mathcal{X}_h\|_{L^p(K)} + \|\nabla_t^a J\mathcal{X}_h\|_{L^p(K)} \\ &\lesssim \|\mu(u)\|_{L^p(K)} + \|d\mu\cdot\boldsymbol{v}_s\|_{L^p(K)} + \|d\mu(u)\cdot\boldsymbol{v}_t\|_{L^p(K)}.\end{aligned}$$

This finishes the proof. □

We also need to estimate the difference between the gauge fields in $\boldsymbol{v}$ and $\boldsymbol{v}'$.

Lemma A.8 (*Recall that* $\eta' ds + \zeta' dt = a - a'$) *one has*

$$\|\eta'\|_{L^p(Z)} + \|\zeta'\|_{L^p(Z)} \le C(M,\Omega,Z)\epsilon^{\frac{2}{p}}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}].$$

Proof Since $\boldsymbol{v}_s'$ is contained in H_X, so $L_{u'}^*\boldsymbol{v}_s' = 0$. Apply $L_{u'}^*$ to (A.6), one obtains

$$L_{u'}^*\mathcal{X}_{\eta'} = L_{u'}^*E_1(\boldsymbol{v}_s) + L_{u'}^*E_2\nabla_s^a(J\mathcal{X}_h). \tag{A.8}$$

Since $\eta' \mapsto L_{u'}^*\mathcal{X}_{\eta'}$ is invertible, and E_1, E_2 are nearly the identity, one has

$$\begin{aligned}|\eta'| &\lesssim |L_{u'}^*E_1(\boldsymbol{v}_s)| + |L_{u'}^*E_2\nabla_s^a(J\mathcal{X}_h)| \\ &\lesssim |L_u^*\boldsymbol{v}_s| + |J\mathcal{X}_h||\boldsymbol{v}_s| + |L_u^*\nabla_s^a(J\mathcal{X}_h)| + |J\mathcal{X}_h||\nabla_s^a(J\mathcal{X}_h)| \lesssim |L_u^*\boldsymbol{v}_s| + |\mu(u)| + |\nabla_s^a J\mathcal{X}_h|.\end{aligned}$$

Therefore by Lemma A.4 and Lemma A.6, one has

$$\begin{aligned}\|\eta'\|_{L^p(Z)} &\lesssim \|d\mu(u)\cdot J\boldsymbol{v}_s\|_{L^p(Z)} + \|\mu(u)\|_{L^p(Z)} + \|\nabla_s^a(J\mathcal{X}_h)\|_{L^p(Z)} \\ &\lesssim \|d\mu(u)\cdot J\boldsymbol{v}_s\|_{L^p(K)} + \|\mu(u)\|_{L^p(Z)} + \|d\mu(u)\cdot\boldsymbol{v}_s\|_{L^p(Z)} \\ &\lesssim \epsilon^{\frac{2}{p}}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}].\end{aligned}$$

Similarly, one can derive the estimate for ζ'.

We also have to estimate the first order derivatives of η' and ζ'.

Lemma A.9 *One has*

$$\epsilon[\|\nabla_s^a\eta'\|_{L^p(Z)} + \|\nabla_s^a\zeta'\|_{L^p(Z)}] \le c(M,\Omega,Z)\epsilon^{\frac{2}{p}}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}].$$

Proof Apply $d\mu(u')$ to (A.6), one has

$$d\mu(u')\cdot E_1(\boldsymbol{v}_s) + d\mu(u')\cdot E_2(\nabla_s^a J\mathcal{X}_h) = 0.$$

Then applying ∇_s^a to the above equation, by Lemma A.1, one obtains

$$\begin{aligned}0 &= \nabla_s^a L_{u'}^* JE_1(\boldsymbol{v}_s) + \nabla_s^a L_{u'}^*(JE_2(\nabla_s^a J\mathcal{X}_h)) \\ &= L_{u'}^*[\nabla_s^a(JE_1(\boldsymbol{v}_s)) + \nabla_s^a JE_2(\nabla_s^a J\mathcal{X}_h)] + \rho(\partial_s u' + \mathcal{X}_\phi, JE_1(\boldsymbol{v}_s) + JE_2(\nabla_s^a J\mathcal{X}_h)) \\ &= L_{u'}^*[JE_1(\nabla_s^a\boldsymbol{v}_s) + \Phi_1(\boldsymbol{v}_s,\boldsymbol{v}_s) + \Phi_2(\boldsymbol{v}_s,\nabla_s^a J\mathcal{X}_h)] \\ &\quad + L_{u'}^*[JE_2(\nabla_s^a\nabla_s^a J\mathcal{X}_h) + \Psi_1(\boldsymbol{v}_s,\nabla_s^a J\mathcal{X}_h) + \Psi_2(\nabla_s^a J\mathcal{X}_h,\nabla_s^a J\mathcal{X}_h)] \\ &\quad + \rho(\partial_s u' + \mathcal{X}_\phi, JE_1(\boldsymbol{v}_s) + JE_2(\nabla_s^a J\mathcal{X}_h)).\end{aligned}$$

Here $\Phi_1, \Phi_2, \Psi_1, \Psi_2$ are tensors that are uniformly bounded, which come from derivatives of JE_1 and JE_2. Using the facts that E_2 is very close to the identity, that $|\boldsymbol{v}_s|$, $|\partial_s u' + \mathcal{X}_\phi|$, and $|\nabla_s^a J\mathcal{X}_h|$ are bounded, one has

$$\|\nabla_s^a \nabla_s^a J\mathcal{X}_h\|_{L^p(Z)} \lesssim \|\nabla_s^a J\mathcal{X}_h\|_{L^p(Z)} + \|\boldsymbol{v}_s\|_{L^p(Z)} + \|\nabla_s^a \boldsymbol{v}_s\|_{L^p(Z)}. \tag{A.9}$$

Then apply $\nabla_s^a L_{u'}^*$ to (A.6), one has

$$\nabla_s^a L_{u'}^* \mathcal{X}_{\eta'} = \nabla_s^a L_{u'}^* E_1(\boldsymbol{v}_s) + \nabla_s^a L_{u'}^* E_2 \nabla_s^a (J\mathcal{X}_h).$$

Using Lemma A.1 again, one has

$$L_{u'}^*[\nabla_s^a \mathcal{X}_{\eta'} - \nabla_s^a E_1(\boldsymbol{v}_s) - \nabla_s^a E_2 \nabla_s^a J\mathcal{X}_h] = \rho[\partial_s u' + \mathcal{X}_\phi, \mathcal{X}_{\eta'} - E_1(\boldsymbol{v}_s) - E_2 \nabla_s^a J\mathcal{X}_h]. \tag{A.10}$$

We know that $\nabla_s^a \mathcal{X}_{\eta'} = L_{u'}(\nabla_s^a \eta') + \nabla_{\partial_s u' + \mathcal{X}_\phi} \mathcal{X}_{\eta'}$ and $L_{u'}^* L_{u'}$ is an isomorphism. Therefore,

$$\begin{aligned}
\|\nabla_s^a \eta'\|_{L^p(Z)} &\lesssim \|L_{u'}^* \nabla_s^a \mathcal{X}_{\eta'}\|_{L^p(Z)} + \|\nabla_{\partial_s u' + \mathcal{X}_\phi} \mathcal{X}_{\eta'}\|_{L^p(Z)} \\
&\lesssim \|L_{u'}^* \nabla_s^a \mathcal{X}_{\eta'}\|_{L^p(Z)} + \|\partial_s u' + \mathcal{X}_\phi\|_{L^\infty} \|\eta'\|_{L^p(Z)} \\
&\lesssim \|\boldsymbol{v}_s\|_{L^p(Z)} + \|\nabla_s^a \boldsymbol{v}_s\|_{L^p(Z)} + \|J\mathcal{X}_h\|_{L^p(Z)} + \|\nabla_s^a J\mathcal{X}_h\| \\
&\quad + \|\nabla_s^a \nabla_s^a J\mathcal{X}_h\|_{L^p(Z)} + \|\eta'\|_{L^p(Z)} \\
&\lesssim \|\boldsymbol{v}_s\|_{L^p(Z)} + \|\nabla_s^a \boldsymbol{v}_s\|_{L^p(Z)} + \|J\mathcal{X}_h\|_{L^p(Z)} + \|\nabla_s^a J\mathcal{X}_h\|_{L^p(Z)} + \|\eta'\|_{L^p(Z)} \\
&\lesssim \epsilon^{\frac{2}{p}-1}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}].
\end{aligned}$$

Here in deriving the third inequality we used (A.10) and in deriving the fourth inequality we used (A.9). In deriving the last inequality, we used the fact that $|\boldsymbol{v}_s|$ is uniformly bounded, Lemma A.4, (A.5), Lemma A.6 and Lemma A.8. Similarly we can derive the estimate for $\nabla^a \zeta'$.

Below is another necessary estimate.

Lemma A.10 *We have*

$$\|\nabla^a \boldsymbol{v}_s'\|_{L^p(K)} + \|\nabla^a \boldsymbol{v}_t'\|_{L^p(K)} \lesssim \epsilon^{\frac{2}{p}-1}[\|\boldsymbol{v}_s\|_{L^2(\Omega)} + \epsilon^{-1}\|\mu(u)\|_{L^2(\Omega)}].$$

Proof We only prove one bound. Apply ∇_s^a to (A.6), one has

$$\nabla_s^a \boldsymbol{v}_s' = -\nabla_s^a \mathcal{X}_{\eta'} + E_1(\nabla_s^a \boldsymbol{v}_s) + E_2(\nabla_s^a \nabla_s^a J\mathcal{X}_h) + \Phi_1(\boldsymbol{v}_s, \boldsymbol{v}_s) + \Phi_2(\boldsymbol{v}_s, \nabla_s^a J\mathcal{X}_h),$$

where Φ_1, Φ_2 are K-invariant tensors that are uniformly bounded. This allows us to prove the desired estimate on $\nabla_s^a \boldsymbol{v}_s'$ from previous results. □

A.4 Convergence

Let $\Omega \subset \boldsymbol{H}$ be an open subset. Let ϵ_n be a sequence of positive numbers converging to zero. Let $z_n = s_n + \boldsymbol{i} t_n : \Omega \to \boldsymbol{H}$ be a sequence of smooth maps that are diffeomorphisms onto their images, that converging uniformly with all derivatives to the identity map. z_n then pulls back a sequence of complex structures j_n and a sequence of volume forms $ds_n \wedge dt_n$ on Ω. Consider a sequence of gauged maps $\boldsymbol{v}_n = (u_n, \phi_n, \psi_n)$ from Ω to X that solve the equation

$$\boldsymbol{v}_{n,s_n} + J\boldsymbol{v}_{n,t_n} = 0, \quad \kappa_n + \frac{1}{\epsilon_n^2}\mu(u_n) = 0. \tag{A.11}$$

Here $\kappa_n = \partial_{s_n}\psi_n - \partial_{t_n}\phi_n + [\phi_n, \psi_n]$. Let $\boldsymbol{v}_\infty = (u_\infty, \phi_\infty, \psi_\infty)$ be a gauged map from Ω to X with image contained in $\mu^{-1}(0)$ that projects down to a holomorphic map with respect to the standard complex structure on Ω. We would like to prove the following theorem.

Theorem A.11 *Suppose u_n converges to u_∞ uniformly on all compact subsets of Ω (a priori no condition on the convergence of the gauge fields). Then there exists $n_0 \geq 1$ such that for all $n \geq n_0$, we can gauge transform $\boldsymbol{v}_n$ to a sequence of vortices (which we still denoted by $\boldsymbol{v}_n$) such that, if we denote*

$$\boldsymbol{v}_n = \exp_{\boldsymbol{v}_\infty} \boldsymbol{\xi}_n, \quad \boldsymbol{\xi}_n = (\xi_n, \alpha_n),$$

then for any compact subset $Z \subset \Omega$,

$$\lim_{n\to\infty}[\|\alpha_n\|_{L^p(Z)} + \epsilon_n\|\nabla^{a_\infty}\alpha_n\|_{L^p(Z)} + \|\xi_n^H\|_{L^{1,p}(Z)} + \epsilon_n^{-1}\|\xi_n^G\|_{L^p(Z)} + \|\nabla^{a_\infty}\xi_n^G\|_{L^p(Z)}] = 0. \quad \text{(A.12)}$$

Here we only present the proof in the special case that $z_n = z$. In the general case the convergence $z_n \to z$ allows us to extend the proof.

To start, we first gauge transform $\boldsymbol{v}_n$ to satisfy the following pointwise gauge-fixing condition. Let $\boldsymbol{v}_n' = (u_n', a_n')$ be the projection of $\boldsymbol{v}_n$ onto $\mu^{-1}(0)$. Namely, there are $h_n : C_R \to \mathfrak{k}$ such that $u_n' = \exp_{u_n} J\mathcal{X}_{h_n}$ is contained in $\mu^{-1}(0)$ and a_n' is pulled back by u_n'. The gauge-fixing condition is that if we write $u_n' = \exp_{u_\infty} \xi_n''$, then

$$\xi_n'' \in u_\infty^* H_X.$$

This is the gauge that allows us to have the estimate in Theorem A.11. To prove it, we compare $\boldsymbol{v}_n$ with $\boldsymbol{v}_n'$ and compare $\boldsymbol{v}_n'$ with $\boldsymbol{v}_\infty$ separately.

Denote $\alpha_n' = a_n - a_n' = \eta_n' ds + \zeta_n' dt$. Then by Lemma A.8 and Lemma A.9, one has

$$\limsup_{i\to\infty}[\|\alpha_n'\|_{L^p(Z)} + \epsilon_n\|\nabla^{a_n}\alpha_n'\|_{L^p(Z)}] \leq c(\epsilon_n)^{\frac{2}{p}}[\|\boldsymbol{v}_{n,s}\|_{L^2(\Omega)} + \epsilon_n^{-1}\|\mu(u_n)\|_{L^2(\Omega)}]. \quad \text{(A.13)}$$

By the uniform bound on energy, the right hand side of (A.13) converges to zero. Similarly, by (A.4), (A.5) and Lemma A.6, one has

$$\epsilon_n^{-1}\|J\mathcal{X}_{h_n}\|_{L^p(Z)} + \|\nabla^{a_n} J\mathcal{X}_{h_n}\|_{L^p(\Omega)} \lesssim (\epsilon_n)^{\frac{2}{p}}. \quad \text{(A.14)}$$

Hence the distance between $\boldsymbol{v}_n$ and $\boldsymbol{v}_n'$ w.r.t. the norm in (A.12) (defined using ∇^{a_n} instead of ∇^{a_∞}) converges to zero.

Lemma A.12 *For any compact subset $Z \subset \Omega$, $\|d_{a_n'}\boldsymbol{v}_n'\|_{L^\infty(Z)}$ is uniformly bounded.*

Proof By the uniform convergence $\boldsymbol{v}_n \to \boldsymbol{v}_\infty$ we know that $\|d_{a_n}\boldsymbol{v}_n\|_{L^\infty(Z)}$ is uniformly bounded (otherwise there will be bubbling). (A.7) implies that $\|\nabla^{a_n} J\mathcal{X}_{h_n}\|_{L^\infty(Z)}$ is uniformly bounded. Then by applying Lemma A.2 for $\boldsymbol{v}_0 = \boldsymbol{v}_n$, $\boldsymbol{v} = \boldsymbol{v}_n'$, one obtains

$$\begin{aligned} -\mathcal{X}_{\eta_n'} &= \boldsymbol{v}_{n,s}' - E_1(u_n, -J\mathcal{X}_{h_n})\boldsymbol{v}_{n,s} - E_2(u_n, -J\mathcal{X}_{h_n})\nabla_s^{a_n} J\mathcal{X}_{h_n}; \\ -\mathcal{X}_{\zeta_n'} &= \boldsymbol{v}_{n,t}' - E_1(u_n, -J\mathcal{X}_{h_n})\boldsymbol{v}_{n,t} - E_2(u_n, -J\mathcal{X}_{h_n})\nabla_t^{a_n} J\mathcal{X}_{h_n}. \end{aligned}$$

Since $\boldsymbol{v}_{n,s}'$ and $\boldsymbol{v}_{n,t}'$ are in $H_X \oplus JK_X$, their sizes are controlled by those of $\boldsymbol{v}_{n,t}$ and $\nabla^{a_\infty} J\mathcal{X}_{h_n}$, both of which are finite.

Now we estimate the distance between $\boldsymbol{v}_n'$ and $\boldsymbol{v}_\infty$. Recall $u_n' = \exp_{u_\infty} \xi_n''$. Denote

$$\alpha_n'' = \eta_n'' ds + \zeta_n'' dt = a_n' - a_\infty.$$

Since $\boldsymbol{v}_n$ converges to $\boldsymbol{v}_\infty$ uniformly on compact subsets, so does $\boldsymbol{v}_n'$. First, we have the decay of $W^{1,p}$-norm of ξ_n'' shown as follows.

Lemma A.13 *One has*

$$\lim_{i\to\infty} [\|\xi_n''\|_{L^p(Z)} + \|\nabla^{a_\infty} \xi_n''\|_{L^p(Z)}] = 0. \tag{A.15}$$

Proof $\|\xi_n''\|_{L^p(Z)} \to 0$ follows from the convergence $\boldsymbol{v}_n$ towards $\boldsymbol{v}_\infty$. The estimate for $\|\nabla^{a_\infty} \xi_n''\|_{L^p(Z)}$ basically follows from Corollary A.7 and elliptic estimate for $\overline{\partial}$ operator. More precisely, by Lemma A.2, one has

$$\boldsymbol{v}_{n,s}' = E_1 \boldsymbol{v}_{\infty,s} + E_2 \nabla_s^{a_\infty} \xi_n'' + \mathcal{X}_{\eta_n''}, \quad \boldsymbol{v}_{n,t}' = E_1 \boldsymbol{v}_{\infty,t} + E_2 \nabla_t^{a_\infty} \xi_n'' + \mathcal{X}_{\zeta_n''}.$$

Hence

$$\boldsymbol{v}_{n,s}' + J\boldsymbol{v}_{n,t}' = E_1 \boldsymbol{v}_{\infty,s} + JE_1 \boldsymbol{v}_{\infty,t} + E_2 \nabla_s^{a_\infty} \xi_n'' + JE_2 \nabla_t^{a_\infty} \xi_n'' + \mathcal{X}_{\eta_n''} + J\mathcal{X}_{\zeta_n''}.$$

Notice that $\boldsymbol{v}_{n,s}'$ and $\boldsymbol{v}_{n,t}'$ are both in H_X. Then projecting on to H_X, one obtains[5)]

$$\begin{aligned} \boldsymbol{v}_{n,s}' + J\boldsymbol{v}_{n,t}' &= P_H E_1 \boldsymbol{v}_{\infty,s} + P_H JE_1 \boldsymbol{v}_{\infty,t} + P_H E_2 \nabla_s^{a_\infty} \xi_n'' + P_H JE_2 \nabla_t^{a_\infty} \xi_n'' \\ &= P_H (JE_1 - E_1 J) \boldsymbol{v}_{\infty,t} + D\xi_n''. \end{aligned}$$

Here the first term above is bounded pointwise by a multiple of $|\xi_n''||\boldsymbol{v}_{\infty,t}|$; D is a Cauchy–Riemann type operator whose zero order term is uniformly bounded. Therefore, by the elliptic estimate for $\overline{\partial}$-operators, using Corollary A.7 and Lemma A.4, for certain precompact open subset $Z' \subset \Omega$ containing Z, one has

$$\begin{aligned} \limsup_{n\to\infty} \|\nabla^{a_\infty} \xi_n''\|_{L^p(Z)} &\lesssim \limsup_{n\to\infty} \|\xi_n''\|_{L^p(Z')} + \limsup_{n\to\infty} \|D\xi_n''\|_{L^p(Z')} \\ &\lesssim \limsup_{n\to\infty} \|\boldsymbol{v}_{n,s}' + J\boldsymbol{v}_{n,t}'\|_{L^p(Z')} + \limsup_{n\to\infty} \||\xi_n''||\boldsymbol{v}_{\infty,t}|\|_{L^p(Z')} \\ &\lesssim \limsup_{n\to\infty} (\|\mu(u_n)\|_{L^p(Z')} + \|d\mu(u_n) \cdot \boldsymbol{v}_{n,s}\|_{L^p(Z')} + \|d\mu(u_n) \cdot \boldsymbol{v}_{n,t}\|_{L^p(Z')}) = 0. \end{aligned}$$

This finishes the proof of (A.15).

Lemma A.14 *One has*

$$\lim_{n\to\infty} [\|\alpha_n''\|_{L^p(Z)} + \epsilon_n \|\nabla^{a_\infty} \alpha_n''\|_{L^p(Z)}] = 0. \tag{A.16}$$

Proof Since a_n' and a_∞ are pulled back from the canonical connection by u_n' and u_∞, and $u_n' = \exp_{u_\infty} \xi_n''$, also using Lemma A.13, one has

$$\lim_{n\to\infty} \|\alpha_n''\|_{L^p(Z)} \lesssim \lim_{n\to\infty} [\|\xi_n''\|_{L^p(Z)} + \|\nabla^{a_\infty} \xi_n''\|_{L^p(Z)}] = 0. \tag{A.17}$$

5) If it is not in the special case that $z_n = z$, then there will be an extra term below. But that term can still be controlled.

For the derivatives, we only estimate $\|\nabla_t^{a_\infty}\eta_n''\|_{L^p(Z)}$. The case for other components can be done similarly, by utilizing Lemma A.3. Apply Lemma A.3 to $\boldsymbol{v}_0 = \boldsymbol{v}_\infty$, $\boldsymbol{v} = \boldsymbol{v}_n'$. Then one has

$$\begin{aligned}\|\nabla_t^{a_\infty}\eta_n''\|_{L^p(Z)} &\lesssim \|a_\infty - a_n'\|_{L^\infty(Z)}\|\eta_n''\|_{L^p(Z)} + \|\nabla_t^{a_n'}\eta_n''\| \\ &\le \|\alpha_n''\|_{L^\infty(Z)}\|\eta_n''\|_{L^p(Z)} + \|L_{u_\infty}^*\mathcal{X}_{\nabla_t^{a_n'}\eta_n''}\|_{L^p(Z)} \\ &\lesssim \|\alpha_n''\|_{L^\infty(Z)}\|\eta_n''\|_{L^p(Z)} + \|\nabla_t^{a_n'}\boldsymbol{v}_{n,s}'\|_{L^p(Z)} + \|\zeta_n''\|_{L^\infty(Z)}\|\eta_n''\|_{L^p(Z)} \\ &\quad + \|\alpha_n''\|_{L^\infty(Z)}\|d_{a_\infty}\boldsymbol{v}_\infty\|_{L^p(Z)} + \|d_{a_\infty}u_\infty\|_{L^\infty(Z)}\|d_{a_\infty}u_\infty\|_{L^p(Z)} \\ &\quad + \|d_{a_\infty}u_\infty\|_{L^\infty(Z)}\|\nabla^{a_\infty}\xi_n''\|_{L^p(Z)} + \|\nabla_t^{a_\infty}\xi_n''\|_{L^\infty}\|\nabla_s^{a_\infty}\xi_n''\|_{L^p(Z)} \\ &\quad + \|\nabla^{a_\infty}d_{a_\infty}\boldsymbol{v}_\infty\|_{L^p(Z)} + \|L_{u_\infty}^*\nabla_t^{a_\infty}\nabla_s^{a_\infty}\xi_n\|_{L^p(Z)}.\end{aligned}$$

We would like to show that the right hand side converges to zero after multiplying ϵ_n. Indeed, Lemma A.12 shows that $\|d_{a_n'}\boldsymbol{v}_n'\|_{L^\infty(Z)}$ is uniformly bounded (u.d. for short). Recall that u_n' projects to a map $\bar{u}_n' : \Omega \to \bar{X}$, whose derivative is the projection of $d_{a_n}'\boldsymbol{v}_n'$. Hence $\|\nabla\bar{u}_n'\|_{L^\infty(Z)}$ is u.d.. Since $u_n' = \exp_{u_\infty}\xi_n''$ with ξ_n'' being in the horizontal distribution, we know that both $\|\nabla^{a_\infty}\xi_n''\|_{L^\infty(Z)}$ and $\|\nabla u_n'\|_{L^\infty(Z)}$ is u.d.. Hence it implies that the gauge fields of $\boldsymbol{v}_n'$ are u.d.. Hence all L^∞-norms appeared in the last long inequality are u.d.. It remains to bound all the L^p-norms that depend on n in the same inequality.

(a) By Lemma A.10 $\|\nabla_t^{a_n'}\boldsymbol{v}_{n,s}'\|_{L^p(Z)} \lesssim (\epsilon_n)^{2/p-1}$.

(b) The bound on $\|\eta_n''\|_{L^p(Z)}$ has been given in (A.17). The bound on $\|\nabla^{a_\infty}\xi_n''\|_{L^p(Z)}$ follows from the bound on its L^∞-norm.

(c) Since $L_{u_\infty}^*\xi_n'' = 0$, by Lemma A.1, we have

$$L_{u_\infty}^*\nabla_t^{a_\infty}\nabla_s^{a_\infty}\xi_n'' = \nabla_t^{a_\infty}(L_{u_\infty}^*\nabla_s^{a_\infty}\xi_n'') - \rho(\boldsymbol{v}_{\infty,t}, \nabla_s^{a_\infty}\xi_n'') = -\nabla_t^{a_\infty}\rho(\boldsymbol{v}_{\infty,s}, \xi_n'') - \rho(\boldsymbol{v}_{\infty,t}, \nabla_s^{a_\infty}\xi_n'').$$

A bound of this term follows easily from the bounds achieved previously.

In summary, $\lim_{i\to\infty}\epsilon_n\|\nabla_t^{a_\infty}\eta_n''\|_{L^p(Z)} = 0$. This finishes the proof of this lemma. □

Now we consider the distance between $\boldsymbol{v}_n$ and $\boldsymbol{v}_\infty$. By (A.13) and Lemma A.14, we know that $\|a_n - a_\infty\|_{L^p(Z)}$ converges to zero. Hence the $W^{1,p}$-norms defined by ∇^{a_n} and ∇^{a_∞} are equivalent. Hence (A.13) and Lemma A.14 imply that

$$\lim_{n\to\infty}[\|a_n - a_\infty\|_{L^p(Z)} + \epsilon_n\|\nabla^{a_\infty}(a_n - a_\infty)\|_{L^p(Z)}] = 0. \tag{A.18}$$

On the other hand, we write

$$u_n = \exp_{u_\infty}\xi_n = -\exp_{u_n'}J\mathcal{X}_{h_n} = -\exp_{\exp_{u_\infty}\xi_n''}J\mathcal{X}_{h_n}.$$

Because of the nonlinearity of the exponential map, $\xi_n \neq \xi_n'' - J\mathcal{X}_{h_n}$. However, we can define a smooth family of function $\Phi_z : H_{X,u_\infty(z)} \oplus \mathfrak{k} \to T_{u_\infty(z)}X$ by

$$\xi_n(z) - \xi_n''(z) + J\mathcal{X}_{h_n}(u_\infty(z)) = \Phi_z(\xi_n''(z), h_n(z)).$$

Moreover, we know that

$$\begin{aligned}&|\Phi_z(\xi_n''(z), h_n(z))| \lesssim |\xi_n''(z)||h_n(z)|, \\ &|\nabla^{a_\infty}\Phi_z(\xi_n''(z), h_n(z))| \lesssim |d_{a_\infty}u_\infty||\xi_n''||h_n(z)| + |\nabla^{a_\infty}\xi_n''(z)||h_n(z)| + |\xi_n''||\nabla^{a_\infty}h_n(z)|.\end{aligned}$$

Roughly speaking, the discrepancy Φ_z is "small". This allows the estimate that

$$\lim_{i\to\infty} \|P_H\xi_n\|_{\tilde{L}^{1,p}_h(Z)} = 0, \quad \lim_{i\to\infty} [\epsilon_n^{-1}\|P_G\xi_n\|_{\tilde{L}^p(Z)} + \|\nabla^{a_\infty} P_G\xi_n\|_{\tilde{L}^p(Z)}] = 0.$$

This finishes the proof of Theorem A.11.

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